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WAKE POTENTIALS OF A RELATIVISTIC CURRENT IN A CAVITY

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The wake potential of a bunch of charged particles is required for the calculation of energy loss and beam stability in high-energy particle accelerators or storage rings. Exact solutions for the wake potential are only known for closed cylindrical cavities, and can be obtained either by mode analysis or in the time domain. The equivalence of these analytic solutions, as well as the good agreement with numerical methods, shows that there is no "missing scalar potential" in the mode analysis, as had been suspected before. The mode analysis can be generalized to cavities of arbitrary shape when the resonant frequencies and loss parameters are known for each mode.

GLOSSARY

$\mathbf{a}_{np}$	eigenfunctions of the vector potential
c	velocity of light
$\mathbf{e}_{r,\theta,z}$	unit vector in the r, θ, or z direction
g	length of the pill-box cavity ("gap" length)
i	$\sqrt{-1}$
j_n	n^{th} root of the Bessel-function J_0
j_λ	current produced by a line charge density λ
k_μ	loss parameter of the mode μ
L	half length of parabolic bunch
n	radial mode number in a pill-box cavity, $n \geq 1$
p	longitudinal mode number in a pill-box cavity, $-\infty < p < \infty$
Q	charge
$q_{np}(t)$	time-dependent coefficients of the vector potential
$r_{np}(t)$	time-dependent coefficients of the scalar potential
R	radius of the pill-box cavity
U_{np}	stored energy in the mode (n, p)
V_{np}	voltage seen by a particle due to the mode (n, p)
W	wake potential (= energy gain in volts)
$W_{d,s,G,p,\lambda}$	wake potential for a point charge (delta function), step current, Gaussian bunch, parabolic bunch, arbitrary line-charge density λ
z_0	distance from the bunch centre or reference point
$Z(\omega)$	impedance
$\lambda(z)$	line-charge density of the driving current
μ	general index for counting resonant modes
ν_{np}	wave number $\nu_{np} = \omega_{np}/c$
σ	standard deviation of a Gaussian bunch
ϕ_{np}	eigenfunctions of the scalar potential
ω_{np}	circular resonant frequencies of a pill-box cavity

1. INTRODUCTION

The wake potential of a bunch of charged particles traversing a resonant cavity is of considerable interest for particle accelerators and storage rings, as it permits the calculation of the coupling impedance—and hence the stability—as well as the evaluation of the energy loss of the bunched beam. The only geometry which permits exact analytic calculations of the wake potential is the closed cylindrical cavity, commonly called the "the pill box". Several different approaches to calculating the wake potential of a bunch of particles traversing a pill-box cavity have been published in the literature,[1–6] but the equivalence of the solutions was not obvious.

Here we compare the solutions obtained by mode analysis and in the time domain with each other and also with a recently published numerical method[7] solving the problem for general rotational-symmetric cavities.

In general, we find complete agreement for the wake potential of bunches with continuous line-charge densities, and there is no "missing scalar potential" in the mode analysis, as had been assumed before. However, for discontinuous charge densities such as delta-function pulses (which can be used as the Green's function for arbitrary charge densities), agreement is found only if one disregards divergent terms which are of no consequence for realistic (continuous) charge densities.

Finally, the mode analysis can be generalized to arbitrary cavities, for which the wake potential is obtained in terms of the loss parameters of each of the resonant modes. The resonant frequencies and loss parameters can be obtained numerically for certain rotationally symmetric cavities with existing computer programs such as KN7C[8] or SUPERFISH.[9] Unfortunately, the series for the wake potential converge rather slowly for positions inside the bunch—which is the case of interest for the coupling impedance—and there the obtainable accuracy is quite limited. However, for positions well behind the bunch, the series converge faster and thus the energy loss can be evaluated more precisely.

2. PILL-BOX CAVITY

In this section the wake potential in a pill-box cavity will be evaluated using the mode concept and the time-domain scheme. Finally these analytical results will be compared with numerical ones.

2.1. *Mode Analysis*

The mode analysis uses the resonant modes of a cavity to compute the wake potential. It is assumed first that the contributions of the free charges, which cannot be taken into account by these modes, vanish. With $z_0 > 0$ as the distance between a point charge Q and a test particle behind it, both travelling at the speed of light along the axis, the mode concept gives the wake potential as an infinite sum[1,2]

$$W_d(z_0) = -2Q \sum_{\mu} k_{\mu} \cos\left(\omega_{\mu} \frac{z_0}{c}\right). \quad (1)$$

The k_{μ} are the loss parameters defined by

$$k_{\mu} = \frac{V_{\mu} V_{\mu}^{*}}{4U_{\mu}}, \quad (2)$$

U_{μ} is the stored energy in the mode μ and V_{μ} is the voltage induced by the point charge. For a pill-box cavity these loss parameters can be given analytically.

The normalized field components are

$$\begin{aligned} E_z{}^{n,p} &= \frac{j_n}{R} J_0\left(j_n \frac{r}{R}\right) \cos\left(\frac{\pi p z}{g}\right) \exp(i\omega_{np} t) \\ E_r{}^{n,p} &= \frac{\pi p}{g} J_1\left(j_n \frac{r}{R}\right) \sin\left(\frac{\pi p z}{g}\right) \exp(i\omega_{np} t) \\ H_\theta{}^{n,p} &= i\omega_{np}\epsilon_0 J_1\left(j_n \frac{r}{R}\right) \cos\left(\frac{\pi p z}{g}\right) \exp(i\omega_{np} t), \end{aligned} \quad (3)$$

where g is the "gap" length of a cavity of radius R, j_n is the n^{th} zero of the Bessel function $J_0(x)$ and $\omega_{np}^2/c^2 = (j_n/R)^2 + (\pi p/g)^2 = \nu_{np}^2$.

Hence the voltage becomes

$$\begin{aligned} V_{np} &= \int_0^g E_z(r = 0, z, t = z/c)\, dz \\ &= \frac{i\nu_{np} R}{j_n} [1 - (-1)^p \exp(i\nu_{np} g)] \end{aligned} \quad (4)$$

and further

$$V_{np} V_{np}{}^{*} = 2\left(\frac{\nu_{np} R}{j_n}\right)^2 \times [1 - (-1)^p \cos(\nu_{np} g)]. \quad (5)$$

The stored energy is given by

$$\begin{aligned} U_{np} &= \frac{\mu_0}{2} \int_0^R \int_0^{2\pi} \int_0^g H_\theta{}^{n,p} H_\theta{}^{*n,p}\, dz\, r\, d\theta\, dr \\ &= \frac{\pi\epsilon_0}{4} \nu_{np}^2 g R^2 J_1{}^2(j_n). \end{aligned} \quad (6)$$

The loss parameters thus are given by

$$k_{np} = \frac{1}{\pi\epsilon_0 g} \frac{2}{1 + \delta_{op}} \times \frac{1 - (-1)^p \cos(\nu_{np} g)}{j_n{}^2 J_1{}^2(j_n)}, \quad (7)$$

where δ_{op} is the Kronecker symbol. The expression for the point charge wake potential becomes

$$W_d(z_0) = \frac{-2Q}{\pi\epsilon_0 g} \sum_{n=1}^{\infty} \sum_{p=-\infty}^{+\infty} \times \frac{1 - (-1)^p \cos(\nu_{np} g)}{j_n{}^2 J_1{}^2(j_n)} \cos(\nu_{np} z_0). \quad (8)$$

(By counting p from $-\infty$ to $+\infty$ rather than from 0 to ∞ we avoid a special factor for $p = 0$.) With $Q = 1$ this expression can be used as the Green's function for an arbitrary charge distribution $\lambda(x)$. The wake potential is then given by

$$W_\lambda(z_0) = \int_0^\infty W_d(x) \lambda(x - z_0)\, dx. \quad (9)$$

2.2. *Time-Domain Analysis*

The electric and magnetic fields induced by a bunch of charged particles traversing a cavity can be derived from the scalar and vector potentials. These potentials can be expressed as infinite series of the products of the eigenmodes of the cavity and of time-dependent factors:[10]

$$\begin{aligned} \phi(\mathbf{r}, t) &= \sum_{\mu} \phi_{\mu}(\mathbf{r}) r_{\mu}(t) \\ \mathbf{A}(\mathbf{r}, t) &= \sum_{\mu} \mathbf{a}_{\mu}(\mathbf{r}) q_{\mu}(t). \end{aligned} \tag{10}$$

The summation extends in general over modes in all three spatial directions ($\mu = m, n, p$). For a beam passing along the axis of a rotationally symmetric cavity however, only azimuthally symmetric fields are excited, and the summation is limited to radial ($1 \leq n \leq \infty$) and axial ($-\infty < p < \infty$) mode numbers.

The eigenmodes are normalized solutions of the homogeneous Helmholtz equations

$$\begin{aligned} [\nabla^2 + \nu_{np}^2]\phi_{np} &= 0 \\ [\nabla^2 + \nu_{np}^2]\mathbf{a}_{np} &= 0, \end{aligned} \tag{11}$$

which fulfil the proper boundary conditions at the cavity walls (assumed to be perfectly conducting for simplicity), and where ν_{np} are the resonant frequencies ($\times\ 2\pi/c$) of the cavity. The time-dependent factors then can be determined from the equations

$$\begin{aligned} & r_{np}(t) \\ &\quad = \epsilon_0 \nu_{np}^2 \int_V \rho(\mathbf{r} - \mathbf{v}t)\phi_{np}(\mathbf{r})\, dV, \\ & \ddot{q}_{np}(t) + \nu_{np}^2 q_{np}(t) \\ &\quad = \frac{1}{\epsilon_0} \int_V \mathbf{J}(\mathbf{r} - \mathbf{v}t)\cdot\mathbf{a}_{np}(\mathbf{r})\, dV, \end{aligned} \tag{12}$$

where $\rho(\mathbf{r})$ is the charge density, and $\mathbf{J}(\mathbf{r}) = \rho\mathbf{v}$ the current density of the bunch moving with velocity $\mathbf{v}$. For convenience, we will restrict our considerations to bunches moving with light velocity along the cavity axis ($\mathbf{v} = c\mathbf{e}_z$). The integration extends generally over the volume of the beam inside the cavity, and reduces to an integral over z for a filamentary beam at the axis (after replacing the volume density ρ by the line density λ). The initial conditions for q_{np} will be chosen such that there are no fields in the cavity before the bunch arrives. In order to include bunches of any length, we take $q_{np}(-\infty) = \dot{q}_{np}(-\infty) = 0$.

The electric field can be obtained from the potentials with the relation

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{13}$$

and hence the axial component on the axis ($r = 0$) of an azimuthally symmetric field ($\partial/\partial\theta = 0$) becomes

$$\begin{aligned} E_z(z, t) = -\sum_{n,p} \Bigg[& \frac{\partial \phi_{np}}{\partial z} r_{np}(t) \\ & + a_{npz}\dot{q}_{np}(t) \Bigg]. \end{aligned} \tag{14}$$

The wake potential at a distance z_0 behind the bunch center is defined as the integral over E_z along the z-axis with $ct = z + z_0$ or

$$W(z_0) = \int_0^g E_z\left(z, \frac{z + z_0}{c}\right) dz. \tag{15}$$

In the Appendix we derive the wake potential from the vector and scalar potentials of a pill-box cavity. In this geometry, the eigenmodes and resonant frequencies are given by closed analytic expressions. For a bunch of the line-charge density $\lambda(z)$ we obtain, in general,

$$\begin{aligned} W(z_0) = & -\frac{1}{\pi\epsilon_0 g} \sum_{n=1} \frac{1}{j_n{}^2 J_1{}^2(j_n)} \\ & \times \sum_{p=-\infty}^{\infty} \Bigg\{ \int_0^{\infty} dx \cos(\nu_{np}x)[2\lambda(x - z_0) \\ & - (-)^p \lambda(x - z_0 + g) \\ & - (-)^p \lambda(x - z_0 - g)] \\ & + \int_{-g}^{0} dx \cos\frac{\pi p x}{g} \\ & \times [\lambda(x - z_0) - (-)^p \lambda(x - z_0 + g)] \Bigg\}. \end{aligned} \tag{16}$$

For continuous charge distributions, we can interchange the order of integration and summation over p. As shown in the Appendix, the wake potential is then given by the much simpler expression

$$W(z_0) = -\frac{2}{\pi\epsilon_0 g}\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\frac{1-(-)^p\cos(\nu_{np}g)}{j_n^2 J^2(j_n)} \times \int_0^\infty \lambda(x-z_0)\cos(\nu_{np}x)\,dx. \quad (17)$$

For discontinuous charge distributions, such as the step or delta-function pulse, this equation yields the expressions which are valid after the discontinuity has left the cavity ($z_0 > g$). For $z_0 < g$, the complete expression Eq. (16) contains a divergent term which is of no consequence for realistic (continuous) distributions, which are always the ultimate aim of the computations.

For the step-function pulse, the infinite sums in Eq. (17) have been summed analytically[3] for $z_0 < (4R^2 + g^2)^{1/2} - g$, i.e., before reflections from the outer cavity wall arrive at the location where the wake potential is evaluated (only for $z_0 > g$). If the divergent term in Eq. (16) is ignored,[5] it yields the same result also for $z_0 < g$ and it thus appears that Eq. (17) may be used even for discontinuous distributions for any value of z_0.

Equation (17) could be reduced further by exchanging the order of integration and summation also for the infinite integral. However, this leads to expressions restricted to $z_0 < (4R^2 + g^2)^{1/2} - g$ discussed above.

We now apply Eq. (17) to a number of typical distributions.

2.2.1. *Delta-function pulse* $\lambda(z) = Q\delta(z)$

$$W_d(z_0) = -\frac{2Q}{\pi\epsilon_0 g} \times \sum_{n,p}\frac{1-(-)^p\cos(\nu_{np}g)}{j_n^2 J_1^2(j_n)} \times \cos(\nu_{np}z_0), \quad z_0 > 0. \quad (18)$$

For $z_0 < (4R^2 + g^2)^{1/2} - g$, these sums can be evaluated analytically and yield

$$W_d(z_0) = \frac{Q}{2\pi\epsilon_0 g} \times \left\{\frac{1}{\frac{z_0}{2g}+\left[\frac{z_0}{2g}\right]} - \frac{1}{\frac{z_0}{2g}+\left[\frac{z_0}{2g}\right]+2}\right\}, \quad (19)$$

where the square brackets stand for the integer part of the term enclosed.

2.2.2. *Step-function pulse* $\lambda(z) = \lambda_0 s(-z)$

where

$$s(z) = \begin{cases} 0 & \text{for } z < 0 \\ \tfrac{1}{2} & \text{for } z = 0 \\ 1 & \text{for } z > 0. \end{cases}$$

Then for $z_0 > 0$:

$$W_s(z_0) = -\frac{2\lambda_0}{\pi\epsilon_0 g} \times \sum_{n,p}\frac{1-(-)^p\cos(\nu_{np}g)\sin(\nu_{np}z_0)}{j_n^2 J_1^2(j_n)\nu_{np}}. \quad (20)$$

Restricting z_0 to be smaller than $(4R^2 + g^2)^{1/2} - g$, one obtains[3]

$$W_s(z_0) = -\frac{\lambda_0}{2\pi\epsilon_0} \times \ln\left\{1 + \frac{1}{\frac{z_0}{2g}+\left[\frac{z_0}{2g}\right]}\right\}, \quad (21)$$

and hence the wake potential for an arbitrary distribution $\lambda(z)$

$$W_\lambda(z_0) = -\frac{1}{2\pi\epsilon_0}\int_0^{z_0}\frac{d\lambda(z)}{dz} \times \ln\left\{1 + \frac{1}{\frac{z_0-z}{2g}+\left[\frac{z_0-z}{2g}\right]}\right\}dz. \quad (22)$$

This expression is valid before the arrival of reflections from the cylindrical cavity wall, i.e., for a limited range of z_0 (which is here counted from the beginning of the bunch).

2.2.3. *Parabolic bunch (half length L)*

$$\lambda(z) = \begin{cases} \frac{3Q}{4L}\left(1-\frac{z^2}{L^2}\right) & \text{for } |z| < L \\ 0 & \text{for } |z| > L. \end{cases}$$

Equation (17) yields

$$W_p = \frac{3Qg}{\pi\epsilon_0 L} \sum_{n,p} \frac{1 - (-)^p \cos \nu_{np} g}{j_n^2 \nu_{np}^2 J_1^2(j_n)} \times \begin{cases} 0 & z_0 < 0 \\ \dfrac{\sin \nu_{np}(z_0 + L)}{\nu_{np} L} - \cos \nu_{np}(z_0 + L) - \dfrac{z_0}{L} & z_0 < 0 \\ 2\left[\dfrac{\sin \nu_{np}(z_0 + L)}{\nu_{np} L} - \cos \nu_{np} L\right] \times \cos \nu_{np} z_0 & z_0 > L. \end{cases} \tag{23}$$

For $z_0 < L < g < (4R^2 + g^2)^{1/2} - g$ these sums yield

$$W_p(z_0) = \frac{3Qg}{4\pi\epsilon_0 L^3}\left[(z_0 + L) - 2(z_0 + g) \times \ln\left(1 + \frac{z_0 + L}{g}\right) + \frac{L^2 - z_0^2}{2g} \times \ln\left(1 + \frac{2g}{z_0 + L}\right)\right]. \tag{24}$$

The same expression is obtained from Eq. (22), which takes the form

$$W_p(z_0) = -\frac{3Q}{4\pi\epsilon_0 L^3}\int_0^z (L - z) \times \ln\left(1 + \frac{2g}{z_0 - z}\right) dz$$

for $z_0 < 2g$ (z_0 counted from the head of the bunch).

2.2.4. *Gaussian charge distribution with standard deviation* σ

$$\lambda(z) = \frac{Q}{\sigma\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma^2}\right).$$

We find from Eq. (22), after evaluation of the integral in Eq. (17),

$$W_G(z_0) = -\frac{Q}{\pi\epsilon_0 g}\exp\left(-\frac{z_0^2}{2\sigma^2}\right) \times \sum_{n,p}\frac{1 - (-)^p \cos \nu_{np} g}{j_n^2 J_1^2(j_n)} \times \mathrm{Re}\left\{w\left(\frac{\nu_{np}\sigma}{\sqrt{2}} - \frac{iz_0}{\sigma\sqrt{2}}\right)\right\}, \tag{25}$$

where $w(z)$ is the complex error function,[12] and Re stands for the real part. No closed expression is at present known to the authors for this sum, but it has been evaluated numerically and is compared with purely numerical results in the next section.

2.3. *Comparison of Results Obtained by Various Methods*

A comparison between Eqs. (8) and (18) for the δ-function wake potential shows that the time domain and the mode analysis yield the *same* analytical expression. A divergent term occurs in Eq. (16) in the time-domain calculations for the case when $z_0 < g$, but has been eliminated in Eq. (17). For realistic, continuous charge distributions no divergence occurs and both methods give exactly the *same* answer for all positions z_0.

Therefore one can conclude that any contributions to the wake potential due to free charges are correctly obtained in the mode concept and thus there is no missing scalar potential contribution as has been suspected in the past.[1,2]

A further comparison was made between the analytic results derived above and numerical results of the computer program BCI,[7] which solves the field equations in the time domain directly by a mesh method, including the effects of free charges.

Figure 1 shows the wake potential in a range of $-4\sigma \le z_0 \le 36\sigma$ for a Gaussian bunch (σ = 2.5 cm) which has passed a pill-box cavity (R = 5 cm, g = 10 cm). An excellent agreement (better than 10^{-3}) can be found for test particles "outside" the bunch ($z_0 \ge 4\sigma$). Although a rough mesh was used in the computer program BCI (11 × 21 points), and only 40 modes in the analytic sum, both results can hardly be distinguished in the range $4\sigma \le z_0 \le 36\sigma$.

"Inside" the bunch ($-4\sigma < z_0 < 4\sigma$) the analytical and numerical results seem to disagree and therefore a second figure is given showing the wake potential in more detail and with increasing precision in both methods. The analytic results (broken lines) approach continuously the numer-

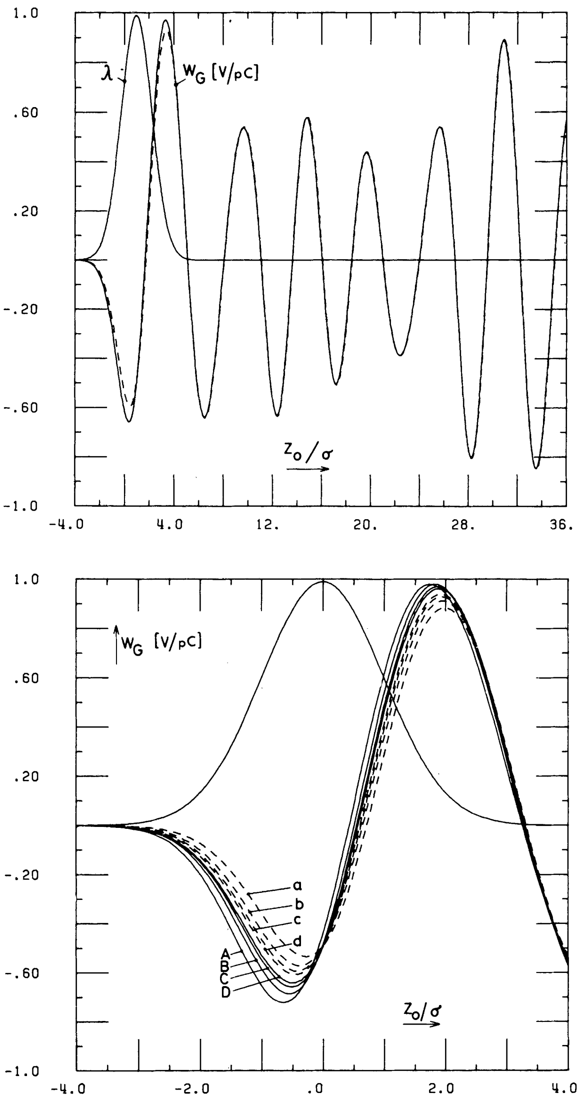

FIGURE 1 (a) The wake potential of a Gaussian bunch ($\sigma = 2.5$ cm) due to a pill-box cavity ($R = 5$ cm, $g = 10$ cm) for $-4\sigma \leq z_0 \leq 36\sigma$
--- mode-analysis results (40 modes)
— results of BCI (11 × 21 mesh).
(b) The wake potential of a Gaussian bunch ($\sigma = 2.5$ cm) due to a pill-box cavity ($R = 5$ cm, $g = 10$ cm) for $-4\sigma \leq z_0 \leq 4\sigma$
--- mode-analysis results for (a) 10 modes, (b) 40 modes, (c) 160 modes, (d) 640 modes
— BCI results for different meshes: (A) 6 × 11, (B) 11 × 21, (C) 21 × 41, (D) 41 × 81.

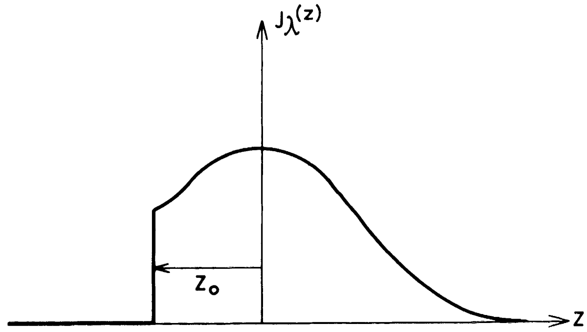

FIGURE 2 The driving current seen by a particle at z_0.

ical results with an increasing number of terms in the sum. The numerical results approach the analytical ones from the opposite side with an increasing number of mesh points. The final difference between the most accurate results in Fig. 1b is less than $\pm 2.5\%$.

The reason for this slow convergence of the results inside the bunch is the behaviour of the Fourier spectrum of the driving current, which is suddenly cut off at z_0 for a beam moving with light velocity (see Fig. 2). Because of causality, a particle at z_0 can only be influenced by fields due to particles in front of itself.)

Inside the bunch, the driving current for the wake potential is a function with a large step, which leads to a Fourier transform proportional to $1/\omega$ over a large range. "Behind" the bunch the step is small and the Fourier transform of the driving term becomes proportional to $\exp(-\omega^2\sigma^2/2c^2)$.

The problem occurs in both methods. In the analytical expressions the terms with high frequencies do not decay sufficiently fast. In the numerical computations the highest frequency which can be included is given by the size of the largest mesh step.[7]

3. GENERAL CAVITIES

From Section 2, we know that the wake potential of a pill box is determined by the eigenmodes of the cavity and by the loss parameters.

We assume that the same representation can be used for a general cavity. The impedance of such a cavity can be represented by an LC network as shown in Fig. 3, which is the sum over all single resonators

$$Z(\omega) = \sum_{m=1}^{\infty} \frac{i\omega L_m}{1 - \omega^2 L_m C_m}\,. \tag{26}$$

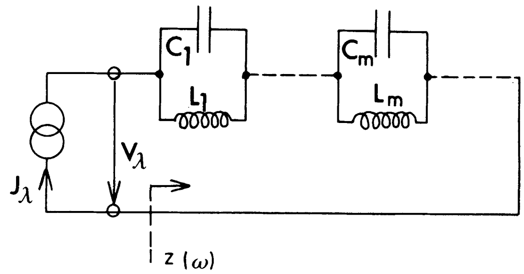

FIGURE 3 An LC network representing the cavity.

It may be rewritten by changing the summation index to μ, running over all single poles

$$Z(\omega) = \sum_{\mu} \frac{-ik_\mu}{\omega - \omega_\mu}. \tag{27}$$

The relations between the cavity parameters ω_μ, k_μ and the network parameters l_μ, C_μ are chosen as

$$C_\mu = \frac{1}{2k_\mu}, L_\mu = \frac{2k_\mu}{\omega_\mu{}^2}, \omega_\mu{}^2 L_\mu C_\mu = 1. \tag{28}$$

The Fourier transform of the bunch current, which is cut off at z_0, is given by (see Fig. 2)

$$j_\lambda(\omega, z_0) = \int_{-z_0/c}^{\infty} j_\lambda(t) e^{i\omega t}\, dt. \tag{29}$$

Hence we get for the wake potential (see Fig. 3)

$$W_\lambda(z_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j_\lambda(\omega. z_0) Z(\omega) \times \exp\left(-i\omega \frac{z_0}{c}\right) d\omega. \tag{30}$$

If $j_\lambda(\omega, z_0)$ has no poles in the complex ω-plane, which is always true for bunches of a finite length and for non-periodic bunches, we can evaluate this integral by the residue method. Then

$$W_\lambda(z_0) = -\sum_{\mu} j_\lambda(\omega_\mu, z_0) k_\mu \exp\left(-i\omega_\mu \frac{z_0}{c}\right). \tag{31}$$

The same result can be obtained by replacing the impedance [Eq. (27)] by

$$\tilde{Z}(\omega) = -2\pi \sum_{\mu} k_\mu \delta(\omega - \omega_\mu), \tag{32}$$

which is much easier to handle.

To find the wake potential at a position z_0 behind a reference point for an arbitrary bunch shape and for an arbitrary cavity, one thus only needs the resonant frequencies ω_μ, the loss parameters k_μ, and the Fourier transforms of the bunch (which is cut off at z_0) evaluated at the resonant frequencies.

For realistic cavities only a limited number of resonant frequencies and loss parameters can be obtained by numerical methods. For z_0 inside the bunch, a wake-potential computation becomes very difficult because of the slow convergence of the sum in Eq. (31). However, the series converge much faster for positions (z_0) well behind the bunch and permit a more accurate calculation of the wake potential by this method.

4. CONCLUSIONS

It has been shown that for a pill-box cavity the time-domain calculation and the mode analysis yield the same analytical expression for the wake potential of realistic bunch shapes.

Extrapolating this result to arbitrary cavities yields an expression for the wake potential as a sum over loss parameters and Fourier transforms. This sum converges very slowly for positions inside the bunch, making it difficult to obtain a precise value for the coupling impedance. However, a good approximation to the wake potential can be obtained after the bunch has passed the cavity, and hence the total energy loss of the bunch passing the cavity can be calculated more accurately.

APPENDIX

WAKE POTENTIAL IN A PILL-BOX CAVITY

(1) *Normalized eigenmodes in a cavity:* gap length g, radius R, $\partial/\partial\theta = 0$:

$$\Phi_{np}(\mathbf{r}) = \sqrt{\frac{2}{\pi g}} \frac{J_0(j_n r/R)}{R J_1(j_n)} \sin\frac{\pi p z}{g}$$

$$\mathbf{a}_{np}(\mathbf{r}) = \frac{1}{\sqrt{\pi g}} \frac{c}{R\omega_{np} J_1(j_n)} \times \begin{cases} \dfrac{\pi p}{g} J_1\left(\dfrac{j_n r}{R}\right) \sin\dfrac{\pi p z}{g}\, \mathbf{e}_r \\ \dfrac{j_n}{R} J_0\left(\dfrac{j_n r}{R}\right) \cos\dfrac{\pi p z}{g}\, \mathbf{e}_z \end{cases} \tag{A1}$$

where $1 \leq n < \infty$, $-\infty < p < \infty$.

The resonant frequencies are given by

$$\nu_{np} = \frac{\omega_{np}}{c} = \left[\left(\frac{j_n}{R}\right)^2 + \left(\frac{\pi p}{g}\right)^2\right]^{1/2}.$$

(2) *Time-dependent factors for a bunch:* line density $\lambda(z)$, moving along the z-axis with light velocity $v = c$.

$$r_{np}(t) = \frac{c^2}{\epsilon_0\omega_{np}^2}\int_0^g \lambda(z - ct)\Phi_{np}(z, r = 0)\,dz$$

$$\ddot{q}_{np}(t) + \omega_{np}^2 q_{np}(t) = \frac{c}{\epsilon_0}\int_0^g \lambda(z - ct) \times a_{npz}(z, r = 0)\,dz = F(t) \quad \text{(A3)}$$

with the initial conditions $q_{np}(-\infty) = \dot{q}_{np}(-\infty) = 0$. The general solution for q_{np} thus is

$$q_{np}(t) = \frac{1}{\omega_{np}}\int_{-\infty}^t F(\tau)\sin\omega_{np}(t - \tau)\,d\tau \quad \text{(A4a)}$$

and for its derivative (which we need for the calculation of the electric field rather than q_{np})

$$\dot{q}_{np}(t) = \int_{-\infty}^t F(\tau)\cos\omega_{np}(t - \tau)\,d\tau. \quad \text{(A4b)}$$

With the eigenmodes of a pill-box cavity [Eq. (A1)] we thus obtain

$$r_{np}(t) = \sqrt{\frac{2}{\pi g}}\frac{c^2}{\epsilon_0\omega_{np}RJ_1(j_n)} \times \int_0^g \lambda(z - ct)\sin\frac{\pi p z}{g}\,dz, \quad \text{(A5)}$$

and after some transformation:[11]

$$\dot{q}_{np}(t) = \frac{1}{\sqrt{\pi g}}\frac{c}{\epsilon_0\omega_{np}j_nJ_1(j_n)} \times \left\{\nu_{np}\int_0^\infty [\lambda(u - ct) - (-)^p\lambda(u - ct + g)] \times \sin(\nu_{np}u)\,du - \frac{\pi p}{g} \times \int_0^g \lambda(u - ct)\sin\frac{\pi p u}{g}\,du\right\}. \quad \text{(A6)}$$

(3) *The longitudinal component of the electric field:* in Eq. (14) the contributions from the scalar potential [Eq. (A5)] and from the second integral in the term of the vector-potential [Eq. (A6)] cancel, and we get simply

$$E_z(z, t) = -\frac{c}{\pi\epsilon_0 gR}\sum_{n,p}\frac{\cos(\pi p z/g)}{\omega_{np}J_1^2(j_n)} \times \int_0^\infty [\lambda(u - ct) - (-)^p\lambda(u - ct + g)] \times \sin(\nu_{np}u)\,du. \quad \text{(A7)}$$

For a delta-function pulse, $\lambda(z) = Q\delta(z)$, this yields

$$E_z(z, t) = -\frac{Qc}{\pi\epsilon_0 gR^2}\sum_{n,p}\frac{\cos(\pi p z/g)}{\omega_{np}J_1^2(j_n)} \times \begin{cases}\sin\omega_{np}t & 0 < ct < g \\ \sin\omega_{np}t - (-)^p\sin\omega_{np}(t - g/c) & ct > g\end{cases} \quad \text{(A8)}$$

while for a Gaussian distribution $\lambda(z) = [Q/\sigma(2\pi)^{1/2}]\exp(-z^2/2\sigma^2)$ one finds

$$E_z(z, t) = -\frac{Q}{2\pi\epsilon_0 gR^2}\sum_{n,p}\frac{\cos(\pi p z/g)}{\nu_{np}J_1^2(j_n)} \times \left[\exp\left(-\frac{c^2t^2}{2\sigma^2}\right) \times \mathrm{Im}\left\{w\left(\frac{\nu_{np}\sigma}{\sqrt{2}} - i\frac{ct}{\sigma\sqrt{2}}\right)\right\} - (-)^p\exp\left(-\frac{(g - ct)^2}{2\sigma^2}\right) \times \mathrm{Im}\left\{w\left(\frac{\nu_{np}\sigma}{\sqrt{2}} + i\frac{g - ct}{\sigma\sqrt{2}}\right)\right\}\right], \quad \text{(A9)}$$

where Im $w(z)$ is the imaginary part of the complex error function w.

(4) *The wake potential at position z_0 behind the center of the bunch:* this is found by using Eqs. (15) and (A7)

$$W(z_0) = -\frac{c}{\pi\epsilon_0 gR^2}\sum_{n,p}\frac{1}{\omega_{np}J_1^2(j_n)} \times \int_0^g dz\cos\frac{\pi p z}{g} \times \int_0^\infty du\sin(\nu_{np}u)[\lambda(u - z + z_0) - (-)^p\lambda(u - z - z_0 + g)]. \quad \text{(A10)}$$

By interchanging the order of integration,[11] one obtains Eq. (16).

Equation (16) can be transformed to

$$
\begin{aligned}
W(z_0) = {} & -\frac{1}{\pi\epsilon_0 g}\sum_{n=1}\frac{1}{j_m^2 J_1^2(j_m)} \\
& \times \sum_{p=-\infty}^{\infty}\left\{\int_0^{\infty} dx\, \lambda(x - z_0)[2\cos(\nu_{np}x)\right. \\
& - (-)^p \cos \nu_{np}(x + g) - (-)^p \\
& \times \cos \nu_{np}(x - g)] + (-)^p \int_{-g}^{0} dx \\
& \times \cos(\nu_{np}x)[\lambda(x + g - z_0) \\
& - \lambda(-x - g - z_0)] \\
& + \int_{-g}^{0} dx \cos\frac{\pi p x}{g}[\lambda(x - z_0) \\
& \left. - (-)^p\lambda(x + g - z_0)]\right\} .
\end{aligned}
\qquad \text{(A11)}
$$

We now interchange the summation over p with the integration in the last two integrals. For $|z| \le g$ we have

$$
\left.
\begin{aligned}
&\sum_p (-)^p \cos(\nu_{np} z) \qquad \text{(A12)} \\
&\sum_p (-)^p \cos\frac{\pi p z}{g} \qquad \text{(A13)}
\end{aligned}
\right\} = \delta\left(\frac{z}{g} + 1\right) + \delta\left(\frac{z}{g} - 1\right) ,
$$

$$
\sum_p \cos\frac{\pi p z}{g} = \delta\left(\frac{z}{g}\right) . \qquad \text{(A14)}
$$

The first of these expressions can be obtained by taking the derivative with respect to z of Eq. (3.14) in Ref. 5, while the other two sums are found in most books on Fourier series. With these expressions, the sums over p of all four terms in the last two integrals of Eq. (A.11) cancel, and we obtain Eq. (17) of the main text by combining the cosines in the first integral with the identity

$$
\begin{aligned}
2\cos\alpha - (-)^p\cos(\alpha + \beta) - (-)^p\cos(\alpha - \beta) \\
= 2\cos\alpha[1 - (-)^p\cos\beta], \qquad \text{(A.15)}
\end{aligned}
$$

where $\alpha = \nu_{np}x$ and $\beta = \nu_{np}g$.

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