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C. Colautti, Beatriz Ruiz, Enrique E. Sicre, Mario Garavaglia

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Walsh functions: analysis of their properties under Fresnel diffraction

C. COLAUTTI, B. RUIZ, E. E. SICRE and M. GARAVAGLIA
Centro de Investigaciones Opticas (CIC), Casilla de Correo 124, 1900 La Plata, Republic of Argentina

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Abstract. The free-propagation properties of the light field diffracted by an aperture, have been investigated by employing a formalism based on the orthogonal and complete set of Walsh functions. We found an interesting link with the well-known self-imaging phenomenon, which can be used to explain the spatial filtering properties of many optical devices. An experimental result is given in order to illustrate this approach.

1. Introduction

The diffraction properties of most optical imaging systems are properly described in terms of Fourier optics. The input signal of such systems is analysed by considering a linear superposition of sinusoidal functions. After taking into account the effect of the system upon each sinusoidal component, the output signal is obtained by performing a suitable synthesis. However, while the analysis of optical systems is simplified by use of sinusoidal functions (i.e. under coherent illumination, the manipulation of the Fourier spectrum can be performed in a relatively easy way), the interaction between diffracted field and apertures is generally complicated. On the other hand, the analysis and synthesis of systems by means of the Walsh–Hadamard transform play an important role in digital image processing, since the operations are performed by using functions that only take values ± 1. For this reason, they suit computer processing and allow fast algorithms to be developed.

It is clear therefore that these two types of function have a well-defined scope in the field of image processing. However, in spite of their discontinuous nature (which seems not to be a natural representation signal for analogical systems), the Walsh functions have the remarkable property of being cyclically periodic [1], i.e. they can be considered as square-wave functions with step-like amplitude changes occurring at periodical intervals. Since spatial periodicity is a sufficient condition for an object aperture to generate the so-called self-imaging phenomenon [2–5], Walsh functions should behave in free propagation of light in a rather similar way as periodic structures, such as the binary Ronchi grating. In this way, the analysis and synthesis of signals performed in a Walsh scheme can be very useful for studying the properties of certain optical systems where both the free propagation of light and the interaction of aperture and diffracted field are relevant.

We start by analysing the behaviour of the Walsh functions under Fresnel diffraction in accordance with their periodicity properties. This analysis will be extended to a more general aperture synthesized as a linear superposition of Walsh functions. Finally, we illustrate this approach with some experimental results.

2. Properties of Walsh functions under Fresnel diffraction

Inside a certain finite domain $|x| < x_0/2$, the orthogonal and complete set of Walsh functions $\{\mathrm{Wal}_n(x);\ n=1,2,\ldots\}$ can be written as [1]

$$\begin{aligned}\mathrm{Wal}_n(x) &= \prod_{k=0}^{m} [R_k(x)]^{g_k}; \quad |x| < x_0/2 \\ &= \prod_{k=0}^{m} [\mathrm{sign}\,(\sin\,(2^k \pi x/x_0))]^{g_k}.\end{aligned} \tag{1}$$

In equation (1), $\{R_k(x)\}$ denotes the set of Rademacher functions, which are periodic square waves of amplitude ± 1, having period $d_k = 2^{1-k}x_0$. The integer m is the rank of the binary expansion of n, in which the g_k are the corresponding bits, 0 or 1, i.e. $n = 2^m g_m + 2^{m-1} g_{m-1} + \ldots + 2^0 g_0$. Since the Rademacher functions form an incomplete set, they cannot serve as an appropriate basis for synthesizing any optical signal.

For our purposes, it is more convenient to rewrite the Walsh functions as defined in the entire space domain, and which are restricted to take only 0 or 1 values, instead of ± 1, i.e.

$$W_n(x) = \tfrac{1}{2}[1 + \mathrm{Wal}_n(x)]\,\mathrm{rect}\,(x/x_0). \tag{2}$$

Next, we consider a binary aperture with an amplitude transmittance given by the Walsh function as written in equation (2). If it is illuminated with a monochromatic plane wave propagating along the z-axis, the diffracted field amplitude $u_n(x;z)$, in the Fresnel region, becomes

$$u_n(x;z) = \int_{-\infty}^{\infty} W_n(x')\exp\,[(\mathrm{i}\pi/\lambda z)(x-x')^2]\,\mathrm{d}x'. \tag{3}$$

By using equations (1) and (2), the following expression for $u_n(x;z)$ results:

$$\begin{aligned}u_n(x;z) = \frac{1}{2}\int_{-\infty}^{\infty} &\left\{1 + K\sum_{k_1}\ldots\sum_{k_m} c_{k_1}^{g_1}\ldots c_{k_m}^{g_m}\exp\left[2\pi\mathrm{i}x'\left(\frac{g_1k_1}{d_1}+\ldots+\frac{g_mk_m}{d_m}\right)\right]\right\} \\ &\times \mathrm{rect}\,(x'/x_0)\exp\,[\mathrm{i}(\pi/\lambda z)(x-x')^2]\,\mathrm{d}x',\end{aligned} \tag{4}$$

where K is a normalization constant. In equation (4), we have replaced the Rademacher functions by their corresponding Fourier expansions, namely the coefficients c_{k_j} which are non-zero only for odd values of k_j. Hence, apart from the Fresnel pattern produced by the uniform finite aperture $(-x_0/2,\ x_0/2)$, we obtain

$$\begin{aligned}u_n(x;z) \simeq (K/2)\exp\,(\mathrm{i}\pi x^2/\lambda z)&\sum_{k_1}\ldots\sum_{k_m} c_{k_1}^{g_1}\ldots c_{k_m}^{g_m} \\ \times \exp&\left[-\mathrm{i}\pi\lambda z\left(\frac{x}{\lambda z}-\frac{g_1k_1}{d_1}-\ldots-\frac{g_mk_m}{d_m}\right)^2\right] \\ * \,\mathrm{sinc}&\left[x_0\left(\frac{x}{\lambda z}-\frac{g_1k_1}{d_1}-\ldots-\frac{g_mk_m}{d_m}\right)\right],\end{aligned} \tag{5}$$

where $*$ denotes convolution.

Next, we analyse the properties of $u_n(x;z)$. If we want to retrieve an amplitude distribution that resembles the Walsh function (as in the self-imaging phenomenon produced by periodic infinite apertures), then it is clear that the effect of the finite size of the aperture on $u_n(x;z)$ should be minimized. To this end, the sinc function array, denoted sinc $(x; k_1, \ldots, k_m)$, must be approximated by δ-functions. Therefore, overlap between different sinc functions should be avoided, except when the respective maxima coincide. The mean width of all the sinc functions is $\Delta x = \lambda z/x_0$, and the separation between successive maxima of sinc $(x; k_p = 1, \ldots, k_{j-1} = 1, k_j, k_{j+1} = 1, \ldots, k_m = 1)$ is $\delta x_j = 2^j \lambda z/x_0$, $p \leqslant j \leqslant m$, for which $R_p(x)$ is the lowest-order Rademacher component that is present in $W_n(x)$. Since $\delta x_j/\delta x_p = 2^{j-p}$, coincidence between the several maxima always occurs, and the minimum distance between adjacent maxima of the whole array becomes $\delta x = \delta x_p = 2^p \lambda z/x_0$. Hence, the validity of the approximation sinc$(\cdot) \simeq \delta(\cdot)$ is assured whenever $\delta x \gg \Delta x$, or equivalently $2^p \gg 1$. Of course, this condition places a limitation on the lowest-order Rademacher function that can be present in $W_n(x)$. Thus, by assuming $2^p \gg 1$, we obtain

$$u_n(x; z) \simeq (K/2) \sum_{k_p} \ldots \sum_{k_m} c_{k_p}^{g_p} \ldots c_{k_m}^{g_m} \times \exp\left[-2\pi i x\left(\frac{g_p k_p}{d_p} + \ldots + \frac{g_m k_m}{d_m}\right)\right] \exp\left[-i\pi\lambda z\left(\frac{g_p k_p}{d_p} + \ldots + \frac{g_m k_m}{d_m}\right)^2\right]. \quad (6)$$

It is convenient to rewrite equation (6) in the following way:

$$u_n(x; z) \simeq (K/2) \sum_{k_p} \ldots \sum_{k_{j-1}} \sum_{k_{j+1}} \ldots \sum_{k_m} c_{k_p}^{g_p} \ldots c_{k_m}^{g_m} \times \exp\left[-2\pi i x \sum_{r \neq j} g_r k_r/d_r\right] \sum_{k_j} c_{k_j} \exp(-2\pi i x k_j/d_j) \times \exp\left[-i\pi\lambda z\left(\sum_{r \neq j} g_r k_r/d_r + k_j/d_j\right)^2\right] \quad (7\,a)$$

or, equivalently

$$u_n(x; z) \simeq (K/2) \sum_{k_p} \ldots \sum_{k_{j-1}} \sum_{k_{j+1}} \ldots \sum_{k_m} c_{k_p}^{g_p} \ldots c_{k_m}^{g_m} \times \exp\left[-2\pi i x \sum_{r \neq j} g_r k_r/d_r\right] \exp\left[-i\pi\lambda z\left(\sum_{r \neq j} g_r k_r/d_r\right)^2\right] \times \sum_{k_j} c_{k_j} \exp(-2\pi i x k_j/d_j) \exp(-i\pi\lambda z k_j^2/d_j^2) \exp\left(-2\pi i \frac{\lambda z k_j}{d_j} \sum_{r \neq j} g_r k_r/d_r\right) \quad (7\,b)$$

From equations (7), we can derive the conditions to be satisfied in order to obtain a self-image of $R_j(x)$; namely

$$\lambda z k_j^2/d_j^2 = 2h \quad \text{and} \quad \frac{\lambda z k_j}{d_j} \sum_{r \neq j} \frac{g_r k_r}{d_r} = q; \quad (8)$$

where h and q are positive integers. Both conditions are fulfilled whenever

$$z^{(j)} = 2\alpha x_0^2/\lambda 2^{p+j-2}, \quad \alpha = 1, 2, 3, \ldots \quad (9)$$

For these values of z, the amplitude distribution given by equation (7) can be considered as a positive self-image of $R_j(x)$, spatially modulated by the remaining defocused Rademacher functions which act as a noise source.

However, by taking into account that

$$z^{(j)} = 2^{m-j} z^{(m)}, \quad p < j < m; \quad (10)$$

i.e. the first self-image of $R_j(x)$ coincides with the $\alpha=2^{m-j}$ self-image of $R_m(x)$, then at the self-image planes associated with $R_p(x)$, all the Rademacher functions are well-focused. Therefore, for the distances

$$z=\alpha z_0=2\alpha x_0^2/\lambda 2^{2p-2}, \quad \alpha=1,2,3,\ldots, \tag{11}$$

we get

$$u_n(x;z)\simeq(K/2)[1+\mathrm{Wal}_n(x)], \tag{12}$$

which is a positive direct-contrast self-image of the Walsh function $W_n(x)$.

3. Interaction between the light field and an aperture mask: an approach based on Walsh functions

We turn now to the interaction between the light field and a given aperture from the viewpoint of the discussed properties of the Walsh functions. Since they form an orthogonal and complete set within a finite domain, any transparency limited by a certain pupil can be considered as synthesized from Walsh functions, instead of sinusoidal functions. If this transparency is the input object of a system illuminated by a coherent plane wave, then each Walsh component behaves in free propagation in a similar way to that treated in § 2. Taking into account the binary nature of these functions, it is rather simple to understand the interaction of the diffracted field with an arbitrary binary aperture placed at a certain distance z (at least in the Kirchhoff approximation, where the product of the light amplitude with the transmittance function of the aperture can be considered for the transmitted field). All the Walsh functions focused at the plane of the aperture mask as positive self-images are transmitted by the system without appreciable distortion or attenuation, while the remaining Walsh components are severely distorted and attenuated. Of course, the interaction is still very complicated for those Walsh components which are not in focus at the plane of the aperture mask. For this reason, it seems appropriate to consider this kind of analysis for all those cases involving several interactions between the diffracted field and the aperture. Thus, if the number of interactions becomes large, the resulting field will be mainly synthesized by the first type of Walsh functions. This approach was used in [6, 7] to explain the spatial filtering properties of some periodic and non-periodic aperture arrangements. However, in such cases, all the apertures were assumed to be synthesized by Rademacher functions which form an incomplete set of periodic functions.

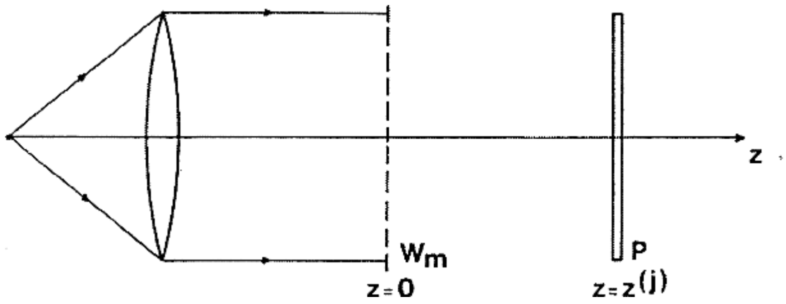
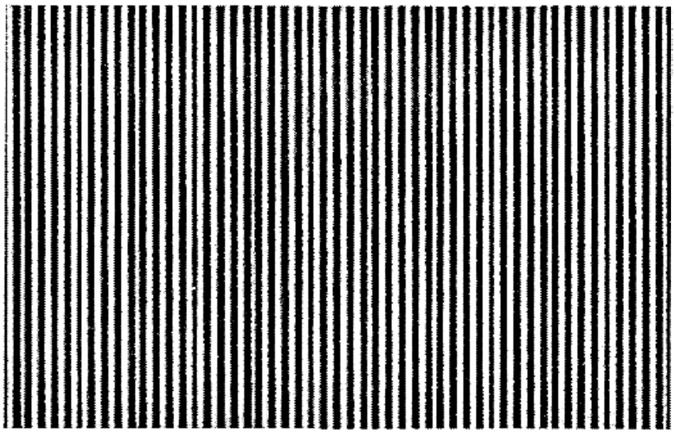
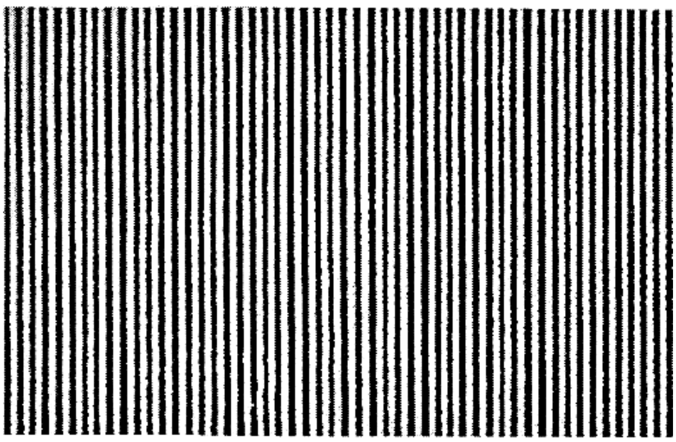


Figure 1. Scheme of the optical device employed for obtaining the self-images of the Walsh functions.



(*a*)



(*b*)

Figure 2. (*a*) Walsh function $\mathrm{Wal}(x) = R_8(x)\, R_9(x)$. (*b*) First self-image of $\mathrm{Wal}(x)$, recorded at $z = 24$ cm.

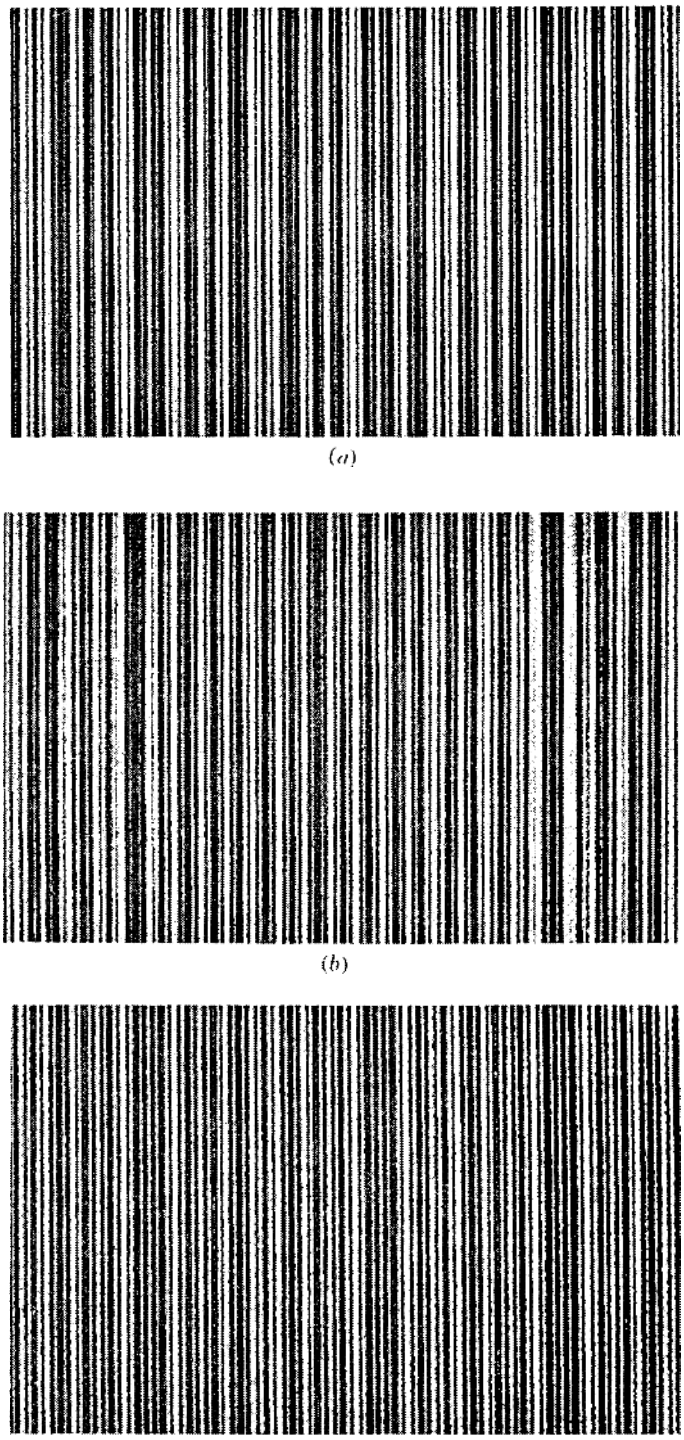


Figure 3. (*a*) Walsh function $\mathrm{Wal}(x) = R_7(x)\, R_8(x)\, R_9(x)$. (*b*) First self-image of $R_8(x)$, and second self-image of $R_9(x)$, recorded at $z = 24$ cm. (*c*) First self-image of $\mathrm{Wal}(x)$, recorded at $z = 48$ cm.

In order to illustrate the results obtained in § 2, we have synthesized an aperture consisting of a single Walsh function (in the normalized form, as given by equation (2)). When illuminated with a coherent plane wave, the aperture gives rise to the self-imaging phenomenon in accordance with equations (7)–(12) (see figure 1). Figure 2 (*a*) shows the Walsh function composed by two Rademacher functions $R_8(x)$ and $R_9(x)$. Figure 2 (*b*) shows the intensity distribution recorded at the first self-image plane of $R_8(x)$, which coincides with the second self-image plane of $R_9(x)$. Hence, the first self-image of the complete Walsh function can be observed. Figure 3 (*a*) shows the Walsh function composed of three Rademacher functions $R_7(x)$, $R_8(x)$ and $R_9(x)$. In figure 3 (*b*) the first self-image of $R_8(x)$ and the second self-image of $R_9(x)$ are well focused but some artifact noise due to the defocused self-image of $R_7(x)$ can be observed. In figure 3 (*c*), the three Rademacher components are in focus, so producing the first self-image of the complete Walsh function. Owing to the fact that a lower-order Rademacher component is present in the latter case, the approximation involved in deriving equation (6) is not well satisfied, and hence the self-image of the Walsh function exhibits more artifact noise than in the former case.

4. Conclusions

From a mathematical point of view, either a Walsh or a Fourier synthesis can be performed for all classes of finite apertures. Which of these approaches is more suited for synthesis and processing purposes is an open question in the field of digital image systems [8].

For optical systems, we can take into account something resembling a signal-to-noise ratio that characterizes the system. If certain Walsh functions are either perfectly cancelled or transmitted by a binary aperture, their associated energy can be considered as a signal level. The remaining Walsh functions present in the field amplitude (which are defocused at the plane of the aperture) will be severely distorted in transmission, so contributing to a noise level. Therefore, a treatment based on a Walsh approach can be suitably carried out for those cases in which several interactions between the diffracted field and a certain aperture mask occur in such a way that the signal level remains practically unchanged, while the noise level becomes increasingly lower. This is the case of several periodic arrays of apertures, such as the in-line analogue of a Fabry–Perot interferometer.

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