WALKING AROUND THE BRAUER TREE

Dedicated to the memory of Hanna Neumann

J. A. GREEN

(Received 22 September 1972)

Communicated by M. F. Newman

1. Introduction

Let G be a finite group, and k a field of finite characteristic p, such that the polynomial $x^{|G|}-1$ splits completely in $k[x]$. Let $\mathbf{B}$ be a kG-block which has defect group D which is cyclic of order p^d ($d \geqq 1$). Brauer showed in a famous paper [2] that, in case $d = 1$, the decomposition matrix of $\mathbf{B}$ is determined by a certain positive integer e which divides $p-1$, and a *tree* Γ, a connected acyclic linear graph of $e+1$ vertices and e edges. Twenty-five years later Dade ([3]) extended Brauer's theorem to the general case.

Dade shows that $\mathbf{B}$ contains $v + (p^d - 1)/e$ simple (= irreducible) ordinary characters $X_1, \cdots, X_e$ and $X_\lambda (\lambda \in \Lambda)$, where Λ is an index set containing $(p^d-1)/e$ elements. B has e simple modular characters $\phi_0, \cdots, \phi_{e-1}$; denote by $\eta_0, \cdots, \eta_{e-1}$ the corresponding projective indecomposable characters.

Put $X_{e+1} = \Sigma_{\lambda \in \Lambda} X_\lambda$. For each $i \in \{0, \cdots, e-1\}$ there is an equation

$$\eta_i = X_{i(1)} + X_{i(2)}, \tag{1.1a}$$

where $i(1), i(2) \in \{1, \cdots, e+1\}$, $i(1) \neq i(2)$ (see [3, section 7]). The Brauer tree Γ of $\mathbf{B}$ is defined to have $\Gamma_v = \{X_1, \cdots, X_{e+1}\}$ as set of vertices, $\Gamma_e = \{\eta_0, \cdots, \eta_{e-1}\}$ as set of edges, and $\eta_i \in \Gamma_e$ is incident with $X_j \in \Gamma_v$ if and only if $X_j \in \{X_{i(1)}, X_{i(2)}\}$.

Let R be a complete discrete valuation ring of characteristic zero, which has k as residue-class field. An *RG-lattice* A is an RG-module which is free and finitely-generated as R-module; A affords a character, which we regard as ordinary character of G. Let $I = \{0, \cdots, e-1\}$, and let $W_i (i \in I)$ be a full set of projective indecomposable RG-modules; these are RG-lattices, and we arrange them so that W_i has character η_i, for all $i \in I$.

The main purpose of this paper is to prove

THEOREM 2. *Let $G, \mathbf{B}, D, \Gamma$ be as above. Then*

(i) *The numbering of $\eta_0, \cdots, \eta_{e-1}$ can be so chosen, that there exists a family $(A_n)_{n\in \mathbf{Z}}$ of RG-lattices, and a permutation δ of the set $I = \{0, \cdots, e-1\}$, such that there exist RG-short exact sequences*

$$\mathbf{E}_{2i} \quad : 0 \to A_{2i+1} \to W_{\delta(i)} \to A_{2i} \quad \to 0$$

$$\mathbf{E}_{2i+1} : 0 \to A_{2i+2} \to W_{i+1} \to A_{2i+1} \to 0,$$

for all $i \in \mathbf{Z}$. Here i is to be taken mod *e, so that $W_n \cong W_{n+e}$ and $A_n \cong A_{n+2e}$, for all $n \in \mathbf{Z}$.*

(ii) *The $2e$ RG-modules $A_0, \cdots, A_{2e-1}$ are mutually non-isomorphic, so that the sequences $\mathbf{E}_n$ ($n \geqq 0$) provide a projective RG-resolution of A_0 which is periodic of period $2e$:*

$$(1.2a) \qquad \cdots \to W_0 \to W_{\delta(e-1)} \to W_{e-1} \to \cdots \to W_{\delta(1)} \to W_1 \to W_{\delta(0)} \to A_0 \to 0.$$

In case $\mathbf{B} = \mathbf{B}_0(G)$ is the principal block of G, we can take $A_0 = R_G$, which is regarded as trivial RG-module.

(iii) *The character P_n of A_n belongs to Γ_v, for all $n \in \mathbf{Z}$.*

(iv) *Suppose that $\mathbf{B}$ is a self-dual block, so that $\bar{\phi}_i \in \mathbf{B}$, for all $i \in I$. Define permutations β, γ of I as follows: $\bar{\phi}_i = \phi_{\beta(i)}$, and $\gamma(i) \equiv \beta\delta(0) - i$ ($i \in I$). Then we have*

$$(1.2b) \qquad \delta = \beta \cdot \gamma.$$

From (i) and (iii) one gets equations

$$(1.2c) \qquad \eta_{\delta(i)} = P_{2i} + P_{2i+1} \text{ and } \eta_{i+1} = P_{2i+1} + P_{2i+2},$$

which show that the edges $\eta_{\delta(i)}$ and η_{i+1} join successive pairs of vertices in P_{2i}, P_{2i+1}, P_{2i+2}. Thus the sequence

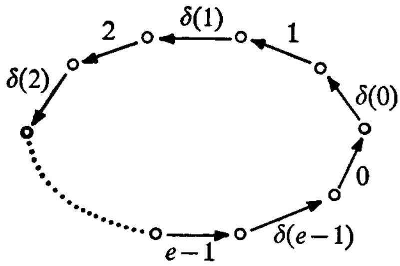
$$P_0, P_1, P_2, \cdots P_{2e-1}, P_0$$

describes a circular "walk" around Γ, accomplished in $2e$ "steps" $P_n \to P_{n+1}$ ($n = 0, \cdots, 2e-1$), so that the e edges of Γ are each traversed exactly twice, in the order

$$(1.2d) \qquad \eta_{\delta(0)}, \eta_1, \eta_{\delta(1)}, \cdots, \eta_{e-1}, \eta_{\delta(e-1)}, \eta_0.$$

It is clear that every vertex must be reached at least once during the walk.

The permutation δ is an interesting invariant of $\mathbf{B}$, since it determines Γ as abstract tree. Indeed Γ must be obtained from the cyclic graph of $2e$ edges, oriented and labelled as shown, by identifying each pair of edges which carry the same label in such a way that the two orientations "cancel each other out".



Of course this procedure can be carried out for any permutation δ of the set I, but will not necessarily give a tree.

We leave the reader to prove the following, as an application of the formula (1.2b) (see also [2, theorem 14]).

Corollary to theorem 2. *Let* $\mathbf{B}$ *be a self-dual block. Then* Γ *is an "open polygon" (i.e. an unbranched chain) if and only if either* (1) *every* ϕ_i *is real, so that* $\beta(i) = i$ *for all* $i \in I$, *or* (2) e *is even, and* $\beta(i) \equiv \frac{1}{2}e + i$ *for all* $i \in I$.

Recently Alperin and Janusz ([1]) have obtained results for the case $\mathbf{B} = \mathbf{B}_0(G)$ which are closely related to those in theorem 2. They show that $A_0 = R_G$ has a projective RG-resolution which is periodic of period $2e$:

$$\cdots \to U(E_{2e-1}) \to \cdots \to U(E_1) \to U(E_0) \to A_0 \to 0.$$

Each $U(E_n)$ is indecomposable, with character E_n, and Alperin and Janusz have observed that

$$(1.3a) \qquad E_0, E_1, E_2 \cdots, E_{2e-1}$$

are the steps in a "walk" around Γ, during which each edge of Γ appears exactly twice; they also give a rule for defining the sequence (1.3a), which is based on Janusz's classification of the indecomposable kG-modules in $\mathbf{B}$. In fact the sequences (1.2d) and (1.3a) must coincide since by Schanuel's lemma ([10, p. 167]) the terms of any minimal projective resolution of A_0 are uniquely determined up to RG-isomorphism.

Theorem 2 is proved in section 7, and the proof is based on theorem 1, which is stated and proved in section 6. Theorem 1 gives information on certain indecomposable kG-modules, and explains how the permutation δ arises. The proof of theorem 1 does not use Janusz's classification, but is based on Dade's description of the indecomposable kH-modules, where $H = N_G(D_{d-1})$ and D_{d-1} is the subgroup of order p of D ([3, section 5]). This information about kH-modules is summarised in section 5; in case $d = 1$, of course, it can be obtained more directly than by Dade's general argument. The passage from H to G, which is made in the proof of theorem 1, uses methods originating with Thompson [12],

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

and developed in [6] by Feit. Sections 2, 3, 4, contain a summary of this general theory, as far as it is needed in this paper.

2. $\mathfrak{S}$-projective maps and modules

Throughout this section, G is an arbitrary finite group, and k an arbitrary commutative ring with 1. Modules, both here and throughout the paper, are right and finitely-generated.

Let $\mathfrak{S} = \{S_\mu | \mu \in M\}$ be a set of subgroups of G. We recall that a kG-module U is said to be *$\mathfrak{S}$-projective* if there exists for each $\mu \in M$ a kS_μ-module A_μ, so that U is isomorphic to a direct summand of $\bigoplus_{\mu \in M} A_\mu^G$. U is *$\mathfrak{S}$-projective-free* if it has no non-zero $\mathfrak{S}$-projective direct summand. (If $\mathfrak{S}$ consists of a single subgroup S, we write S-projective, S-projective-free rather than $\mathfrak{S}$-projective, $\mathfrak{S}$-projective-free, and make a similar convention for all the notation of this section.)

If U, V are kG-modules let (U, V) be the k-module of all k-maps from U to V. If $\theta \in (U, V)$ and $g \in G$, let $\theta^g \in (U, V)$ be the map $u \mapsto ((ug^{-1})\theta)g$ $(u \in U)$. For any subgroup S of G, write

$$(U, V)_S = \{\theta \in (U, V) \,|\, \theta^s = \theta, \text{ all } s \in S\}.$$

Thus $(U, V)_S$ is the set of all kS-maps from U to V.

The *relative norm map* $T_{S,G} : (U, V)_S \to (U, V)_G$ is the k-map defined by $T_{S,G}(\sigma) = \Sigma \sigma^g$, for $\sigma \in (U, V)_S$, the sum being over a transversal of the coset-space $G/S = \{Sg \,|\, g \in G\}$.

Definition. A kG-map $\theta : U \to V$ is *$\mathfrak{S}$-projective* if it belongs to

$$(U, V)_{\mathfrak{S}, G} = \sum_{\mu \in M} T_{S_\mu, G}((U, V)_{S_\mu}).$$

The $\mathfrak{S}$-projective kG-maps form an "ideal" in the category $\mathscr{M}(kG)$ of all kG-modules and all kG-maps, i.e.

(2.1) *Let U, V, W be kG-modules, and let $\theta \in (U, V)_G$, $\phi \in (V, W)_G$. Then $\theta\phi$ is $\mathfrak{S}$-projective, if either θ or ϕ is $\mathfrak{S}$-projective.*

In particular $(U, U)_{S,G}$ is an ideal, in the usual sense, of the k-algebra $(U, U)_G$. For proofs of these facts, which are nearly trivial, see [7, section 1].

The following theorem of Higman and Dress gives the connection between $\mathfrak{S}$-projectivity of maps and $\mathfrak{S}$-projectivity of modules.

(2.2) (See [4, lemma 51.2] and [5, theorem 1]) *A kG-module U is $\mathfrak{S}$-projective, if and only if the identity map ι_U on U is an $\mathfrak{S}$-projective kG-map. This is equivalent to $(U, U)_G = (U, U)_{\mathfrak{S},G}$.*

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

DEFINITION. For any kG-modules U, V let $(U, V)_G^{\mathfrak{S}}$ denote the k-module $(U, V)_G / (U, V)_{\mathfrak{S}, G}$.

We can now rewrite (2.2) as

(2.3) *A kG-module U is $\mathfrak{S}$-projective if and only if* $(U, U)_G^{\mathfrak{S}} = 0$.

3. Some homological lemmas

These are some elementary pieces of homological algebra which are useful in calculating $(U, V)_G^{\mathfrak{S}}$ in the very special case $\mathfrak{S} = \{1\}$, where 1 is the unit subgroup of G. In this case we use the term *projective* (instead of 1-projective). We shall also *assume k is a field*. Throughout this section U, V denote arbitrary kG-modules.

(3.1) *Let $\pi : Q \to V$ be a projective presentation of V, i.e. Q is a projective kG-module and π is a surjective kG-map. Let $\theta \in (U, V)_G$. Then θ is projective if and only if there exists $\phi \in (U, Q)_G$ such that $\theta = \phi\pi$.*

PROOF. If such a ϕ exists, then $\theta = \phi\pi = \phi \cdot \iota_Q \cdot \pi$. But ι_Q is is projective by (2.2), so θ is projective by (2.1). Conversely, suppose that θ is projective. Then $\theta = T_{1,G}(\alpha)$ for some $\alpha \in (U, V)$. Since V is a free k-module, there exists $\beta \in (V, Q)$ such that $\beta\pi = \iota_V$. Then by a trivial calculation $\theta = \phi\pi$, where $\phi = T_{1,G}(\alpha\beta)$.

The category $\mathscr{M}(kG)$ is self-dual, by means of the contragredient functor which takes U to the usual dual kG-module $U^* = (U, k)$. Since U^* is free, if and only if U is free, the classes of projective and injective kG-modules coincide. Thus (3.1) automatically gives a dual version

(3.1*) *Let $\pi' : U \to Q'$ be an injective embedding of U, i.e. Q' is an injective (= projective) kG-module and π' is an injective kG-map. Let $\theta \in (U, V)_G$. Then θ is projective if and only if there exists $\phi \in (Q', V)_G$ such that $\theta = \pi'\phi$.*

Denote by $\Phi(U)$ the *Frattini submodule of U*, i.e. the intersection of all maximal submodules of U; denote by $\Sigma(U)$ the *socle of U*, the sum of all minimal submodules of U. We recall that a projective presentation $\pi : Q \to U$ is *minimal* if Q is minimal (among all such presentations of U); such minimal presentations always exist, and have the property $\operatorname{Ker} \pi \leqq \Phi(Q)$.

(3.2) *Suppose that U is projective-free, and that $\theta : U \to V$ is a surjective kG-map. Then θ is not projective unless $\theta = 0$.*

PROOF. Let $\pi : Q \to V$ be a minimal projective presentation of V. If θ is projective, then by (3.1) there exists $\phi \in (U, Q)_G$ such that $\theta = \phi\pi$. Thus $(\operatorname{Im} \phi)\pi = \operatorname{Im} \theta = V$, hence $\operatorname{Im} \phi + \operatorname{Ker} \pi = Q$. But $\operatorname{Ker} \pi \leqq \Phi(Q)$, hence $\operatorname{Im} \phi + \Phi(Q) = Q$, so by a standard property of Frattini modules, $\operatorname{Im} \phi = Q$. This implies that

U has a direct summand isomorphic to Q (see [4, lemma 45.2]), and since U is projective-free, we must have $Q = 0$. Therefore $V = 0$, and $\theta = 0$.

The dual version is

(3.2*) *Suppose that V is projective-free, and that $\theta : U \to V$ is an injective kG-map. Then θ is not projective unless $\theta = 0$.*

(3.3) *$(U, V)_{1,G} = 0$ in each of the two cases*

(1) U projective-free and V simple, or

(2) U simple and V projective-free.

PROOF. Suppose if possible that $\theta \in (U, V)_{1,G}$ and $\theta \neq 0$. In case (1), θ must be surjective since V is simple; this contradicts (3.2). Similarly, case (2) leads to a contradiction of (3.2*).

(3.3) has as immediate corollary

(3.4) *If U is simple and non-projective, then $(U, U)_G^1 \cong (U, U)_G$ as k-algebras.*

We conclude this section with a note on Hellers's Ω-*functor* (Heller, [8]). If U is a kG-module, define ΩU to be the kernel of a minimal projective presentation $\pi : Q \to U$, so that there is a kG-short exact sequence

$$0 \to \Omega U \to Q \xrightarrow{\pi} U \to 0. \tag{3.5a}$$

By Schanuel's lemma ([10, p. 167]) ΩU is defined up to isomorphism by (3.5a), and the fact that π is minimal. We can make exactly the same construction for RG-lattices, where R is the ring defined in section 1. Heller has proved

(3.5) *Let U be a kG-module, or a RG-lattice. Then if U is indecomposable and non-projective, so is ΩU.*

We say that a kG-module U can be "lifted" to an RG-lattice M, if $\bar{M} \cong U$, where $\bar{M} = M/\mathfrak{p} M$ and $\mathfrak{p}$ is the maximal ideal of R. Every projective kG-module Q, can be lifted to a projective RG-lattice P.

(3.6) *Suppose there is a kG-short exact sequence*

$$0 \to V \to Q \to U \to 0 \tag{3.6a}$$

with Q projective, and that Q, U can be lifted to P, M as above. Then V can be lifted to an RG-lattice N, and there is an RG-short exact sequence

$$0 \to N \to P \to M \to 0, \tag{3.6b}$$

which, in an obvious sense, "lifts" (3.6b).

PROOF. This follows easily from the projective property of P.

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

4. The functors f and g

In this section D is any p-subgroup of G, and H any subgroup of G which contains $N_G(D)$. Define

$$\mathfrak{X} = \{D \cap D^g \mid g \in G \backslash H\}, \qquad \mathfrak{Y} = \{H \cap D^g \mid g \in G \backslash H\},$$

so that these are sets of subgroups of H. We state the results below for the case where k is a field of characteristic p; they hold also with k replaced by the ring R of section 1. Proofs of (4.1) to (4.4) are in [7].

(4.1) (i) *Let U be a D-projective kG-module. Then there exists a $\mathfrak{Y}$-projective-free kH-module fU, and a $\mathfrak{Y}$-projective kH-module U_0, such that*

(4.1a) $$U_H \cong fU \oplus U_0.$$

(ii) *Let L be a D-projective kH-module. Then there exists an $\mathfrak{X}$-projective-free kG-module gL, and an $\mathfrak{X}$-projective kG-module L_0, such that*

(4.1b) $$L^G \cong gL \oplus L_0.$$

Notice that fU, gL are determined up to isomorphism, by the Krull-Schmidt theorem.

Now define $\mathfrak{A}$ to be the set of all subgroups S of D, which are not G-conjugate to a subgroup of any X in $\mathfrak{X}$.

(4.2) *Suppose that U, L above are both indecomposable, and have vertices D_0, $D_1 \in \mathfrak{A}$. Then fU, gL are both indecomposable, and have vertices D_0, D_1 respectively. Moreover*

(4.2a) $$g(fU) \cong U, \text{ and}$$

(4.2b) $$f(gL) \cong L.$$

This shows that f, g determine a one-one, vertex-preserving correspondence between the kG-isomorphism classes of indecomposable kG-modules with vertex in $\mathfrak{A}$, and the kH-isomorphism classes of indecomposable kH-modules with vertex in $\mathfrak{A}$. This *module correspondence* determined by (G, H, D) is closely connected with Brauer's famous *block correspondence* (see [4, theorem 58.3]) between the kG-blocks of defect group D, and the kH-blocks of defect group D.

(4.3) *Let U, L be as in* (4.2). *Let $\mathbf{B}, \mathbf{B}'$ be, respectively, kG,kH-blocks of defect group D, which correspond under Brauer's block correspondence. Then*

(4.3a) *U belongs to $\mathbf{B}$ if and only if fU belongs to $\mathbf{B}'$, and*

(4.3b) *L belongs to $\mathbf{B}'$ if and only if gL belongs to $\mathbf{B}$.*

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

(4.3) is proved in [7, theorem 5.8]. (It is clear from (4.2) that each of (4.3a), (4.3b) implies the other.)

Finally we may apply f and g to maps, in a functorial way. For each kG-module U, choose a decomposition (4.1a), with projection $p_U : U_H \to fU$ and injection $i_U : fU \to U_H$. If U, V are kG-modules and $\theta : U \to V$ is a kG-map, we define the kH-map $f\theta : fU \to fV$ by

$$f\theta = i_U \cdot \theta_H \cdot p_V.$$

Here θ_H is θ, regarded as kH-map.

(4.4) *Let U, V be D-projective kG-modules, and let $\theta : U \to V$ be an arbitrary kG-map. Then*

(i) $f\iota_U = \iota_{fU}$.

(ii) *$\theta \mapsto f\theta$ induces a k-isomorphism*

(4.4a) $$(U, V)_G^{\mathfrak{X}} \cong (fU, fV)_H^{\mathfrak{X}}.$$

In case $U = V$, this is an isomorphism of k-algebras.

One can also define $g\alpha : gL \to gM$, for any kH-map $\alpha : L \to M$ (L, M kH-modules), and prove an analogue to (4.4). However we do not need this, and so we do not give it.

Our final lemma (4.5) shows that f, g "commute" with the functor Ω. We leave the proof to the reader. (In fact we shall need only the formula (4.5*b*), and only in a case where $\mathfrak{X} = \{1\}$. This can be proved by an application of Schanuel's lemma.)

(4.5) *Let U be a D-projective kG-module, and L a D-projective kH-module. Then*

(4.5a) $$f\Omega U \cong \Omega f U, \text{ and}$$

(4.5b) $$g\Omega L \cong \Omega g L.$$

5. Indecomposable kH-modules in $\mathbf{B}'$

From now on, $G, \mathbf{B}, D, k, R$ are as in section 1. Let $H = N_G(D_{d-1})$, where D_{d-1} is the subgroup of D of order p. Since $H \geqq N_G(D)$, the correspondences of section 4 apply.

Let $\mathbf{B}'$ be the kH-block corresponding to $\mathbf{B}$. In this section we give Dade's results on indecomposable kH-modules in $\mathbf{B}'$.

Let $C = C_G(D_{d-1})$[1]. Dade shows ([3, section 1]) that there is a kC-block $\mathbf{b}$

(1) The table below gives equivalents in [3], for notations used in section 5.

Section 5:	d	q	D_{d-1}	H	C	F	$\mathbf{B}'$	$\mathbf{b}$
[3]:	a	p^a	D_{a-1}	N_{a-1}	C_{a-1}	$E.C_{a-1}$	B_{a-1}	b_{a-1}

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

such that $\mathbf{b}^H = \mathbf{B}'$, and that all such kC-blocks are conjugate in H. The stabilizer F of $\mathbf{b}$ in H has the form $F = E \cdot C$, where E is a certain subgroup of $N_G(D)$; F/C is cyclic of order e dividing $p-1$, because it is a subgroup of H/C which is isomorphic to a subgroup of $\mathrm{Aut}(D_{d-1})$. We may use this as our definition of e, or alternatively use Dade's definition in [3, section 1], and use [3, lemma 1.4].

Write $q = p^d$, and $I = \{0, 1, \cdots, e-1\}$.

(**5.1**) (i) $\mathbf{B}'$ *contains e simple kH-modules* $S_i\,(i \in I)$, *such that every simple kH-module in* $\mathbf{B}'$ *is isomorphic to exactly one* S_i. *Let* $T_i\,(i \in I)$ *be projective indecomposable kH-modules, numbered so that* $T_i/\Phi(T_i) \cong S_i\,(i \in I)$.

(ii) *There is a multiplicative isomorphism* $\sigma \mapsto \bar{\sigma}$ $(\sigma \in D)$ *of D into the centre of kC, such that if* α *is a generator of D then for any* $i \in I$

$$T_i > T_i(\bar{\alpha}-1) > T_i(\bar{\alpha}-1)^2 > \cdots > T_i(\bar{\alpha}-1)^{q-1} > T_i(\bar{\alpha}-1)^q = 0 \qquad (5.1a)$$

is the unique kH-composition series of T_i.

(iii) *Every indecomposable kH-module in* $\mathbf{B}'$ *is isomorphic to exactly one of the following*

$$T_{i,\nu} = T_i/T_i(\bar{\alpha}-1)^\nu \, (i \in I, \nu \in \{1, \cdots, q\}).$$

In particular $S_i \cong T_{i,1}$ *and* $T_i \cong T_{i,q}$, *for all* $i \in I$.

Proof. For each $\sigma \in D$, take $\bar{\sigma}$ to be the residue class mod $\mathfrak{p} \cdot \mathfrak{O}C$ of the element $\tilde{\sigma}$ defined in [3, (5.3)]. We may interpret $\bar{\sigma}$ as an element in kC. All parts of (5.1) now follow from [3, section 5].

(**5.2**) *Let* S, S' *be simple kH-modules in* $\mathbf{B}'$, *such that* $S_F \cong S'_F$. *Then* $S \cong S'$.

Proof. Any simple component of S_C or S'_C is the unique (up to isomorphism) simple kC-module in some H-conjugate of $\mathbf{b}$, and hence has F as its stabilizer in H. Now (5.2) follows from Clifford's theorem (see [4, theorem 14.1]).

We need information on the order of the composition factors in (5.1a). Let α be a generator of D, and let $\alpha_1 = \alpha^{p^{d-1}}$. Since H normalises $D_{d-1} = \langle \alpha_1 \rangle$, there is a linear representation $\psi : H \to k^*$, given by

$$\psi(h) = n(h) \cdot 1_k \; (h \in H), \qquad (5.3a)$$

where $n(h) \in \mathbf{Z}$ is defined up to congruence mod p by

$$\alpha_1^h = \alpha_1^{n(h)} \qquad (h \in H). \qquad (5.3b)$$

Evidently $\psi(c) = 1$, for all $c \in C$, and hence $\psi^{[H:C]} = 1$, the trivial representation of H.

Now let z be an element of E. Since z normalises D,

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

(5.3c) $$\alpha^z = \alpha^{n(z)},$$

where $n(z) \in \boldsymbol{Z}$ is defined up to congruence mod p^d. Since (5.3c) implies $\alpha_1^z = \alpha_1^{n(z)}$, there is no conflict between (5.3b) and (5.3c). From [3, (5.3)] we find that $(\bar{\sigma})^z = \overline{\sigma^z}$, for all $\sigma \in D$. Taking $\sigma = \alpha$ and using (5.3c) we have

(5.3d) $$\bar{\alpha}^z = \bar{\alpha}^{n(z)} \qquad (z \in Z).$$

In the next theorem, $\psi^\nu (\nu \in \boldsymbol{Z})$ denotes, abusively, a 1-dimensional kH-module which affords the representation ψ^ν; $\otimes = \otimes_k$ is the usual "external tensor product" of kH-modules. In particular, $\psi^\nu \cong \psi \otimes \cdots \otimes \psi$ (ν factors ψ), if $\nu > 0$.

(5.4) (i) *Let* $S_{i,\nu} = T_i(\bar{\alpha} - 1)^\nu / T_i(\bar{\alpha} - 1)^{\nu+1}$, *for given* $i \in I$ *and* $\nu \in \{0 \cdots, q-1\}$. *Then* $S_{i,\nu} \cong \psi^\nu \otimes S_i$.

(ii) *Write* $S_n = \psi^n \otimes S_0$, *for all* $n \in \boldsymbol{Z}$. *Then* $S_m \cong S_n$ *if and only if* $m \equiv n$ *mod* e, *and we can take* $S_0, \cdots, S_{e-1}$ *for the set of simple kH-modules in* $\mathbf{B}'$ *mentioned in* (5.1) (i).

(iii) *With the notation just given, the composition factors of* T_i *(see* (5.1*a*)) *are* $S_i, S_{i+1}, \cdots, S_{i+q-1} \cong S_i$.

Proof. (i) Let $t \in T_i$ and $z \in E$. Then using (5.3a) – (5.3d) one finds

$$t(\bar{\alpha} - 1)^\nu z = tz(\bar{\alpha}^{n(z)} - 1)^\nu = tz(\bar{\alpha} - 1)^\nu (1 + \bar{\alpha} + \cdots + \bar{\alpha}^{n(z)-1})^\nu$$
$$\equiv tz(\bar{\alpha} - 1)^\nu \cdot n(z)^\nu \mod T_i(\bar{\alpha} - 1)^{\nu+1},$$

and hence

(5.4a) $$t(\bar{\alpha} - 1)^\nu z \equiv \psi^\nu(z) \cdot tz(\bar{\alpha} - 1)^\nu \mod T_i(\bar{\alpha} - 1)^{\nu+1},$$

for all $z \in E$. But (5.4a) holds also for all $z \in C$, since then z commutes with $\bar{\alpha}$, and $\psi(z) = 1$. Hence (5.4a) holds for all z in $F = E \cdot C$ and it is easy to deduce that $(S_{i,\nu})_F \cong (\psi^\nu \otimes S_i)_F$. Therefore $S_{i,\nu} \cong \psi^\nu \otimes S_i$ by (5.2).

(ii) Since $\psi^{[H:C]} = 1$, we have $S_m \cong S_n$ if $m \equiv n \mod [H:C]$. Now $[H:C]$ divides $p - 1$, therefore is less than q, and so it follows from (i) that all the modules

(5.4b) $$S_0, S_1, S_2, \cdots$$

are composition factors of T_0, and hence lie in $\mathbf{B}'$. Conversely, let S be any simple kH-module in $\mathbf{B}'$. By Brauer's theorem [4, theorem 46.2] there exists a finite sequence of elements $i_0, i_1, \cdots, i_r \in I$, such that $S_{i_0} = S_0$, $S_{i_r} \cong S$, and for each $\rho \in \{1, \cdots, r\}$, S_{i_ρ} is a composition factor of $T_{i_{\rho-1}}$. Then (i) shows that for each ρ, $S_{i_\rho} \cong \psi^{\nu(\rho)} \otimes S_{i_{\rho-1}}$ for some $\nu(\rho) \in \boldsymbol{Z}$. Therefore $S \cong \psi^n \otimes S_0$ for some $n \in \boldsymbol{Z}$, and hence S is isomorphic to some module in (5.4b). We now know (by (5.1)(i)) that (5.4b) contains exactly e non-isomorphic modules, and all the statements of (ii) follow.

(iii) is an immediate consequence of (i), (ii) and (5.1a).

(5.4) (ii) allows us to adopt, as we shall do henceforth, the following

NOTATION. S_i is defined for all $i \in \mathbf{Z}$, in such a way that $S_i \cong \psi^i \otimes S_0$; hence $S_{i+e} \cong S_i$ $(i \in \mathbf{Z})$. Similarly $T_i, T_{i,v}$ are defined for all $i \in \mathbf{Z}$, in such a way that (5.1) holds, and $T_{i+e} \cong T_i$, $T_{i+e,v} \cong T_{i,v}$ $(i \in \mathbf{Z},\ v \in \{1, \cdots q\})$.

(5.5) *Let* $i \in I$, $v \in \{1, \cdots, q\}$. *Then*

(i) $T_{i,v}$ *is projective if and only if* $v = q$.

(ii) *There is a kH-short exact sequence*

$$0 \to T_{i+v,q-v} \to T_{i,q} \to T_{i,v} \to 0. \tag{5.5a}$$

(iii) *If* $1 \leqq v \leqq q-1$, $\Omega T_{i,v} \cong T_{i+v,q-v}$.

(iv) $\Omega^2 S_i \cong S_{i+1}$.

PROOF. (i) and (ii) follow from (5.1), (5.4). (iii) follows from the definition of the Ω-functor, and the fact that $T_{i,q} \to T_{i,v}$ is a minimal projective presentation if $1 \leqq v \leqq q-1$. Then (iv) follows from (iii) and the fact that $S_i \cong T_{i\,1}$, all i.

We conclude with a remark on the dimensions of the modules S_i. Let $\dim S_0 = N_0$. Then $\dim S_i = N_0$ for all i, since $S_i \cong \psi^i \otimes S_0$. Let $\pi(n)$, for any positive integer n, denote the exponent of the highest power of p which divides n. A general theorem of Brauer on blocks ([4, theorem 61.6 (2)]) now gives at once

$$\pi(N_0) = \pi(|H|) - d. \tag{5.6}$$

6. The permutation δ

Take $G, D, H, \mathbf{B}, \mathbf{B}'$ as in section 5. We shall apply the module correspondence of section 4, using the notation $f, g, \mathfrak{X}, \mathfrak{Y}, \mathfrak{A}$ there. In our case $\mathfrak{X} = \{1\}$, since it is clear that $D \cap D^g = 1$ for any $g \in G \backslash H$. In general $\mathfrak{Y} \neq \{1\}$. The set $\mathfrak{A}$ consists of all the subgroups of D except 1.

THEOREM 1. (i) $\mathbf{B}$ *contains e simple kG-modules* $V_i (i \in I)$, *such that every simple kG-module in* $\mathbf{B}$ *is isomorphic to exactly one* V_i. *Let* $\bar{W}_i$ $(i \in I)$ *be projective indecomposable kG-modules such that* $\bar{W}_i / \Phi(\bar{W}_i) \cong V_i (i \in I)$.

(ii) *The numbering of the* $V_i (i \in I)$ *can be arranged so that*

(6.1a) $\quad (fV_j, S_i)_H \cong (V_j, gS_i)_G \cong k$ *or zero, according as* $i = j$ *or* $i \neq j$, *and there is a permutation* δ *of* I *such that*

(6.1b) $\quad (S_i, fV_j)_H \cong (gS_i, V_j)_G \cong k$ *or zero, according as* $\delta(i) = j$ *or* $\delta(i) \neq j$, *for all* $i, j \in I$.

(iii) *For each* $i \in I$ *there exist kG-short exact sequences*

$$\mathsf{F}_{2i}: \quad 0 \to \Omega g S_i \to \bar{W}_{\delta(i)} \to g S_i \to 0.$$

$$\mathsf{F}_{2i+1}: 0 \to g S_{i+1} \to \bar{W}_{i+1} \to \Omega g S_i \to 0,$$

Theorem 1 is proved below, in a series of lemmas. Let $V_j \, (j \in J)$ be a full set of mutually non-isomorphic simple kG-modules in B, indexed by a suitable finite set J. All the S_i and V_j are D-projective, since they belong to blocks with D as defect group ([4, theorem 54.10]). On the other hand no S_i, V_j is projective, since a simple, projective kG-module S (for any finite group G), must lie in a block of defect group 1[(2)]. Therefore each S_i, V_j has vertex in $\mathfrak{A}$, and we can apply (4.2), (4.3) to prove

(6.2) *fV_j is indecomposable, non-projective and lies in B'. gS_i is indecomposable, non-projective and lies in B.*

We have now, for any $i \in I, j \in J$, that $(S_i, fV_j)_H^1 \cong (gS_i, V_j)_G^1$ by (4.2), (4.4a). But (3.3) gives also $(S_i, fV_j)_H^1 \cong (S_i, fV_j)_H$, and $(gS_i, V_j)_G^1 \cong (gS_i, V_j)_G$. This proves the first part of (6.3) below, and the second part is proved similarly.

(6.3) $(S_i, fV_j)_H \cong (gS_i, V_j)_G$ *and* $(fV_j, S_i)_H \cong (V_j, gS_i)_G$.

(6.4) *There is a map $h : J \to I$ such that for all $i \in I, j \in J$,*

(6.4a) $h(j) = i$ *if and only if* $(fV_j, S_i)_H \neq 0$. *Moreover h is a bijection, and hence* $|J| = |I| = e$.

PROOF. Take any $j \in J$. By (6.2), (5.1), (5.5) we have $fV_j \cong T_{h(j), v(j)}$ for some $h(j) \in I$ and some $v(j) \in \{1, \cdots, q-1\}$. Since $T_{h(j), v(j)}$ is uniserial, with "top" composition factor $S_{h(j)}$, one has by Schur's lemma

(6.4b) $(fV_j, S_i)_H \cong k$ *or zero, according as* $h(j) = i$ *or* $h(j) \neq i$.

This establishes the existence of h, and proves (6.4a).

Now suppose $i \in I$ is given. Take any minimal submodule S of gS_i. Since S is in B, there exists $j \in J$ such that $V_j \cong S$. This implies $(V_j, gS_i)_G \neq 0$, hence $(fV_j, S_i)_H \neq 0$ by (6.3). Thus $h(j) = i$. We have now proved that h is surjective.

Suppose $j, j' \in J$ are such that $h(j) = i = h(j')$. Then $fV_j = T_{i,v}$ and $fV_{j'} = T_{i,v'}$ for some $v, v' \in \{1, \cdots, q-1\}$. We may assume $v \geqq v'$. But then there exists a surjective kH-map $\theta : T_{i,v} \to T_{i,v'}$, and by (3.2) θ is not projective. Therefore $(fV_j, fV_{j'})_H^1 \neq 0$. By (4.4a) $(V_j, V_{j'})_G^1 \neq 0$, hence $(V_j, V_{j'})_G \neq 0$. Schur's lemma now gives $j = j'$. Therefore h is a bijection.

We have now proved part (i) of theorem 1. From now on we take $J = I$ and

---

(2) Because S is projective, its dimension is divisible by p^a $(a = \pi(|G|))$; also S can be lifted to an RG-lattice M, for the same reason. The character χ of M must be simple, because S is simple. Since p^a divides $\chi(1)$, χ, and hence also S, lies is a block of defect zero, i.e. of defect group 1.

arrange notation of the V so that the map $h: I \to I$ is the identity. This means that we have for each $j \in I$

(6.4c) $fV_j = T_{j\,\nu(j)}$, *for some* $\nu(j) \in \{1, \cdots, q-1\}$.

Formula (6.1a) in theorem 1 is now a consequence of (6.4b) and (6.3). But we may now prove a "dual" version of (6.4), which in the present notation will show that there is a map $\delta: I \to I$, such that for all $i, j \in I$, we have $\delta(i) = j$ if and only if $(S_i, fV_j)_H \neq 0$; moreover δ is a bijection, i.e. it is a permutation of I. The proof is exactly parallel to that of (6.4). In place of (6.4b) one has, using the uniseriality of $fV_j = T_{j\,\nu(j)}$,

(6.4d) $(S_i, fV_j)_H \cong k$ *or zero, according as* $\delta(i) = j$ *or* $\delta(i) \neq j$.

Now (6.4d) and (6.3) yield (6.1b), and we have proved part (ii) of theorem 1.

(6.1a), (6.1b) tell us that, for all $i \in I$,

$$\Sigma(gS_i) \cong V_i \text{ and } gS_i/\Phi(gS_i) \cong V_{\delta(i)}.$$

From the second of these, and using the projective property of $\bar{W}_{\delta(i)}$, one can make a projective presentation $\bar{W}_{\delta(i)} \to gS_i$. This must be minimal, since $\bar{W}_{\delta(i)}$ is indecomposable. Therefore there exists a kG-short exact sequence $\mathbf{F}_{2i}$.

From $\Sigma(gS_{i+1}) \cong \bar{W}_{i+1}$,[3] and using the injective property of $\bar{W}_{i+1}$, one can make an injective embedding $gS_{i+1} \to \bar{W}_{i+1}$. Hence there is some kG-module V, and a short exact sequence

(6.4e) $$0 \to gS_{i+1} \to \bar{W}_{i+1} \to V \to 0.$$

On the other hand we deduce from (4.5b) and (5.5)(iv) that $\Omega(\Omega gS_i) \cong g(\Omega^2 S_i) \cong gS_{i+1}$. So there is a short exact sequence

$$0 \to gS_{i+1} \to \bar{W} \to \Omega gS_i \to 0,$$

where $\bar{W}$ is projective. The dual form of Schanuel's lemma now gives $V \cong \Omega gS_i$, and hence we can take (6.4e) as $\mathbf{F}_{2i+1}$. We have now proved all parts of theorem 1.

REMARK. The integers $\nu(j)$ in (6.4c) satisfy the condition

(6.5a) *For each* $j \in I$, *either* $1 \leqq \nu(j) \leqq e$, *or* $q - e \leqq \nu(j) \leqq q - 1$.

This result is due to Feit, and is proved by applying a lemma of Passman [6, lemma 4] to the condition $(T_{j\,\nu(j)}, T_{j,\nu(j)})_H^1 \cong k$, which in turn follows from Schur's lemma, and (4.4a) in the case $U = V = V_j$. Now $gT_{j\,\nu(j)} \cong V_j$, by (6.2), (4.2a), so that

---

(3) We extend the definition of $\bar{W}_i$ to any $i \in Z$, by the convention $\bar{W}_{i+e} = \bar{W}_i$ $(i \in Z)$. See section 7.

(6.5b) $$T_{j,v(j)}{}^G \cong V_j \oplus U_0,$$

where U_0 is a projective kG-module. By (5.1), (5.5), (5.6) we have $\dim T_{j,v(j)}{}^G = v(j) \cdot N$, where $N = N_0 \cdot [G : H]$, and

(6.5c) $$\pi(N) = a - d \quad (a = \pi(|G|)).$$

On the other hand, $\dim U_0 \equiv 0 \bmod p^a$ ([4, lemma 59.6]). So taking dimensions on both sides of (6.5b), we have

(6.5d) $$\dim V_j \equiv v(j) \cdot N \bmod p^a.$$

Taking (6.5d) with (6.5a), we get a strengthened form of a theorem of Rothschild [11]. Feit points out in [6], that this theorem shows that all the V_j have vertex D. However this is not necessary for the proof of theorem 2.

7. Proof of theorem 2

Define projective indecomposable RG-lattices W_i, such the module $\bar{W}_i$ in theorem 1 lifts to W_i, for all $i \in I$. Then define W_i for all $i \in \boldsymbol{Z}$, by the rule $W_i = W_{i+e}$ for all $i \in \boldsymbol{Z}$. With the notations of section 5, we can say that the kG-short exact sequences $\mathbf{F}_{2i}$, $\mathbf{F}_{2i+1}$ of theorem 1 exist for all $i \in \boldsymbol{Z}$, and that $\mathbf{F}_n \cong \mathbf{F}_{n+2e}$ ($n \in \boldsymbol{Z}$), with the usual isomorphism of short exact sequences. Write $B_{2i} = gS_i$, $B_{2i+1} = \Omega g S_i$, all $i \in \boldsymbol{Z}$.

(7.1) *Let M be an RG-lattice and m a fixed element of $\boldsymbol{Z}$ such that $\bar{M} \cong B_m$. Then we can construct RG-lattices A_n and sequences $\mathbf{E}_n$, with $A_m = M$, and $\bar{A}_n \cong B_n$ and $\mathbf{E}_n$ "lifts" $\mathbf{F}_n$, for all $n \in \boldsymbol{Z}$.*

Proof. By (3.6) we can lift $\mathbf{F}_m$ to an RG-short exact sequence $\mathbf{E}_m$ in which $A_m = M$, and $\bar{A}_{m+1} \cong B_{m+1}$. Now we can repeat the process with $m+1$ in place of m. So we define, inductively, $\mathbf{E}_n$ for all $n \geqq m$. Now we can apply (3.6) to the "dual" of $\mathbf{F}_{m-1}$, and regard the result as the dual of an RG sequence $\mathbf{F}_{m-1}$, which lifts F_{m-1}. Proceeding in this way, we can define $\mathbf{E}_n$ for all $n < m$.

(7.2) *Let M, A_n be as in* (7.1), *and assume that the character P_m of $M = A_m$ lies in Γ_v. Then the character P_n of A_n lies in Γ_v, for all $n \in \boldsymbol{Z}$. Also $P_{n+2e} = P_n$, for all $n \in \boldsymbol{Z}$.*

Proof. We have equations (1.2c), for all $i \in \boldsymbol{Z}$. Taking these together with equations (1.1a), it is clear that all the P_n lie in Γ_v, as soon as P_m does. To prove the final statement of (7.2), suppose first that $n = 2i$ ($i \in \boldsymbol{Z}$). From (1.2c) we have

$$\begin{aligned} \eta_{\delta(i)} &= P_n + P_{n+1}, \\ \eta_{i+1} &= P_{n+1} + P_{n+2}, \\ &\cdots \quad \cdots \end{aligned}$$

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

$$\eta_{\delta(i+e-1)} = P_{-+2e-2} + P_{n+2e-1},$$

$$\eta_{i+e} = P_{n+2e-1} + P_{n+2e}.$$

Form the alternating sum of these $2e$ equations. We find $0 = P_n - P_{n+2e}$, as required. A similar argument works in the case $n = 2i+1$ ($i \in \boldsymbol{Z}$).

(7.3) *Let M, A_n be as in* (7.2). *Then $A_{n+2e} \cong A_n$, for all $n \in \boldsymbol{Z}$.*

PROOF. Let $i \in \boldsymbol{Z}$, and let K be the quotient field of R. From (1.1a) it is clear that $K \otimes_R W_i$ has unique KG-submodules, $Y_{i(1)}$, $Y_{i(2)}$ say, with characters $X_{i(1)}$, $X_{i(2)}$ respectively, and that these are the only KG-submodules of $K \otimes_R W_i$ which have characters in Γ_v. Thus $W_i \cap Y_{i(1)}$, $W_i \cap Y_{i(2)}$ are the only R-pure RG-submodules of W_i, which have characters in Γ_v. Fix $n \in \boldsymbol{Z}$. Then $\mathbf{E}_{n-1}$ shows that A_n is isomorphic to an R-pure RG-submodule of some W_i, similarly $\mathbf{E}_{n+2e-1}$ shows that A_{n+2e} is isomorphic to an R-pure RG-submodule of $W_{i+e} = W_i$. But (7.2) shows that these submodules of W_i have the same character. So they coincide, i.e. $A_{n+2e} \cong A_e$.

(7.1), (7.2), (7.3) allow us to prove parts (i) and (iii) of theorem 1, as soon as we have an RG-lattice M such that $\bar{M} \cong B_m$ for some m, and M has character in Γ_v. If $\mathbf{B} = \mathbf{B}_0(G)$, the principal block of G, we just take $M = A_0 = R_G$. In general we proceed as follows.

Let M be the indecomposable RG-lattice in $\mathbf{B}$, which has character X_1, and is defined in [3, p. 39]. By [3, lemma 6.2] there is an indecomposable RH-lattice L such that

$$M_H = L \oplus Q_1 \oplus \cdots \oplus Q_t,^{(4)} \tag{7.4}$$

where the Q_j are indecomposable RH-modules, each of which is either projective, or lies in a block other than $\mathbf{B}'$. Both $\bar{M}, \bar{L}$ are shown to be indecomposable and it follows by taking (7.4) mod $\mathfrak{p}$, and using (4.1), (4.2), (4.3) that $\bar{L} \cong f\bar{M}$, and hence $\bar{M} \cong g\bar{L}$. But ([3, p. 39]) L has character either $\tilde{X}_i$ ($i \in I$) or $\sum \tilde{X}_\lambda$, where $\tilde{X}_0, \cdots, \tilde{X}_{e-1}$, $\tilde{X}_\lambda$ ($\lambda \in \Lambda$) are the ordinary characters of $\mathbf{B}'$; now [3, theorem 4] and (5.5)(iii) show that $\bar{L}$ is isomorphic either to S_i, or to ΩS_i, for some $i \in I$. Therefore $\bar{M}$ is isomorphic either to gS_i or to $\Omega g S_i$, i.e. to some B_m. Now (7.1), (7.2), (7.3) allow us to prove parts (i) and (iii) of theorem 1.

To prove part (ii) of theorem 2, observe that the construction used for (7.1) shows that $\bar{A}_0, \bar{A}_1, \cdots, \bar{A}_{2e-2}, \bar{A}_{2e-1}$ are isomorphic to

$$gS_0, \Omega g S_0, \cdots, g S_{e-1}, \Omega g S_{e-1}.$$

These modules are mutually non-isomorphic, provided $q = |D| \neq 2$, for in this case S_0, $\Omega S_0, \cdots, S_{e-1}$, ΩS_{e-1} are easily seen to be non-isomorphic. Hence A_0,

(4) M, L, Q, are denoted by corresponding script capitals in [3].

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

$\cdots, A_{2e-1}$ are non-isomorphic, which proves (ii) in case $q \neq 2$. If $q = 2$, then $A_0 \cong A_1$, but an *ad hoc* argument proves that $A_0 \not\cong A_1$ anyway. For we have $p = 2$, $e = 1$ and $(q-1)/e = 1$. Therefore equations (1.1a) reduce to the single equation $\eta_0 = X_1 + X_2$. But (7.2) gives $\eta_0 (= \eta_{\delta(0)}) = P_0 + P_1$. Hence $\{P_0, P_1\} = \{X_1, X_2\}$, which shows that $P_0 \neq P_1$, therefore $A_0 \not\cong A_1$. So (ii) holds in all cases.

It remains to prove part (iv) of theorem 2. We assume $\mathbf{B}$ is self-dual, and that β is the permutation of I given by $\bar{\Phi}_i = \phi_{\beta(i)}$. Evidently $\beta = \beta^{-1}$. We have also

$$\bar{W}_i^* \cong \bar{W}_{\beta(i)} \tag{7.5}$$

for all $i \in I$, and by an obvious extension of β, we can say that (7.5) holds for all $i \in \mathbf{Z}$. Apply the dual functor to $\mathbf{F}_{2i-1}$, and use (7.5). We get the kH-short exact sequence

$$0 \to (\Omega g S_i)^* \to \bar{W}_{\beta(i)} \to (g S_i)^* \to 0. \tag{7.6}$$

It is trivial that the functor g commutes with $*$, hence $\mathbf{B}'$ is self-dual, and so $S_0^* \cong S_m$ for some $m \in I$. Then by (5.5) one has for any $i \in I$, $S_i^* \cong (\psi^i \otimes S_0)^* \cong \psi^i \otimes S_m \cong S_{m-i}$. Hence $(gS_i)^* \cong gS_{m-i}$. Now compare (7.6) with

$$\mathbf{F}_{2(m-i)} : 0 \to gS_{m-i} \to \bar{W}_{\delta(m-i)} \to \Omega g S_{m-i} \to 0.$$

Schanuel's lemma gives $\bar{W}_{\beta(i)} \cong \bar{W}_{\delta(m-i)}$, and therefore $\beta(i) \equiv \delta(m-i) \bmod e$, for all $i \in \mathbf{Z}$. Replace i by $m - i$; we have

$$\delta(i) \equiv \beta(m-i) \bmod e, \text{ all } i \in \mathbf{Z}. \tag{7.7}$$

If we put $i = 0$ in (7.7), we find $\delta(0) = \beta(m)$, so $m = \beta^{-1}\delta(0) = \beta\delta(0)$. Thus (7.7) is the formula (1.2b) which we want.

References

[1] J. Alperin and H. Janusz, *Resolutions and periodicity*, Proc. Amer. Math. Soc. 37 (1973), 403–406.

[2] R. Brauer, 'Investigations on group characters', *Ann. of Math.* 42 (1941), 936–958.

[3] E. C. Dade, 'Blocks with cyclic defect groups', *Ann. of Math.* 84 (1966), 20–48.

[4] L. Dornhoff, *Group Representation Theory*, Marcel Dekker, New York, 1972.

[5] A. Dress, 'Vertices of integral representations', *Math. Z.* 114 (1970), 159–169.

[6] W. Feit, Some properties of the Green correspondence, Theory of Finite Groups, Symposium, Harvard Univ., Cambridge, Mass., 1968.

[7] J. A. Green, 'Relative module categories for finite groups', *J. Pure Applied Algebra* 2 (1972), 371–393.

[8] A. Heller, 'The loop-space functor in homological algebra', *Trans. Amer. Math. Soc.* 96 (1960), 382–394.

[9] G. Janusz, 'Indecomposable modules for finite groups', *Ann. of Math.* 89 (1969), 209–241.

https://doi.org/10.1017/S1446788700016761 Published online by Cambridge University Press

[10] I. Kaplansky, *Fields and Rings*, University of Chicago Press, Chicago, Ill., 1969.
[11] B. Rothschild, 'Degrees of irreducible modular characters of blocks with cyclic defect groups', *Bull. Amer. Math. Soc.* 73 (1967), 102–104.
[12] J. Thompson, 'Vertices and sources', *J. Algebra* 6 (1967), 1–6.

Mathematics Institute
University of Warwick
Coventry CV4 7AL
England