J. Korean Math. Soc. **51** (2014), No. 3, pp. 509–525
http://dx.doi.org/10.4134/JKMS.2014.51.3.509

w-INJECTIVE MODULES AND w-SEMI-HEREDITARY RINGS

Fanggui Wang and Hwankoo Kim

Abstract. Let R be a commutative ring with identity. An R-module M is said to be w-projective if $\mathrm{Ext}^1_R(M,N)$ is GV-torsion for any torsion-free w-module N. In this paper, we define a ring R to be w-semi-hereditary if every finite type ideal of R is w-projective. To characterize w-semi-hereditary rings, we introduce the concept of w-injective modules and study some basic properties of w-injective modules. Using these concepts, we show that R is w-semi-hereditary if and only if the total quotient ring $T(R)$ of R is a von Neumann regular ring and $R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R. It is also shown that a connected ring R is w-semi-hereditary if and only if R is a Prüfer v-multiplication domain.

1. Introduction

Throughout, R denotes a commutative ring with identity 1 and $E(M)$ denotes the injective hull (or envelope) of an R-module M. And let us regard that the v-, t- and w-operation are well-known star-operations on domains. For unexplained terminologies and notations, we refer to [3, 14, 15].

Prüfer v-multiplication domains (PVMD for short) have received a good deal of attention in much literature. A domain R is called a *PVMD* if every nonzero finitely generated ideal I is t-invertible, that is, there is a fractional ideal B of R such that $(IB)_t = R$, equivalently, $(IB)_w = R$. A natural question arises as follows: How do we extend the study on PVMDs to commutative rings with zero divisors. There are at least two methods for doing this. One is to replace the quotient field of a domain R with the total quotient ring $T(R)$ and to define $A^{-1} = \{x \in T(R) \,|\, xA \subseteq R\}$ for an R-submodule A of $T(R)$. In this case, we must consider regular ideals of R and we get the notion of so-called Prüfer v-multiplication rings (PVMRs for short) for which every finitely generated regular ideal of R is t-invertible (see [13]). The other is to replace the quotient field of a domain R with the ring $Q_0(R)$ of so-called finite fractions of R and to define $A^{-1} = \{x \in Q_0(R) \mid xA \subseteq R\}$ for an R-submodule A of $Q_0(R)$, where

Received August 28, 2013; Revised February 24, 2014.
2010 *Mathematics Subject Classification.* 13A15, 13C11, 13D07.
Key words and phrases. w-projective module, w-flat module, w-injective module, finite type, w-semi-hereditary ring.

$Q_0(R)$ is the subring of $T(R[X])$ consisting of elements $\frac{\sum_{i=0}^{n} a_i X^i}{\sum_{i=0}^{n} b_i X^i} \in T(R[X])$ with $a_i b_j = a_j b_i$ for all i, j. Recall an ideal I of R is called *semi-regular* if there is a finitely generated subideal B of I with $\operatorname{ann}(I) = 0$. In the second case, we must consider semi-regular ideals of R and we get the notion of Q_0-Prüfer v-multiplication rings (Q_0-PVMRs for short) for which every finitely generated semi-regular ideal of R is t-invertible (see [11]).

Since the concept of w-modules appeared in [22], we note the method of the w-operation on domains is effective for commutative rings with zero divisors. Let J be an ideal of R. Following [22], J is called a *Glaz-Vasconcelos ideal* (a GV-ideal for short) if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \operatorname{Hom}_R(J, R)$ is an isomorphism (also see [5]). Note that the set $\operatorname{GV}(R)$ of GV-ideals of R is a multiplicative system of ideals of R. Let M be an R-module. Define

$$\operatorname{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \operatorname{GV}(R)\}.$$

Thus $\operatorname{tor}_{\mathrm{GV}}(M)$ is a submodule of M. Now M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\operatorname{tor}_{\mathrm{GV}}(M) = M$ (resp., $\operatorname{tor}_{\mathrm{GV}}(M) = 0$). An R-module M is GV-torsion if and only if $M_{\mathfrak{m}} = 0$ for any maximal w-ideal $\mathfrak{m}$ of R (see [21]). A GV-torsion-free module M is called a *w-module* if $\operatorname{Ext}_R^1(R/J, M) = 0$ for any $J \in \operatorname{GV}(R)$. Then projective modules and reflexive modules are w-modules. In a recent paper [23], it was shown that flat modules are w-modules. For any GV-torsion-free module M,

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \operatorname{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M. It is clear that a GV-torsion-free module M is a w-module if and only if $M_w = M$. Note that in the language of torsion theories, the w-envelope for modules coincides with the $\operatorname{tor}_{\mathrm{GV}}$-injective envelope with respect to the torsion theory whose torsion modules are the GV-torsion modules and the torsion-free modules are the GV-torsion-free modules. Thus the w-operation theory is a bridge closely connecting torsion theory with multiplicative ideal theory.

The notions of w-projective modules and w-flat modules appeared first in [16] when R is a domain. In [20], the notion of w-projective modules was extended to arbitrary commutative rings. Recall that a ring R is called *semi-hereditary* if every finitely generated ideal of R is projective. Endo [2] proved that a ring R is semi-hereditary if and only if the total quotient ring of R is a von Neumann regular ring and $R_{\mathfrak{p}}$ is a valuation domain for any maximal ideal $\mathfrak{p}$ of R. We also define a ring R to be *w-semi-hereditary* if every finite type ideal of R is w-projective. It follows from [20, Theorem 4.13] that a w-semi-hereditary ring is certainly a Q_0-PVMR, and therefore, a PVMR.

In this paper, we introduce the concept of w-injective modules and study their properties. As in the classical homological algebra, we also give with the help of the notions above a systematical characterization of w-semi-hereditary rings.

2. Preliminaries

Let M and N be R-modules and let $f : M \to N$ be a homomorphism. Following [19], f is called a *w-monomorphism* (resp., *w-epimorphism, w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) for any maximal w-ideal $\mathfrak{m}$ of R. A sequence $A \to B \to C$ of modules and homomorphisms is called *w-exact* if the sequence $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact for any maximal w-ideal $\mathfrak{m}$ of R. In [16], a finite type module M means a torsion-free module with $M_w = B_w$ for some finitely generated submodule B of M. In [22] the notion of finite type modules was enlarged to GV-torsion-free modules. In [19] the notion of finite type modules has been redefined. An R-module M is said to be *of finite type* if there exists a finitely generated free R-module F and a w-epimorphism $g : F \to M$. Similarly, an R-module M is said to be *of finitely presented type* if there exists a w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_1 and F_0 are finitely generated free.

An R-module M is called a *w-flat* module if the induced map $1 \otimes f : M \otimes_R A \to M \otimes_R B$ is a w-monomorphism for any w-monomorphism $f : A \to B$. Certainly, a GV-torsion modules is w-flat.

For easy reference, we list some of the results on w-flat modules which will be used frequently.

Theorem 2.1 ([8, Theorem 3.3])**.** *The following statements are equivalent for a module M:*

(1) *M is w-flat.*
(2) *For any w-exact sequence $0 \to A \to B \to C \to 0$, the sequence*

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is w-exact.
(3) *$M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for any maximal w-ideal $\mathfrak{m}$ of R.*
(4) *$\mathrm{Tor}_1^R(M, N)$ is a GV-torsion module for any R-module N.*
(5) *$\mathrm{Tor}_n^R(M, N)$ is a GV-torsion module for any R-module N and any $n \geqslant 1$.*
(6) *The natural homomorphism $M \otimes_R I \to IM$ is a w-isomorphism for any ideal I of R.*
(7) *The natural homomorphism $M \otimes_R I \to IM$ is a w-isomorphism for any finite type ideal I of R.*
(8) *The natural homomorphism $M \otimes_R I \to M$ is a w-monomorphism for any finite type ideal I of R.*
(9) *The natural homomorphism $M \otimes_R I \to M$ is a w-monomorphism for any ideal I of R.*

Remark. The notion of w-flat modules appeared first in [16] in which a torsion-free module M over a domain R is called w-flat if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for any maximal w-ideal of R. From Theorem 2.1, we see that this notion has been extended. For example, let R be a domain and let J be a GV-ideal of R such that $J \neq R$. Thus R/J is GV-torsion, and therefore is a w-flat module, but not torsion-free.

Proposition 2.2 ([8, Proposition 3.4]). *Let $0 \to A \to F \to M \to 0$ be w-exact, where F is a* GV*-torsion-free w-flat module and A is a submodule of F. Then the following statements are equivalent*:

(1) *M is w-flat.*
(2) *$A_w \cap (IF)_w = (IA)_w$ for any ideal I of R.*
(3) *$A_w \cap (IF)_w = (IA)_w$ for any finitely generated ideal I of R.*

Proposition 2.3 ([8, Proposition 3.9]). *Let M be an R-module and let $\{A_i \mid i \in \Gamma\}$ be a direct system of w-flat submodules of M over a directed index set Γ. Then $\varinjlim A_i$ is w-flat.*

Let M be an R-module and set $L(M) = (M/\mathrm{tor}_{\mathrm{GV}}(M))_w$. Recall from [20] that M is called *w-projective* if $\mathrm{Ext}_R^1(L(M), N)$ is GV-torsion for every torsion-free w-module N. When M is of finite type, we have that M is w-projective if and only if $\mathrm{Ext}_R^1(M, N)$ is GV-torsion for every torsion-free w-module N (see [20, Theorem 2.16]).

Proposition 2.4. *Every w-projective module is w-flat.*

Proof. This follows from [20, Theorem 2.5] and Theorem 2.1. □

We record some results on w-projective modules for subsequent usage.

Lemma 2.5 ([20, Proposition 2.3]). *Let M and M' be R-modules and let $f : M \to M'$ be a w-isomorphism. Then M is w-projective if and only if M' is w-projective.*

Lemma 2.6 ([20, Theorem 2.18]). *Every w-projective module of finite type is of finitely presented type.*

Lemma 2.7 ([20, Proposition 2.17]). *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a w-exact sequence. If A and C are w-projective of finite type, then B is w-projective of finite type.*

Proposition 2.8 ([20, Theorem 2.7]). *Let M be an R-module of finitely presented type. Then M is w-projective if and only if $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$ for any maximal w-ideal $\mathfrak{m}$ of R.*

3. w-injective modules

In this section, we introduce the concept of w-injective modules and study their properties.

Definition 3.1. An R-module E is said to be *w-injective* if

$$0 \to \mathrm{Hom}_R(C, L(E)) \to \mathrm{Hom}_R(B, L(E)) \to \mathrm{Hom}_R(A, L(E)) \to 0$$

is w-exact for any w-exact sequence $0 \to A \to B \to C \to 0$.

Example 3.2. Certainly, if $f : M \to N$ is a w-isomorphism, then M is w-injective if and only if N is w-injective. In particular, GV-torsion modules are w-injective. Therefore, a w-injective module is not necessarily an injective module.

In the following, we give characterizations of w-injective modules, which are similar to those of injective modules.

Theorem 3.3. *The following statements are equivalent for a w-module E.*

(1) *E is w-injective.*

(2) $0 \to \mathrm{Hom}_R(C,E) \to \mathrm{Hom}_R(B,E) \to \mathrm{Hom}_R(A,E) \to 0$ *is w-exact for any exact sequence* $0 \to A \to B \to C \to 0$.

(3) $\mathrm{Ext}^1_R(M,E)$ *is* GV*-torsion for any module M.*

(4) $\mathrm{Ext}^n_R(M,E)$ *is* GV*-torsion for any module M and any integer $n \geqslant 1$.*

Proof. (1)⇒(2). Since E is a w-module, we have $L(E) = E$.

(2)⇒(3). Let $0 \to A \to F \to M \to 0$ be exact, where F is free. This follows by comparing the w-exact sequence $0 \to \mathrm{Hom}_R(M,E) \to \mathrm{Hom}_R(F,E) \to \mathrm{Hom}_R(A,E) \to 0$ with the exact sequence $0 \to \mathrm{Hom}_R(M,E) \to \mathrm{Hom}_R(F,E) \to \mathrm{Hom}_R(A,E) \to \mathrm{Ext}^1_R(M,E) \to 0$.

(3)⇒(1). Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be w-exact. Set $C_1 = \mathrm{Im}(g)$ and $C_2 = C/C_1$. Then $0 \to C_1 \to C \to C_2 \to 0$ is exact and C_2 is GV-torsion. Since

$$0 \to \mathrm{Hom}_R(C_2,E) \to \mathrm{Hom}_R(C,E) \to \mathrm{Hom}_R(C_1,E) \to \mathrm{Ext}^1_R(C_2,E)$$

is exact and $\mathrm{Hom}_R(C_2,E) = \mathrm{Ext}^1_R(C_2,E) = 0$, we have $\mathrm{Hom}_R(C_1,E) \cong \mathrm{Hom}_R(C,E)$.

Set $A_1 = \ker(f)$ and $B_1 = \mathrm{Im}(f)$. Then A_1 is GV-torsion and $0 \to A_1 \to A \to B_1 \to 0$ is exact. By the same argument, we have $\mathrm{Hom}_R(B_1,E) \cong \mathrm{Hom}_R(A,E)$.

Set $B_2 = \ker(g)$. Then $0 \to B_2 \to B \to C_1 \to 0$ is exact. Hence $0 \to \mathrm{Hom}_R(C_1,E) \to \mathrm{Hom}_R(B,E) \to \mathrm{Hom}_R(B_2,E) \to \mathrm{Ext}^1_R(C_1,E)$ is exact. Note that $(B_1+B_2)/B_1$ and $(B_1+B_2)/B_2$ are GV-torsion. Thus we have $\mathrm{Hom}_R(B_1,E) = \mathrm{Hom}_R(B_1+B_2,E) = \mathrm{Hom}_R(B_2,E)$. Hence $0 \to \mathrm{Hom}_R(C,E) \to \mathrm{Hom}_R(B,E) \to \mathrm{Hom}_R(A,E) \to 0$ is w-exact.

(3)⇒(4). Let $n > 1$ and let $0 \to A \to F \to M \to 0$ be exact, where F is free. Then $\mathrm{Ext}^n_R(M,E) \cong \mathrm{Ext}^{n-1}_R(A,E)$. Hence $\mathrm{Ext}^n_R(M,E)$ is GV-torsion by induction.

(4)⇒(3). This is trivial. □

Corollary 3.4. *A module E is w-injective if and only if* $\mathrm{Ext}^1_R(M,L(E))$ *is GV-torsion for any module M; if and only if* $\mathrm{Ext}^n_R(M,L(E))$ *is GV-torsion for any module M and for all $n \geqslant 1$.*

Corollary 3.5. *Let E be a* GV*-torsion-free injective module. Then E is a w-injective w-module.*

In [21, Theorem 1.3(1)], it is shown that an R-module N is GV-torsion if and only if $\mathrm{Hom}_R(N,E)=0$ for any GV-torsion-free module E. (Note that this result is well known in torsion theory.) It is also known that for a hereditary torsion theory τ, an R-module N is τ-torsion if and only if $\mathrm{Hom}_R(N,E(M))=0$ for any τ-torsion-free module M [6, Proposition 1.2]. The following result is a variant of these results.

Theorem 3.6. *An R-module N is* GV*-torsion if and only if* $\mathrm{Hom}_R(N,E)=0$ *for any w-injective w-module E.*

Proof. Certainly, if E is a w-injective w-module and N is GV-torsion, then $\mathrm{Hom}_R(N,E)=0$. Conversely, set $T=\mathrm{tor}_{\mathrm{GV}}(N)$ and $C=N/T$. Then C is GV-torsion-free. By [19, Proposition 1.1], $E=E(C)$ is also GV-torsion-free. Hence E is a w-injective w-module E by Corollary 3.5. Therefore, $\mathrm{Hom}_R(N,E)=0$ by hypothesis. Since $0\to\mathrm{Hom}_R(C,E)\to\mathrm{Hom}_R(N,E)$ is exact, we have that $\mathrm{Hom}_R(C,E)=0$, and hence the inclusion map $C\hookrightarrow E$ is the zero homomorphism. So $C=0$, and hence N is GV-torsion. □

It is well known that an R-module E is injective if and only if $\mathrm{Hom}_R(-,E)$ is an exact functor. The corresponding result for w-injective modules is the following:

Theorem 3.7. *A sequence $0\to A\xrightarrow{f}B$ is w-exact if and only if, for any w-injective w-module E, the sequence* $\mathrm{Hom}_R(B,E)\to\mathrm{Hom}_R(A,E)\to 0$ *is w-exact.*

Proof. It is sufficient to show the "if" part. Set $A_1=\ker(f)$, $B_1=\mathrm{Im}(f)$ and $C=\mathrm{cok}(f)$. Then $0\to A_1\to A\to B_1\to 0$ and $0\to B_1\to B\to C\to 0$ are exact. Hence the sequences $\mathrm{Hom}_R(B_1,E)\to\mathrm{Hom}_R(A,E)\to\mathrm{Hom}_R(A_1,E)\to 0$ and $\mathrm{Hom}_R(B,E)\to\mathrm{Hom}_R(B_1,E)\to 0$ are w-exact. Consider the following commutative diagram with w-exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(B,E) & \longrightarrow & \mathrm{Hom}_R(A,E) & \longrightarrow & 0 & \longrightarrow & 0\\
\downarrow & & \| & & \downarrow & & \\
\mathrm{Hom}_R(B_1,E) & \longrightarrow & \mathrm{Hom}_R(A,E) & \longrightarrow & \mathrm{Hom}_R(A_1,E) & \longrightarrow & 0
\end{array}$$

Then $\mathrm{Hom}_R(A_1,E)$ is a GV-torsion module by w-Five Lemma (see [19, Lemma 1.1]). Now we show that A_1 is GV-torsion. Take $\overline{A_1}=A_1/\mathrm{tor}_{\mathrm{GV}}(A_1)$ and $E=E(\overline{A_1})$. Then

$$0\to\mathrm{Hom}(\overline{A_1},E)\to\mathrm{Hom}(A_1,E)\to\mathrm{Hom}(\mathrm{tor}_{\mathrm{GV}}(A_1),E)$$

is exact. By Theorem 3.6, $\mathrm{Hom}(\mathrm{tor}_{\mathrm{GV}}(A_1),E)=0$. Since $\mathrm{Hom}(A_1,E)$ is GV-torsion, $\mathrm{Hom}(\overline{A_1},E)$ is GV-torsion. In particular, the canonical injection $i:\overline{A_1}\to E$ is a GV-torsion element, so $\overline{A_1}=0$. Hence A_1 is GV-torsion. Therefore $0\to A\to B$ is w-exact. □

The Injective Production Lemma states that if M is a flat R-module and N is an injective R-module, then $\mathrm{Hom}_R(M, N)$ is injective. The following is the w-theoretic analogue of this result.

Theorem 3.8. *Let M be w-flat and let E be a w-injective w-module. Then $\mathrm{Hom}_R(M, E)$ is w-injective.*

Proof. Let $0 \to A \to B \to C \to 0$ be w-exact. Then $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is w-exact since M is w-flat. Because E is a w-injective w-modules, we have that

$$0 \to \mathrm{Hom}_R(M \otimes_R C, E) \to \mathrm{Hom}_R(M \otimes_R B, E) \to \mathrm{Hom}_R(M \otimes_R A, E) \to 0$$

is w-exact by Theorem 3.3. Note that $\mathrm{Hom}_R(M, E)$ is also a w-module. By the Adjoint Isomorphism Theorem, we have that $\mathrm{Hom}_R(M, E)$ is w-injective. □

We say that an R-module M is *divisible* if $M = sM$ for all non-zero-divisors s of R.

Proposition 3.9. *Let E be a w-injective w-module. Then E is divisible.*

Proof. Let s be a non-zero-divisor of R. Then $\mathrm{Ext}^1_R(R/(s), E)$ is GV-torsion by Theorem 3.3. Since s is a non-zero-divisor, sE is also a w-module. By [14] and [22, Theorem 2.7], $\mathrm{Ext}^1_R(R/(s), E) \cong E/sE$ is GV-torsion-free, which implies $E/sE = 0$, that is, $E = sE$. Hence E is divisible. □

In [1], it is shown that a domain R is a Krull domain if and only if every divisible w-module is injective. Hence we have the following:

Corollary 3.10. *If R is a Krull domain, then every w-injective w-module is injective.*

By combining Corollary 3.10 with Corollary 3.5, one sees readily that over Krull domains the class of all w-injective w-modules and the class of all GV-torsion-free injective modules are identical.

Let A, B and C be R-modules. Consider the natural homomorphism

$$\eta : A \otimes_R \mathrm{Hom}_R(B, C) \to \mathrm{Hom}_R(\mathrm{Hom}_R(A, B), C),$$

by $\eta(a \otimes f)(g) = f(g(a))$ for $a \in A$, $f \in \mathrm{Hom}_R(B, C)$ and $g \in \mathrm{Hom}_R(A, B)$.

Lemma 3.11. *Let A be finitely generated.*

(1) *If A is projective, then η is an isomorphism.*
(2) *If C is a w-injective w-module, then η is a w-epimorphism.*

Proof. (1) This is well known.

(2) Let $g : F \to A \to 0$ be exact, where F is finitely generated free. Then $0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(F, B)$ is exact. As C is a w-injective w-module, we

have the following commutative diagram with w-exact rows:

$$\begin{array}{ccccc} F\otimes_R \operatorname{Hom}_T(B,C) & \longrightarrow & A\otimes_R \operatorname{Hom}_T(B,C) & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow\eta_A & & \\ \operatorname{Hom}_T(\operatorname{Hom}_R(F,B),C) & \longrightarrow & \operatorname{Hom}_T(\operatorname{Hom}_R(A,B),C) & \longrightarrow & 0 \end{array}$$

Hence η_A is a w-epimorphism by [19, Lemma 1.1]. □

Lemma 3.12. *Let S be the set of all non-zero-divisors of R. Suppose M is a finitely generated torsion-free R-module such that M_S is a projective $T(R)$-module.*

(1) *There is a finitely generated free R-module F such that $M \subseteq F$ and $(F/M)_S$ is a projective $T(R)$-module.*

(2) *If M is w-flat, N is a divisible module, and E is a w-injective w-module, then*

$$\eta : M\otimes_R \operatorname{Hom}_R(N,E) \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,N),E)$$

is a w-isomorphism. Moreover, $\operatorname{Ext}_R^1(F/M,N)$ is GV*-torsion.*

Proof. (1) Since M_S is a projective $T(R)$-module, M_S is a summand of a finitely generated free $T(R)$-module G, that is, $G = M_S \oplus N$ for some $T(R)$-module N. Let $x_1,\dots,x_n$ be an $T(R)$-basis of G. Set $F = Rx_1+\cdots+Rx_n$. Then F is a free R-module and $F_S = G$. Since M is finitely generated, there is $s\in S$ such that $sM\subseteq F$. Since M is torsion-free, $M \to sM \subseteq F$ is a monomorphism and $(F/M)_S \cong N$ is a projective $T(R)$-module.

(2) By (1), we have an exact sequence $0\to M\to F\to F/M\to 0$, where F is finitely generated free and $(F/M)_S$ is a projective $T(R)$-module. Hence

$$0\to \operatorname{Hom}_R(F/M,N)\to \operatorname{Hom}_R(F,N)\to \operatorname{Hom}_R(M,N)\to \operatorname{Ext}_R^1(F/M,N)\to 0$$

is exact. Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & \operatorname{Tor}_1^R(F/M,\operatorname{Hom}_R(N,E)) & \longrightarrow & M\otimes_R \operatorname{Hom}_R(N,E) & \longrightarrow & F\otimes_R \operatorname{Hom}_R(N,E) \\ & \eta_1\downarrow & & \downarrow\eta_M & & \downarrow\eta_F \\ 0 \to & \operatorname{Hom}_R(\operatorname{Ext}_R^1(F/M,N),E) & \to & \operatorname{Hom}_R(\operatorname{Hom}_R(M,N),E) & \to & \operatorname{Hom}_R(\operatorname{Hom}_R(F,N),E), \end{array}$$

where η_F is an isomorphism and η_M is a w-epimorphism by Lemma 3.11. Hence η_1 is a w-epimorphism.

Let L be a torsion-free module. Then $0\to L\to L_S$ is exact. Since L_S is a $T(R)$-module, $\operatorname{Tor}_1^R(F/M,L_S)$ is a $T(R)$-module. Thus we have

$$\operatorname{Tor}_1^R(F/M,L_S) = \operatorname{Tor}_1^R(F/M,L_S)_S \cong \operatorname{Tor}_1^{T(R)}(F_S/M_S,L_S) = 0.$$

Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 \longrightarrow \operatorname{Tor}_1^R(F/M,L) & \longrightarrow & M\otimes_R L & \longrightarrow & F\otimes_R L \\ \downarrow & & \downarrow & & \downarrow \\ 0 = \operatorname{Tor}_1^R(F/M,L_S) & \longrightarrow & M\otimes_R L_S & \longrightarrow & F\otimes_R L_S \end{array}$$

Since M is w-flat, $M \otimes_R L \to M \otimes_R L_S$ is a w-monomorphism and

$$\mathrm{Tor}_1^R(F/M, L)$$

is GV-torsion for any torsion-free R-module L. Since N is divisible, it is routine to verify that $L = \mathrm{Hom}_R(N, E)$ is torsion-free. Hence

$$\mathrm{Tor}_1^R(F/M, \mathrm{Hom}_R(N, E))$$

is GV-torsion. Thus η_M is a w-monomorphism. □

Let M and N be R-modules. Let S be a multiplicatively closed set of R. Consider the natural homomorphism

$$\theta : \mathrm{Hom}_R(M, N)_S \to \mathrm{Hom}_{R_S}(M_S, N_S),$$

by

$$\theta(\frac{f}{s})(\frac{x}{1}) = \frac{f(x)}{s},$$

for $s \in S$, $x \in M$, and $f \in \mathrm{Hom}_R(M, N)$. It is well known that if M is finitely generated, then θ is a monomorphism and that if M is finitely presented, then θ is an isomorphism.

Lemma 3.13 ([18, Theorem 3.4.8])**.** *Let S be the set of all non-zero-divisors of R. If M is finitely generated and N is torsion-free, then θ is an isomorphism.*

Lemma 3.14 ([20, Theorem 3.12])**.** *Let M be a w-projective module of finite type and let $\mathfrak{p}$ be a prime w-ideal of R. Set $S = R \setminus \mathfrak{p}$. If N is a torsion-free w-module, then θ is an isomorphism.*

Lemma 3.15 ([22, Theorem 2.8])**.** *Let M be a module and let N be a w-module. Then* $\mathrm{Hom}_R(M, N)$ *is a w-module. Especially, reflexive modules are w-modules.*

Lemma 3.16 ([20, Theorem 1.6])**.** *Let M be a finitely generated module and let N be a GV-torsion-free module. Then* $\mathrm{Hom}_R(M, N)_w = \mathrm{Hom}_R(M, N_w)$.

Lemma 3.17. *Let S be the set of all non-zero-divisors of R and let N be a torsion-free w-module. Then N_S as an R-module is a w-module. In particular, $T(R)$ is a w-module.*

Proof. Since N_S is an essential extension of N, we have $E(N_S) = E(N)$. Let $J \in \mathrm{GV}(R)$ and $x \in E(N)$ with $Jx \subseteq N_S$. Since J is finitely generated, there is $s \in S$ with $Jsx \subseteq N$. Thus $sx \in N$, and hence $x \in N_S$. □

Lemma 3.18. *Let S be the set of non-zero-divisors of R and let N be a w-module over $T(R)$. Then N as an R-module is a w-module.*

Proof. Note that N is certainly a GV-torsion-free R-module because $J_S \in \mathrm{GV}(T(R))$ for every $J \in \mathrm{GV}(R)$. Let E denote the injective hull of N as an R_S-module. Then it is easy to see that E is certainly the injective hull of N as an R-module. Let $J \in \mathrm{GV}(R)$ and $x \in E$ with $Jx \subseteq N$. Then $J_S x \subseteq N_S = N$. Hence $x \in N$ by hypothesis. Therefore, N as an R-module is a w-module. □

Theorem 3.19. *Let S be the set of all non-zero-divisors of R and let B be a finitely generated module.*

(1) *If B is w-projective, then B_S is a w-projective $T(R)$-module.*
(2) *If B is w-flat and torsion-free, and B_S is a projective $T(R)$-module, then B is a w-projective R-module.*

Proof. (1) Let N be a w-module over $T(R)$. Then N as an R-module is a torsion-free w-module by Lemma 3.18. For any R-module X, it is clear that $\mathrm{Hom}_R(X,N)=\mathrm{Hom}_R(X,N)_S$.

Let $\mathfrak{m}$ be a maximal w-ideal of $T(R)$ and set $\mathfrak{p}=\mathfrak{m}\cap R$. Then $\mathfrak{p}$ is a prime w-ideal of R and $\mathfrak{p}\cap S=\varnothing$. Hence $S\subseteq R\setminus\mathfrak{p}$ and $\mathfrak{m}=\mathfrak{p}_S$. Therefore we have $T(R)_{\mathfrak{m}}=(R_S)_{\mathfrak{p}_S}\cong R_{\mathfrak{p}}$, which implies that $L_{\mathfrak{m}}\cong L_{\mathfrak{p}}$ for any $T(R)$-module L.

Let Y be of finite type. Then there is a finitely generated submodule Z of Y such that Y/Z is GV-torsion. Note that $J_S\in\mathrm{GV}(T(R))$ for any $J\in\mathrm{GV}(R)$. Hence Y_S/Z_S is GV-torsion over $T(R)$. Therefore, $\mathrm{Hom}_R(Y,N)=\mathrm{Hom}_R(Z,N)$ and $\mathrm{Hom}_{T(R)}(Y_S,N)=\mathrm{Hom}_{T(R)}(Z_S,N)$. By Lemma 3.13, we have

$$\mathrm{Hom}_R(Z,N)=\mathrm{Hom}_R(Z,N)_S\cong\mathrm{Hom}_{T(R)}(Z_S,N).$$

Hence we have $\mathrm{Hom}_R(Y,N)=\mathrm{Hom}_{T(R)}(Y_S,N)$.

Let $0\to A\to F\to B\to 0$ be an exact R-sequence, where F is finitely generated free. By Lemma 2.6, A is of finite type. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(F,N)_{\mathfrak{p}} & \longrightarrow & \mathrm{Hom}_R(A,N)_{\mathfrak{p}} & \longrightarrow & \mathrm{Ext}^1_R(B,N)_{\mathfrak{p}} & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{T(R)}(F_S,N)_{\mathfrak{m}} & \longrightarrow & \mathrm{Hom}_{T(R)}(A_S,N)_{\mathfrak{m}} & \longrightarrow & \mathrm{Ext}^1_{T(R)}(B_S,N)_{\mathfrak{m}} & \longrightarrow & 0
\end{array}$$

The two vertical arrows on the left are isomorphisms by the same argument above. Hence the vertical arrow on the right is an isomorphism. So

$$\mathrm{Ext}^1_{T(R)}(B_S,N)$$

is GV-torsion over $T(R)$. Therefore, B_S is w-projective over $T(R)$.

(2) Let N be a torsion-free w-module and let $0\to N\to E_1\to C\to 0$ be exact, where E_1 is the injective hull of N. Thus C and E_1 are GV-torsion-free and divisible. Then the sequence

$$0\to\mathrm{Hom}_R(B,N)\to\mathrm{Hom}_R(B,E_1)\to\mathrm{Hom}_R(B,C)\to\mathrm{Ext}^1_R(B,N)\to 0$$

is exact. Let E be a w-injective w-module over R. Then

$$0\to\mathrm{Hom}_R(C,E)\to\mathrm{Hom}_R(E_1,E)\to\mathrm{Hom}_R(N,E)\to 0$$

is w-exact. Since B is w-flat, we have the following commutative diagram with w-exact rows

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & B \otimes_R \mathrm{Hom}_R(C,E) & \longrightarrow & B \otimes_R \mathrm{Hom}_R(E_1,E) \\ & & \downarrow & & \downarrow \eta_C & & \downarrow \eta_{E_1} \\ 0 & \longrightarrow & \mathrm{Hom}_R(\mathrm{Ext}^1_R(B,N),E) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(B,C),E) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(B,E_1),E). \end{array}$$

By Lemma 3.12, η_C and η_{E_1} are w-isomorphisms. Hence

$$\mathrm{Hom}_R(\mathrm{Ext}^1_R(B,N),E)$$

is GV-torsion. Thus $\mathrm{Ext}^1_R(B,N)$ is GV-torsion by Theorem 3.6. Therefore B is w-projective. □

4. *w*-semi-hereditary rings

Recall that a *semi-hereditary* ring is a ring in which all (nonzero) finitely generated ideals are projective. It is well known that over a domain, an ideal is projective if and only if it is invertible. Thus a semi-hereditary domain is a Prüfer domain, which is generalized to the concept of PVMDs: A domain is a PVMD if all nonzero finitely generated ideals are t-invertible, equivalently w-invertible. In this section, we generalize this concept to commutative rings with zero divisors, and characterize some related rings.

Lemma 4.1. (1) *Let R be a reduced ring (i.e., $nil(R) = 0$) and let A and B be ideals of R. Then $A \cap B = 0$ if and only if $AB = 0$.*

(2) *If $R_{\mathfrak{m}}$ is a domain for any maximal w-ideal $\mathfrak{m}$ of R, then R is reduced.*

Proof. (1) Suppose $AB = 0$. Let $x \in A \cap B$. Then $x^2 = 0$. Thus we have $x = 0$ since R is reduced.

(2) Let $N = nil(R)$. Then $N_{\mathfrak{m}} = 0$ for any maximal w-ideal $\mathfrak{m}$ of R by hypothesis. Hence N is GV-torsion. Because R is GV-torsion-free, $N = 0$. □

Lemma 4.2. *Let $R = R_1 \times R_2$ be a product decomposition of rings.*

(1) *Let $J = J_1 \times J_2$ be an ideal of R. Then $J \in GV(R)$ if and only if $J_i \in GV(R_i)$ for $i = 1, 2$.*

(2) *Let $M = M_1 \times M_2$ be a GV-torsion-free R-module. Then M_{w_i} is GV-torsion-free over R_i and $M_w = (M_1)_{w_1} \times (M_2)_{w_2}$, where M_{w_i} denotes the w-envelope of M_i over R_i.*

Proof. This is routine. □

Lemma 4.3. *Let $a \in R$.*

(1) *If a is a zero divisor, then* $\mathrm{ann}(\mathrm{ann}(a)) \neq R$.

(2) *If a is not a unit, then $(a)_w \neq R$. In other words, if $(a)_w = R$, then a is a unit, and therefore $(a) = R$.*

Proof. (1) As a is a zero divisor, $\mathrm{ann}(a) \neq 0$. Thus $\mathrm{ann}(\mathrm{ann}(a)) \neq R$.

(2) If a is a non-zero-divisor, then it is clear that $(a)_w = (a) \neq R$. If a is a zero divisor, then $a \in \mathrm{ann}(\mathrm{ann}(a)) \neq R$ by (1). Since $\mathrm{ann}(\mathrm{ann}(a))$ is a w-ideal of R, we have $(a)_w \subseteq \mathrm{ann}(\mathrm{ann}(x)) \neq R$. □

Now we give new characterizations of von Neumann regular rings.

Theorem 4.4. *The following statements are equivalent for a ring R.*

(1) *R is a von Neumann regular ring.*
(2) *Every R-module is w-flat.*
(3) *For any $a \in R$, $a \in (a^2)_w$.*
(4) *If I is a finitely generated ideal of R, then $I \subseteq (I^2)_w$.*
(5) *$R_{\mathfrak{m}}$ is a field for any maximal w-ideal $\mathfrak{m}$ of R.*

Proof. (2)⇒(3). For any $a \in R$, since R/Ra is w-flat and $0 \to Ra \to R \to R/Ra \to 0$ is exact, we have $((a) \cap (a))_w = (a)_w = (a^2)_w$ by Proposition 2.2. Hence $a \in (a^2)_w$.

(3)⇒(5). By hypothesis, we have $(a)_{\mathfrak{m}} = (a^2)_{\mathfrak{m}}$. Hence $R_{\mathfrak{m}}$ is a local von Neumann regular ring. Hence $R_{\mathfrak{m}}$ is a field.

(5)⇒(2). This follows from Theorem 2.1.

(3)⇒(4). Let $I = Ra_1 + \cdots + Ra_n$. Then there is $J \in \mathrm{GV}(R)$ such that $Ja_i \in (a_i^2)$ for each i. Hence $JI \subseteq I^2$. Therefore $I \subseteq (I^2)_w$.

(4)⇒(3) and (1)⇒(2). These are trivial.

(3)+(5)⇒(1). By Lemma 4.1, R is reduced. Let $a \in R$ and set $I = \mathrm{ann}(a)$. Then there is $J \in \mathrm{GV}(R)$ such that $Ja \subseteq (a^2)$. Thus, for any $c \in J$, $ca = ra^2$ for some $r \in R$. Hence $c - ra \in I$, that is, $J \subseteq I + (a) = I \oplus (a)$. Then $(I + (a))_w = I \oplus (a)_w = R$, which implies that $(a)_w$ is generated by an idempotent element e. Set $R_1 = (a)_w = Re$. Then R_1 is a ring with the identity e. Denote by I_{w_1} the w-envelope of an ideal I of R_1. By Lemma 4.2, $(a)_w = (a)_{w_1} = R_1$. By Lemma 4.3, $(a) = R_1 = Re$. Hence, R is a von Neumann regular ring. □

From Theorem 4.4, it is not necessary to define "w-von Neumann regular rings".

Definition 4.5. A ring R is said to be *w-semi-hereditary* if every finite type ideal of R is w-projective; equivalently, every finitely generated ideal of R is w-projective.

Certainly, semi-hereditary rings and PVMDs (Prüfer v-multiplication domains) are w-semi-hereditary. Following [19], an R-module M is called *w-coherent* if M is of finite type and each finite type submodule of M is of finitely presented type; a ring R is called *w-coherent* if R is w-coherent as an R-module. Also it is shown that a ring R is w-coherent if and only if every finitely generated ideal of R is of finitely presented type; if and only if every finite type submodule of a free module is of finitely presented type [19, Theorem 3.1]. Since w-projective modules of finite type are of finitely presented type by Lemma 2.6, every w-semi-hereditary ring is w-coherent.

Proposition 4.6 ([20, Proposition 2.9])**.** *Let I be a nonzero nil ideal of R. Then I is not w-projective.*

Corollary 4.7. *Let R be w-semi-hereditary. Then R is reduced.*

Proof. Let u be a nilpotent element of R. Then $I = (u)$ is w-projective by hypothesis. Hence $I = 0$ by Proposition 4.6. Hence $u = 0$. □

Proposition 4.8. *Let $R = R_1 \times R_2$. Then R is w-semi-hereditary if and only if R_1 and R_2 are w-semi-hereditary.*

Proof. This is straightforward. □

Next, we will consider the w-operation analogue of rings with weak global dimension less than or equal to one. The weak global dimension is the measure of flatness of modules over R. A few characterizations of rings with weak global dimension less than or equal to one can be found in [4, 12]. The following is the w-theoretic analogue of these results.

Theorem 4.9. *The following statements are equivalent for a ring R.*

(1) *Every submodules of a w-flat module is w-flat.*
(2) *Every finite type submodule of a w-flat module is w-flat.*
(3) *Every finitely generated submodule of a w-flat module is w-flat.*
(4) *Every finitely generated ideal of R is w-flat.*
(5) *Every ideal of R is w-flat.*
(6) *Every finite type ideal of R is w-flat.*
(7) *$R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R.*

Proof. (1)⇒(2)⇒(3)⇒(4). These are trivial.

(4)⇒(5). Let I be an ideal of R. Then $I = \bigcup B$, where B ranges over the set of all finitely generated subideals of I. Hence I is w-flat by Proposition 2.3

(5)⇒(7). Let I be any ideal of R. Then I is w-flat. Hence $I_{\mathfrak{m}}$ is a flat ideal of $R_{\mathfrak{m}}$, which implies that every ideal of $R_{\mathfrak{m}}$ is flat. Then $R_{\mathfrak{m}}$ is a valuation domain.

(7)⇒(1). This is clear.

(4)⇔(6). This is trivial. □

Let us call a commutative ring R a ring with w-$w.gl.dim(R) \leqslant 1$ if any of the equivalent conditions of Theorem 4.9 is satisfied. In fact, for a commutative ring R, w-$w.gl.dim(R)$ can be defined analogously by making the following substitutions: flat module (w-flat module) and flat dimension (w-flat dimension).

Corollary 4.10. *Let R be a w-semi-hereditary ring. Then every ideal of R is w-flat.*

Proof. This follows from Proposition 2.4 and Theorem 4.9. □

There are several characterizations of semi-hereditary rings in literature (cf., [2, 4, 7, 12]). In particular, it is well known that R is semi-hereditary if and only if every finitely generated submodule of a projective module is projective [4, Theorem 1.4.3] and that R is semi-hereditary if and only if $T(R)$ is a von

Neumann regular ring and $R_{\mathfrak{m}}$ is a valuation domain for any maximal ideal $\mathfrak{m}$ of R [2, Theorem 2]. The following and Theorem 4.14 are the w-theoretic analogue of these characterizations.

Theorem 4.11. *The following statements are equivalent for a ring R.*

(1) *R is w-semi-hereditary.*
(2) *Every finite type submodule of a free module is w-projective.*
(3) *Every finitely generated submodule of a free module is w-projective.*
(4) *Every finitely generated ideal of R is w-projective.*

Proof. (1)⇒(2). Let F be a free module and let M be a finite type submodule of F. Without loss of generality, we assume that $F = R^n$ is finitely generated. The assertion is proved by induction on n. If $n = 1$, then M is a finite type ideal of R. Hence M is w-projective. Suppose $n > 1$. Let $p : R^n \to R$ be the n-th projection and set $I = p(M)$. Thus I is of finite type by [19, Proposition 1.3]. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \cap R^{n-1} & \longrightarrow & M & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R^{n-1} & \longrightarrow & R^n & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

Since I is w-projective, I is of finitely presented type by Lemma 2.6. Thus $M \cap R^{n-1}$ is of finite type. Then $M \cap R^{n-1}$ is w-projective by induction. Hence M is w-projective by Lemma 2.7.

(2)⇒(3)⇒(4). These are trivial.

(4)⇒(1). Let I be a finite type ideal of R. Then I is w-isomorphic to a finitely generated subideal B of I. Hence I is w-projective by hypothesis and Lemma 2.5. □

Lemma 4.12. *Let R be a ring such that $R_{\mathfrak{m}}$ is a domain for any maximal w-ideal $\mathfrak{m}$ of R. Let a be a nonzero element of R and set $I = \mathrm{ann}(a)$. If I is of finite type, then I is generated by an idempotent element.*

Proof. Set $J = \mathrm{ann}(I)$. Thus $IJ = 0$, and so $I \cap J = 0$ by Lemma 4.1. Hence $I + J$ is a w-ideal of R. If $I + J \neq R$, then there is a maximal w-ideal $\mathfrak{m}$ of R such that $I + J \subseteq \mathfrak{m}$. Note that $\frac{a}{1} \neq 0$ in $R_{\mathfrak{m}}$, otherwise there is $s \in R \setminus \mathfrak{m}$ such that $sa = 0$, and hence $I \not\subseteq \mathfrak{m}$. Thus $I_{\mathfrak{m}} = 0$ since $R_{\mathfrak{m}}$ is a domain. Since I is a finite type w-ideal, $sI = 0$ for some $s \in R \setminus \mathfrak{m}$, that is, $s \in \mathrm{ann}(I)$, which contradicts $\mathrm{ann}(I) \subseteq \mathfrak{m}$. Then $I + J = I \oplus J = R$. This completes the proof. □

Theorem 4.13. *Let R be a w-semi-hereditary ring. If every non-zero-divisor of R is a unit, then R is a von Neumann regular ring.*

Proof. If R is not a von Neumann regular ring, then there is a maximal w-ideal $\mathfrak{m}$ of R such that $R_{\mathfrak{m}}$ is not a field. Thus there exists a prime subideal $\mathfrak{p}$ of $\mathfrak{m}$ such that $\mathfrak{p}_{\mathfrak{m}} \neq 0$. Let $a \in \mathfrak{m} \setminus \mathfrak{p}$ with $\frac{a}{1} \neq 0$ and write $I = \mathrm{ann}(a)$. Then

$0 \to I \to R \to Ra \to 0$ is exact. Since R is w-semi-hereditary, then I is of finite type. Because each localization of R at a maximal w-ideal $\mathfrak{q}$ of R is a valuation domain by Theorem 4.9, we have that $I = Re$ for some idempotent element e by Lemma 4.12. Set $s = e - a$. By $ea = 0 \in \mathfrak{p}$, we have $e \in \mathfrak{p} \subset \mathfrak{m}$. Hence $s = e - a \in \mathfrak{m}$, therefore, s is not a unit.

Since $Ia = 0$, $I \cap Ra = 0$ by Lemma 4.1. Let $x \in R$ with $sx = ex - ax = 0$. Since $ex = ax \in I \cap Ra$, we have $ex = ax = 0$. Thus $x \in I$, whence $x = re = re^2 = ex = 0$. Thus s is a non-zero-divisor. Therefore, s is a unit by hypothesis, a contradiction. Hence $R_{\mathfrak{m}}$ is a field. Thus R is a von Neumann regular ring by Theorem 4.4. □

Theorem 4.14. *The following statements are equivalent for a ring R:*

(1) *R is w-semi-hereditary.*
(2) *$T(R)$ is a von Neumann regular ring and $R_{\mathfrak{p}}$ is a valuation domain for any prime w-ideal $\mathfrak{p}$ of R.*
(3) *$T(R)$ is a von Neumann regular ring and $R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R.*
(4) *R is a w-coherent ring with w-$w.gl.dim(R) \leqslant 1$.*

Proof. (1)⇒(2). By Corollary 4.10 and Theorem 4.9, it is sufficient to show that $T(R)$ is a von Neumann regular ring. Let S be the set of all non-zero-divisors of R. Then $T(R) = R_S$. Let A be a finitely generated ideal of $T(R)$. Then there is a finitely generated ideal B of R such that $A = B_S$. Let $\mathfrak{m}$ be a maximal w-ideal of $T(R)$ and set $\mathfrak{p} = \mathfrak{m} \cap R$. Then $\mathfrak{p} \cap S = \varnothing$. Hence $S \subseteq R \setminus \mathfrak{p}$ and $\mathfrak{m} = \mathfrak{p}_S$. Then $A_{\mathfrak{m}} \cong (B_S)_{\mathfrak{p}_S} \cong B_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ is a valuation domain, $B_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$. By Proposition 2.8, B is w-projective over R. Consequently, A is w-projective over $T(R)$ by Theorem 3.19(1). Hence $T(R)$ is w-semi-hereditary. By Theorem 4.13, $T(R)$ is a von Neumann regular ring.

(2)⇒(3). This is trivial.

(3)⇒(1). Let I be a finitely generated ideal of R. For any maximal w-ideal $\mathfrak{m}$ of R, $I_{\mathfrak{m}}$ is free as an $R_{\mathfrak{m}}$-module by hypothesis. Hence I is a w-flat ideal. Since $T(R) = R_S$ is a von Neumann regular ring, I_S is a projective ideal of $T(R)$. By Theorem 3.19(2), I is w-projective, and hence R is w-semi-hereditary.

(1) ⇒ (4). The first assertion is in the remark before Proposition 4.6, while the second assertion follows from Theorem 4.9 and (1) ⇔ (3).

(4) ⇒ (1). Let I be a finite type ideal of R. Then by Theorem 4.9, I is w-flat. Since w-$w.gl.dim(R) \leqslant 1$, again by Theorem 4.9, $R_{\mathfrak{m}}$ is a valuation domain for every maximal w-ideal $\mathfrak{m}$ of R. Thus $I_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-free for every maximal w-ideal $\mathfrak{m}$ of R. Moreover since R is w-coherent, I is of finitely presented type. Thus by Proposition 2.8, I is w-projective, and so R is w-semi-hereditary. □

Recall that a ring R is said to be *connected* if $\mathrm{Spec}(R)$ is a connected topological space.

Theorem 4.15. *Let R be a connected ring. Then R is w-semi-hereditary if and only if R is a PVMD.*

Proof. Let R be a connected w-semi-hereditary ring. It is sufficient to show that R is a domain. Let a be a nonzero element and set $I = \text{ann}(a)$. Because Ra is finitely generated w-projective, I is of finite type by Lemma 2.6. Hence I is generated by an idempotent element e by Lemma 4.12. Since R is connected and $a \neq 0$, we have $e = 0$, and hence $I = 0$. Hence R is a domain. □

Following Lucas [9, 10, 11], we denote by $Q_0(R)$ the ring of finite fractions of R. In [20], an R-module A is said to *have w-rank n* if $A_\mathfrak{m} \cong R_\mathfrak{m}^n$ for any maximal w-ideal $\mathfrak{m}$ of R. And recall that an R-module A is said to be *w-invertible* if the trace map $\tau : A \otimes_R A^* \to R$ is a w-isomorphism. In [20], it was shown that A is w-invertible if and only if M is of finite type and has w-rank 1, that is, $A_\mathfrak{m} \cong R_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. If A is a submodule of $Q_0(R)$, then A is w-invertible if and only if there is a submodule B of $Q_0(R)$ such that $(AB)_w = R$.

Proposition 4.16. *The following statements are equivalent for a commutative ring R.*

(1) *Every nonzero finite type ideal of R is w-invertible.*
(2) *Every nonzero finitely generated ideal of R is w-invertible.*
(3) *Every nonzero finite type torsion-free module is w-projective and has finite w-rank.*
(4) *Every nonzero finitely generated torsion-free module is w-projective and has finite w-rank.*
(5) *R is a PVMD.*

Proof. (1)⇔(2) and (3)⇔(4). These follow from [20, Theorem 4.15].

(4)⇒(2). This is trivial.

(2)⇒(5). Let a be a nonzero element of R and set $I = \text{ann}(a)$. Then Ra is w-projective and $(Ra)_\mathfrak{m} \cong R_\mathfrak{m}$. Then $I_\mathfrak{m} = 0$, which implies $I = 0$ since I is a w-module. Thus R is a domain, and hence R is a PVMD.

(5)⇒(3). This follows from [17, Theorem 3.9]. □

Acknowledgements. The authors would like to thank the referee for a careful reading and many helpful suggestions which improve the paper. This work was partly supported by the National Natural Science Foundation of China (Grant No. 11171240) and by the Research Foundation for Doctor Programme (Grant No. 20125134110001). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).

References

[1] S. El Baghdadi, H. Kim, and F. Wang, *Injective-like modules over Krull domains*, preprint.
[2] S. Endo, *On semi-hereditary rings*, J. Math. Soc. Japan **13** (1961), 109–119.
[3] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker Inc., 1972.

[4] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, vol. 1371. Springer, Berlin, 1989.
[5] S. Glaz and W. V. Vasconcelos, *Flat ideals. II*, Manuscripta Math. **22** (1977), no. 4, 325–341.
[6] J. S. Golan, *Localizations of Noncommutative Rings*, Marcel Dekker, New York, 1975.
[7] R. Griffin, *Prüfer rings with zero divisors*, J. Reine Anger. Math. **239** (1969), 55–67.
[8] H. Kim and F. Wang, *On LCM-stable modules*, J. Algebra Appl. **13** (2014), 1350133, 18 pages.
[9] T. G. Lucas, *Strong Prüfer rings and the ring of finite fractions*, J. Pure Appl. Algebra **84** (1993), no. 1, 59–71.
[10] ______, *The Mori property in rings with zero divisors*, in Rings, modules, algebras, and abelian groups, 379–400, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, 2004.
[11] ______, *Krull rings, Prüfer v-multiplication rings and the ring of finite fractions*, Rocky Mountain J. Math. **35** (2005), no. 4, 1251–1326.
[12] J. Marot, *Sur le anneaux universellement Japonais*, Ph.D. Thesis, Universite de Paris-Sud, 1977.
[13] R. Matsuda, *Notes on Prüfer v-multiplication rings*, Bull. Fac. Sci. Ibaraki Univ. Ser. A **12** (1980), 9–15.
[14] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979.
[15] J. R. Silvester, *Introduction to Algebraic K-Theory*, Chapman and Hall, 1981.
[16] F. Wang, *w-projective modules and w-flat modules*, Algebra Colloq. **4** (1997), no. 1, 111–120.
[17] ______, *w-modules over a PVMD*, Proc. ISTAEM, Hong Kong, 117–120, 2001.
[18] ______, *Commutative Rings and Star-Operation Theory*, (in Chinese), Science Press, Beijing, 2006.
[19] ______, *Finitely presented type modules and w-coherent rings*, J. Sichuan Normal Univ. **33** (2010), 1–9.
[20] F. Wang and H. Kim, *Two generalizations of projective modules and their applications*, submitted.
[21] F. Wang and J. Zhang, *Injective modules over w-Noetherian rings*, Acta Math. Sinica (Chin. Ser.) **53** (2010), no. 6, 1119–1130.
[22] H. Yin, F. Wang, X. Zhu, and Y. Chen, *w-Modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222.
[23] S. Q. Zhao, F. Wang, and H. L. Chen, *Flat modules over a commutative ring are w-modules*, (in Chinese), J. Sichuan Normal Univ. **35** (2012), 364–366.

Fanggui Wang
College of Mathematics
Sichuan Normal University
Chengdu, Sichuan 610068, P. R. China
E-mail address: wangfg2004@163.com

Hwankoo Kim
School of Computer and Information Engineering
Hoseo University
Asan 336-795, Korea
E-mail address: hkkim@hoseo.edu