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WANG'S MULTIPLICITY RESULT FOR SUPERLINEAR (p,q)–EQUATIONS WITHOUT THE AMBROSETTI–RABINOWITZ CONDITION

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Abstract. We consider a nonlinear elliptic equation driven by the sum of a p–Laplacian and a q–Laplacian, where $1 < q \le 2 \le p < \infty$ with a $(p-1)$–superlinear Carathéodory reaction term which doesn't satisfy the usual Ambrosetti–Rabinowitz condition. Using variational methods based on critical point theory together with techniques from Morse theory, we show that the problem has at least three nontrivial solutions; among them one is positive and one is negative.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \ge 1$, with C^2 boundary $\partial\Omega$. In this paper we deal with the following (p,q)–equation:

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(z,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $1 < q \le 2 \le p < \infty$, $\mu \ge 0$ and for every $r \in (1,\infty)$ Δ_r denotes the usual r–Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for any $x \in \mathbb{R}$ the map $z \mapsto f(z,x)$ is measurable, and for a.e. $z \in \Omega$ the map $x \mapsto f(z,x)$ is continuous). We assume that for a.e. $z \in \Omega$, the function $f(z,\cdot)$ exhibits a $(p-1)$–superlinear growth near $\pm\infty$; but to express this $(p-1)$–superlinearity of $f(z,\cdot)$, instead of using the usual (in such cases) Ambrosetti–Rabinowitz condition (the AR–condition for short), we employ an alternative condition which involves the function

$$\sigma(z,x) = f(z,x)x - pF(z,x),$$

where $F(z,x) = \int_0^x f(z,s)\,ds$. In this way, we are able to incorporate in our framework of analysis "superlinear" forcing terms with "slow" growth near $\pm\infty$, which fail to satisfy the AR–condition. We recall that the AR–condition says that there exist $\tau > p$ and $M > 0$ such that

$$0 < \tau F(z,x) \le f(z,x)x \quad \text{for a.e. } z \in \Omega \text{ and all } |x| \ge M. \tag{1.2}$$

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A direct integration of (1.2) in the case that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous (otherwise see [21]) produces the following weaker condition:

$$F(z,x) \geq c_1|x|^\tau \quad \text{for a.e. } z \in \Omega, \text{ all } |x| \geq M \text{ and some } c_1 > 0. \tag{1.3}$$

It is clear from (1.3) that the AR–condition, although natural and very useful in verifying the Palais–Smale condition (the PS–condition for short) for the energy functional associated to the problem, is somewhat restrictive and excludes from consideration several interesting nonlinearities. For this reason, there have been efforts to remove hypothesis (1.2). For an overview of the relevant literature in this direction, we refer to the pioneering papers of Liu–Wang [17] and of Li–Wang–Zeng [14] (where the analogue of **H**(ii) in Section 3 below was introduced for $p = 2$), and to the more recent ones of Li–Yang [15], Liu [16] and Miyagaki–Souto [19].

We mention that (p,q)–equations arise as the steady state of a general reaction–diffusion system of the form

$$u_t = \operatorname{div}(K(u)Du) + h(z,u), \tag{1.4}$$

where $K(u) = |Du|^{p-2} + |Du|^{q-2}$. Such equations arise in the study of many phenomena in physical sciences. In these applications u describes a concentration, the first term on the right–hand side of (1.4) corresponds to the diffusion with a diffusion coefficient $K(u)$ and the second term on the right–hand side of (1.4) is related to sources and loss processes. Typically, in such applications, $h(z,u)$ is a polynomial of u with variable coefficients; see Cherfils–Il'yasov [4] and He–Li [10]. Moreover, the $(p,2)$-equation is important in quantum physics for the existence of solitons; see Benci–D'Avenia–Fortunato–Pisani [2]. Recently, (p,q)–equations were studied by Cingolani–Degiovanni [5], Figueiredo [7], Li–Guo [13] and Medeiros–Perera [18]. Of the aforementioned works, only Li–Guo [13] have a $(p-1)$–superlinear reaction. More precisely, their right–hand side term is $f(z,x) = |x|^{p^*-2}x + \mu|x|^{r-2}x$ with

$$1 < r \leq p < p^* = \begin{cases} \frac{N}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geq N, \end{cases}$$

and they prove the existence of infinitely many solutions when $\mu \in (0, \mu_0)$, for some suitable $\mu_0 \in (0,1]$.

Our result here is an extension to (p,q)–equations of a "three solutions theorem" for superlinear semilinear equations (i.e. when $p = 2$ and $\mu = 0$) of Wang [27], Mugnai [22] and Rabinowitz–Su–Wang [25]. But in [27] and [25] the reaction f belongs to $C^1(\overline{\Omega} \times \mathbb{R})$ and satisfies the AR–condition, while in [22] f was assumed to be a Carathéodory function giving a linking structure to the associated functional.

Our approach combines variational methods based on critical point theory with Morse theory and truncation techniques, which allow us to treat the coexistence of a singular and a degenerate operator. In the next section, for the reader's convenience, we briefly recall some of the main mathematical tools which we shall use in the sequel.

2. Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\phi \in C^1(X)$; we say that ϕ satisfies the "Cerami condition" (the "C–condition" for short) if the following holds:

$$\begin{gathered}\text{every sequence } \{x_n\}_{n\geq 1} \subseteq X \text{ such that}\\ \{\phi(x_n)\}_{n\geq 1} \subseteq \mathbb{R} \text{ is bounded and } (1+\|x_n\|)\phi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,\\ \text{admits a strongly convergent subsequence.}\end{gathered}$$

This compactness condition is, in general, weaker than the Palais–Smale condition. However, as was shown by Bartolo–Benci–Fortunato [1], it is sufficient to have a deformation theorem for ϕ from which one can deduce the minimax theory for certain critical values of ϕ. In particular, we can state the following theorem, known in literature as the "Mountain Pass Theorem".

Theorem 1. *If $\phi \in C^1(X)$ satisfies the C–condition, there exist x_0, $x_1 \in X$ and $\rho > 0$ with $\|x_0 - x_1\| > \rho$ and*

$$\max\left\{\phi(x_0), \phi(x_1)\right\} < \inf\left\{\phi(x) \,:\, \|x - x_0\| = \rho\right\} = \eta_\rho,$$

and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \phi(\gamma(t)),$$

where $\Gamma := \Big\{\gamma \in C([0,1], X) \,:\, \gamma(0) = x_0,\, \gamma(1) = x_1\Big\}$. Then $c \geq \eta_\rho$ and c is a critical value for ϕ.

For a given $\phi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following notation:

$$\phi^c := \Big\{x \in X \,:\, \phi(x) \leq c\Big\}, \quad \dot{\phi}^c := \Big\{x \in X \,:\, \phi(x) < c\Big\},$$

$$K_\phi := \Big\{x \in X \,:\, \phi'(x) = 0\Big\} \text{ and } K_\phi^c = \Big\{x \in K_\phi \,:\, \phi(x) = c\Big\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}–relative singular homology group with integer coefficients for the pair (Y_1, Y_2). The critical groups of ϕ at an isolated critical point $x_0 \in X$ with $\phi(x_0) = c$ (i.e. $x_0 \in K_\phi^c$) are defined by

$$C_k(\phi, x) := H_k(\phi^c \cap U, \phi^c \cap U \setminus \{x\}) \quad \text{for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\phi \cap \phi^c \cap U = \{x\}$. The excision property of singular homology implies that this definition is independent of the particular choice of the neighborhood U.

Now, suppose that $\phi \in C^1(X)$ satisfies the C–condition and $\inf \phi(K_\phi) > -\infty$. Let $c < \phi(K_\phi)$; then the critical groups of ϕ at infinity are defined by

$$C_k(\phi, \infty) := H_k(X, \phi^c) \quad \text{for all } k \geq 0.$$

The Second Deformation Theorem (see, for example, Gasinski–Papageorgiou [8, p. 628]) implies that the definition above is independent of the particular choice of the level $c < \inf \phi(K_\phi)$. Moreover, if $c < \inf \phi(K_\phi)$, then $C_k(\phi, \infty) = H_k(X, \dot{\phi}^c)$ for all $k \geq 0$. To see this, let $\xi < c < \inf \phi(K_\phi)$. Then ϕ^ξ is a deformation retract of $\dot{\phi}^c$ (see, for example, Granas–Dugundji [9, p. 407]), and so

$$H_k(X, \phi^\xi) = H_k(X, \dot{\phi}^c) \text{ for all } k \geq 0,$$

so that

$$C_k(\phi, \infty) = H_k(X, \dot{\phi}^c) \text{ for all } k \geq 0,$$

as claimed.

If K_ϕ is finite, we can define

$$M(t,x) := \sum_{k\geq 0} \operatorname{rank} C_k(\phi,x)t^k \quad \text{for all } t\in\mathbb{R},\ x\in K_\phi$$

and

$$P(t,\infty) := \sum_{k\geq 0} \operatorname{rank} C_k(\phi,\infty)t^k \quad \text{for all } t\in\mathbb{R}.$$

The Morse relation says that

$$\sum_{x\in K_\phi} M(t,x) = P(t,\infty) + (1+t)Q(t) \text{ for all } t\in\mathbb{R}, \tag{2.1}$$

where $Q(t) := \sum_{k\geq 0}\beta_k t^k$ is a formal series with nonnegative integer coefficients (see, for example, Chang [3, p. 339]).

In the analysis of problem (1.1), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space $C_0^1(\overline{\Omega}) = \Big\{u\in C^1(\overline{\Omega}) \,:\, u_{|\partial\Omega}=0\Big\}$. This is an ordered Banach space with positive cone

$$C_+ := \Big\{u\in C_0^1(\overline{\Omega}) \,:\, u(z)\geq 0 \text{ for all } z\in\overline{\Omega}\Big\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \Big\{u\in C_+ \,:\, u(z)>0 \text{ for all } z\in\Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega}<0\Big\},$$

where $n(\cdot)$ denotes the outward unit normal to $\partial\Omega$.

Recall that the negative Dirichlet p–Laplacian, denoted by $(-\Delta_p, W_0^{1,p}(\Omega))$, has first eigenvalue $\hat{\lambda}_{1,p}$ which is positive, isolated and simple (i.e., it is a principal eigenvalue), and admits the following variational characterization:

$$\hat{\lambda}_{1,p} = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} \,:\, u\in W_0^{1,p}(\Omega),\ u\neq 0\right\}. \tag{2.2}$$

In (2.2) the infimum is attained at the corresponding one dimensional eigenspace; in the sequel by $\hat{u}_{1,p}$ we denote the L^p–normalized (i.e., $\|\hat{u}_{1,p}\|_p=1$) eigenfunction associated to $\hat{\lambda}_{1,p}$. It is clear from (2.2) that we may always assume that $\hat{u}_{1,p}\geq 0$ in Ω. Actually, by nonlinear regularity theory (see, for example, Gazinzki–Papageorgiou [8, pp. 737-738]) and the nonlinear maximum principle of Vazquez [26] (see also Pucci–Serrin [24, p. 120]), we have that $\hat{u}_{1,p}\in\operatorname{int} C_+$.

For every $r\in(1,\infty)$, let $A_r : W_0^{1,r}(\Omega)\to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ $(1/r+1/r'=1)$ be the nonlinear map defined by

$$\langle A_r(u),v\rangle = \int_\Omega |Du|^{r-2}(Du,Dv)_{\mathbb{R}^N}\,dz \qquad \text{for all } u,\ v\in W_0^{1,r}(\Omega). \tag{2.3}$$

Proposition 2. *The map $A_r : W_0^{1,r}(\Omega)\to W^{-1,r'}(\Omega)$ defined by (2.3) is bounded, continuous, strictly monotone (strongly monotone if $r\geq 2$), and hence maximal monotone and of type $(S)_+$, i.e.,*

$$\text{if } u_n \rightharpoonup u \text{ in } W_0^{1,r}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A_r(u_n),u_n-u\rangle\leq 0,$$
$$\text{then } u_n\to u \text{ in } W_0^{1,r}(\Omega).$$

Finally we mention that throughout this work, for every $x \in \mathbb{R}$ we set $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. For every $u \in W_0^{1,p}(\Omega)$, we set $\|u\| = \|Du\|_p$, and we know that $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$ and $|u| = u^+ + u^-$. Observe that the notation $|\cdot|$ will also be used to denote the $\mathbb{R}^N$–norm; it will always be clear from the context which norm is used. Lastly, $|\cdot|_N$ denotes the Lebesgue measure in $\mathbb{R}^N$.

3. Hypotheses and solutions of constant sign

The hypotheses on the reaction f are the following:

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.e. $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in (p, p^*)$ such that
$$|f(z, x)| \le a(z) + c|x|^{r-1} \quad \text{for a.e. } z \in \Omega \text{ and for all } x \in \mathbb{R};$$

(ii) $\displaystyle\lim_{x\to\pm\infty} \frac{F(z, x)}{|x|^p} = \infty$ uniformly for a.e. $z \in \Omega$;

(iii) if $\sigma(z, x) = f(z, x)x - pF(z, x)$, then there exists $\beta^* \in L^1(\Omega)_+$ such that
$$\sigma(z, x) \le \sigma(z, y) + \beta^*(z) \text{ for a.e. } z \in \Omega, \text{ all } 0 \le x \le y \text{ or all } y \le x \le 0;$$

(iv) there exists $\theta \in L^\infty(\Omega)_+$, $\theta \le \hat{\lambda}_{1,p}$, $\theta \ne \hat{\lambda}_{1,p}$, and $\eta > 0$ such that
$$\limsup_{x\to 0} \frac{pF(z, x)}{|x|^p} \le \theta(z) \text{ uniformly for a.e. } z \in \Omega$$
and
$$\liminf_{x\to 0} \frac{f(z, x)}{|x|^{p-2}x} \ge -\eta \text{ uniformly for a.e. } z \in \Omega.$$

Remark 1. 1. Hypothesis **H**(ii) implies that for a.e. $z \in \Omega$ the map $F(z, \cdot)$ is p–superlinear near $\pm\infty$.

2. Hypotheses **H**(ii),(iii) together imply that
$$\lim_{x\to\pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = \infty \text{ uniformly for a.e. } z \in \Omega;$$
see Li–Yang [15, Lemma 2.4]. Therefore, for a.e. $z \in \Omega$, $f(z, \cdot)$ is $(p-1)$–superlinear near $\pm\infty$.

3. Hypothesis **H**(iii) is a quasimonotonicity condition on σ, and it is satisfied, for example, if there exists $M > 0$ such that for a.e. $z \in \Omega$ the map
$$x \mapsto \frac{f(z, x)}{x^{p-1}} \text{ is increasing when } x \ge M \text{ and}$$
$$x \mapsto \frac{f(z, x)}{|x|^{p-2}x} \text{ is decreasing when } x \le -M;$$
see Li–Yang [15].

Example 1. The following functions satisfy hypotheses **H** (for the sake of simplicity we drop the z–dependence):
$$f_1(x) = |x|^{r-2}x + \theta|x|^{\tau-2}x$$
with $1 < \tau \le p < r < p^*$, $\theta \in \mathbb{R}$ and $\theta < \hat{\lambda}_{1,p}$ when $\tau = p$, and
$$f_2(x) = \begin{cases} \theta|x|^{p-2}x - \theta\left(\frac{p-1}{p}\right)|x|^{r-2}x & \text{if } |x| \le 1, \\ \theta|x|^{p-2}x\Big(\log|x| + \frac{1}{p}\Big) & \text{if } |x| > 1, \end{cases}$$

where $1 < p < r < p^*$ and $\theta < \hat{\lambda}_{1,p}$. Note that f_1 satisfies the AR–condition (1.2), but f_2 does not.

We introduce the following "positive" and "negative" truncations of f:

$$f_+(z,x) = f(z,x^+) \text{ and } f_-(z,x) = f(z,-x^-) \text{ for a.e. } z \in \mathbb{R} \text{ and all } x \in \mathbb{R},$$

and both are Carathéodory functions. We also set $F_\pm(z,x) = \int_0^x f_\pm(z,s)\,ds$ and consider the C^1 functionals $\phi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\phi_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{q}\|Du\|_q^q - \int_\Omega F_\pm(z,u)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$. Moreover, let $\phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional associated to problem (1.1) defined by

$$\phi(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{q}\|Du\|_q^q - \int_\Omega F(z,u)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$.

From now on we will assume, without loss of generality, that $q < p$.

Proposition 3. *If hypotheses* $\mathbf{H}$(i),(ii),(iii) *hold, then $\phi_\pm$ satisfy the C–condition.*

Proof. We perform the proof for ϕ_+, the proof for ϕ_- being similar. Thus, let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\phi_+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \geq 1 \text{ and} \tag{3.1}$$

$$(1 + \|u_n\|)\phi'_+(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \ (1/p + 1/p' = 1) \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$\left|\langle A_p(u_n), h\rangle + \mu\langle A_q(u_n), h\rangle - \int_\Omega f_+(z,u_n)h\,dz\right| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{3.3}$$

for all $h \in W_0^{1,p}(\Omega)$ and with $\varepsilon_n \to 0$ as $n \to \infty$.

In (3.3) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\|Du_n^-\|_p^p + \mu\|Du_n^-\|_q^q \leq \varepsilon_n \quad \text{for all } n \geq 1,$$

so that

$$u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.4}$$

Now in (3.3) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$-\|Du_n^+\|_p^p - \mu\|Du_n^+\|_q^q + \int_\Omega f_+(z,u_n^+)u_n^+\,dz \leq \varepsilon_n \text{ for all } n \geq 1. \tag{3.5}$$

From (3.1) and (3.4) we have

$$\|Du_n^+\|_p^p + \frac{\mu p}{q}\|Du_n^+\|_q^q - p\int_\Omega F(z,u_n^+)\,dz \leq M_2 \tag{3.6}$$

for some $M_2 > 0$ and all $n \geq 1$. Adding (3.5) and (3.6) we obtain

$$\mu\left(\frac{p}{q} - 1\right)\|Du_n^+\|_q^q + \int_\Omega \sigma(z,u_n^+)\,dz \leq M_3 \tag{3.7}$$

for some $M_3 > 0$ and all $n \geq 1$.

Claim: $\{u_n^+\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ *is bounded.* We argue by contradiction, so we assume that the Claim is not true. Then, by passing to a subsequence, if necessary, we may assume that $\|u_n^+\| \to \infty$. We set $y_n = u_n^+/\|u_n^+\|$, $n \geq 1$, and thus $\|y_n\| = 1$ for all $n \geq 1$. Hence, we may assume that

$$y_n \rightharpoonup y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega), \text{ with } y \geq 0. \tag{3.8}$$

First we assume that $y \neq 0$. Thus, if $Z(y) = \{z \in \Omega \,:\, y(z) = 0\}$, then $|\Omega \backslash Z(y)|_N > 0$ and $u_n^+(z) \to \infty$ for a.e. $z \in \Omega \setminus Z(y)$ as $n \to \infty$. Invoking hypothesis $\mathbf{H}$(ii), we have

$$\frac{F(z, u_n^+(z))}{\|u_n^+\|^p} = \frac{F(z, u_n^+(z))}{(u_n^+(z))^p} y_n(z)^p \to \infty \text{ for a.e. } z \in \Omega \setminus Z(y),$$

and so, by Fatou's Lemma,

$$\int_\Omega \frac{F(z, u_n^+(z))}{\|u_n^+\|^p}\, dz \to \infty \text{ as } n \to \infty. \tag{3.9}$$

From (3.1) and (3.4) we have

$$-\frac{1}{p}\|D_n^+\|_p^p - \frac{\mu}{q}\|Du_n^+\|_q^q + \int_\Omega F(z, u_n^+)\, dz \leq M_4$$

for some $M_4 > 0$ and all $n \geq 1$, and so

$$-\frac{1}{p} - \frac{\mu}{q}\frac{1}{\|u_n^+\|^{p-q}} + \int_\Omega \frac{F(z, u_n^+(z))}{\|u_n^+\|^p}\, dz \leq \frac{M_4}{\|u_n^+\|^p} \text{ for all } n \geq 1.$$

Passing to the limit as $n \to \infty$, using the fact that $q < p$, we obtain

$$\limsup_{n\to\infty} \int_\Omega \frac{F(z, u_n^+(z))}{\|u_n^+\|^p}\, dz \leq M_5 \text{ for some } M_5 > 0. \tag{3.10}$$

Comparing (3.9) and (3.10) we reach a contradiction. This takes care of the case $y \neq 0$.

Now, suppose that $y \equiv 0$. Let us consider the continuous functions $\gamma_n : [0,1] \to \mathbb{R}$ defined by

$$\gamma_n(t) = \phi(tu_n^+) \text{ for all } t \in [0,1] \text{ and all } n \geq 1,$$

and define $t_n \in [0,1]$ such that

$$\gamma_n(t_n) = \max\{\gamma_n(t) \,:\, t \in [0,1]\}. \tag{3.11}$$

Now, for $\lambda > 0$, let $v_n = (2\lambda)^{1/p} y_n \in W_0^{1,p}(\Omega)$. Then $v_n \to 0$ in $L^r(\Omega)$ by (3.8) and the fact that we are assuming $y \equiv 0$. By Krasnoselskii's Theorem and $\mathbf{H}$(i) we have

$$\int_\Omega F(z, v_n)\, dz \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Since $\|u_n^+\| \to \infty$ as $n \to \infty$, we can find an integer $n_0 \geq 1$ such that $\frac{(2\lambda)^{1/p}}{\|u_n^+\|} \in (0,1)$ for all $n \geq n_0$. Then, by (3.11),

$$\gamma_n(t_n) \geq \gamma_n\left(\frac{(2\lambda)^{1/p}}{\|u_n^+\|}\right) \text{ for all } n \geq n_0.$$

Thus

$$\begin{aligned}
\phi(t_n u_n^+) &\geq \phi((2\lambda)^{1/p} y_n) = \phi(v_n) \\
&\geq \frac{2\lambda}{p} - \int_\Omega F(x, v_n)\, dz \text{ (recall that } \|y_n\| = \|Dy_n\|_p = 1\ \forall\, n \geq 1) \\
&\geq \frac{2\lambda}{p} - \frac{\lambda}{p} = \frac{\lambda}{p} \text{ for all } n \geq n_1 \geq n_0 \text{ by (3.12).}
\end{aligned} \tag{3.13}$$

Since $\lambda > 0$ is arbitrary, from (3.13) we infer that

$$\phi(t_n u_n^+) \to \infty \quad \text{as } n \to \infty. \tag{3.14}$$

Since $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \geq 1$, by virtue of hypothesis **(H)**(iii) we get

$$\int_\Omega \sigma(z, t_n u_n^+)\, dz \leq \int_\Omega \sigma(z, u_n^+)\, dz + \|\beta^*\|_1 \text{ for all } n \geq 1. \tag{3.15}$$

Moreover, by (3.1) and (3.4), we find $M_6 > 0$ such that

$$\phi(0) = 0 \text{ and } \phi(u_n^+) = \phi_+(u_n) \leq M_6 \text{ for all } n \geq 1.$$

These facts together with (3.14) imply that $t_n \in (0,1)$ for all $n \geq n_2 \geq 1$. Thus, (3.11) implies that

$$\begin{aligned}
0 = t_n \frac{d}{dt}\phi(t_n u_n^+)\bigg|_{t=t_n} &= \langle \phi'(t_n u_n^+), t_n u_n^+ \rangle \\
&= \|D(t_n u_n^+)\|_p^p + \mu \|D(t_n u_n^+)\|_q^q - \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)\, dz
\end{aligned}$$

for all $n \geq 1$, that is,

$$\|D(t_n u_n^+)\|_p^p + \mu \|D(t_n u_n^+)\|_q^q - \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)\, dz = 0 \text{ for all } n \geq n_2. \tag{3.16}$$

Now, we return to (3.15) and use (3.16); then

$$\|D(t_n u_n^+)\|_p^p + \mu \|D(t_n u_n^+)\|_q^q - p\int_\Omega F(z, t_n u_n^+)\, dz \leq \int_\Omega \sigma(z, u_n^+)\, dz + \|\beta^*\|_1,$$

for all $n \geq n_2$, so that

$$\begin{aligned}
p\phi(t_n u_n^+) &\leq \mu\left(\frac{p}{q} - 1\right)\|D(t_n u_n^+)\|_q^q + \int_\Omega \sigma(z, u_n^+)\, dz + \|\beta^*\|_1 \\
&\leq \mu\left(\frac{p}{q} - 1\right)\|Du_n^+\|_q^q + \int_\Omega \sigma(z, u_n^+)\, dz + \|\beta^*\|_1
\end{aligned}$$

for all $n \geq n_2$, since $t_n \in (0,1)$. Thus, by (3.14),

$$\mu\left(\frac{p}{q} - 1\right)\|Du_n^+\|_q^q + \int_\Omega \sigma(z, u_n^+)\, dz \to \infty \text{ as } n \to \infty. \tag{3.17}$$

Comparing (3.7) and (3.17) we reach a contradiction, and the Claim follows.

The Claim and (3.4) imply that $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ is bounded. So we may assume that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{3.18}$$

Now, in (3.3) we choose $h = u_n - u$, pass to the limit as $n \to \infty$ and use (3.18), obtaining

$$\lim_{n\to\infty} \langle A_p(u_n) + \mu A_q(u_n), u_n - u \rangle = 0. \tag{3.19}$$

From the monotonicity of A_q we have

$$\langle A_q(u), u_n - u\rangle \le \langle A_q(u_n), u_n - u\rangle,$$

and so

$$\langle A_p(u_n) + \mu A_q(u), u_n - u\rangle \le \langle A_p(u_n) + \mu A_q(u_n), u_n - u\rangle.$$

Since $q < p$, we have that $u_n \to u$ in $W_0^{1,q}(\Omega)$. Thus by (3.18) and (3.19) we obtain

$$\limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \le 0,$$

and by Proposition 2 we finally get that $u_n \to u$ in $W_0^{1,p}(\Omega)$. This proves that ϕ_+ satisfies the C–condition, and similarly we proceed to prove that ϕ_- verifies the C–condition as well. □

With minor changes in the proof above we can show (we omit the straightforward details):

Proposition 4. *If hypotheses* **H**(i),(ii),(iii) *hold, then ϕ satisfies the C–condition.*

Proposition 5. *If hypotheses* **H**(i),(iv) *hold, then $u = 0$ is a strict local minimizer for the functionals $\phi_\pm$ and for ϕ.*

Proof. We perform the proof for ϕ_+, the proofs for ϕ_- and ϕ being similar.

Hypotheses **H**(i),(iv) imply that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(z,x) \le \frac{1}{p}\big[\theta(z) + \varepsilon\big]|x|^p + C_\varepsilon |x|^r \quad \text{for a.e. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.20}$$

For every $u \in W_0^{1,p}(\Omega)$ we thus have

$$\begin{aligned}
\phi_+(u) &= \frac{1}{p}\|Du\|_p^p + \frac{\mu}{q}\|Du\|_q^q - \int_\Omega F_+(z,u)\,dz \\
&= \frac{1}{p}\|Du^+\|_p^p + \frac{1}{p}\|Du^-\|_p^p + \frac{\mu}{q}\|Du\|_q^q - \int_\Omega F_+(z,u)\,dz \\
&\ge \frac{1}{p}\|Du^+\|_p^p - \frac{1}{p}\int_\Omega \theta(u^+)^p dz - \frac{\varepsilon}{p\hat{\lambda}_{1,p}}\|u^+\|^p + \frac{1}{p}\|u^-\|^p - c_1\|u\|^r \\
&\qquad \text{for some } c_1 > 0 \text{ by (3.20)} \\
&\ge \frac{1}{p}\left[\left(c_2 - \frac{\varepsilon}{\hat{\lambda}_{1,p}}\right)\|u^+\|^p + \|u^-\|^p\right] - c_1\|u\|^r \\
&\qquad \text{for some } c_2 > 0 \text{ by [23, Lemma 5.1.3, p. 356].}
\end{aligned} \tag{3.21}$$

Choosing $\varepsilon \in (0, c_2\hat{\lambda}_{1,p})$, from (3.21) we have

$$\phi_+(u) \ge c_3\|u\|^p - c_1\|u\|^r \quad \text{for some } c_3 > 0. \tag{3.22}$$

Since $p < r$, from (3.22) it follows that there exists $\rho > 0$ sufficiently small so that

$$\phi_+(u) \ge \phi_+(0) = 0 \quad \text{for all } u \in W_0^{1,p}(\Omega) \text{ with } \|u\| \le \rho,$$

i.e. $u = 0$ is a local minimizer for ϕ_+. Similarly for ϕ_- and ϕ. □

Now we are ready to produce two nontrivial constant sign solutions. The approach is variational and is based on Theorem 1, the Mountain Pass Theorem.

Proposition 6. *If hypotheses* **H** *hold, then problem* (1.1) *has at least two nontrivial constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$.*

Proof. From Proposition 5 we know that $u = 0$ is a local minimizer of ϕ_+. We may assume that it is an isolated critical point for ϕ_+ (otherwise we have a whole sequence of distinct nontrivial positive solutions). From the final part of the proof of Proposition 5, we can find $\rho \in (0,1)$ so small that

$$0 = \phi_+(0) < \inf\left\{\phi_+(u) \,:\, \|u\| = \rho\right\} := \eta_+. \tag{3.23}$$

By virtue of hypotheses $\mathbf{H}$(i),(ii), for any $\varepsilon > 0$ there exists $c_4 = c_4(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{\varepsilon}{p}x^p - c_4 \quad \text{for a.e. } z \in \Omega \text{ and for all } x \geq 0. \tag{3.24}$$

Recalling that $\hat{u}_{1,p} \in \operatorname{int} C_+$ is the L^p–normalized principal eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see (2.2)) for every $t > 0$ we have

$$\begin{aligned} \phi_+(t\hat{u}_{1,p}) &= \frac{t^p}{p}\hat{\lambda}_{1,p} + \frac{t^q}{q}\|D\hat{u}_{1,p}\|_q^q - \int_\Omega F(z, t\hat{u}_{1,p})\, dz \\ &\leq \frac{t^p}{p}(\hat{\lambda}_{1,p} - \varepsilon) + \frac{t^q}{q}\|D\hat{u}_{1,p}\|_q^q + c_4|\Omega|_N \end{aligned} \tag{3.25}$$

by (3.24). Choosing $\varepsilon > \hat{\lambda}_{1,p}$, since $q < p$, from (3.25) it follows that

$$\phi_+(t\hat{u}_{1,p}) \to -\infty \quad \text{as } t \to \infty. \tag{3.26}$$

From (3.23), (3.26) and Proposition 3, we get that we can apply Theorem 1 (the Mountain Pass Theorem), and so we obtain $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\phi_+(0) = 0 < \eta_+ \leq \phi_+(u_0) \tag{3.27}$$

and

$$\phi_+'(u_0) = 0. \tag{3.28}$$

From (3.27) we see that $u_0 \neq 0$, while (3.28) yields

$$A(u_0) + \mu A_q(u_0) = N_{f_+}(u_0), \tag{3.29}$$

where $N_{f_+}(u)(\cdot) = f_+(\cdot, u(\cdot))$ for all $u \in W_0^{1,p}(\Omega)$. On (3.29) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain $u_0 \geq 0$, $u_0 \neq 0$, so that (3.29) becomes

$$A(u_0) + \mu A_q(u_0) = N_f(u_0),$$

where $N_f(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W_0^{1,p}(\Omega)$, i.e.

$$\begin{cases} -\Delta_p u_0(z) - \mu\Delta_q u_0(z) = f(z, u_0(z)) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$ for all $y \in \mathbb{R}^N$. Then (3.30) becomes

$$\begin{cases} -\operatorname{div} a(Du_0(z)) = f(z, u_0(z)) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

and we can apply Theorem IV.7.1 of Ladyzhenskaya–Uraltseva [11], so that $u_0 \in L^\infty(\Omega)$.

Note that $a \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ and that for every $y \in \mathbb{R}^N \setminus \{0\}$ we have

$$\nabla a(y) = |y|^{p-2}\left(I + (p-2)\frac{y \otimes y}{|y|^2}\right) + \mu|y|^{q-2}\left(I + (q-2)\frac{y \otimes y}{|y|^2}\right).$$

Thus, by the Cauchy–Schwarz inequality, since $q-2 \le 0$, we find

(3.31)
$$\begin{aligned}(\nabla a(y)\xi,\xi)_{\mathbb{R}^N} &= |y|^{p-2}\left(|\xi|^2+(p-2)\frac{(y,\xi)^2_{\mathbb{R}^N}}{|y|^2}\right)+\mu|y|^{q-2}\left(|\xi|^2+(q-2)\frac{(y,\xi)^2_{\mathbb{R}^N}}{|y|^2}\right)\\ &\ge (|y|^{p-2}+\mu(q-1)|y|^{q-2})|\xi|^2.\end{aligned}$$

Moreover,

$$|\nabla a(y)| \le (p-1)|y|^{p-2}+\mu(q-1)|y|^{q-2} \le (p-1)(|y|^{p-2}+\mu|y|^{q-2}). \tag{3.32}$$

Thus, setting $g(t)=t^{p-1}+\mu t^{q-1}$, $t \ge 0$, we have that $g \in C^1(0,\infty)$ and conditions (1.2), (1.10a) and (1.10b) of Lieberman [12] are satisfied and Theorem 1.7 therein applies. Hence, by (3.30)–(3.32), we have that $u_0 \in C_+ \setminus \{0\}$.

Hypotheses $\mathbf{H}$(i), (iv) imply that we can find $\eta_1 > \eta$ such that

$$f(z,x) \ge -\eta_1 x^{p-1} \text{ for a.e. } z \in \Omega \text{ and all } x \in [0, \|u_0\|_\infty]. \tag{3.33}$$

Thus, from (3.30) and (3.33) we have

$$-\Delta_p u_0 - \mu\Delta_q u_0 + \eta_1 u_0^{p-1} = f(z,u_0)+\eta_1 u_0^{p-1} \ge 0 \text{ a.e. in } \Omega.$$

Invoking [24, Theorem 5.4.1], we infer that $u_0(z) > 0$ for all $z \in \Omega$ (recall that $u_0 \ne 0$). Finally, we can apply [24, Theorem 5.5.1] and conclude that $u_0 \in \operatorname{int} C_+$.

Similarly, working with ϕ_-, we find $v_0 \in W_0^{1,p}(\Omega)$, $v_0 \le 0$, $v_0 \ne 0$, another constant sign solution of (1.1). □

4. Three solutions theorem

In this section we generate a third nontrivial solution for problem (1.1) by using Morse Theory. We start by computing the critical groups of ϕ at infinity.

Proposition 7. *If hypotheses* $\mathbf{H}$ *hold, then* $C_k(\phi,\infty)=0$ *for all* $k \ge 0$.

Proof. By virtue of hypothesis $\mathbf{H}$)(ii), for every $u \in W_0^{1,p}(\Omega)\setminus\{0\}$, we have

$$\phi(tu) \to -\infty \quad \text{as } t \to \infty. \tag{4.1}$$

Moreover, hypothesis $\mathbf{H}$(iii) implies that for every $u \in W_0^{1,p}(\Omega)$ we have

$0=\sigma(z,0)\le\sigma(z,u^+(z))+\beta^*(z)$ and $0=\sigma(z,0)\le\sigma(z,-u^-(z))+\beta^*(z)$ a.e. in Ω,

so that

$$0=\sigma(z,0)\le\sigma(z,u(z))+\beta^*(z) \text{ a.e. in } \Omega,$$

and thus

$$-\sigma(z,u(z)) = pF(z,u(z))-f(z,u(z))u(z) \le \beta^*(z) \text{ a.e. in } \Omega. \tag{4.2}$$

Now let $u \in W_0^{1,p}(\Omega)$ and $t>0$; we have

(4.3)
$$\begin{aligned}\frac{d}{dt}\phi(tu) &= \langle\phi'(tu),u\rangle = \frac{1}{t}\langle\phi'(tu),tu\rangle\\ &= \frac{1}{t}\left(\|D(tu)\|_p^p+\mu\|D(tu)\|_q^q-\int_\Omega f(z,tu)tu\,dz\right)\\ &\le \frac{1}{t}\left(\|D(tu)\|_p^p+\mu\|D(tu)\|_q^q-\int_\Omega pF(z,tu)\,dz+\|\beta^*\|_1\right) \quad \text{(by (4.2))}\\ &\le \frac{1}{t}\big(p\phi(tu)+\|\beta^*\|_1\big) \quad (\text{since } q<p).\end{aligned}$$

By (4.1) we see that, if $u \neq 0$, for $t > 0$ large enough we have $\phi(tu) \leq \tau_0 < -\frac{\|\beta^*\|_1}{p}$, and so from (4.3) it follows that

$$\frac{d}{dt}\phi(tu) < 0 \quad \text{for } t > 0 \text{ large enough.} \tag{4.4}$$

Let $\partial B_1 = \left\{u \in W_0^{1,p}(\Omega) : \|u\| = 1\right\}$. For $u \in \partial B_1$ we can find a unique $\beta(u) > 0$ such that $\phi(\beta(u)u) = \tau_0$, and by virtue of the Implicit Function Theorem (applicable by (4.4)), we find in particular that $\beta \in C(\partial B_1)$. We extend β on $W_0^{1,p}(\Omega) \setminus \{0\}$ as follows:

$$\beta_0(u) = \frac{1}{\|u\|}\beta\left(\frac{u}{\|u\|}\right) \quad \text{for all } u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Evidently $\beta_0 \in C(W_0^{1,p}(\Omega) \setminus \{0\})$ and $\phi(\beta_0(u)u) = \tau_0$. Moreover, if $\phi(u) = \tau_0$, then $\beta_0(u) = 1$. Therefore, we set

$$\hat{\beta}(u) = \begin{cases} 1 & \text{if } \phi(u) < \tau_0, \\ \beta_0(u) & \text{if } \phi(u) \geq \tau_0. \end{cases} \tag{4.5}$$

From (4.5) and the previous remarks we immediately have that $\hat{\beta}_0 \in C(W_0^{1,p}(\Omega) \setminus \{0\})$.

Now we introduce the homotopy $h : [0,1] \times (W_0^{1,p}(\Omega) \setminus \{0\}) \to W_0^{1,p}(\Omega) \setminus \{0\}$ defined by

$$h(t,u) = (1-t)u + t\hat{\beta}_0(u)u.$$

Thus we have

$$h(0,u) = u \quad \text{and} \quad h(1,u) = \hat{\beta}_0(u)u \in \phi^{\tau_0} \quad \text{for all } u \in W_0^{1,p}(\Omega) \setminus \{0\}$$

and $h(t,\cdot)_{|\phi^{\tau_0}} = Id_{|\phi^{\tau_0}}$ for all $t \in [0,1]$; see (4.5). These facts imply that ϕ^{τ_0} is a strong deformation retract of $W_0^{1,p}(\Omega) \setminus \{0\}$. Using the radial retraction $\hat{r}(u) = u/\|u\|$ for all $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we see that ∂B_1 is a retract of $W_0^{1,p}(\Omega) \setminus \{0\}$, and the latter is deformable into ∂B_1. Thus, by Theorem XV.6.5 (p. 325) of Dugundji [6], we infer that ∂B_1 is a deformation retract of $W_0^{1,p}(\Omega) \setminus \{0\}$. It follows that ϕ^{τ_0} and ∂B_1 are homotopy equivalent, and so

$$H_k(W_0^{1,p}(\Omega), \phi^{\tau_0}) = H_k(W_0^{1,p}(\Omega), \partial B_1) \quad \text{for all } k \geq 0. \tag{4.6}$$

Since $W_0^{1,p}(\Omega)$ is infinite dimensional, the set ∂B_1 is contractible in itself. Hence

$$H_k(W_0^{1,p}(\Omega), \partial B_1) = 0 \quad \text{for all } k \geq 0; \text{ see [9, p. 389]},$$

and so, by (4.6),

$$H_k(W_0^{1,p}(\Omega), \phi^{\tau_0}) = 0 \quad \text{for all } k \geq 0.$$

By choosing $\tau_0 < -\|\beta^*\|_1/p$ and $|\tau_0|$ large, we get $C_k(\phi, \infty) = 0$ for all $k \geq 0$, as desired. □

Now we provide an analogous result for the functionals $\phi_\pm$.

Proposition 8. *If hypotheses* **H** *hold, then* $C_k(\phi_+, \infty) = C_k(\phi_-, \infty) = 0$ *for all* $k \geq 0$.

Proof. Let $\psi_+ = \phi_{+|C_0^1(\bar{\Omega})}$. Nonlinear regularity theory (see Lieberman [12]) implies that $K_{\phi_+} \subset C_0^1(\overline{\Omega})$, and in fact, using [24, Theorem 5.4.1, p. 111], we have that $K_{\phi_+} \subset C_+$. Thus $K_{\psi_+} = K_{\phi_+} = K \subset C_+$.

Since $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$, using Theorem 16 of Palais [20], we have

$$H_k(W_0^{1,p}(\Omega), \dot{\phi}_+^a) = H_k(C_0^1(\overline{\Omega}), \dot{\psi}_+^a) \text{ for all } k \geq 0, \tag{4.7}$$

and recalling that $a < \inf_K \psi_+ = \inf_K \phi_+$, we have

$$H_k(W_0^{1,p}(\Omega), \dot{\phi}_+^a) = C_k(\phi_+, \infty) \text{ and } H_k(C_0^1(\overline{\Omega}), \dot{\psi}_+^a) = C_k(\psi_+, \infty) \text{ for all } k \geq 0,$$

and thus, by (4.6),

$$C_k(\phi_+, \infty) = C_k(\psi_+, \infty) \text{ for all } k \geq 0. \tag{4.8}$$

Therefore, according to (4.8), in order to prove the proposition for ϕ_+, it suffices to show that

$$C_k(\phi_+, \infty) = C_k(C_0^1(\overline{\Omega}), \psi_+^a) = 0 \text{ for all } k \geq 0.$$

To this purpose, we consider the following two sets:

$$\partial B_1^C = \left\{u \in C_0^1(\overline{\Omega}) \,:\, \|u\|_{C_0^1(\overline{\Omega})} = 1\right\} \text{ and } \partial B_{1,+}^C = \left\{u \in \partial B_1^C \,:\, u^+ \not\equiv 0\right\}.$$

Then, we consider the homotopy $h_+ : [0,1] \times \partial B_{1,+}^C \to \partial B_{1,+}^C$, defined by

$$h_+(t,u) = \frac{(1-t)u + t\hat{u}_{1,p}}{\|(1-t)u + t\hat{u}_{1,p}\|_{C_0^1(\overline{\Omega})}} \text{ for all } (t,u) \in [0,1] \times \partial B_{1,+}^C.$$

We have

$$h_+(1,u) = \frac{\hat{u}_{1,p}}{\|\hat{u}_{1,p}\|_{C_0^1(\overline{\Omega})}} \in \partial B_{1,+}^C,$$

and so $\partial B_{1,+}^C$ is contractible in itself.

Recall that hypothesis $\mathbf{H}$(ii) implies that for all $u \in \partial B_{1,+}^C$ we have

$$\psi_+(tu) \to -\infty \text{ as } t \to \infty. \tag{4.9}$$

From (4.2) we have

$$-f_+(z,u(z))u(z) \leq \beta^*(z) - pF_+(z,u(z)) \text{ a.e. in } \Omega. \tag{4.10}$$

Let $\langle \cdot,\cdot \rangle_C$ denote the duality brackets for the pair $\left(C_0^1(\overline{\Omega})^*, C_0^1(\overline{\Omega})\right)$. Fixing $u \in \partial B_{1,+}^C$, for all $t > 0$ we have

$$\begin{aligned}
\frac{d}{dt}\psi_+(tu) &= \langle \psi_+'(tu), u \rangle_C \text{ (by the chain rule)} \\
&= \frac{1}{t}\langle \psi_+'(tu), tu \rangle \\
&= \frac{1}{t}\left\{\|D(tu)\|_p^p + \mu\|D(tu)\|_q^q - \int_\Omega f_+(z,tu)(tu)\,dx\right\} \\
&\leq \frac{1}{t}\left\{\|D(tu)\|_p^p + \frac{p\mu}{q}\|D(tu)\|_q^q - \int_\Omega pF_+(z,tu)\,dz + \|\beta^*\|_1\right\} \\
&\qquad \text{(by (4.10) and since } q \leq p) \\
&= \frac{1}{t}\left\{p\phi_+(tu) + \|\beta^*\|_1\right\}.
\end{aligned} \tag{4.11}$$

From (4.9) it follows that for $t > 0$ large enough, we have $\phi_+(tu) < -\|\beta^*\|_1/p$, and so (4.11) implies that

$$\frac{d}{dt}\psi_+(tu) < 0 \text{ for } t > 0 \text{ large.} \tag{4.12}$$

Now, let $\overline{B_1^C} = \left\{u \in C_0^1(\overline{\Omega}) \,:\, \|u\|_{C_0^1(\overline{\Omega})} \le 1\right\}$ and $a \in \mathbb{R}$ such that

$$a < \min\left\{-\frac{\|\beta^*\|_1}{p}, \inf_{B_1^C} \hat{\psi}_+\right\}. \tag{4.13}$$

From (4.12) it follows that there exists a unique $\lambda(u) \ge 1$ such that

$$\psi_+(tu) = \begin{cases} > a & \text{if } t \in [0, \lambda(u)), \\ = a & \text{if } t = \lambda(u), \\ < a & \text{if } t > \lambda(u). \end{cases} \tag{4.14}$$

The Implicit Function Theorem (see (4.12)) implies that $\lambda : \partial B_{1,+}^C \to [1, \infty)$ is continuous. Moreover, by (4.13) and (4.14), we have

$$\psi_+^a = \left\{tu \,:\, u \in \partial B_{1,+}^C, t \ge \lambda(u)\right\}. \tag{4.15}$$

We set

$$D_+ = \left\{tu \,:\, u \in \partial B_{1,+}^C, t \ge 1\right\}.$$

Then from (4.15) we see that $\psi_+^a \subset D_+$. Now, consider the deformation $\hat{h}_+ : [0,1] \times D_+ \to D_+$ defined by

$$\hat{h}_+(s, tu) = \begin{cases} (1-s)tu + s\lambda(u)u & \text{if } t \in [1, \lambda(u)], \\ tu & \text{if } t > \lambda(u), \end{cases}$$

for all $s \in [0,1]$, $t \ge 1$ and $u \in \partial B_{1,+}^C$. Then $\hat{h}_+(0, tu) = tu$, $\hat{h}_+(1, tu) \in \psi_+^a$ (see (4.14)) and $\hat{h}_+(s, \cdot)_{|\psi_+^a} = Id_{|\psi_+^a}$ for all $s \in [0,1]$, as it is clear from (4.15). Thus it follows that ψ_+^a is a deformation retract of D_+, and so

$$H_k(C_0^1(\overline{\Omega}), D_+) = H_k(C_0^1(\overline{\Omega}), \psi_+^a) \text{ for all } k \ge 0; \tag{4.16}$$

see [9, E.2, p. 406].

Next, consider the deformation $\tilde{h}_+ : [0,1] \times D_+ \to D_+$ defined by

$$\tilde{h}_+(s, tu) = (1-s)tu + s\frac{tu}{\|tu\|_{C_0^1(\overline{\Omega})}} \text{ for all } s \in [0,1],\ t \ge 1 \text{ and } u \in \partial B_{1,+}^C.$$

Using $\tilde{h}_+$ and Theorem XV.6.5 (p. 325) of Dugundji [6], we infer that $\partial B_{1,+}^C$ is a deformation retract of D_+. Hence

$$H_k(C_0^1(\overline{\Omega}), D_+) = H_k(C_0^1(\overline{\Omega}), \partial B_{1,+}^C) \text{ for all } k \ge 0. \tag{4.17}$$

From (4.16) and (4.17) it follows that

$$H_k(C_0^1(\overline{\Omega}), \psi_+^a) = H_k(C_0^1(\overline{\Omega}), \partial B_{1,+}^C) \text{ for all } k \ge 0. \tag{4.18}$$

Recalling that $\partial B_{1,+}^C$ is contractible in itself, we have (see Granas–Dugundji [9], Propositions (4.9) and (4.10), p. 389)

$$H_k(C_0^1(\overline{\Omega}), \partial B_{1,+}^C) = 0 \text{ for all } k \ge 0;$$

hence, by (4.18),

$$H_k(C_0^1(\overline{\Omega}), \psi_+^a) = 0 \text{ for all } k \geq 0,$$

so that

$$H_k(\psi_+, \infty) = 0 \text{ for all } k \geq 0,$$

and by (4.8),

$$C_k(\phi_+, \infty) = 0 \text{ for all } k \geq 0.$$

Analogous considerations for ϕ_- conclude the proof of the proposition. $\square$

Using the proposition above, we can compute the critical groups of the energy functional ϕ at the two constant sign smooth solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ found in Proposition 6.

Proposition 9. *If hypotheses* **H** *hold and if* $K_\phi = \{0, u_0, v_0\}$, *then* $C_k(\phi, u_0) = C_k(\phi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \geq 0$, *where* δ *denotes the usual Kronecker symbol.*

Proof. First we compute the critical groups of ϕ_+ at u_0. Note that $K_{\phi_+} \subseteq C_+$ and that $\phi'_{+|C_+} = \phi'_{|C_+}$. Therefore, $K_{\phi_+} = \{0, u_0\}$. Let $\tau < 0 < \lambda < \eta_+$ (see (3.23)), and consider the following triple of sets:

$$\phi_+^\tau \subseteq \phi_+^\lambda \subseteq W_0^{1,p}(\Omega).$$

For this triple of sets we consider the long exact sequence of singular homology groups, so we have

$$\ldots \to H_k(W_0^{1,p}(\Omega), \phi_+^\tau) \xrightarrow{i_*} H_k(W_0^{1,p}(\Omega), \phi_+^\lambda) \xrightarrow{\partial_*} H_{k-1}(\phi_+^\lambda, \phi_+^\tau) \to \ldots \tag{4.19}$$

for all $k \geq 1$. Here i_* is the group homomorphism induced by the inclusion $i : (W_0^{1,p}(\Omega), \phi_+^\tau) \to (W_0^{1,p}(\Omega), \phi_+^\lambda)$ and ∂_* is the boundary homomorphism. From (4.19) and the Rank Theorem we have, using the exactness of (4.19),

$$\begin{aligned} \operatorname{rank} H_k(W_0^{1,p}(\Omega), \phi_+^\lambda) &= \operatorname{rank} \mathit{ker}\, \partial_* + \operatorname{rank} \mathit{Im}\, \partial_* \\ &= \operatorname{rank} \mathit{Im}\, i_* + \operatorname{rank} \mathit{Im}\, \partial_*. \end{aligned} \tag{4.20}$$

Recall that $K_{\phi_+} = \{0, u_0\}$ and $\tau < 0 < \lambda < \eta_+ \leq \phi_+(u_0)$; see (3.27). Hence, Proposition 8 gives

$$H_k(W_0^{1,p}(\Omega), \phi_+^\tau) = C_k(\phi_+, \infty) = 0 \text{ for all } k \geq 0,$$

and thus, from (4.19), we find

$$\mathit{Im}\, i_* = \{0\}. \tag{4.21}$$

Moreover, since $0 < \lambda < \eta_+ \leq \phi_+(u_0)$, we have

$$H_k(W_0^{1,p}(\Omega), \phi_+^\lambda) = C_k(\phi_+, u_0) \text{ for all } k \geq 0; \tag{4.22}$$

see, for example, Chang [3, p. 338].

Finally, since $\tau < 0 = \phi_+(0) < \lambda < \eta_+ \leq \phi_+(u_0)$ and u_0 is a local minimizer of ϕ_+ (see Proposition 5), we have

$$H_{k-1}(\phi_+^\lambda, \phi_+^\tau) = C_{k-1}(\phi_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.23}$$

Now, we return to (4.21) and use (4.21), (4.22) and (4.23), obtaining

$$\operatorname{rank} C_1(\phi_+, u_0) \leq 1. \tag{4.24}$$

From the proof of Proposition 6 we know that $u_0 \in \operatorname{int} C_+$ is a critical point of ϕ_+ of mountain pass type, hence $C_1(\phi_+, u_0) \neq 0$. This, combined with (4.24), and since in (4.19) all terms are trivial for $k \geq 2$ (see (4.21)) we infer that

$$C_k(\phi_+, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.25}$$

Now consider the homotopy $h : [0,1] \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$h(t,u) = (1-t)\phi(u) + t\phi_+(u) \text{ for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Claim: *There exists* $\rho \in (0,1)$ *such that* u_0 *is the unique critical point of* $\{h(t,\cdot)\}_{t\in[0,1]}$ *in* $\overline{B_\rho(u_0)} = \{u \in W_0^{1,p}(\Omega) : \|u - u_0\| \leq \rho\}$.

We argue by contradiction. Thus, suppose that the Claim is not true; then we can find two sequences $\{t_n\}_{n\geq 1} \subset [0,1]$ and $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that

$$t_n \to t_0 \text{ in } [0,1],\ u_n \to u_0 \text{ in } W_0^{1,p}(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \text{ for all } n \geq 1. \tag{4.26}$$

From the equality in (4.26) we have

$$A_p(u_n) + \mu A_q(u_n) = (1-t_n)N_f(u_n) + t_n N_{f_+}(u_n) \text{ for all } n \geq 1,$$

that is,
(4.27)

$$\begin{cases} -\Delta_p u_n - \mu \Delta_q u_n = (1-t_n) f(z,u_n) + t_n f_+(z,u_n) & \text{a.e. in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{for all } n \geq 1.$$

Invoking the regularity result of Lieberman [12], we can find $\alpha \in (0,1)$ and $\hat{M} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq \hat{M} \text{ for all } n \geq 1. \tag{4.28}$$

Since $C^{1,\alpha}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, from (4.28), and recalling (4.26), it follows that

$$u_n \to u_0 \text{ in } C^1(\overline{\Omega}).$$

Hence, u_0 belonging to $\operatorname{int} C_+$, we have that $u_n \in \operatorname{int} C_+$ for all n larger than a suitable $n_0 \geq 1$.

Note that $N_{f|C_+} = N_{f_+|C_+}$. Thus, from (4.27) we have

$$\begin{cases} -\Delta_p u_n - \mu \Delta_q u_n = f(z,u_n) & \text{a.e. in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{for all } n \geq n_0,$$

and so $\{u_n\}_{n\geq n_0} \subset \operatorname{int} C_+$ is a sequence of distinct positive smooth solutions of (1.1), in contradiction with the hypothesis that $K_\phi = \{0, u_0, v_0\}$. This proves the Claim.

Finally, recall that ϕ_+ and ϕ satisfy the C–condition by Propositions 3 and 4 above. This fact and the Claim permit the use of the homotopy invariance of critical groups (see, for example, Chang [3, p. 334]), and so we have

$$C_k(h(0,\cdot), u_0) = C_k(h(1,\cdot), u_0) \text{ for all } k \geq 0,$$

that is

$$C_k(\phi, u_0) = C_k(\phi_+, u_0) \text{ for all } k \geq 0,$$

so that, by (4.25),

$$C_k(\phi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0.$$

Similarly, using ϕ_-, first we show that $C_k(\phi_-, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$, and then, via the homotopy invariance of critical groups, we show that $C_k(\phi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. $\square$

Now we are ready for the full multiplicity theorem ("three solutions theorem") for problem (1.1).

Theorem 10. *If hypotheses* **H** *hold, then problem* (1.1) *has at least three nontrivial smooth solutions* $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ *and* $w_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$.

Proof. From Proposition 6, we know that problem (1.1) has two nontrivial constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$. Assume that $K_\phi = \{0, u_0, v_0\}$. Then from Proposition 9 we have

$$C_k(\phi, u_0) = C_k(\phi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.29}$$

Moreover, from Proposition 5, we have that u_0 is a local minimizer for ϕ; hence

$$C_k(\phi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.30}$$

Finally, from Proposition 7 we have

$$C_k(\phi, \infty) = 0 \text{ for all } k \geq 0. \tag{4.31}$$

From (4.29), (4.30), (4.31) and the Morse relation for $t = -1$ (see (2.1)), we have

$$2(-1)^1 + (-1)^0 = 0,$$

a contradiction. Thus there exists $w_0 \in K_\phi \setminus \{0, u_0, v_0\}$. As before, the regularity result of Lieberman [12] implies that $w_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$. $\square$

Remark 2. In a celebrated paper, Wang [27] proved a three solutions theorem for semilinear Dirichlet problems (i.e. $p = 2$ and $\mu = 0$) with a superlinear reaction $f(z, \cdot)$; more precisely, he assumed that $f \in C^1(\overline{\Omega}, \mathbb{R})$, f satisfies the AR–condition and $f'_x(z, 0) = 0$ for every $z \in \overline{\Omega}$. We see that Theorem 10, even in the very special case $p = 2$ and $\mu = 0$, is more general than the multiplicity theorem of Wang [27].

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