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WANDERING DOMAINS AND NONTRIVIAL REDUCTION IN NON-ARCHIMEDEAN DYNAMICS

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ABSTRACT. Let K be a non-archimedean field with residue field k, and suppose that k is not an algebraic extension of a finite field. We prove two results concerning wandering domains of rational functions $\phi \in K(z)$ and Rivera-Letelier's notion of nontrivial reduction. First, if ϕ has nontrivial reduction, then assuming some simple hypotheses, we show that the Fatou set of ϕ has wandering components by any of the usual definitions of "components of the Fatou set". Second, we show that if k has characteristic zero and K is discretely valued, then the existence of a wandering domain implies that some iterate has nontrivial reduction in some coordinate.

The theory of complex dynamics in dimension one, founded by Fatou and Julia in the early twentieth century, concerns the action of a rational function $\phi \in \mathbb{C}(z)$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Any such ϕ induces a partition of the sphere into the closed Julia set $\mathcal{J}_\phi$, where small errors become arbitrarily large under iteration, and the open Fatou set $\mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{J}_\phi$. There is also a natural action of ϕ on the connected components of $\mathcal{F}_\phi$, taking a component U to $\phi(U)$, which is also a connected component of the Fatou set. In 1985, using quasiconformal methods, Sullivan [36] proved that $\phi \in \mathbb{C}(z)$ has no wandering domains; that is, for each component U of $\mathcal{F}_\phi$, there are integers $M \geq 0$ and $N \geq 1$ such that $\phi^M(U) = \phi^{M+N}(U)$. We refer the reader to [1], [14], [26] for background on complex dynamics.

In the past two decades, there have been a number of investigations of dynamics over complete metric fields other than $\mathbb{R}$ or $\mathbb{C}$. All such fields are *non-archimedean*; that is, the metric on the field K satisfies the *ultrametric* triangle inequality

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all } x, y, z \in K.$$

Herman and Yoccoz [20] first considered dynamics over such fields in a study of linearization at fixed points, in part to discover which properties of complex

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dynamical systems are specific to archimedean fields and which are more general. The question of comparing archimedean and non-archimedean dynamics has continued to drive the field, as have questions arising in number theory in the study of rational dynamics; see [4], [5], [6], [11], [21], [28], [30], [31], [35].

In particular, it is natural to ask how the dynamical properties of Fatou components extend to the non-archimedean setting. In [3], the author proved a no wandering domains theorem over p-adic fields, assuming some weak hypotheses. That theorem relied heavily on the fact that the residue field k (see below) of a p-adic field K is an algebraic extension of the finite field $\mathbb{F}_p$. In fact, for non-archimedean fields K without such a residue field, it is easy to construct rational functions with wandering domains; see Example 6 and [6, Example 2]. The aim of this paper is to classify all such wandering domains.

We fix the following notation.

K	a complete non-archimedean field with absolute value $\lvert\cdot\rvert$
$\hat{K}$	an algebraic closure of K
$\mathbb{C}_K$	the completion of $\hat{K}$
$\mathcal{O}_K$	the ring of integers $\{x \in K : \lvert x\rvert \le 1\}$ of K
k	the residue field of K
$\mathcal{O}_{\mathbb{C}_K}$	the ring of integers $\{x \in \mathbb{C}_K : \lvert x\rvert \le 1\}$ of $\mathbb{C}_K$
$\hat{k}$	the residue field of $\mathbb{C}_K$
$\mathbb{P}^1(\mathbb{C}_K)$	the projective line $\mathbb{C}_K \cup \{\infty\}$

Recall that the absolute value $|\cdot|$ extends in unique fashion to $\hat{K}$ and to $\mathbb{C}_K$. Recall also that the residue field k is defined to be $\mathcal{O}_K/\mathcal{M}_K$, where $\mathcal{M}_K$ is the maximal ideal $\{x \in K : |x| < 1\}$ of $\mathcal{O}_K$. The residue field $\hat{k}$ is defined similarly. There is a natural inclusion of the residue field k into $\hat{k}$, making $\hat{k}$ an algebraic closure of k. We refer the reader to [18], [24], [32], [34] for surveys of non-archimedean fields.

The best known complete non-archimedean field is $K = \mathbb{Q}_p$, the field of p-adic rational numbers (for any fixed prime number p). Its algebraic closure is $\hat{K} = \overline{\mathbb{Q}}_p$, and the completion $\mathbb{C}_K$ is frequently denoted $\mathbb{C}_p$. The ring of integers is $\mathcal{O}_K = \mathbb{Z}_p$, with residue field $k = \mathbb{F}_p$ (the field of p elements), and $\hat{k}$ is the algebraic closure $\overline{\mathbb{F}}_p$. Note that $\operatorname{char} K = 0$, but $\operatorname{char} k = p$. Thus, we say the characteristic of $\mathbb{Q}_p$ is 0, but the *residue characteristic* of $\mathbb{Q}_p$ is p.

As another example, if L is any abstract field, then $K = L((T))$, the field of formal Laurent series with coefficients in L, is a complete non-archimedean field with $\mathcal{O}_K = L[[T]]$ (the ring of formal Taylor series) and $k = L$. In this case, $\operatorname{char} K = \operatorname{char} k = \operatorname{char} L$. The absolute value $|\cdot|$ on K may be defined by $|f| = 2^{-n}$, where $n \in \mathbb{Z}$ is the least integer for which the T^n term of the formal Laurent series f has a nonzero coefficient. Dynamics over such a function field K has applications to the study of one-parameter families of functions defined over the original field L; see, for example, [23].

In the study of non-archimedean dynamics of one-variable rational functions, we consider a rational function $\phi \in K(z)$, which acts on $\mathbb{P}^1(\mathbb{C}_K)$ in the same way that a complex rational function acts on the Riemann sphere. In [2], [3], [6], the author defined non-archimedean Fatou and Julia sets. The wandering domains we will study are components of the Fatou set, which is an open subset of $\mathbb{P}^1(\mathbb{C}_K)$.

Of course, to study wandering domains, we must first have an appropriate notion of "connected components" of subsets of $\mathbb{P}^1(\mathbb{C}_K)$, which is a totally disconnected topological space. Several definitions have been proposed in the literature, and each is useful in slightly different settings, just as connected components and path-connected components are related but distinct notions. We will consider four definitions in this paper, all of which are closely related and which frequently coincide with one another.

In [2], [3], [6], the author proposed two analogues of "connected components" of the Fatou set: D-components and analytic components, both of which we will define in Section 1. Rivera-Letelier proposed an alternate definition in [30], [31]. His definition was stated only over the p-adic field $\mathbb{C}_p$; in Section 1, we define an equivalent version of his components, which we call dynamical components, for all non-archimedean fields. We also will propose a fourth analogue, called dynamical D-components, which will actually be useful for proving things about the other three types of components.

Our first main result generalizes the aforementioned wandering domain from [6, Example 2], for the function $\phi(z) = (z^3 + (1+T)z^2)/(z+1) \in K(z)$, where $K = \mathbb{Q}((T))$. In that example, the wandering domain U and all of its forward images $\phi^n(U)$ are open disks of the form $D(a,1)$, with $|a| \leq 1$, where $D(a,r)$ denotes the open disk of radius r about a. In fact, the map ϕ has the property that for all but finitely many of the disks $D(a,1)$ with $|a| \leq 1$, the image $\phi(D(a,1))$ is just $D(\phi(a),1)$; in Rivera-Letelier's language [30], [31], ϕ has *nontrivial reduction*. Equivalently, ϕ has a fixed point in Rivera-Letelier's "hyperbolic space" $\mathbb{H}$, which is essentially the same space as the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_K)$; see [10], [33]. Both $\mathbb{H}$ and $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_K)$ have been used with increasing frequency in the study of the mapping properties and dynamics of non-archimedean rational functions. Readers familiar with either space will recognize the wandering domains we will construct as connected components of the full space $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_K)$ with a single type II point removed. However, for simplicity, we will present our arguments without reference to Berkovich spaces.

We will define nontrivial reduction precisely in Section 2. We will then present Theorem 4.2 and Example 6, which imply the following result.

THEOREM A. *Let K be a non-archimedean field with residue field k, where k is not an algebraic extension of a finite field.*

(a) *Let $\phi \in K(z)$ be a rational function of nontrivial reduction $\overline{\phi}$ with $\deg \overline{\phi} \geq 2$. Then ϕ has a wandering dynamical component U such that for every integer $n \geq 0$, $\phi^n(U)$ is an open disk of the form $D(b_n, 1)$,* *with* $|b_n| \leq 1$*. U is also a wandering dynamical D-component; moreover, if the Julia set of* ϕ *intersects infinitely many residue classes of* $\mathbb{P}^1(\mathbb{C}_K)$*, then U is also a wandering D-component, and a wandering analytic component.*

(b) *There exist functions* $\phi \in K(z)$ *satisfying all of the hypotheses of part* (a).

Theorem 4.2 is an even stronger result: that under the hypotheses of Theorem A, there are actually infinitely many different grand orbits of wandering domains of the form $D(b, 1)$. In addition, Theorem 4.3 will give sufficient conditions for the Julia set of ϕ to intersect infinitely many residue classes.

Theorem A and Theorem 4.2 apply to maps with reduction $\overline{\phi}$ of degree at least two. If ϕ has a nontrivial reduction $\overline{\phi}$ of degree one, the situation is a bit more complicated. Examples 7–10 will show that in some such cases there is a wandering domain of the form $D(b, 1)$, and in other cases there is not.

Given Theorem A, we can construct still more wandering domains over appropriate fields K, as follows. If $\phi \in K(z)$ is a rational function and $g \in \mathrm{PGL}(2, \mathbb{C}_K)$ is a linear fractional transformation, suppose that some conjugated iterate $\psi(z) = g \circ \phi^n \circ g^{-1}(z)$ has nontrivial reduction of degree at least two. Even if the original map ϕ has trivial reduction, Theorem 4.2 shows that ϕ has a wandering domain, because ψ does.

The existence of rational functions with wandering domains should not come as a surprise for fields K satisfying the hypotheses of Theorem A. For example, the infinite residue field prevents K from being locally compact, allowing plenty of room for the various iterates of the wandering domain to coexist. Thus, the impact of Theorem A is not so much the fact that wandering domains exist, but that they may be produced by such simple reduction conditions.

Perhaps more interesting than the existence of such wandering domains is our next theorem, which shows that for certain fields K, the *only* wandering domains possible for rational functions are those described above. That is, any wandering domain for such a field *must* come from a nontrivial reduction.

Theorem B. *Let K be a non-archimedean field with residue field k. Suppose that K is discretely valued and that* $\operatorname{char} k = 0$*. Let* $\phi(z) \in K(z)$ *be a rational function, and suppose that some* $U \subset \mathbb{P}^1(\mathbb{C}_K)$ *is a wandering domain (analytic, dynamical, D-component, or dynamical D-component) of* ϕ*. Then there are integers* $M \geq 0$ *and* $N \geq 1$ *and a change of coordinate* $g \in \mathrm{PGL}(2, \mathbb{C}_K)$ *with the following property:*

Let $\psi(z) = g \circ \phi^N \circ g^{-1}(z) \in \mathbb{C}_K(z)$*. Then* $D(0, 1)$ *is the component of the Fatou set of* ψ *containing* $g(\phi^M(U))$*, and* ψ *has nontrivial reduction.*

The clause "$D(0, 1)$ is the component of the Fatou set of ψ containing $g(\phi^M(U))$" is equivalent to "$g(\phi^M(U)) = D(0, 1)$" if we are dealing with analytic or dynamical components. On the other hand, if U is a D-component or dynamical D-component, then it is possible that $\phi^M(U)$ is a proper subset of a (dynamical) D-component. We will prove a slightly stronger version of Theorem B in Theorem 5.1 of Section 5, showing the function g can be defined over a finite extension of K.

In [7], [9], it was shown that rational functions, including polynomials, may have wandering domains if $K = \mathbb{C}_K$ and if $\operatorname{char} k > 0$. (The hypotheses of the no wandering domains theorems of [2], [3], [6] require the field of definition K to be locally compact, whereas $\mathbb{C}_K$ is not locally compact.) By contrast, Theorem B shows that rational functions have no wandering domains (besides those arising from a nontrivial reduction, as in Example 6) if the field of definition K is discretely valued and has residue characteristic *zero*. However, an algebraically closed non-archimedean field (such as $\mathbb{C}_K$) cannot be discretely valued. Thus, one is naturally led to ask the following open questions:

(1) If K is locally compact (implying both that K is discretely valued and has residue characteristic $p > 0$), do there exist polynomial or rational functions $\phi \in K(z)$ with wandering domains?
(2) If K is complete and *algebraically closed*, with residue characteristic zero, do there exist functions $\phi \in K(z)$ with wandering domains other than those arising from a nontrivial reduction?

The results of [2], [3], [6] suggest that the answer to the first question above is probably “no”. Meanwhile, we know of no progress on the second question.

The outline of the paper is as follows. We will begin in Section 1 with a review of some dynamical terminology and the definitions of the various types of Fatou components. In Section 2, we will introduce Rivera-Letelier’s notion of nontrivial reduction and state three important lemmas. In Section 3, we will recall a few facts from the theory of diophantine height functions. Heights will be used only in the proof of Lemma 4.1; the reader unfamiliar with the theory may skip both Section 3 and the proof of Lemma 4.1 without loss of continuity. Finally, in Section 4 we will prove Theorem A, and in Section 5 we will prove Theorem B. We also include an appendix on the relevant terminology and fundamental properties of rational functions and some non-archimedean analysis.

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1. Dynamical terminology and Fatou components

Let X be a set, and let $f : X \to X$ be a function. For any $n \geq 1$, we write $f^1 = f$, $f^2 = f \circ f$, and in general, $f^{n+1} = f \circ f^n$; we also define f^0 to be the identity function on X. Let $x \in X$. We say that x is *fixed* if $f(x) = x$; that x is *periodic of period* $n \geq 1$ if $f^n(x) = x$; that x is *preperiodic* if $f^m(x)$ is periodic for some $m \geq 0$; or that x is *wandering* if x is not preperiodic. Note that all fixed points are periodic, and all periodic points are preperiodic. We define the *forward orbit* of x to be the set $\{f^n(x) : n \geq 0\}$; the *backwards orbit* of x to be $\bigcup_{n\geq 0} f^{-n}(x)$; and the *grand orbit* of x to be

$$\{y \in X : \exists\, m, n \geq 0 \text{ such that } f^m(x) = f^n(y)\}.$$

Equivalently, the grand orbit of x is the union of the backwards orbits of all points in the forward orbit of x. We say a grand orbit S is *preperiodic* if it contains a preperiodic point, or S is *wandering* otherwise. Note that S is preperiodic (respectively, wandering) if and only if every point in S is preperiodic (respectively, wandering).

Suppose X, Y are metric spaces. Recall that the family F of functions from X to Y is *equicontinuous* at $x \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon$ for all $f \in F$ and for all $x' \in X$ satisfying $d(x, x') < \delta$. (The key point is that the choice of ε is independent of f.)

Now consider $X = \mathbb{P}^1(\mathbb{C}_K)$ and $f = \phi \in \mathbb{C}_K(z)$. The *Fatou set* of ϕ is the set $\mathcal{F} = \mathcal{F}_\phi$ consisting of all points $x \in \mathbb{P}^1(\mathbb{C}_K)$ for which $\{\phi^n : n \geq 0\}$ is equicontinuous on some neighborhood of x, with respect to the spherical metric (see, for example, [28, Section 5]) on $\mathbb{P}^1(\mathbb{C}_K)$. The *Julia set* $\mathcal{J} = \mathcal{J}_\phi$ of ϕ is the complement $\mathcal{J} = \mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{F}$. Clearly the Fatou set is open, and the Julia set is closed. It is easy to show that $\phi(\mathcal{F}) = \phi^{-1}(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}_{\phi^n} = \mathcal{F}_\phi$ for all $n \geq 1$, and similarly for the Julia set.

Intuitively speaking, the Fatou set is the region where small errors stay small under iteration, while the Julia set is the region of chaos. Note that because $\mathbb{P}^1(\mathbb{C}_K)$ is not locally compact, the Arzelà-Ascoli theorem fails, which is why non-archimedean Fatou and Julia sets are defined in terms of equicontinuity instead of normality.

It is easy to verify (using, for example, [8, Lemma 2.7] or other well known lemmas on non-archimedean power series) that if $U \subset K$ is a disk and if $R > 0$ such that for all $n \geq 0$, $\phi^n(U)$ is a subset of $\mathbb{C}_K$ of diameter at most R, then $U \subset \mathcal{F}_\phi$. It follows that if $V \subset K$ is any open set and if $R > 0$ such that for all $n \geq 0$, $\phi^n(V) \subset K$ and $\operatorname{diam}(\phi^n(V)) \leq R$, then $V \subset \mathcal{F}_\phi$. Another criterion, due to Hsia [21] (see also [8, Theorem 3.7]) states that if $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a disk such that $\bigcup_{n\geq 0} \phi^n(U)$ omits at least two points of $\mathbb{P}^1(\mathbb{C}_K)$, then $U \subset \mathcal{F}_\phi$. Clearly Hsia's criterion also extends to arbitrary open sets V in place of U.

Using the language of affinoids from Section A.3 of the Appendix, we now define components of non-archimedean Fatou sets.

Definition 1.1. Let $\phi \in \mathbb{C}_K(z)$ be a rational function with Fatou set $\mathcal{F}$, and let $x \in \mathcal{F}$.

(a) The *analytic component* of $\mathcal{F}$ containing x is the union of all connected affinoids W in $\mathbb{P}^1(\mathbb{C}_K)$ such that $x \in W \subset \mathcal{F}$.
(b) The *D-component* of $\mathcal{F}$ containing x is the union of all disks U in $\mathbb{P}^1(\mathbb{C}_K)$ such that $x \in U \subset \mathcal{F}$.

(c) The *dynamical component* of $\mathcal{F}$ containing x is the union of all rational open connected affinoids W in $\mathbb{P}^1(\mathbb{C}_K)$ such that $x \in W$ and the set

$$\mathbb{P}^1(\mathbb{C}_K) \setminus \left(\bigcup_{n \geq 0} \phi^n(W) \right)$$

is infinite.

(d) The *dynamical D-component* of $\mathcal{F}$ containing x is the union of all rational open disks U in $\mathbb{P}^1(\mathbb{C}_K)$ such that $x \in U$ and the set

$$\mathbb{P}^1(\mathbb{C}_K) \setminus \left(\bigcup_{n \geq 0} \phi^n(U) \right)$$

is infinite.

Clearly all of these components are open sets. Because finite unions of overlapping connected affinoids or disks are again connected affinoids or disks (or all of $\mathbb{P}^1(\mathbb{C}_K)$), the relation "$y$ is in the component of $\mathcal{F}$ containing x" is an equivalence relation between x and y, for each of the four types of components. Note that by Hsia's criterion, any dynamical component or dynamical D-component must in fact be contained in the Fatou set, so the terminology "component of $\mathcal{F}$" is not misleading. D-components must be either disks, all of $\mathbb{P}^1(\mathbb{C}_K)$, or all but one point of $\mathbb{P}^1(\mathbb{C}_K)$. Dynamical D-components must be either open disks, all of $\mathbb{P}^1(\mathbb{C}_K)$, or all but one point of $\mathbb{P}^1(\mathbb{C}_K)$. Analytic and dynamical components may be more complicated geometrically. Frequently, two or more types of components coincide in a particular case.

Analytic components and D-components were first defined in [2], [3]. Dynamical components were defined by Rivera-Letelier in [30], [31]; he called them simply "components", and he used a different, but equivalent, definition. Dynamical D-components are new to the literature.

For any of the four types of components, if $x \in \mathcal{F}_\phi$ and if V is the component containing x, then $\phi(V)$ is contained in the component containing $\phi(x)$, by of the mapping properties discussed in Sections A.2 and A.3 of the appendix. Thus, ϕ induces an action Φ_D on the set of D-components by

$$\Phi_D(U) = \text{the D-component containing } \phi(U).$$

Similarly, ϕ induces actions Φ_{an} on the set of analytic components, Φ_{dyn} on the set of dynamical components, and Φ_{dD} on the set of dynamical D-components. Thus, we can discuss fixed components, wandering components, grand orbits of components, etc., for each of the four types.

For analytic and dynamical components, it can be shown [3], [30] that $\Phi_{an}(V) = \phi(V)$ and $\Phi_{dyn}(V) = \phi(V)$. For D-components and dynamical D-components, the corresponding equalities usually hold; but occasionally, the containment may be proper. Fortunately, by Lemma A.5, for any given

$\phi \in \mathbb{C}_K(z)$ of degree d, there are at most $d-1$ D-components U for which there exists a D-component V with $\Phi_D(V) = U$ but $\phi(V) \subsetneq U$. The analogous statement also holds for dynamical D-components.

The following examples should help to clarify how each of the four types of components behaves. We omit the proofs of most of the claims in the following examples. Details concerning similar examples may be found in, for example, [2], [3], [4], [6], [30], [31].

Example 1. Let $n \geq 2$, and let $\phi(z) = z^n$. Then it is easy to show that $\mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}_K)$. (The same is true in the more general situation that ϕ has good reduction; see Section 2.) It follows immediately that there is only one D-component and only one analytic component, namely the full set $\mathbb{P}^1(\mathbb{C}_K)$.

On the other hand, all disks of the form $D(\alpha, 1)$ for $\alpha \in \mathbb{C}_K$ with $|\alpha| \leq 1$, as well as the disk $\mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}(0,1)$ at ∞, are dynamical components and dynamical D-components of the Fatou set. Indeed, any strictly larger open disk or affinoid U will have the property that $\bigcup_{n\geq 0} \phi^n(U)$ omits at most the two points 0 and ∞. (Cf. Lemma 2.4.)

Example 2. Let $p = \operatorname{char} K \geq 0$, let $c \in K$ with $0 < |c| < 1$, let $d \geq 2$ be an integer not divisible by p, and let $\phi(z) = z^d - c^{-d}$. Writing $U_0 = \overline{D}(0, |c|^{-1})$, it is easy to see that for $x \in \mathbb{P}^1(\mathbb{C}_K) \setminus U_0$, the iterates $\phi^n(x)$ approach ∞. Thus, the Julia set $\mathcal{J}_\phi$ is contained in U_0. In fact, one can check that for any $n \geq 0$, if we set $U_n = \phi^{-n}(U_0)$, then U_n is the disjoint union of d^n disks, each of radius $|c|^{n-1}$, with d such disks in each of the d^{n-1} disks of U_{n-1}. It follows easily that $\mathcal{J}_\phi = \bigcap_{n\geq 0} U_n$. The dynamical D-component and the D-component of $\mathcal{F}_\phi$ containing ∞ are both the open disk $\mathbb{P}^1(\mathbb{C}_K) \setminus U_0$. On the other hand, the analytic and the dynamical component at ∞, which also coincide in this case, are both the more complicated set $\mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{J}_\phi$, which is the whole Fatou set.

Example 3. Let $p = \operatorname{char} K$, and assume that $p > 0$. Let $d \geq 2$ be an integer not divisible by p, and let $c \in K$ with $|p|^{1/(pd-1)} < |c| < 1$. Let $\phi(z) = z^{pd} - c^{-pd}$. Writing $U_0 = \overline{D}(0, |c|^{-1})$ and $V_0 = \mathbb{P}^1(\mathbb{C}_K) \setminus U_0$, it is again easy to see that for $x \in V_0$, the iterates $\phi^n(x)$ approach ∞. Defining $V_n = \phi^{-n}(V_0)$ for $n \geq 0$, the set of points which are attracted to ∞ under iteration is $V = \bigcup_{n\geq 0} V_n$, which is a complicated union of affinoids (and which is not itself an affinoid). However, for $z \in U_0$, we have $|\phi'(z)| < 1$, from which it follows easily that $\mathcal{J}_\phi = \emptyset$. Thus, the D-component and analytic component of $\mathcal{F}_\phi$ containing ∞ are both $\mathbb{P}^1(\mathbb{C}_K)$ itself, the dynamical component is V, and the dynamical D-component is V_0.

Example 4. Let $c \in K$ with $0 < |c| < 1$, let $d \geq 3$ be an integer, and let $\phi(z) = cz^d + z^{d-1} + z$. Clearly $\phi(\overline{D}(0,1)) = \overline{D}(0,1)$. There is also a repelling fixed point at $z = -1/c$, and the backwards orbit of $-1/c$ includes points of absolute value $|c|^{-1/d^n}$ for arbitrary small $n \geq 0$. Therefore, any connected affinoid strictly containing $\overline{D}(0,1)$ must intersect the Julia set.

The dynamical D-component and the dynamical component of $\mathcal{F}_\phi$ containing 0 are both the open disk $D(0,1)$. However, the analytic component and the D-component of 0 are both the larger closed disk $\overline{D}(0,1)$.

EXAMPLE 5. Let $b, c \in K$ with $0 < |c| < |b| = |b-1| = 1$, and let

$$\phi(z) = \frac{bz(z+c)(z+c^2)}{(z+bc)(z+c^3)(cz+1)^2}.$$

Then $\phi(0) = 0$ is a repelling fixed point, and $\phi(\infty) = 0$. Let V be the annulus $D(0,|c|^{-1}) \setminus \overline{D}(0,|c|)$; then for all $z \in V$, we have $|\phi(z) - bz| < |z|$. In particular, $\overline{\phi}(z) = \overline{b}z$, and $\phi(V) = V$. One can also show that there are infinitely many disks of the form $D(\alpha, |c|)$ and $D(\beta, |c|^{-1})$ with $|\alpha| = |c|$ and $|\beta| = |c|^{-1}$ which contain preimages of 0; thus, $\mathcal{J}_\phi$ intersects infinitely many such disks. It follows that the analytic and dynamical components of $\mathcal{F}_\phi$ containing 1 are both the annulus V. However, the D-component and dynamical D-component are both the open disk $D(1,1)$.

In addition, for any $\alpha \in K$ with $|\alpha| = |c|$, write $W_\alpha = D(\alpha, |c|)$. There are infinitely many such disks for which there are distinct integers $n > m \geq 0$ such that $\phi^m(W_\alpha) = \phi^n(W_\alpha)$. For any such α, the analytic component, D-component, dynamical component, and dynamical D-component all coincide and are equal to W_α.

2. Nontrivial reduction

As is well known, the natural projection $\mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_K}/\mathcal{M}_{\mathbb{C}_K} = \hat{k}$ induces a reduction map red : $\mathbb{P}^1(\mathbb{C}_K) \to \mathbb{P}^1(\hat{k})$. Given $\overline{a} \in \mathbb{P}^1(\hat{k})$, the associated *residue class*, which we shall denote $W_{\overline{a}} \subset \mathbb{P}^1(\mathbb{C}_K)$, is the preimage

$$W_{\overline{a}} = \mathrm{red}^{-1}(\overline{a}).$$

Any such class is either an open disk $W_{\overline{a}} = D(a,1)$ with $a \in \mathbb{C}_K$ and $|a| \leq 1$, or else it is the disk at infinity, $W_{\overline{\infty}} = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}(0,1)$.

Given a rational function $\phi \in \mathbb{C}_K(z)$ and a residue class $W_{\overline{a}}$, it will be useful to know whether or not $\phi(W_{\overline{a}})$ is again a residue class. To do so, we recall the following definition of Rivera-Letelier [31], which generalizes the notion of good reduction first stated by Morton and Silverman [27].

DEFINITION 2.1. Let $\phi \in \mathbb{C}_K(z)$ be a nonconstant rational function. Write ϕ as f/g, with $f, g \in \mathcal{O}_{\mathbb{C}_K}[z]$, such that at least one coefficient of f or g has absolute value 1. Denote by $\overline{f}$ and $\overline{g}$ the reductions of f and g in $\hat{k}[z]$. Let $\overline{h} = \gcd(\overline{f}, \overline{g}) \in \hat{k}[z]$, let $\overline{f}_0 = \overline{f}/\overline{h}$, and let $\overline{g}_0 = \overline{g}/\overline{h}$. We say that ϕ has *nontrivial reduction* if $\overline{f}_0$ and $\overline{g}_0$ are not both constant. In that case, we define $\overline{\phi} = \overline{f}_0/\overline{g}_0 \in \hat{k}(z)$. If $\deg \overline{\phi} = \deg \phi$, we say ϕ has *good reduction*.

If ϕ and ψ have nontrivial reductions $\overline{\phi}$ and $\overline{\psi}$, then $\phi \circ \psi$ has nontrivial reduction $\overline{\phi} \circ \overline{\psi}$. Rivera-Letelier showed that the above definition of good reduction is equivalent to Morton and Silverman's original definition. His analysis is summarized in the following two lemmas. The proofs, stated for the field $\mathbb{C}_p$, but which apply to arbitrary $\mathbb{C}_K$, appear in [30, Proposition 2.4].

LEMMA 2.2. *Let $\phi \in \mathbb{C}_K(z)$ be a rational function. Then ϕ has nontrivial reduction if and only if there are (not necessarily distinct) points $\overline{a}, \overline{b} \in \mathbb{P}^1(\hat{k})$ such that $\phi(W_{\overline{a}}) = W_{\overline{b}}$.*

LEMMA 2.3. *Let $\phi \in \mathbb{C}_K(z)$ be a rational function of nontrivial reduction $\overline{\phi} \in \hat{k}(z)$. Then there is a finite set $T \subset \mathbb{P}^1(\hat{k})$ such that*

$$\phi(W_{\overline{a}}) = W_{\overline{\phi}(\overline{a})} \quad \textit{for all} \quad \overline{a} \in \mathbb{P}^1(\hat{k}) \setminus T,$$

and

$$\phi(W_{\overline{a}}) = \mathbb{P}^1(\mathbb{C}_K) \quad \textit{for all} \quad \overline{a} \in T.$$

Moreover, ϕ has good reduction if and only if $T = \emptyset$.

Given $\phi \in \mathbb{C}_K(z)$ of nontrivial reduction and its set $T \subset \mathbb{P}^1(\hat{k})$ from Lemma 2.3, we call classes $W_{\overline{a}}$ of elements $\overline{a} \in T$ the *bad classes*, and we call the remaining classes the *good classes*. The bad classes are precisely those classes that contain *both* a zero and a pole of ϕ; that is, they are the classes $W_{\overline{a}}$ corresponding to linear factors $(z - \overline{a})$ of $\overline{h} = \gcd(\overline{f}, \overline{g})$ in Definition 2.1.

The following lemma will be needed to prove Theorem 4.2. We provide a sketch of the proof, using methods similar to those used by Rivera-Letelier.

LEMMA 2.4. *Let $\phi \in K(z)$ be a rational function of nontrivial reduction. Let $\overline{a} \in \mathbb{P}^1(\hat{k})$ be a point of ramification of $\overline{\phi}$ which is also fixed by $\overline{\phi}$. Let $0 < r < 1$, and let $a \in \mathbb{P}^1(\mathbb{C}_K)$ be a point in the residue class $W_{\overline{a}}$. If $\overline{a} \neq \infty$, let U be the annulus $D(a, 1) \setminus \overline{D}(a, r)$; if $\overline{a} = \infty$, let U be the image of $D(1/a, 1) \setminus \overline{D}(1/a, r)$ under the map $z \mapsto 1/z$. Then the set*

$$W_{\overline{a}} \setminus \left(\bigcup_{n \geq 0} \phi^n(U) \right)$$

contains at most one point.

Proof (Sketch). After a $\mathrm{PGL}(2, \mathcal{O}_{\hat{K}})$-change of coordinates, we may assume that $a = 0$. If $\overline{0}$ is a good class, then the hypotheses imply that for $z \in D(0, 1)$, $\phi(z)$ is given by a power series

$$\phi(z) = \sum_{i=0}^{\infty} c_i z^i$$

with all $|c_i| \leq 1$, with $|c_0|, |c_1| < 1$, and with $|c_m| = 1$ for some minimal $m \geq 2$. (The conditions on c_1 and c_m come from the ramification hypothesis; they imply that the reduction $\overline{\phi}$ looks like $\overline{c}_m z^m + \overline{c}_{m+1} z^{m+1} + \ldots$.) Solving $\phi(z) = z$, it follows easily that $D(0,1)$ contains a fixed point b; without loss, $b = 0$, so that $c_0 = 0$. Then for any $0 < s < 1$, solving the power series equations $\phi(z) = x$ for $x \in D(0,1) \setminus \overline{D}(0, s^m)$ shows that

$$D(0,1) \setminus \overline{D}(0, s^m) \subseteq \phi\left(D(0,1) \setminus \overline{D}(0,s)\right).$$

Thus, for any nonzero $x \in D(0,1)$, there must be an integer $n \geq 0$ such that $x \in \phi^n(U)$. Hence, $D(0,1) \setminus \bigcup \phi^n(U) \subseteq \{0\}$. (In dynamical language, 0 is an attracting fixed point with basin containing $D(0,1)$.)

If $\overline{0}$ is a bad class, then $D(0,1)$ contains finitely many poles, so that for $z \in D(0,1)$, $\phi(z)$ may be written as

$$\phi(z) = \left(\sum_{i=0}^{\infty} c_i z^i\right) + \sum_{j=1}^{M} \frac{A_j}{(z - \alpha_j)^{e_j}},$$

with the same conditions as before on $\{c_i\}$, and with $|A_j|, |\alpha_j| < 1$. Again, we may change coordinates so that $c_0 = 0$, although this time, 0 itself may not be a fixed point. Let $R = \max\{|\alpha_j|\} < 1$. Then for any $s \in [R, 1)$,

$$D(0,1) \setminus \overline{D}(0, s^m) \subseteq \phi\left(D(0,1) \setminus \overline{D}(0,s)\right),$$

as before. Thus, $\phi^n(U)$ contains a pole for some $n \geq 0$; further computations show that $\phi^{n+1}(U) = \mathbb{P}^1(\mathbb{C}_K)$. □

3. Canonical heights

To prove our existence result (Theorem 4.2), we will need a few facts from the theory of diophantine height functions. We present the required statements without proof; instead, we refer the reader to [25, Chapters 2–4] for more details. The results of this section will be used only in the technical proof of Lemma 4.1. The reader may therefore wish to skip ahead to the application of Lemma 4.1 in the proof of Theorem 4.2.

Let k_0 be either $\mathbb{Q}$ or else the field $L(T)$ of rational functions in one variable defined over an arbitrary field L. Let k be a finite extension of k_0, and let $\hat{k}$ be an algebraic closure of k.

The standard *height function* $h_0 : k_0 \to \mathbb{R}_{\geq 0}$ is given by

$$h_0\left(\frac{f}{g}\right) = \max\{\deg f, \deg g\} \tag{1}$$

if $k_0 = L(T)$ and $f, g \in L[T]$ are relatively prime polynomials, or

$$h_0\left(\frac{m}{n}\right) = \max\{\log|m|, \log|n|\} \tag{2}$$

if $k_0 = \mathbb{Q}$ and $m, n \in \mathbb{Z}$ are relatively prime integers. Considering k_0 as a subset of $\mathbb{P}^1(\hat{k})$ in the natural way, the height function h_0 extends to

$$h : \mathbb{P}^1(\hat{k}) \to \mathbb{R}_{\geq 0}$$

with the property that for any rational function $\overline{\phi} \in k(z)$ of degree $d \geq 1$, there is a real constant $C = C_{\overline{\phi}} \geq 0$ such that

$$\text{for all } x \in \mathbb{P}^1(\hat{k}), \quad \left| h\left(\overline{\phi}(x)\right) - dh(x) \right| \leq C.$$

For a *fixed* function $\overline{\phi}(z)$ of degree $d \geq 2$, Call and Silverman [13] introduced a related *canonical* height function

$$\hat{h} = \hat{h}_{\overline{\phi}} : \mathbb{P}^1(\hat{k}) \to \mathbb{R}_{\geq 0}$$

generalizing a construction of Néron [29] and Tate [37]. The key property of $\hat{h}$ is that there is a real constant $C' = C'_{\overline{\phi}} \geq 0$ such that for all $x \in \mathbb{P}^1(\hat{k})$,

$$\hat{h}(\overline{\phi}(x)) = d\hat{h}(x) \quad \text{and} \quad \left| \hat{h}(x) - h(x) \right| \leq C', \tag{3}$$

where h is the standard height function described above. Note that by (3), a preperiodic point x of $\overline{\phi}$ must have canonical height $\hat{h}_{\overline{\phi}}(x) = 0$.

The following lemma is not directly concerned with heights, but it applies to fields k of the type we have been considering in this section. It can be proven using the fact that such a field contains a Dedekind ring of integers $\mathcal{O}_k$ with infinitely many prime ideals.

LEMMA 3.1. *Let k_0 be either $\mathbb{Q}$ or the function field $L(T)$ for some field L, and let k be an algebraic extension of k_0. Let $c \in k^*$ be such that $c^n \neq 1$ for all $n \geq 1$, and define $\overline{\phi}(z) = cz$. Then there exists an infinite sequence $\{x_i : i \in \mathbb{Z}\}$ of wandering points in $\mathbb{P}^1(k)$ such that for any distinct $i, j \in \mathbb{Z}$, x_i and x_j lie in different grand orbits of $\overline{\phi}$.*

4. Existence of wandering domains

Our strategy for constructing wandering domains of $\phi(z) \in K(z)$ begins with finding wandering *points* in $\mathbb{P}^1(\hat{k})$ of the reduction $\overline{\phi}(z) \in k(z)$. The following lemma shows that outside of trivial counterexamples, such points always exist. As mentioned in the previous section, the reader may wish to skip the proof of the Lemma to see its application in the proof of Theorem 4.2.

LEMMA 4.1. *Let k be a field, and let $\overline{\phi}(z) \in k(z)$ be a nonconstant rational function. Suppose that for every $n \geq 1$, $\overline{\phi}^n$ is not the identity function. Then the following five statements are equivalent:*

(a) *k is an algebraic extension of a finite field.*

(b) *Only finitely many wandering grand orbits of* $\overline{\phi}$ *intersect* $\mathbb{P}^1(k)$.
(c) *There are only finitely many wandering grand orbits of* $\overline{\phi}$ *in* $\mathbb{P}^1(\hat{k})$.
(d) *There are no wandering grand orbits of* $\overline{\phi}$ *intersecting* $\mathbb{P}^1(k)$. *That is, all points in* $\mathbb{P}^1(k)$ *are preperiodic under* $\overline{\phi}$.
(e) *There are no wandering grand orbits of* $\overline{\phi}$ *in* $\mathbb{P}^1(\hat{k})$. *That is, all points in* $\mathbb{P}^1(\hat{k})$ *are preperiodic under* $\overline{\phi}$.

Proof. (i) Clearly (e) implies (d) implies (b), and (e) implies (c) implies (b).

To show (a) implies (e), suppose that k is an algebraic extension of a finite field. Then we may assume that $\hat{k} \cong \overline{\mathbb{F}}_p$, an algebraic closure of the field $\mathbb{F}_p$ of p elements, for some prime number p. Given $x \in \mathbb{P}^1(\hat{k})$, there is some $r \geq 1$ such that $x \in \mathbb{P}^1(\mathbb{F}_{p^r})$ and all of the (finitely many) coefficients of $\overline{\phi}(z)$ also lie in $\mathbb{F}_{p^r}$. Since $\mathbb{P}^1(\mathbb{F}_{p^r})$ is a finite set which is mapped into itself by $\overline{\phi}$, x must be preperiodic, proving the implication.

The remaining (and substantive) part of the proof is to show that (b) implies (a). Suppose that k is not an algebraic extension of a finite field; we must show that $\mathbb{P}^1(k)$ intersects infinitely many wandering grand orbits of $\overline{\phi}$.

(ii) We will now reduce to the case that k is a finite extension either of $\mathbb{Q}$ or of the function field $L(T)$, for some field L.

Clearly, k is a field extension of L_0, where $L_0 = \mathbb{Q}$ if $\operatorname{char} k = 0$, or $L_0 = \mathbb{F}_p$ if $\operatorname{char} k = p > 0$. If k/L_0 is an algebraic extension, then by hypothesis, k must be an algebraic extension of $\mathbb{Q}$.

On the other hand, if k/L_0 is a transcendental extension, then there is a nonempty transcendence basis $B \subset k$ such that k is an algebraic extension of $L_0(B)$. (See, for example, [22, Theorem 8.35].) Pick $T \in B$, let $B' = B \setminus \{T\}$, and let $L = L_0(B')$, so that $L(T) \cong L_0(B)$, and T is transcendental over L. In that case, then, k is an algebraic extension of the function field $L(T)$.

We may now assume that k is a *finite* extension of either $\mathbb{Q}$ or $L(T)$. After all, the finitely many coefficients of $\overline{\phi}$ are each algebraic over $\mathbb{Q}$ or $L(T)$, so there is a single finite extension that contains all of them.

Write $k_0 = \mathbb{Q}$ or $k_0 = L(T)$ as appropriate, so that k is a finite extension of k_0. Let $d = \deg \overline{\phi}$. We consider two cases: that $d = 1$, or that $d \geq 2$.

(iii) If $d = 1$, then by a change of coordinates, we may assume that either $\overline{\phi}(z) = z + 1$ or $\overline{\phi}(z) = cz$, for some $c \in k^*$. (If there are two distinct fixed points, move one to 0 and one to ∞, to get $\overline{\phi}(z) = cz$. If there is only one, move it to ∞ and then scale to get $\overline{\phi}(z) = z+1$.) If $\overline{\phi}(z) = z+1$ and $\operatorname{char} k = p > 0$, then $\overline{\phi}^p(z) = z$, contradicting the hypotheses. If $\overline{\phi}(z) = z + 1$ and $\operatorname{char} k = 0$, then $\mathbb{Q} \subset k$, so that there are clearly infinitely many wandering grand orbits; for example, there is one such orbit for each element of $\mathbb{Q} \cap [0, 1)$. On the other hand, if $\overline{\phi}(z) = cz$, then by hypothesis, $c^n \neq 1$ for all $n \geq 1$. Therefore, we have infinitely many wandering grand orbits by Lemma 3.1.

(iv) For the remainder of the proof, suppose $d \geq 2$. Define the canonical height function $\hat{h} = \hat{h}_{\overline{\phi}}$ as in Section 3, and let $C' = C'_{\overline{\phi}} \geq 0$ be the corresponding constant in inequality (3). Let $M = 1 + 2C'$.

We claim that for any real number $r \geq 0$, there exists $x \in \mathbb{P}^1(k)$ satisfying $\hat{h}(x) \in (r, r+M]$. Indeed, by the definition of the height function h_0 in equations (1) and (2), there is some $x \in k_0$ such that $h(x) \in (r+C', r+C'+1]$. Because $|\hat{h}(x) - h(x)| \leq C'$, it follows that $\hat{h}(x) \in (r, r+M]$, proving the claim.

(v) Let $N \geq 1$ be any positive integer; we will show that $\overline{\phi}$ has at least N distinct wandering grand orbits which intersect $\mathbb{P}^1(k)$.

Let I be the real interval $I = (MN, 2MN]$. By (iv), there are at least N different points $x \in \mathbb{P}^1(k)$ such that $\hat{h}(x) \in I$. Recall that the preperiodic points all have canonical height zero; so if $\hat{h}(x) \in I$, then x must be wandering. Thus, it suffices to show that if $x, y \in \mathbb{P}^1(k)$ are two points with $\hat{h}(x), \hat{h}(y) \in I$ but $\hat{h}(x) \neq \hat{h}(y)$, then x and y must lie in different grand orbits.

Suppose not. Then there exist points $x, y \in \mathbb{P}^1(k)$ with $\hat{h}(x), \hat{h}(y) \in I$ but $\hat{h}(x) \neq \hat{h}(y)$, and integers $n \geq m \geq 0$ such that $\overline{\phi}^m(x) = \overline{\phi}^n(y)$. Thus, we have $d^m \hat{h}(x) = d^n \hat{h}(y)$, and therefore $\hat{h}(x) = d^{n-m}\hat{h}(y)$. Since $\hat{h}(x) \neq \hat{h}(y)$, we must have $m < n$. Hence,

$$2MN < 2\hat{h}(y) \leq d^{n-m}\hat{h}(y) = \hat{h}(x) \leq 2MN,$$

because $MN < \hat{h}(y), \hat{h}(x) \leq 2MN$. This contradiction completes the proof. $\square$

We are now prepared to state and prove our existence theorem, which immediately implies part (a) of Theorem A.

THEOREM 4.2. *Let K be a non-archimedean field with residue field k, where k is not an algebraic extension of a finite field. Let $\phi \in K(z)$ be a rational function of nontrivial reduction $\overline{\phi}$, and suppose that $\deg \overline{\phi} \geq 2$. Then there is an infinite set $\{\overline{b}_i : i \in \mathbb{Z}\} \subset \mathbb{P}^1(\hat{k})$ such that $\phi^n(W_{\overline{b}_i}) = W_{\overline{\phi}^n(\overline{b}_i)}$ for every $n \geq 0$ and $i \in \mathbb{Z}$, and such that all iterates $\overline{\phi}^n(\overline{b}_i)$ are distinct. Furthermore:*

(a) *Each $W_{\overline{b}_i}$ is a wandering dynamical component and a wandering dynamical D-component for ϕ, and each $W_{\overline{b}_i}$ lies in a different grand orbit of such components.*
(b) *If the Julia set $\mathcal{J}$ of ϕ intersects at least two different residue classes $W_{\overline{a}_1}$, $W_{\overline{a}_2}$, then each $W_{\overline{b}_i}$ is also a wandering D-component, and each $W_{\overline{b}_i}$ lies in a different grand orbit of such components.*
(c) *If $\mathcal{J}$ has nonempty intersection with infinitely many different residue classes, then each $W_{\overline{b}_i}$ is a wandering analytic component for ϕ, and each $W_{\overline{b}_i}$ lies in a different grand orbit of such components.*

Proof. (i) Let $\{\overline{c}_1, \dots, \overline{c}_m\} \subset \mathbb{P}^1(\hat{k})$ represent the finitely many bad residue classes for $\overline{\phi}$. We claim that there is an infinite set $\{\overline{b}_i : i \in \mathbb{Z}\} \subset \mathbb{P}^1(\hat{k})$ such that no $\overline{b}_i$ is preperiodic under $\overline{\phi}$, such that $\overline{\phi}^n(\overline{b_i})$ avoids the $\overline{c}_j$'s, and such that for any distinct $i, j \in \mathbb{Z}$, the grand orbits of $\overline{b}_i$ and $\overline{b}_j$ under $\overline{\phi}$ are distinct.

To prove the claim, note that by Lemma 4.1, there are points $\{\overline{b}'_i : i \in \mathbb{Z}\}$ in $\mathbb{P}^1(\hat{k})$, each with infinite forward orbit under $\overline{\phi}$, such that no two lie in the same grand orbit. For each $i \in \mathbb{Z}$, let N_i be the largest nonnegative integer n such that $\overline{\phi}^n(\overline{b}'_i)$ equals some $\overline{c}_j$, or else $N_i = -1$ if no such n exists. Then $\overline{b}_i = \overline{\phi}^{N_i+1}(\overline{b}'_i)$ for each i satisfies the claim.

It follows immediately from Lemma 2.3 that for all $i \in \mathbb{Z}$ and all $n \geq 0$, $\phi^n(W_{\overline{b}_i}) = W_{\overline{\phi}^n(\overline{b}_i)}$. Thus, each $W_{\overline{b}_i}$ wanders and lies in the Fatou set of ϕ. Moreover, $W_{\overline{b}}$ is a rational open disk, and therefore it must be contained in a single component of the Fatou set, by any of the four definitions of components. Thus, it suffices to only show that each $W_{\overline{b}_i}$ is the full Fatou component, for each of the four types.

(ii) Fix $\overline{b} = \overline{b}_i$ for some $i \in \mathbb{Z}$. Let V_{dD} be the dynamical D-component containing $W_{\overline{b}}$, V_{dyn} the dynamical component, V_D the D-component, and V_{an} the analytic component.

If $V_{dyn} \supsetneq W_{\overline{b}}$, then V_{dyn} contains a connected open affinoid U such that $U \supsetneq W_{\overline{b}}$. Write $U = \mathbb{P}^1(\mathbb{C}_K) \setminus (D_1 \cup \dots \cup D_m)$ where $D_1, \dots, D_m$ are disjoint closed disks. Because U *properly* contains a residue class, each disk D_i either is contained in $\mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}(0,1)$ or has radius strictly less than 1. Thus, U must contain all but finitely many residue classes. Define the finite (and possibly empty) sets

$$T_1 = \left\{ \overline{a} \in \mathbb{P}^1(\hat{k}) : W_{\overline{a}} \not\subset U \right\}$$

and

$$T_2 = \left\{ \overline{a} \in T_1 : \overline{\phi}^{-n}(\overline{a}) \subseteq T_1 \text{ for all } n \geq 0 \right\}.$$

That is, T_2 is the set of all points $\overline{a} \in \mathbb{P}^1(\hat{k})$ none of whose preimages $\overline{c}$ under any $\overline{\phi}^n$ have class $W_{\overline{c}}$ contained in U. Because T_2 is finite and $\overline{\phi}^{-1}(T_2) \subset T_2$, every element of T_2 must be periodic under $\overline{\phi}$. Thus, there is some integer $m \geq 1$ such that $\overline{\phi}^m$ fixes every element of T_2; it follows that for every $\overline{a} \in T_2$, $\overline{\phi}^{-m}(\overline{a}) = \{\overline{a}\}$. But $\overline{\phi}^m$ has degree larger than 1, and therefore every element of T_2 is a fixed ramification point of $\overline{\phi}^m$.

Let $\tilde{U} = \bigcup_{n \geq 0} \phi^n(U)$. For any $\overline{a} \notin T_2$, there is some $\ell \geq 0$ such that $\phi^\ell(U) \supset W_{\overline{a}}$; hence $\tilde{U} \supset W_{\overline{a}}$. On the other hand, for $a \in T_2$, the intersection $U \cap W_{\overline{a}}$ contains an annulus of the sort described in Lemma 2.4. By that lemma, then, $\tilde{U}$ must contain all but at most one point of $W_{\overline{a}}$. Thus, $\tilde{U}$ contains all but finitely many points of $\mathbb{P}^1(\mathbb{C}_K)$, which contradicts the definition of a dynamical component. Therefore $V_{dyn} = W_{\overline{b}}$.

From the definitions, we have $W_{\overline{b}} \subseteq V_{dD} \subseteq V_{dyn}$. Thus, $V_{dD} = W_{\overline{b}}$ also.

(iii) Next, under the assumption that $\mathcal{J}$ intersects at least two different residue classes, suppose that $V_D \supsetneq W_{\overline{b}}$. Then V_D contains a disk $U \supsetneq W_{\overline{b}}$. Such a disk must contain all but one residue class, and therefore U must intersect the Julia set, which is impossible. Therefore $V_D = W_{\overline{b}}$.

Similarly, if $\mathcal{J}$ intersects infinitely many classes $W_{\overline{a}}$, and if $V_{an} \supsetneq W_{\overline{b}}$, then V_{an} contains a connected affinoid $U \supsetneq W_{\overline{b}}$. U must contain all but finitely many residue classes, which is impossible because then U would intersect $\mathcal{J}$. Hence $V_{an} = W_{\overline{b}}$. $\square$

The following theorem shows that the condition that the Julia set intersects infinitely many different residue classes holds frequently.

THEOREM 4.3. *Let K be a non-archimedean field with residue field k, let $p = \operatorname{char} k \geq 0$, let $\phi \in K(z)$ be a rational function of nontrivial reduction $\overline{\phi}$, and let $\mathcal{J} \subset \mathbb{P}^1(\mathbb{C}_K)$ be the Julia set of ϕ. Suppose either that $\overline{\phi}$ is separable and of degree at least two, or that there is a separable map $\overline{\psi} \in k(z)$ of degree at least two and an integer $r \geq 1$ such that $\overline{\phi}(z) = \overline{\psi}(z^{p^r})$. If $\mathcal{J}$ intersects at least three different residue classes of $\mathbb{P}^1(\mathbb{C}_K)$, then $\mathcal{J}$ intersects infinitely many different residue classes in $\mathbb{P}^1(\mathbb{C}_K)$.*

Proof. Because $\overline{\psi}$ is separable and of degree at least two, then by the Riemann-Hurwitz formula (see [19, Corollary 2.4], for example) at most two points of $\mathbb{P}^1(\hat{k})$ have only one preimage each under $\overline{\psi}$. Given any $N \geq 3$, then, and any set $S_N \subset \mathbb{P}^1(\hat{k})$ of N distinct points, the number of points in $\overline{\psi}^{-1}(S_N)$ must be strictly greater than N.

Applying this fact inductively to $\overline{\phi}(z) = \overline{\psi}(z^{p^r})$, we see that, given any three distinct points $\overline{c}_1, \overline{c}_2, \overline{c}_3 \in \mathbb{P}^1(\hat{k})$, there are infinitely many points $\overline{a} \in \mathbb{P}^1(\hat{k})$ which eventually map to some $\overline{c}_i$ under some $\overline{\phi}^n$.

Meanwhile, by Lemma 2.3, for any class $W_{\overline{a}}$, we have $\phi(W_{\overline{a}}) \supseteq W_{\overline{\phi}(\overline{a})}$. Thus, if $\mathcal{J}$ intersects at least three residue classes, it must intersect infinitely many residue classes. $\square$

To show that wandering domains coming from nontrivial reduction actually exist, we present the following example, which is just a generalization of [6, Example 2]. Our example proves part (b) of Theorem A.

EXAMPLE 6. Let K be a non-archimedean field with residue field k that is not an algebraic extension of a finite field. If $m \geq 2$ is an integer not divisible by $\operatorname{char} k$, let $\Psi_m(z)$ denote the m-th cyclotomic polynomial. For example, if $\operatorname{char} k \neq 2$, we may choose $m = 2$ and hence $\Psi_m(z) = z + 1$; if $\operatorname{char} k = 2$, we may choose $m = 3$ and hence $\Psi_m(z) = z^2 + z + 1$. In either case, $\overline{\Psi}_m$ has distinct roots in $\hat{k}$, and if $\zeta \in \mathbb{C}_K$ is any root, then $\overline{\zeta} \neq 1$ but $\zeta^m = 1$.

If $T \in K$ is any element satisfying $0 < |T| < 1$, define the rational function

$$\phi(z) = z^m + \frac{T}{\Psi_m(z)} = \frac{z^m \Psi_m(z) + T}{\Psi_m(z)}.$$

Then ϕ has nontrivial reduction $\overline{\phi}(z) = z^m \in \hat{k}[z]$, which is separable and of degree $m \geq 2$. The only bad residue classes are the roots of $\overline{\Psi}_m$ in $\hat{k}$. Hence, given $\overline{a} \in \mathbb{P}^1(\hat{k})$ which is not a root of $\overline{\Psi}_m$, we have $\phi(W_{\overline{a}}) = W_{\overline{a}^m}$.

Moreover, we claim that the Julia set of ϕ intersects infinitely many distinct residue classes. To show this, let $\zeta \in \mathbb{C}_K$ be a root of Ψ_m. First, we can easily check that ϕ has a fixed point $\alpha \in \mathbb{C}_K$ with $|\alpha - \zeta| = |T|$. Indeed, substituting $w = z - \zeta$ in the equation $\phi(z) = z$ gives a polynomial in $\mathcal{O}_{\mathbb{C}_K}[w]$ with linear coefficient $(1-\zeta)\Psi'_m(\zeta)$ (which has absolute value 1) and constant term T. Second, we compute $|\phi'(\alpha)| = |T|^{-1} > 1$, so that α is a repelling fixed point and hence lies in the Julia set. Furthermore, because $\overline{\phi}(z) = z^m$ is separable, with no ramification points in $\mathbb{P}^1(\hat{k})$ besides 0 and ∞, the set $\{\overline{\zeta}\} \cup \overline{\phi}^{-1}(\overline{\zeta})$ consists of at least three points. Finally, the corresponding residue classes each contain preimages of α, and hence they intersect the Julia set. By Theorem 4.3, our claim is valid.

It follows by Theorem 4.2 that ϕ has infinitely many grand orbits of wandering components (of all four types). More precisely, for any $b \in K$ such that $\{\overline{b}^n\}_{n \in \mathbb{Z}}$ is an infinite subset of k, the class $W_{\overline{b}}$ is a wandering domain. After all, no iterate $\overline{\phi}(\overline{b})$ is ever one of the bad classes $\overline{\zeta}$ (otherwise, all future iterates of $\overline{b}$ would be $\overline{1}$), and those iterates are all distinct.

As another example, if the field K satisfies the hypotheses of Theorem 4.2, then for $n \geq 2$, the function $\phi(z) = z^n$ of Example 1 has wandering dynamical components and wandering dynamical D-components. However, the unique analytic component and D-component, namely $\mathbb{P}^1(\mathbb{C}_K)$, is not wandering.

As mentioned in the introduction, sufficient conditions for residue classes of $\mathbb{P}^1(\mathbb{C}_K)$ to be wandering domains are more complicated if the map has a nontrivial reduction of degree one. The remaining examples of this section are of functions of reduction degree one, all defined over the field $K = \mathbb{Q}((T))$, whose residue field $\mathbb{Q}$ is not an algebraic extension of a finite field.

EXAMPLE 7. Let $\phi(z) = z + 1 + T/z \in \mathbb{Q}((T))$. Then ϕ has nontrivial reduction $\overline{\phi}(z) = z + 1$ of degree one. The disk $D(0,1)$ contains the repelling fixed point $-T$; it follows that $D(-m,1)$ intersects the Julia set for every integer $m \geq 0$. On the other hand, the disk $U = D(1,1)$ satisfies $\phi^n(U) = D(n+1,1)$ for every integer $n \geq 0$, so that U lies in the Fatou set. Moreover, any strictly larger affinoid containing U must contain one of the disks $D(-m,1)$ and hence must intersect the Julia set. So U is a wandering analytic component, wandering D-component, wandering dynamical component, and wandering dynamical D-component.

Example 8. Let $\phi(z) = Tz^2 + z + 1 \in \mathbb{Q}((T))$. Then ϕ has nontrivial reduction $\overline{\phi}(z) = z+1$ of degree one, and as in the previous example, the disk $U = D(1,1)$ lies in the Fatou set and is wandering; in fact, the same is true of every disk $D(b,1)$ for $|b| \leq 1$. However, all these disks are contained in the single disk $D(0, 1/|T|)$, which is fixed. Thus, although the smaller disks are wandering, none of them is large enough to be a component of the Fatou set.

In fact, $\phi = h \circ \psi \circ h^{-1}$, where $h(z) = Tz$ and $\psi(z) = z^2 + z + T$, which is a map of good reduction, having reduction $\overline{\psi}(z) = z^2 + z$ of degree two. By Theorem 4.2, ψ does have wandering dynamical components and wandering dynamical D-components. Therefore ϕ also has such wandering components, though they are *not* the residue classes $D(b,1)$ that we considered at first. Moreover, the whole of $\mathbb{P}^1(\mathbb{C}_K)$ forms a single D-component and a single analytic component; hence, there are no wandering analytic or D-components.

Example 9. Let $b = 2$ and $c = T$ in Example 5, so that

$$\phi(z) = \frac{2z(z+T)(z+T^2)}{(z+2T)(z+T^3)(Tz+1)^2},$$

which has nontrivial reduction $\overline{\phi}(z) = 2z$ of degree one. As in Example 5, let $V = D(0, |T|^{-1}) \setminus \overline{D}(0, |T|)$. All of the residue classes $D(a,1)$ (for $|a| = 1$) are contained in the Fatou set; in fact, every such residue class $D(a,1)$ is a wandering D-component and a wandering dynamical D-component. On the other hand, as we saw before, the affinoid V, which contains all the disks $D(a,1)$, is both a fixed analytic component and a fixed dynamical component.

Example 10. Let $b = -1$ and $c = T$ in Example 5, so that

$$\phi(z) = \frac{-z(z+T)(z+T^2)}{(z-T)(z+T^3)(Tz+1)^2},$$

which has nontrivial reduction $\overline{\phi}(z) = -z$ of degree one. Again, all of the residue classes $D(a,1)$ (for $|a| = 1$) are contained in the Fatou set and are both D-components and dynamical D-components. This time, however, all those disks are fixed by ϕ^2, so none of them is wandering. As before, the open affinoid $V = D(0, |T|^{-1}) \setminus \overline{D}(0, |T|)$ contains all the disks $D(a,1)$ and is both a fixed analytic component and fixed dynamical component.

5. Residue characteristic zero

We now prove Theorem B. The following theorem is a slightly stronger result, showing that the desired conjugacy is defined over a certain finite extension of K.

Theorem 5.1. *Let K be a discretely valued non-archimedean field with residue field k and residue characteristic* $\operatorname{char} k = 0$. *Let $\phi \in K(z)$ be a rational function, and suppose that U is a wandering analytic component,* *wandering D-component, wandering dynamical D-component, or wandering dynamical component of* ϕ*. Let* $L \subset \mathbb{C}_K$ *be any finite extension of* K *such that* U *contains a point of* $\mathbb{P}^1(L)$*. Then there is a change of coordinates* $g \in \mathrm{PGL}(2, L)$ *and there are integers* $M \geq 0$ *and* $N \geq 1$ *such that* $\psi(z) = g \circ \phi^N \circ g^{-1}(z)$ *has nontrivial reduction,* $D(0,1)$ *is a wandering component (of the same type as* U*) of* ψ*, and* $g(\phi^M(U)) \subset D(0,1)$*.*

Note that a field L satisfying the required properties always exists. Indeed, the algebraic closure $\hat{K}$ of K is dense in $\mathbb{P}^1(\mathbb{C}_K)$, so that the open set U must contain some $a \in \hat{K}$. Then $L = K(a)$ is a finite extension of K.

Proof. We devote the bulk of the proof to the case that U is a wandering dynamical D-component.

(i) Let $a \in U \cap \mathbb{P}^1(L)$. Write $U_n = \Phi^n_{dD}(U)$ to simplify notation; recall that $\Phi^n_{dD}(U)$ is the dynamical D-component containing $\phi^n(U)$.

We may assume without loss that $U_n \subset \overline{D}(0,1)$ for every $n \geq 0$. To do so, make a $\mathrm{PGL}(2, L)$-change of coordinates to move a to ∞ and U to a set containing $\mathbb{P}^1 \setminus \overline{D}(0,1)$. Because U is wandering, it follows that $U_n \subset \overline{D}(0,1)$ for every $n \geq 1$. Finally, replace U by U_1, and we have the desired scenario.

Let L' be a finite extension of L such that $\mathbb{P}^1(L')$ contains all critical points and all poles of ϕ in $\mathbb{P}^1(\mathbb{C}_K)$. L' is discretely valued, because it is only a finite extension of the original field K. Thus, there is a real number $0 < \varepsilon < 1$ such that $|(L')^*| = \{\varepsilon^m : m \in \mathbb{Z}\}$.

Furthermore, there are only finitely many $n \geq 0$ such that U_n contains a critical point, and by Lemma A.5, only finitely many $n \geq 0$ such that $\Phi_{dD}(U_n) \neq \phi(U_n)$. Thus, by replacing U by $U_{M'}$ for some $M' \geq 0$, we may assume for all $n \geq 0$ that $U_n = \phi^n(U)$, and that U_n contains no critical points and no poles.

Write $r_n = \mathrm{diam}(U_n) > 0$, so that $U_n = D(\phi^n(a), r_n)$, for each $n \geq 0$. (Recall that the diameter and the radius of a non-archimedean disk are the same; see Section A.2.) By Lemmas A.3 and A.4, because each U_n contains no critical points or poles, there are integers $\ell_n \in \mathbb{Z}$ such that $r_n = \varepsilon^{\ell_n} r_0$.

(ii) We now claim that $r_0 \in |(L')^*|$. To prove the claim, suppose not. Because L' is discretely valued, there exists a real number $s_0 > r_0$ such that no $x \in L'$ satisfies $r_0 \leq |x| < s_0$. For every $n \geq 0$, let $s_n = r_n \cdot s_0/r_0$. By the fact that $r_n = \varepsilon^{\ell_n} r_0$ and $|(L')^*| = \{\varepsilon^m\}$, it follows that no $x \in L'$ satisfies $r_n \leq |x| < s_n$.

Let $V_n = \phi^n(D(a, s_0))$, for all $n \geq 0$. We will now show, by induction on n, that V_n is an open disk of radius (i.e., diameter) s_n that contains no critical points or poles. For $n = 0$, V_0 is an open disk of radius s_0 by definition, and it contains no critical points or poles because $V_0 \cap L' = U_0 \cap L'$ by our choice of s_0. Assuming the claim is true for $n \geq 0$, then by Lemmas A.3 and A.4, $\mathrm{diam}(V_{n+1})/\mathrm{diam}(V_n) = \mathrm{diam}(U_{n+1})/\mathrm{diam}(U_n)$, since V_n contains no critical points or poles. It follows that V_{n+1} is a set of diameter s_{n+1}. Thus, V_{n+1} certainly omits at least two points of $\mathbb{P}^1(\mathbb{C}_K)$; by Lemma A.1, then, it is an open disk. Because no $x \in L'$ satisfies $r_{n+1} \leq |x| < s_{n+1}$, we have $V_{n+1} \cap L' = U_{n+1} \cap L'$, and therefore V_{n+1} contains no critical points or poles, completing the induction.

Since each U_n is contained in $\overline{D}(0,1)$, then $s_n \leq s_0/r_0$ for every $n \geq 0$. Therefore,

$$\phi^n(V_0) = V_n \subseteq \overline{D}(0, s_0/r_0)$$

for all $n \geq 0$. Because $U = U_0 \subsetneq V_0$, we have contradicted the assumption that U is a dynamical D-component.

Thus, $r_0 \in |(L')^*|$, as claimed. It follows that $r_n \in |(L')^*|$ for all $n \geq 0$.

(iii) For all $n \geq 0$, let $\overline{U}_n = \overline{D}(\phi^n(a), r_n)$, so that $U_n \subsetneq \overline{U}_n \subset \overline{D}(0,1)$.

We claim that for infinitely many $n \geq 0$, $\overline{U}_n$ contains a pole or a critical point of ϕ. To prove the claim, suppose only finitely many of the $\overline{U}_n$ contained poles or critical points, and replace U by $U_{M'}$ (for some appropriate $M' \geq 0$) so that no $\overline{U}_n$ contains a pole or critical point. Because $|(L')^*| = \{\varepsilon^m\}$ is discrete and $r_n \in |(L')^*|$, the larger disk $D(\phi^n(a), r_n/\varepsilon)$ also contains no poles or critical points for any $n \geq 0$.

For all $n \geq 0$, define $V'_n = \phi^n(D(a, r_0/\varepsilon))$. By an induction argument similar to that in part (ii) above, $V'_n = D(\phi^n(a), r_n/\varepsilon)$. Because V'_0 is an open disk that properly contains U, and because $\phi^n(V'_0) \subseteq D(0, 1/\varepsilon)$, we have contradicted the assumption that U is a dynamical D-component, thus proving the claim.

(iv) Next, we claim that either $\overline{U}_n$ contains a pole for infinitely many $n \geq 0$, or else there exist $M \geq 0$ and $N \geq 1$ such that $\overline{U}_M = \overline{U}_{M+N}$.

If only finitely many of the $\overline{U}_n$ contain poles, then by (iii), infinitely many of them contain critical points. As there are only finitely many critical points, there must be integers $M \geq 0$ and $N \geq 1$ such that $\overline{U}_M \cap \overline{U}_{M+N}$ is nonempty, and such that for all $n \geq M$, $\overline{U}_n$ contains no poles. Replacing U by U_M and ϕ by ϕ^N, we may assume that $M = 0$ and $N = 1$. By Lemma A.2 and the fact that $\overline{U}_n$ contains no poles, $\phi(\overline{U}_n) = \overline{U}_{n+1}$ for all $n \geq 0$. Because $\overline{U}_0$ and $\overline{U}_1$ are disks in $\mathbb{C}_K$, either $\overline{U}_0 \supsetneq \overline{U}_1$ or $\overline{U}_0 \subseteq \overline{U}_1$.

If $\overline{U}_0 \supsetneq \overline{U}_1$, then because $|(L')^*| = \{\varepsilon^m\}$, we must have $r_1 \leq \varepsilon r_0 < r_0$. Let $V'' = D(\phi(a), r_0)$. Since $V'' \subset \overline{U}_0$, we have $\phi(V'') \subset \phi(\overline{U}_0) = \overline{U}_1 \subset \overline{U}_0$; by induction, we get $\phi^n(V'') \subset \overline{U}_0$ for all $n \geq 0$. However, V'' is an open disk that properly contains U_1, contradicting the supposition that $U_1 = \Phi_{dD}(U)$ is a dynamical D-component.

If $\overline{U}_0 \subseteq \overline{U}_1$, then by the fact that $\phi(\overline{U}_n) = \overline{U}_{n+1}$ for every $n \geq 0$, we have

$$\overline{U}_0 \subseteq \overline{U}_1 \subseteq \overline{U}_2 \subseteq \cdots .$$

If all the inclusions are proper, then

$$r_0 < r_1 < r_2 < \cdots .$$

Because $|(L')^*| = \{\varepsilon^m\}$, we must have $r_n > 1$ for some $n \geq 0$, contradicting the assumption that every U_n is contained in $\overline{D}(0,1)$. Thus, for some $n \geq 0$, we have $\overline{U}_n = \overline{U}_{n+1}$, and the claim is proven. (In fact, we would have $\overline{U}_0 = \overline{U}_1$, but we do not need that result here.)

(v) Consider the case that $\overline{U}_n$ contains a pole for infinitely many $n \geq 0$. Since there are only finitely many poles, there must be some pole $y \in \mathbb{P}^1(L')$ and an infinite set I of nonnegative integers such that $y \in \overline{U}_n = \overline{D}(\phi^n(a), r_n)$ for all $n \in I$. Pick $s > 0$ so that $\phi(D(y,s)) \subset \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}(0,1)$. By our initial assumptions, no U_n can intersect $D(y,s)$, or else U_{n+1} would not be contained in $\overline{D}(0,1)$. Thus, $s < r_n \leq 1$ for all $n \in I$.

However, we also know that $r_n = \varepsilon^{\ell_n} r_0 \in |(L')^*|$ for all $n \geq 0$. As n ranges over I, then, there are only finitely many possible values that r_n can attain. At least one must be attained infinitely often. In particular, there are integers $M \geq 0$ and $N \geq 1$ such that $M, M+N \in I$ and $r_M = r_{M+N}$. Since y lies in both U_M and U_{M+N}, we have $\overline{U}_M = \overline{U}_{M+N}$.

(vi) By (iv) and (v), then, there exist integers $M \geq 0$ and $N \geq 1$ such that $\overline{U}_M = \overline{U}_{M+N}$. Thus, $r_M = r_{M+N}$, and $|\phi^M(a) - \phi^{M+N}(a)| \leq r_M$. Because $U_M \cap U_{M+N} = \emptyset$, we must in fact have $|\phi^M(a) - \phi^{M+N}(a)| = r_M$. Therefore $r_M \in |L^*|$, since $a \in \mathbb{P}^1(L)$ and $\phi \in K(z) \subseteq L(z)$.

Let $g \in \mathrm{PGL}(2,L)$ be the unique linear fractional transformation satisfying $g(\infty) = \infty$, $g(\phi^M(a)) = 0$, and $g(\phi^{M+N}(a)) = 1$. Thus, $g(U_M) = D(0,1)$ and $g(U_{M+N}) = D(1,1)$. Let $\psi = g \circ \phi^N \circ g^{-1}$. By Lemma 2.2, ψ has nontrivial reduction, and the remaining conclusions of the theorem follow as well, at least for the case of dynamical D-components.

(vii) Finally, suppose that U is a wandering D-component, wandering analytic component, or wandering dynamical component containing a point $a \in \mathbb{P}^1(L)$. Let U' be the dynamical D-component containing a. Then $U' \subseteq U$, and therefore U' is wandering.

For any integer $n \geq 0$, define $U_n = \Phi^n(U)$ (where Φ is Φ_D, Φ_{an}, or Φ_{dyn}, as appropriate), and define $U'_n = \Phi_{dD}(U')$. Choose g, M, and N for ϕ as in the theorem applied to U'. It suffices to show that $U_M = U'_M$.

Suppose not; then $U'_M \subsetneq U_M$, and therefore $g(U'_M) \subsetneq g(U_M)$. Thus, $g(U_M)$ contains an affinoid strictly containing the residue class $g(U'_M)$; hence, $g(U_M)$ contains all but finitely many of the residue classes $D(b,1)$. However, for every $n \geq 0$, $g(U'_{M+nN})$ is a residue class. In particular, $g(U_M)$ contains $g(U'_{M+nN})$ for some $n \geq 1$. Thus, $U_M \cap U_{M+nN}$ is nonempty, contradicting the wandering assumption and proving the theorem. $\square$

Appendix A. Rational functions and non-archimedean analysis

A.1. General properties of rational functions. We recall some basic facts about rational functions $\phi \in L(z)$, for an abstract field L with algebraic closure $\hat{L}$. A point $x \in L$ is called a *pole* of ϕ if $\phi(x) = \infty$. We may define the derivative $\phi'(z)$ away from the poles by the usual formal differentiation rules; if L has a metric structure, then the formal definition of ϕ' agrees with the difference quotient definition of ϕ'.

If $x \in \mathbb{P}^1(\hat{L})$ maps to $\phi(x)$ with multiplicity greater than one (i.e., if $\phi'(x) = 0$), we say x is a *critical point* or *ramification point* of ϕ. After a coordinate change in the domain and range, we may assume that $x = \phi(x) = 0$, and we may expand ϕ locally about 0 as a power series

$$\phi(z) = \sum_{n=1}^{\infty} c_n z^n.$$

We say that x *maps to* $\phi(x)$ *with multiplicity* m if m is the smallest integer such that $c_m \neq 0$. Note that if $\operatorname{char} L = p > 0$, the multiplicity might *not* be the same as the number of the first nonzero derivative at x. For example, if $\phi(z) = z^p$ where $\operatorname{char} L = p$, then $\phi'(z) = 0$, but every point x maps to its image with multiplicity p, not infinite multiplicity.

If $\phi'(z)$ is not identically zero, we say ϕ is *separable*. If $\operatorname{char} L = 0$, then all nonconstant rational functions are separable. If $\operatorname{char} L = p > 0$, then $\phi \in L(z)$ is separable if and only if ϕ cannot be written as $\phi(z) = \psi(z^p)$ for any $\psi \in L(z)$.

A function $\phi \in L(z)$ may be written as $\phi = f/g$, where $f, g \in L[z]$ are relatively prime polynomials. The *degree* $\deg \phi$ is defined to be

$$\deg \phi = \max\{\deg f, \deg g\}.$$

Any point $y \in \mathbb{P}^1(\hat{L})$ has exactly $\deg \phi$ preimages in $\phi^{-1}(y)$, counting multiplicity. If ϕ is separable of degree d, then ϕ has exactly $2d - 2$ critical points in $\mathbb{P}^1(\hat{L})$, counting multiplicity. (Here, the multiplicity of a critical point x is the multiplicity of x as a root of the equation $\phi'(z) = 0$. Usually, this multiplicity is $e_x - 1$, where x maps to $\phi(x)$ with multiplicity e_x. However, if $\operatorname{char} L = p > 0$, and if $p \mid e_x$, then the multiplicity of x as a critical point will be strictly greater than $e_x - 1$. See [19, IV.2] for more details.)

A.2. Non-archimedean analysis. Given $a \in \mathbb{C}_K$ and $r > 0$, we denote by $D(a, r)$ and by $\overline{D}(a, r)$ the open disk and the closed disk, respectively, of radius r centered at a. (We will follow the convention that all disks have *positive* radius by definition, so that singleton sets and the empty set are *not* considered to be disks.) By the non-archimedean triangle inequality, any point of such a disk may be considered a center, and if $U_1, U_2 \subset \mathbb{C}_K$ are two overlapping disks, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. Moreover, if $U \subset \mathbb{C}_K$ is an open or closed disk of radius r, then r is also the diameter of U; that is,

$$r = \operatorname{diam}(U) = \sup\{|x - y| : x, y \in U\}.$$

The set $|K^*| = \{|x| : x \in K \setminus \{0\}\} \subset \mathbb{R}_{>0}$ may be a discrete subset of $\mathbb{R}_{>0}$; if so, we say that K is *discretely valued*. In that case, there is a real number $0 < \varepsilon < 1$ such that $|K^*| = \{\varepsilon^m : m \in \mathbb{Z}\}$.

Meanwhile, the set $|\mathbb{C}_K^*| = \{|x| : x \in \mathbb{C}_K \setminus \{0\}\}$ must be dense in $\mathbb{R}_{>0}$, but it need not contain all positive real numbers. For example, $|\mathbb{C}_p^*| = \{p^q : q \in \mathbb{Q}\}$. Therefore, we say that a disk U is *rational* if $\operatorname{diam}(U) \in |\mathbb{C}_K^*|$, and U is *irrational* otherwise. If $a \in \mathbb{C}_K$ and $r \in |\mathbb{C}_K^*|$, then $D(a,r) \subsetneq \overline{D}(a,r)$; but if $r \in (\mathbb{R}_{>0} \setminus |\mathbb{C}_K^*|)$, then $D(a,r) = \overline{D}(a,r)$. Thus, every disk is exactly one of the following three types: rational open, rational closed, or irrational. The distinctions between the three indicate metric properties, but not topological properties; *all* disks are both open and closed as *topological sets*.

More generally, a set $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a *rational open disk* if either $U \subset \mathbb{C}_K$ is a rational open disk or $\mathbb{P}^1(\mathbb{C}_K) \setminus U$ is a rational closed disk. Similarly, $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a *rational closed disk* if either $U \subset \mathbb{C}_K$ is a rational closed disk or $(\mathbb{P}^1(\mathbb{C}_K) \setminus U) \subset \mathbb{C}_K$ is a rational open disk; and $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is an *irrational disk* if either $U \subset \mathbb{C}_K$ is an irrational disk or $(\mathbb{P}^1(\mathbb{C}_K) \setminus U) \subset \mathbb{C}_K$ is an irrational disk. There is a natural spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$ (see, for example, [5], [8], [28]), but not all the disks we have just defined in $\mathbb{P}^1(\mathbb{C}_K)$ are disks with respect to the spherical metric.

If $U_1, U_2 \subset \mathbb{P}^1(\mathbb{C}_K)$ are disks such that $U_1 \cap U_2 \neq \emptyset$ and $U_1 \cup U_2 \neq \mathbb{P}^1(\mathbb{C}_K)$, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. In particular, both $U_1 \cap U_2$ and $U_1 \cup U_2$ are also disks; and if U_1 and U_2 are both rational closed (respectively, rational open, irrational), then so are $U_1 \cap U_2$ and $U_1 \cup U_2$.

The group $\mathrm{PGL}(2, \mathbb{C}_K)$ acts on $\mathbb{P}^1(\mathbb{C}_K)$ by linear fractional transformations. Any $g \in \mathrm{PGL}(2, \mathbb{C}_K)$ maps rational open disks to rational open disks, rational closed disks to rational closed disks, and irrational disks to irrational disks.

The following four lemmas concern the action of non-archimedean rational functions on disks. In fact, all four apply more generally to power series on disks, though we do not need to define the necessary terminology of rigid analyticity to state the lemmas. We omit the proofs, which are easy applications of the Weierstrass Preparation Theorem, Newton polygons, and other fundamentals of non-archimedean analysis. Some proofs may be found in [8]; see any of [12, Chapter 5], [15, Chapter II], [24, Chapter IV], or [32, Chapter 6] for the theory surrounding such results.

LEMMA A.1. *Let $U \subset \mathbb{P}^1(\mathbb{C}_K)$ be a disk, and let $\phi \in \mathbb{C}_K(z)$ be a rational function. Suppose that $\mathbb{P}^1(\mathbb{C}_K) \setminus \phi(U)$ contains at least two points. Then $\phi(U)$ is a disk of the same type (rational closed, rational open, or irrational) as U.*

LEMMA A.2. *Let $a, b \in \mathbb{C}_K$, let $r, s > 0$, and let $\phi \in \mathbb{C}_K(z)$ be a rational function with no poles in $\overline{D}(a,r)$, such that $\phi(D(a,r)) = D(b,s)$. Then $\phi(\overline{D}(a,r)) = \overline{D}(b,s)$.*

LEMMA A.3. *Let $U \subset \mathbb{C}_K$ be a disk, let $a \in U$, and let $\phi \in \mathbb{C}_K(z)$ be a rational function with no poles in U. Then the following two statements are equivalent.*

(a) *ϕ is one-to-one on U.*
(b) *For all $x, y \in U$, $|\phi(x) - \phi(y)| = |\phi'(a)| \cdot |x - y|$.*

LEMMA A.4. *Let K be a non-archimedean field with residue field k, and suppose that* $\operatorname{char} k = 0$*. Let $U \subset \mathbb{C}_K$ be a disk, and let $\phi \in \mathbb{C}_K(z)$ be a rational function. Then the following two statements are equivalent.*

(a) *ϕ is one-to-one on U.*
(b) *ϕ has no critical points in U.*

Lemma A.4 is needed only in parts (ii) and (iii) of the proof of Theorem 5.1, and it is the only use in that proof of the hypothesis that $\operatorname{char} k = 0$. (The lemma is also quoted in part (i) of the same proof, but its use there can be avoided if desired.)

A.3. Rigid analysis. We will need some basic facts and definitions from the non-archimedean theory of rigid analysis. We refer the reader to [12, Part C] or [16] for detailed background, or to [17] for a broader (but still technical) overview of the subject; however, the discussion that follows is mostly self-contained.

A *connected affinoid* is a set $W \subset \mathbb{P}^1(\mathbb{C}_K)$ of the form

$$W = \mathbb{P}^1(\mathbb{C}_K) \setminus (U_1 \cup U_2 \cup \cdots \cup U_N),$$

where $N \geq 0$, and where the $\{U_i\}$ are pairwise disjoint disks. If each U_i is rational open, we say W is a *connected rational closed affinoid*; if each U_i is rational closed, we say W is a *connected rational open affinoid*; and if each U_i is irrational, we say W is a *connected irrational affinoid*.

If W_1 and W_2 are connected affinoids, and if $W_1 \cap W_2 \neq \emptyset$, then $W_1 \cap W_2$ and $W_1 \cup W_2$ are also connected affinoids. In that case, if W_1 and W_2 are both rational closed (respectively, rational open, irrational), then so are $W_1 \cap W_2$ and $W_1 \cup W_2$.

In general, an *affinoid* is a finite union of connected affinoids. However, we will not need that notion in this paper. Note that our definition allows the full set $\mathbb{P}^1(\mathbb{C}_K)$ and the empty set $\emptyset$ to be considered connected affinoids, while traditional rigid analysis does not. Also note that we consider $\mathbb{P}^1(\mathbb{C}_K)$ to be a connected affinoid of all three types. Every other connected affinoid is at most one of the three types; or, it may be none of them, if, for example, U_1 is a rational open disk and U_2 is a rational closed disk.

Intuitively, connected affinoids are supposed to behave like connected sets, even though *topologically*, all subsets of $\mathbb{P}^1(\mathbb{C}_K)$ are totally disconnected. In particular, it is well known (as can be shown using standard rigid analysis techniques) that if $\phi \in \mathbb{C}_K(z)$ is a rational function of degree d, and if W is a connected affinoid, then:

- $\phi(W)$ is also a connected affinoid. Moreover, if W is rational closed (respectively, rational open, irrational), then so is $\phi(W)$.

- $\phi^{-1}(W)$ is a disjoint union of connected affinoids $V_1, \dots, V_N$, with $1 \le N \le d$. For every $i = 1, \dots, N$, ϕ maps V_i onto W. Moreover, if W is rational closed (respectively, rational open, irrational), then so are $V_1, \dots, V_N$.

The following lemma shows that for any given rational function ϕ, most disks $U \subset \mathbb{P}^1(\mathbb{C}_K)$ have preimage $\phi^{-1}(U)$ consisting simply of a finite union of disks. It appeared as [2, Lemma 3.1.4], but we include a partial proof here for the convenience of the reader.

Lemma A.5. *Let $U_1, \dots, U_n \subset \mathbb{P}^1(\mathbb{C}_K)$ be disjoint disks, and let $\phi \in K(z)$ be a rational function of degree $d \ge 1$. Suppose that for each $i = 1, \dots, n$, the inverse image $\phi^{-1}(U_i)$ is not a finite union of disks. Then $n \le d - 1$.*

Proof (Sketch). If U_i is an open disk, then it can be written as a nested union $\bigcup_{m \ge 1} V_m$ of rational closed disks V_m, with $V_m \subset V_{m+1}$. If each $\phi^{-1}(V_m)$ is a union of at most d disks, then the same is true of $\phi^{-1}(U_i)$. Thus, we may assume that each U_i is a rational closed disk.

Let $W = \mathbb{P}^1(\mathbb{C}_K) \setminus (U_1 \cup \cdots \cup U_n)$. Then W is a rational open connected affinoid. By the discussion above, the inverse image $\phi^{-1}(W)$ is a disjoint union of at most d rational open connected affinoids. Thus, as we leave to the reader to verify, the complement $\mathbb{P}^1(\mathbb{C}_K) \setminus \phi^{-1}(W)$ is a union of some rational closed disks and at most $d - 1$ connected affinoids which are not disks. (Note, for example, that if V is a closed affinoid that is neither a disk nor all of $\mathbb{P}^1(\mathbb{C}_K)$, then the complement of V consists of at least two connected components. Thus, if $\mathbb{P}^1(\mathbb{C}_K) \setminus \phi^{-1}(W)$ consisted of d or more non-disk connected components, then $\phi^{-1}(W)$ would consist of at least $d+1$ connected components.) However, the complement of $\phi^{-1}(W)$ is precisely the disjoint union $\bigcup_{i=1}^{n} \phi^{-1}(U_i)$. It follows that $n \le d - 1$. □

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