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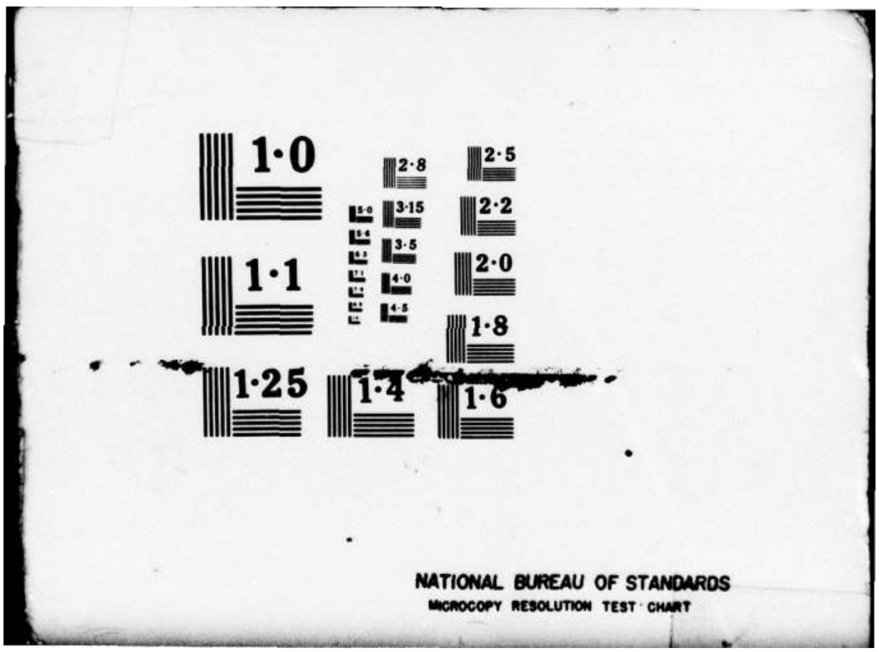
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WAITING TIME IN A CONTINUOUS REVIEW (s,S) INVENTORY SYSTEM WITH CONSTANT LEAD TIMES

TECHNICAL REPORT

BY

W. KARL KRUSE

SEPTEMBER 1978

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The waiting time distribution in an (s,S) continuous review inventory system with constant lead times is derived in this paper. The demand process is assumed to be a renewal process, and demand sizes are iid integer valued random variables. Some relationships are given between waiting time and some common inventory measures.

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INTRODUCTION

Consider an inventory system using a continuous review (s,S) policy with constant lead time of size T. The time between successive demands is idd with distribution function $H(\cdot)$ and pdf $h(\cdot)$. Likewise, demand sizes are iid integer valued random variables with probability function $b(\cdot)$. All demands are backlogged until filled. We will derive the distribution of customer waiting time. Since it is reasonable, we take $s \geq -1$, which means that no customer will ever wait more than T.

While the (s,S) continuous review inventory system has been greatly studied, there has been little work on customer waiting time. Sherbrooke [3] derived the waiting time distribution for the special case (S-1,S) system subject to compound Poisson demands. Simon [4] derived an expression for the expected wait in the (s,S) system when the demand process is simple Poisson which can be shown to be a particularization of $L = \lambda W$.

The most general analysis of the (s,S) continuous review system has been done by Sahin [2] who developed expressions for both the time dependent and stationary distributions of net inventory, i.e. on hand minus backorders, and inventory position, i.e. net inventory plus on order, using a renewal - theoretic structure. His advancement over earlier work was in permitting demand size to be a random variable. Urbach [6] also analyzed a similar system under the condition that no more than one order is outstanding. He was interested in the case of random lead times.

For the sake of presentation, we first develop the waiting time distribution for the case of unit demand size. We then use the logic and some of the results derived to extend to the case of random demand size.

Formulas suitable for computation are given in the paper only for the limiting stationary distribution of customer waiting time. These are derived in Appendix A. Since we will later be using Laplace Transforms, we denote (s,S) by (R,R+Q) to avoid confusion with the Laplace variable "s".

Notation and Some Preliminaries

The following notation is used.

$A(t)$ = inventory position at time t = on hand + on order - backorders at t (also called assets)

$d(t_1,t_2)$ = demand quantity in $[t_1,t_2)$

$W(t)$ = waiting time of a customer who arrives at t.

T = constant lead time

θ = expected demand size

μ = expected time between demands

$b_n(\cdot)$ = probability function of the sum of n demand sizes, i.e. the n-fold convolution of $b(\cdot)$.

$h_n(\cdot)$ = n-fold convolution of $h(\cdot)$

$H_n(x) = \int_o^y h_n(y)dy$

$\tilde{h}(s)$ = Laplace transform of $h(\cdot)$

$\tilde{H}(s)$ = Laplace transform of $H(\cdot)$

$\tilde{h}_n(s)$ = Laplace transform of $h_n(\cdot) = h(s)^n$

$\tilde{H}_n(s)$ = Laplace transform of $H_n(\cdot) = \frac{\tilde{h}(s)^n}{s} = \tilde{H}(s)\tilde{h}(s)^{n-1}$

Waiting Time for Unit Demand Size

Since the lead time is constant, all of the suppliers assets at y, i.e. A(y), will be available to be issued to customers by y + T; and any assets ordered after y will not be available until after y + T. This means that a customer who arrives at t will wait $\leq \tau$ iff he receives one of the assets on account at $t + \tau - T$. The customer will get an item from $A(t+\tau -T)$ only if the previous demands for those assets, $d(t+\tau -T,t)$, are less than $A(t+\tau -T)$. We have then that

$$(1) \qquad \Pr[W(t) \leq \tau] = \sum_{a=R+1}^{R+Q} \Pr[W(t) \leq \tau \mid A(t+\tau-T) = a] \Pr[A(t+\tau-T) = a]$$

$$= \sum_{a=R+1}^{R+Q} \Pr[d(t+\tau-T,t) < a \mid A(t+ -T) = a, \text{ Demand at } t] \Pr[A(t+\tau-T) = a]$$

$$= \sum_{a=R+1}^{R+Q} \Pr[A(t+\tau-T) = a,\ d(t+\tau-T,t) < a \mid \text{Demand at } t]$$

For finite time, $A(t+\tau-T)$ and $d(t+\tau-T,t)$ may be dependent random variables since knowledge of $A(t+\tau-T)$ may provide information about the demands from the start of the inventory system until $t+\tau-T$ which, in turn, affect the likelihood of $d(t+\tau-T,t)$. However, in Appendix A we show that $A(t+\tau-T)$ and $d(t+\tau-T,t)$ are independent in the steady state, and with the given condition of a demand at t have probability functions

$$\Pr[A=a] = \frac{1}{Q}\ ;\ a = R+1, R+2, \ldots R+Q$$

and

$$\lim_{t\to\infty} \Pr[d(t+\tau-T,t) < d \mid \text{demand at } t] = 1-H_d(T-\tau)$$

In other words, the demands in the $T-\tau$ units preceeding the present customers arrival form an ordinary renewal process in the steady state. Letting $F_W(\cdot)$ denote the steady state distribution of waiting time we then have from (1) that

$$(2) \qquad F_w(\tau) = \frac{1}{Q} \sum_{k=1}^{Q} [1 - H_{R+k}(T-\tau)] \; ; \; 0 \leq \tau < T$$

$$F_w(T) = 1$$

As a matter of interest a special case of the results in Appendix B is that $E(W) = E(B)/\lambda$ where

$E(W)$ = expected waiting time

$E(B)$ = expected steady state backorders

and $1/\lambda$ = expected time between demands. Of course, this is simply an example of $L = \lambda W$.

Waiting Time for Random Demand Size

For the unit demand size case the meaning of waiting time was obvious. In extending to random demand size, we have the problem of defining customer wait. For example, what is the wait when a customer who demanded 10 units receives five units immediately, but waits, say 10 days, before receiving the other five units? Recognizing that the definition of waiting time should depend upon the context in which the statistic is to be used, we avoid the problem of defining waiting time by deriving the distribution of wait separately for each unit in the demand. Later we show the richness of this approach by demonstrating how this distribution can be used to develop several common performance measures.

As before we take a demand arrival to occur at t, but allow the demand size U to be ≥ 1. Each unit in the demand is identified by an index j from 1 to U. The j^{th} unit will wait $\leq \tau$ iff the demands preceding the j^{th} unit which are vying for $A(t+\tau-T)$ are less than $A(t+\tau-T)$. In this case, those demands are the j-1 units of the present demand plus $d(t+\tau-T,t)$. So

$$(3)\quad Pr[j^{th} \text{ unit waits} \le \tau] = \sum_{a=R+1}^{R+Q} Pr[d(t+\tau-T,t) \le a-j \,|\, A(t+\tau-T = a, \text{Demand at } t] \cdot Pr[A(t+\tau-T) = a]$$

$$= \sum_{a=R+1}^{R+Q} Pr[A(t+\tau-T) = a,\ d(t+\tau-T,t) \le a-j \,|\, \text{Demand at } t]$$

Again, as is shown in Appendix A, $A(t+\tau-T)$ and $d(t+\tau-T,t)$ are independent in the steady state. Several authors, [2], [6], and [8], have shown how to compute the steady state distribution of assets. As in the unit demand size case, the number of demand occurrences in the preceding $T-\tau$ units form an ordinary renewal process in the steady state. That is

$$p(d) = \lim_{t\to\infty} Pr[d(t+\tau-T,t) = d \,|\, \text{Demand at } t]$$

$$= \sum_{n=1}^{d} b_n(d)[H_n(T-\tau) - H_{n+1}(T-\tau)];\ d \ge 1$$

and

$$p(0) = \lim_{t\to\infty} Pr[d(t+\tau-T,t) = 0 \,|\, \text{Demand at } t] = 1 - H(T-\tau)$$

Letting ${}_jF_w(\cdot)$ denote the steady state distribution function of waiting time for the j^{th} unit we have from (3) that

$$(4)\quad {}_jF_w(\tau) = \sum_{a=R+1}^{R+Q} Pr[A=a]\, P(a-j)\ ;\ 0 \le \tau < T$$

$${}_jF_w(T) = 1$$

where $P(a-j) = \sum_{k=0}^{a-j} p(k)$ if $a - j \ge 0$

$= 0$ otherwise

Waiting Time and Some Common Inventory Measures

Expected number of units backordered and initial fill, i.e. the fraction of demand satisfied without backorder, are two commonly used inventory measures. Waiting time relates to each of these in a similar way. We show in Appendix C that

$$\text{Expected Units Backordered} = \frac{\Theta}{\mu} \int_0^T \sum_{j=1}^{\infty} \left(1 - \frac{B(j-1)}{\Theta}\right)\left(1 - {}_jF_w(\tau)\right) d\tau$$

and

$$\text{Initial Fill} = \sum_{j=1}^{\infty} \left(\frac{1-B(j-1)}{\Theta}\right) \left({}_jF_w(0)\right)$$

where

$$B(j-1) = \sum_{k=1}^{j-1} b(k)$$

Both have the common term $\frac{1-B(j-1)}{\Theta}$ which has a simple interpretation. Suppose a series of N demands is observed and the units in each of the demands are indexed as before. Let $n_i(N)$ be the number of demands of size i in the N demands. Then $\phi(N,j) = \sum_{i=j}^{\infty} n_i(N) / \sum_{i=1}^{\infty} i\, n_i(N)$ is the fraction of total units in the N demands which have index j. In the limit as N goes to infinity we get

$$(j) = \lim_{N\to\infty} \phi(N,j) = \lim_{N\to\infty} \sum_{i=j}^{\infty} \frac{n_i(N)}{N} \Big/ \sum_{i=1}^{\infty} \frac{i\, n_i(N)}{N}$$

$$= \sum_{i=j}^{\infty} b(i)/\Theta \quad = \quad \frac{1-B(j-1)}{\Theta}$$

which is just the proportion of units demanded which have index j. Thus, both expected units backordered and initial fill are equivalent to taking the corresponding measures for each possible unit in a demand, weighting by the proportion of times that unit occurs, and averaging.

There is no particular advantage to actually computing the above measures using ${}_jF_w(\cdot)$. However, this does indicate how simply some measures are able to be expressed with ${}_jF_w(\cdot)$. Other examples are the probability that a demand is completely filled without waiting and the expected number of demands backordered when a demand is counted as backordered until completely filled.

APPENDIX A

DISTRIBUTION OF A(t) AND d(t,t+z) GIVEN A DEMAND AT t + z

The R,R+Q inventory system is observed from time 0 to t+z. Arbitrarily, we take the demand process to begin at time 0 with A(0) = R+Q. We will find

$$\lim_{t\to\infty} [\Pr[A(t) = a,\ d(t,t+z) = d \mid A(0) = R+Q,\ \text{Demand at } t+z]$$

The expression "Demand at t+z" is used to mean that a demand occurs within dz of t+z. Let N(0,t) be the number of demand occurrences in (0,t) and N(t,t+z) be the number of demand occurrences in t,t+z. Given the values of N(0,t), N(t,t+z), and A(0), then A(t) and d(t,t+z) are uniquely determined by the sequence of demand sizes associated with those demand occurrences. If N(0,t) = m, there is a countable though, in general, infinite number of demand size sequences which will result in A(t) = a given A(0) = R+Q. Call X(m,a,R+Q) the set of all such sequences. And, if N(t,t+z) = n then d(t,t+z) = d iff the n demand sizes sum to d. We have then

(A1) $$G(a,d,z,t) = \Pr[A(t) = a,\ d(t,t+z) = d \mid A(0) = R+Q,\ \text{Demand at } t+z]$$

$$= \sum_{n=0}^{d} \sum_{m=0}^{\infty} \Pr[A(t) = a,\ d(t,t+z) = d \mid A(0) = R+Q,\ \text{Demand at } t+z,\ N(0,t) = m,\ N(t,t+z) = n]$$

$$\Pr[N(0,t) = m,\ N(t,t+z) = n \mid A(0) = R+Q,\ \text{Demand at } t+z]$$

$$= \sum_{n=0}^{d} b_n(d) \sum_{m=0}^{\infty} \frac{\Pr[X(m,a,R+Q)]\,\Pr[N(o,t) = m,\ N(t,t+z) = n,\ \text{Demand at } t+z]}{r(t+z)dz}$$

$$= F(a,d,z,t,R+Q)\,dz/\,r(t+z)dz$$

where $r(t+z)$ = pdf of a demand occurrence at $t+z$

(also called the renewal density)

and $Pr[X(m,a,R+Q)]$ = probability

a demand size sequence from the set $X(m,a,R+Q)$ occurs.

Consider $P(m,n,z,t) = Pr[N(0,t) = m, N(t,t+z) = n, \text{Demand at } t+z]$

For $m \geq 1$ we have

$$P(m,n,z,t) = \int_{y=o}^{t} P(m-1,n,z,t-y)\, H(y)dy$$

and on taking Laplace transforms we get

$$\tilde{P}(m,n,z,s) = \tilde{P}(m-1,n,z,s)\tilde{h}(s)$$

Applying this recursively from $m = 1$ yields

(A2) $$\tilde{P}(m,n,z,s) = \tilde{P}(0,n,z,s)[\tilde{h}(s)]^m \quad ; \quad m \geq 1$$

Consequently, the Laplace transform of $F(a,d,z,t)$ is

$$\tilde{F}(a,d,z,s) = \sum_{n=o}^{d} b_n(d)\, \tilde{P}(o,n,z,s) \sum_{m=o}^{\infty} Pr[X(m,a,R+Q)][\tilde{h}(s)]^m$$

$$= \tilde{F}_1(s)\, \tilde{F}_2(s)$$

where

$$\tilde{F}_1(s) = \sum_{n=o}^{d} b_n(d)\, \tilde{P}(o,n,z,s)$$

and

$$\tilde{F}_2(s) = \sum_{m=o}^{\infty} Pr[X(m,a,R+Q)][\tilde{h}(s)]^m$$

Now

$$\lim_{t\to\infty} G(a,d,z,t) = \lim_{t\to\infty} F(a,d,z,t)dz / \lim_{t\to\infty} r(t+z)dz$$

$$= \lim_{s\to o} \left(\frac{s\tilde{F}_1(s)}{s}\right) \lim_{s\to o} (s\tilde{F}_2(s)) / \lim_{t\to\infty} r(t+z)$$

$$= \left[\lim_{t\to\infty} \int_o^t \sum_{n=o}^{d} b_n(d)\tilde{P}(o,n,z,y)dy\right]\left[\frac{\lim_{t\to\infty} \sum_{m=o}^{\infty} Pr[X(m,a,R+Q)h_m(t)}{\lim_{t\to\infty} r(t+z)}\right]$$

provided the separate limits exist.

Consider the first term

$$\lim_{t\to\infty}\int_{o}^{t}\sum_{n=o}^{d} b_n(d)P(o,n,z,y)dy = \sum_{n=o}^{d} b_n(d)\lim_{t\to\infty}\int_{o}^{t} P(o,n,z,y).$$

But $$\lim_{t\to\infty}\int_{o}^{t} P(o,n,z,y)dy = \lim_{t\to\infty}\int_{y=o}^{t}\int_{x=o}^{z} h_n(x)h(y+z-x)dxdy$$

$$= \int_{x=o}^{z} h_n(x)\ \lim_{t\to\infty}\int_{y=z-x}^{t+z-x} h(y)dy\ \ dx$$

$$= \quad h_n(x)(1-H(z-x))\ dx$$

$$= h_n * (1-H) = H_n(z) - H_{n+1}(z)$$

which is just the probability that the number of renewals in an interval z for an ordinary renewal process equals n.

Now consider the second term.

$$\lim_{t\to\infty}\ \sum_{m=o}^{\infty} Pr[X(m,a,R+Q)][h_m(t)]/\lim_{t\to\infty} r(t+z)$$

$$= \lim_{t\to\infty}\frac{\text{pdf of a transition to asset state a at t}}{\text{pdf of a demand occurrence at t+z}}$$

$$= \frac{1}{\mu_a}\ /\ \frac{1}{\mu} = \frac{\mu}{\mu_a}$$

where

μ_a = average time between transitions to state a

and μ = average time between demands.

and which follows because the number of transitions to state a is itself a renewal process. Moreover, since μ = expected amount of time the system spends in state a between transitions to state a, then

$$\mu/\mu_a = \lim_{t\to\infty} Pr[A(t) = a] = Pr[A = a]$$

Summarizing then, we have shown that

$$\lim_{t\to\infty} G(a,d,z,t) = Pr[A=a] \sum_{n=o}^{d} b_n(d)\ [H_n(z) - H_{n+1}(z)]$$

This contrasts to Sahin's [2] result for the steady state distribution of A(t) and d(t,t+z) without the condition of a demand at t+z. This condition changes the demand process in a interval of length z from a equilibrium demand renewal process to an ordinary demand renewal process.

APPENDIX B

RELATIONSHIP OF WAITING TIME TO EXPECTED UNITS BACKORDERED AND INITIAL FILL

From Sahin's work [2] we have for the stationary system that

$$\text{(B1)}\quad \text{Expected units backordered} = E[B_u] = \sum_{j=o}^{\infty} \Pr[\text{Units Backordered} > j]$$

$$= \sum_{j=o}^{\infty} \sum_{a=r+1}^{R+Q} \Pr[A=a] \sum_{n=1}^{\infty} [H'_n(T) - H'_{n+1}(T)]\, C_n(a+j)$$

where $H'_n(T) = \frac{1}{\mu} \int_o^T (1-H(y))\, H_{n-1}(T-y)\,dy$

and $C_n(k) = 1 - \sum_{i=1}^{k} b_n(i) = 1 - \beta_n(k) \quad ; \quad n \leq k$

$= 1$; otherwise

Letting $Z(T) = E[B_u]$ and taking Laplace transforms we get after some algebra that

$$\text{(B2)}\quad \tilde{Z}(S) = \frac{\theta}{\mu s} \sum_{a=R+1}^{R+Q} (1-\tilde{H}(s)) \sum_{j=o}^{\infty} \frac{C(a+j)}{\theta} + \sum_{n=1}^{\infty} (\tilde{H}(s)^n - \tilde{H}(s)^{n+1}) \sum_{j=o}^{\infty} \frac{[C_{n+1}(a+j) - C_n(a+j)]}{\theta}$$

where we have used that the Laplace transform of $H'_n(t)$ is $\frac{(1-\tilde{H}(s))\tilde{H}(s)^{n-1}}{\mu s}$

Returning to the time domain we get

$$\text{(B3)} \quad E[B_y] = \frac{\theta}{\mu} \int_0^T \sum_{a=R+1}^{R+Q} [(1-H(T-\tau)) \sum_{j=o}^{\infty} \frac{C(A+j)}{\theta} + \sum_{n=1}^{\infty} (H_n(T-\tau) - H_{n+1}(T-\tau)) \sum_{j=o}^{\infty} \frac{C_{n+1}(a+j) - C_n(a+j))}{\theta}]$$

It is possible to show by algebraic manipulation of (B3) that

$$E[B_u] = \sum_{j=1}^{\infty} \frac{1-B(j-1)}{\theta} \frac{\theta}{\mu} \int_0^T (1-{}_jF_w(\tau)d\tau$$

but it is more appealing to argue the above result straight from the terms in (B3). The term $\sum_{j=o}^{\infty} \frac{C_{n+1}(a+j) - C_n(a+j)}{\theta}$ is the proportion of units in the $n+1^{st}$ demand which wait longer than τ given that $A(t+\tau-T) = a$, and that the number of demand occurrences in $(t+\tau-T,t) = n$. In this perspective, the $n+1^{st}$ demand is the customer arriving at t. By probablistically weighting over all values of n, we get the proportion of units in the arriving customers demand which must wait longer than τ. But $\sum_{j=1}^{\infty} \frac{1-B(j-1)}{\theta} \int_0^T (1-{}_jF_w(\tau))d\tau$, by the arguments given in the report is the same proportion.

Initial fill, IF, is the fraction of total demand which is filled without wait. Thus

$$IF = \lim_{t\to\infty} \frac{\text{Number of units filled without wait}}{\text{Number of units demanded}}$$

$$= \frac{\lim_{t\to\infty} \sum_{i=1}^{N(t)} n_i}{\lim_{t\to\infty} \sum_{i=1}^{N(t)} d_i}$$

where

$N(t)$ = number of demands in period t

d_i = size of i_{th} demand; $i = 1, 2, \ldots N(t)$

and n_i = amount of i^{th} demand filled without wait; $i = 1, 2 \ldots N(t)$

So

$$IF = \lim_{t \to \infty} \sum_{i=1}^{N(t)} \frac{n_i}{N(t)} \Big/ \lim_{t \to \infty} \sum_{i=1}^{N(t)} d_i / N(t).$$

$$= \frac{\text{Expected number of units filled without wait per demand}}{\theta}$$ with probability 1.

Now the expected number of units filled without wait per demand is $\sum_{j=1}^{\infty}$ [Prob that j or more units in a demand are filled without wait] which equals $\sum_{j=1}^{\infty} ({}_jF_n(0))(1-B(j-1))$. In other words, for j or more units to be filled immediately, there must be at least j units demanded and at least the j^{th} unit does not wait. Consequently

$$IF = \sum_{j=1}^{\infty} \frac{(1-B(j-1))({}_jF_w(0))}{\theta}$$

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