# WARING RANK OF BINARY FORMS, HARMONIC CROSS-RATIO AND GOLDEN RATIO 

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#### Abstract

We discuss the Waring rank of binary forms of degree 4 and 5 , without multiple factors, and point out unexpected relations to the harmonic cross-ratio, $j$-invariants and the golden ratio. These computations of ranks for binary forms are used to show that the combinatorics of a line arrangement in the complex projective plane does not determine the Waring rank of the defining equation even in very simple situations.


## 1. Introduction

For the general question of symmetric tensor decomposition we refer to [3, 6, 8, 11, 10, 15, 16, 18, 19, 20, 21, 23, as well as to the extensive literature quoted at the references in [3] and [16]. Consider the graded polynomial ring $S=\mathbb{C}[x, y]$, let $S_{d}$ denote the vector space of homogeneous polynomials of degree $d$ in $S$, and let $f \in S_{d}$ be a binary form of degree $d$. We consider the Waring decomposition

$$
\begin{equation*}
(\mathcal{D}) \quad f=\ell_{1}^{d}+\cdots+\ell_{r}^{d}, \tag{1.1}
\end{equation*}
$$

where $\ell_{j} \in S_{1}$ are linear forms in $x, y$, and $r$ is minimal, in other words $r=\operatorname{rank} f$ is the Waring rank of $f$. Hence, the nonzero binary form $f$ has Waring rank $r=$ rank $f=1$ if and only if $f$ is the power of a linear form. Note that the Waring rank of a form $f$ of degree $d$ depends only on the corresponding class $[f]$ in $\mathbb{P}\left(S_{d}\right)$, and even on the corresponding $S L_{2}(\mathbb{C})$-orbit of $[f]$ in $\mathbb{P}\left(S_{d}\right)$. It is clear that two forms $f$ and $f^{\prime}$ in $S_{d}$ such that

$$
\begin{equation*}
k=\operatorname{rank} f=\operatorname{rank} f^{\prime} \in\{1,2\} \tag{1.2}
\end{equation*}
$$

give rise to the same $S L_{2}(\mathbb{C})$-orbit in $\mathbb{P}\left(S_{d}\right)$. The rank two binary forms are discussed in detail in 4].

In this note we discuss the Waring ranks of binary quartics and binary quintics, assuming they have distinct factors. For binary quartics the generic rank is 3 . We describe precisely the quartic forms of rank 2 in terms of the harmonic cross-ratio of the corresponding roots in $\mathbb{P}^{1}$, and explain why all the other binary quartics with distinct factors have rank 3, see Theorem 3.1. For binary quintics with distinct factors, those of rank 2 are closely related to the golden ratio. The generic binary quintics still have Waring rank 3, and there is an algebraic curve parametrizing the binary quintics with distinct factors and with rank 4, see Theorem 4.1.

[^0]In the final section we use the previous results to show that the combinatorics of a line arrangement $\mathcal{A}: F(x, y, z)=0$ in $\mathbb{P}^{2}$ does not determine the Waring rank of $F$ even in very simple situations, namely when $F(x, y, z)=z f(x, y)$, see Theorem 5.1 and Example 5.2.

We would like to thank Alessandro Oneto for kindly drawing our attention to several key results in [7], and to Laura Brustenga i Moncusí for useful informations concerning [4]. Computations with CoCoa [9] and Singular [12] also played a key role in our results.

## 2. Sylvester's Theorem

The Waring rank $r=\operatorname{rank} f$ can be described as follows. Let $Q=\mathbb{C}[X, Y]$, where $X=\partial_{x}$ and $Y=\partial_{y}$. Then $Q$ is the ring of differential operators with constant coefficients and acts on $S$ in the obvious way. For a binary form $f \in S$, we consider the ideal of differential operators in $Q$ killing $f$, namely

$$
\begin{equation*}
\operatorname{Ann}(f)=\{q \in Q: q \cdot f=0\} \tag{2.1}
\end{equation*}
$$

also denoted by $f^{\perp}$ and called the apolar ideal of $f$. Note that $\operatorname{Ann}(f)$ is a graded ideal, whose degree $s$ homogeneous component is given by $\operatorname{ker}[f]$, where

$$
\begin{equation*}
[f]: Q_{s} \rightarrow S_{d-s} \tag{2.2}
\end{equation*}
$$

is the morphism $g \mapsto g \cdot f$. The matrix of this linear map with respect to the obvious monomial bases in $Q_{s}$ and $S_{d-s}$ is called the catalecticant matrix $C(f)_{s}$ of $f$ in degree $s$.

Example 2.1. As a example, if we take $d=4$ and write

$$
f=a_{0} x^{4}+4 a_{1} x^{3} y+6 a_{2} x^{2} y^{2}+4 a_{3} x y^{3}+a_{4} y^{4}
$$

then

$$
C(f)_{2}=12\left(\begin{array}{ccc}
a_{0} & a_{1} & a_{2} \\
2 a_{1} & 2 a_{2} & 2 a_{3} \\
a_{2} & a_{3} & a_{4}
\end{array}\right) .
$$

The following result is perhaps well known.
Lemma 2.2. The graded ideal $\operatorname{Ann}(f) \subset Q$ of the binary form $f$ of degree $d$ satisfies the following.
(1) $\operatorname{Ann}(f)_{0} \neq 0$ if and only if $f=0$.
(2) $\operatorname{Ann}(f)_{0}=0$ and $\operatorname{Ann}(f)_{1} \neq 0$ if and only if $f=x^{d}$ after a linear change of coordinates.
(3) $\operatorname{Ann}(f)_{1}=0$ and $\operatorname{Ann}(f)_{2}=\mathbb{C} \ell^{2}$ for some $\ell \in Q_{1}$ if and only if $f=x^{d-1} y$ after a linear change of coordinates.

The following result goes back to Sylvester [24]. See also [10].

Theorem 2.3. For a binary form $f$ of degree $d$, the apolar ideal $A n n(f)$ is a complete intersection, namely there are two binary forms $g_{1}$ and $g_{2}$ in $Q$ such that $\operatorname{Ann}(f)=$ $\left(g_{1}, g_{2}\right)$. The degrees $d_{j}$ of $g_{j}$ for $j=1,2$ satisfy $d_{1}+d_{2}=d+2$. Moreover, if we assume $d_{1} \leq d_{2}$, then the Waring rank $r=\operatorname{rank} f$ is determined as follows.
(1) If the binary form $g_{1}$ has no multiple factors, then $r=d_{1}$.
(2) Otherwise, $r=d_{2}$.

According to Lemma 2.2, the interesting case is

$$
2 \leq d_{1} \leq d_{2}
$$

In this case we have the following, see also the Introduction in [5].
Theorem 2.4. If the binary form $f$ of degree $d$ satisfies rank $f \geq 2$, then

$$
\operatorname{rank} f \leq d
$$

and the equality holds if and only if $f=x^{d-1} y$ after a linear change of coordinates. Moreover, for a generic binary form $f$ of degree d one has

$$
\operatorname{rank} f=\left\lfloor\frac{d+2}{2}\right\rfloor=\left\lceil\frac{d+1}{2}\right\rceil .
$$

Proof. The first claim follows from Lemma 2.2, (3) and Sylvester's Theorem 2.3. The second claim is a special case of Alexander-Hirschowitz results in [2].
Example 2.5. When $d=3$, a binary form $f$ has $\operatorname{rank} f=2$ if and only if $f$ has no multiple factor, and then $f$ is projectively equivalent to the binary form $x^{3}+y^{3}$.

## 3. Binary quartics and the harmonic cross-ratio

In this section we investigate the Waring rank of binary forms of degree 4 having no multiple factor. If we write

$$
f=a_{0} x^{4}+4 a_{1} x^{3} y+6 a_{2} x^{2} y^{2}+4 a_{3} x y^{3}+a_{4} y^{4}
$$

then the determinant

$$
T(f)=\operatorname{det}\left(\begin{array}{ccc}
a_{0} & a_{1} & a_{2} \\
a_{1} & a_{2} & a_{3} \\
a_{2} & a_{3} & a_{4}
\end{array}\right)=a_{0} a_{2} a_{4}+2 a_{1} a_{2} a_{3}-a_{2}^{3}-a_{0} a_{3}^{2}-a_{1}^{2} a_{4}
$$

which is, up to a constant factor, just $\operatorname{det} C(f)_{2}$ from Example 2.1, is called classically the Hankel determinant, and the induced function on $S_{4}$ given by $f \mapsto \operatorname{det} C(f)_{2}$ is, up to a constant factor, the catalecticant from classical Invariant Theory, see [14], p. 10. In particular, the catalecticant is invariant with respect to the group $S L_{2}(\mathbb{C})$. Another invariant of the binary form $f$ is given by

$$
S(f)=a_{0} a_{4}-4 a_{1} a_{3}+3 a_{2}^{2}
$$

see [1, 14]. Using these two invariants, one defines

$$
j(f)=\frac{S(f)^{3}}{S(f)^{3}-27 T(f)^{2}}
$$

It is known that two binary quartics $f$ and $f^{\prime}$, without multiple factors and regarded as points in $\mathbb{P}\left(S_{4}\right)$, are in the same $S L_{2}(\mathbb{C})$-orbits if and only if

$$
\begin{equation*}
j(f)=j\left(f^{\prime}\right) \tag{3.1}
\end{equation*}
$$

We have the following.
Theorem 3.1. The Waring rank of a quartic binary form $f$ having no multiple factor is 2 if and only if $j(f)=1$. Otherwise rank $f=3$.

Proof. Up to projective equivalence a quartic binary form $f$ having no multiple factor can be written as

$$
\begin{equation*}
f=x y(x+y)(x+t y) \tag{3.2}
\end{equation*}
$$

with $t \in \mathbb{C} \backslash\{0,1\}$. Note that one has, using the above formulas,

$$
j(f)=\frac{4}{27} \frac{\left(t^{2}-t+1\right)^{3}}{t^{2}(t-1)^{2}} .
$$

Since $f$ having no multiple factor, it is clear that rank $f \geq 2$ by Lemma 2.2. On the other hand, Theorem 2.4 implies that

$$
\operatorname{rank} f \leq d-1=3
$$

Moreover, Lemma2.2 (3) and our hypothesis that $t \in \mathbb{C} \backslash\{0,1\}$, implies that rank $f=$ 2 if and only if $\operatorname{Ann}(f)_{2} \neq 0$. Using the formula for the catalecticant $C(f)_{2}$ given in Example 2.1 and the formula for $f$ in (3.2), it follows that $\operatorname{det} C(f)_{2}=0$ exactly for $t \in\left\{-1, \frac{1}{2}, 2\right\}$. For all these three values of $t$ we get $j(f)=1$.

Corollary 3.2. The quartic binary forms $f$ having no multiple factor and with Waring rank 2 form a single $S L_{2}(\mathbb{C})$-orbit in $\mathbb{P}\left(S_{4}\right)$. More precisely, rank $f=2$ if and only if the four roots of $f$, regarded as points in the projective line $\mathbb{P}^{1}$, have a harmonic cross-ratio.

Proof. It is known that $j(f)=1$ corresponds exactly to the case when the four roots of $f$, regarded as points in the projective line $\mathbb{P}^{1}$, have a harmonic cross-ratio. Recall also our remark related to (1.2) in the Introduction.

Remark 3.3. The fact that a binary quartic has Waring rank 2 when the catalecticant $C(f)_{2}$ vanish and the relation to harmonic cross-ration is stated as a remark in [22], see middle of page 29, with a reference to an exercise in Gurevich book [17], namely Exercise 25.7. We leave the interested reader to compare the two different approaches and to notice the distinct terminology used by various authors.

## 4. Binary quintics and the golden ratio

In this section we investigate the Waring rank of binary forms of degree 5 having no multiple factor. Up to projective equivalence such a form $f$ can be written as

$$
\begin{equation*}
f_{s, t}=x y(x+y)(x+s y)(x+t y)=x y(x+y)\left(x^{2}+S x y+P y^{2}\right)=f_{S, P}, \tag{4.1}
\end{equation*}
$$

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with $s, t \in \mathbb{C} \backslash\{0,1\}$ and $s \neq t$. Here $S=s+t$ and $P=s t$. Recall that the golden ratio

$$
\varphi^{+}=\frac{1+\sqrt{5}}{2}
$$

is the positive root of the equation $z^{2}-z-1=0$. We have the following.
Theorem 4.1. The Waring rank of the quintic binary form $f_{s, t}$ having no multiple factor is 2 if and only if the pair $(s, t)$ is one of the following 12 pairs
$\left(\varphi^{ \pm}, 1+\varphi^{ \pm}\right),\left(1+\varphi^{ \pm}, \varphi^{ \pm}\right),\left(-\varphi^{ \pm}, 1+\varphi^{ \pm}\right),\left(1+\varphi^{ \pm},-\varphi^{ \pm}\right),\left(-1+\varphi^{ \pm}, \varphi^{ \pm}\right),\left(\varphi^{ \pm},-1+\varphi^{ \pm}\right)$, where $\varphi^{ \pm}$are the two roots of the equation $z^{2}-z-1=0$. Otherwise $3 \leq \operatorname{rank} f \leq 4$. More precisely, the rank of the form $f_{S, P}$ is 4 exactly when the pair $(S, P)$ is a zero of the polynomial

$$
\begin{gathered}
\Delta(S, P)=-4 S^{12}+12 S^{11} P+S^{10} P^{2}-22 S^{9} P^{3}+S^{8} P^{4}+12 S^{7} P^{5}-4 S^{6} P^{6}+ \\
+12 S^{11}+30 S^{10} P-202 S^{9} P^{2}+84 S^{8} P^{3}+292 S^{7} P^{4}-78 S^{6} P^{5}-102 S^{5} P^{6}+36 S^{4} P^{7}+ \\
+S^{10}-202 S^{9} P+190 S^{8} P^{2}+1176 S^{7} P^{3}-1198 S^{6} P^{4}-1234 S^{5} P^{5}+666 S^{4} P^{6}+188 S^{3} P^{7}- \\
-83 S^{2} P^{8}-22 S^{9}+84 S^{8} P+1176 S^{7} P^{2}-2640 S^{6} P^{3}-2264 S^{5} P^{4}+5392 S^{4} P^{5}+1236 S^{3} P^{6}- \\
-1532 S^{2} P^{7}+130 S P^{8}+8 P^{9}+S^{8}+292 S^{7} P-1198 S^{6} P^{2}-2264 S^{5} P^{3}+9312 S^{4} P^{4}- \\
-1924 S^{3} P^{5}-8100 S^{2} P^{6}+1860 S P^{7}+77 P^{8}+12 S^{7}-78 S^{6} P-1234 S^{5} P^{2}+5392 S^{4} P^{3}- \\
-1924 S^{3} P^{4}-10570 S^{2} P^{5}+8010 S P^{6}+120 P^{7}-4 S^{6}-102 S^{5} P+666 S^{4} P^{2}+1236 S^{3} P^{3}- \\
-8100 S^{2} P^{4}+8010 S P^{5}-410 P^{6}+36 S^{4} P+188 S^{3} P^{2}-1532 S^{2} P^{3}+ \\
+1860 S P^{4}+120 P^{5}-83 S^{2} P^{2}+130 S P^{3}+77 P^{4}+8 P^{3} .
\end{gathered}
$$

Proof. As in the proof above, we see that rank $f \geq 2$ and the equality holds if and only if the catalecticant $C(f)_{2}$ has not maximal rank 3. A direct computation shows that

$$
C(f)_{2}=\left(\begin{array}{ccc}
0 & 4 & 2(s+t+1) \\
12 & 6(s+t+1) & 6(s+t+s t) \\
6(s+t+1) & 6(s+t+s t) & 12 s t \\
2(s+t+s t) & 4 s t & 0
\end{array}\right)
$$

Using the software SINGULAR, we see that the ideal of 3-minors of this matrix has as zero set exactly the 12 pairs $(s, t)$ listed above. Assume now that the catalecticant $C(f)_{2}$ has maximal rank 3 , which implies that $g_{1}$, the generator of $\operatorname{Ann}(f)$ of minimal degree has degree $d_{1}=3$. It follows that in this case $\operatorname{rank}(f) \geq 3$. If $g_{1}=a X^{3}+$ $3 b X^{2} Y+3 c X Y^{2}+d Y^{3}$ is in $\operatorname{Ann}(f)_{3}$, it follows that $(a, 3 b, 3 c, d)$ is in the kernel of the matrix $C(f)_{3}$, and hence in the kernel of the matrix

$$
\left(\begin{array}{cccc}
0 & 2 & 1+S & S+P \\
2 & 1+S & S+P & 2 P \\
1+S & S+P & 2 P & 0
\end{array}\right)
$$

obtained by dividing the rows in $C(f)_{3}$ by 6,12 and 6 . Let $m_{i}$ be the determinant of the $3 \times 3$ matrix obtained from this matrix by deleting the $i$-th column. Then, since the matrix $C(f)_{3}$ is essentially the transpose of the matrix $C(f)_{2}$, we know that at least one of the minors $m_{i}$ is not zero. It follows that one can take

$$
(a, 3 b, 3 c, d)=\left(m_{1},-m_{2}, m_{3},-m_{4}\right),
$$

and in this way $a, b, c, d$ become polynomials in $S, P$. We define $\alpha=a c-b^{2}, \beta=$ $a d-b c, \gamma=b d-c^{2}$ and

$$
\Delta(S, P)=\beta^{2}-4 \alpha \gamma
$$

Then it is known that the binary cubic form $g_{1}$ has no multiple factors if and only if $\Delta(S, P) \neq 0$. In this case $\operatorname{rank} f_{S, P}=3$, and otherwise $\operatorname{rank} f_{S, P}=4$. This follows from Sylvester's Theorem 2.3, recalling that $d_{1}+d_{2}=d+2=7$ in our case.

Remark 4.2. Note that a binary form of Waring rank two has necessarily distinct factors, see [4, Corollary 4.1.1].

To the quintic form $f_{s, t}$ above we can associated 5 binary quartic forms without multiple factors, namely $h_{1}=f_{s, t} / x, h_{2}=f_{s, t} / y, h_{3}=f_{s, t} /(x+y), h_{4}=f_{s, t} /(x+s y)$ and $h_{5}=f_{s, t} /(x+t y)$. It is known that the $S L_{2}(\mathbb{C})$-orbit of $f_{s, t}$ in $\mathbb{P}\left(S_{5}\right)$ is determined by the unordered list of 5 complex numbers

$$
j\left(f_{s, t}\right):=\left(\left(j\left(h_{1}\right), j\left(h_{2}\right), j\left(h_{3}\right), j\left(h_{4}\right), j\left(h_{5}\right)\right)\right.
$$

see [1, Theorem 13].
Corollary 4.3. The quintic binary forms $f$ with Waring rank 2 form a single $S L_{2}(\mathbb{C})$-orbit in $\mathbb{P}\left(S_{5}\right)$. More precisely, $\operatorname{rank} f=2$ if and only if $f$ has distinct factors and

$$
j(f)=\left(\frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}\right) .
$$

Proof. The fact that the 12 pairs $(s, t)$ listed in Theorem 4.1 give rise to a single $S L_{2}(\mathbb{C})$-orbit in $\mathbb{P}\left(S_{5}\right)$ follows from our general remark related to (1.2) in Introduction. A direct computation shows that

$$
j\left(f_{\varphi^{+}, 1+\varphi^{+}}\right)=\left(\frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}, \frac{2^{5}}{3^{3}}\right)
$$

and this completes the proof.
Remark 4.4. It is shown in [4, Theorem 4.14] that there are exactly

$$
N_{d}=\binom{d-1}{2}
$$

distinct forms in $\mathbb{P}\left(S_{d}\right)$ which are multiple of a fixed cubic form $c \in S_{3}$ with distinct factors, say $c=x y(x+y)$. For $d=4$ we get $N_{4}=3$, which explains why we get 3 values for $t$ in the proof of Theorem 3.1 above. Similarly, for $d=5$ we get $N_{5}=6$, and the corresponding 6 forms are those listed at the beginning of the proof of Corollary 4.3 above. These specific binary forms are related to the map $\Gamma$, the dihedral cover for the cubic $x y(x+y)$, see [4, Definition 4.7].

Example 4.5. Consider the quintic binary form

$$
f=x y(x+y)\left(x^{2}+y^{2}\right)
$$

corresponding to the case $t=i, s=-i$ with $i^{2}=-1$. The corresponding pair $(S, P)=(s+t, s t)$ is now $(0,1)$ and clearly $\Delta(0,1)=0$. The corresponding form $g_{1}=(Y-X)(X+Y)^{2}$ has a multiple factor, and hence

$$
\operatorname{rank} x y(x+y)\left(x^{2}+y^{2}\right)=4
$$

## 5. On the Waring rank of some ternary forms

Let $f \in S_{d}=\mathbb{C}[x, y]_{d}$ be a binary form of degree $d$ and rank rank $f \geq 2$, and consider the ternary form $F=z f \in R_{d+1}$, where $R=\mathbb{C}[x, y, z]$. Assume, using Theorem [2.3, that $\operatorname{Ann}(f)=\left(g_{1}, g_{2}\right)$ such that

$$
2 \leq d_{1}=\operatorname{deg} g_{1} \leq d_{2}=\operatorname{deg} g_{2} \text { and } d_{1}+d_{2}=d+2
$$

Then it is clear that $\operatorname{Ann}(F)$ in the ring $T=\mathbb{C}[X, Y, Z]$, where $Z$ corresponds to $\partial_{z}$, it is given by

$$
\left(g_{1}, g_{2}, Z^{2}\right)
$$

The following result is a special case of [7, Theorem 4.14]. We include a proof, essentially the same as the proof given in [7. Theorem 4.14], just for the reader's convenience.

Theorem 5.1. The Waring rank of the ternary form $F=z f(x, y)$ is exactly $d_{1} d_{2}$, and all the linear forms $\ell_{j}$ occurring in a minimal length Waring decomposition (1.1) have the forms $\ell_{j}=a_{j} x+b_{b} y+c_{j} z$ with $c_{j} \neq 0$ for all $j=1, \ldots, r=d_{1} d_{2}$.

Proof. Since $\operatorname{Ann}(f)$ is a complete intersection, it follows that $g_{2}$ can be chosen without multiple factors. With such a choice, we claim that the ideal

$$
I=\left(g_{1}(X, Y)+Z^{d_{1}}, g_{2}(X, Y)\right) \subset A n n(F) \subset T
$$

is a smooth complete intersection $V$, containing $d_{1} d_{2}$ simple points in $\mathbb{P}^{2}$. Take a point $(p: q: r) \in \mathbb{P}^{2}$ in the zero set of this ideal $I$. Note that $r \neq 0$, since the equations

$$
g_{1}(X, Y)=g_{2}(X, Y)=0
$$

have only the trivial solution $(p, q)=(0,0)$ in $\mathbb{C}^{2}$. Hence we can take $r=1$ and compute the Jacobian matrix of the mapping $\left(g_{1}(X, Y)+Z^{d_{1}}, g_{2}(X, Y)\right)$ at the point ( $p: q: 1$ ). This matrix has rank 2, due to the fact that $g_{2}$ was supposed without multiple factors, and hence $g_{2}(p, q)=0$ implies that the gradient of $g_{2}$ at $(p, q)$ is non-zero. It follows that

$$
I(V)=I \subset \operatorname{Ann}(F)
$$

It is known that the Waring rank rank $F$ is the minimal cardinality of a finite set $W$ in $\mathbb{P}^{2}$ such that $I(W) \subset \operatorname{Ann}(F)$. The set $V$ constructed above shows that this minimal number is $|W| \leq d_{1} d_{2}$. Let $W^{\prime}=W \backslash L$, where $L$ is the line $Z=0$. Then one has the following
(1) The cardinality $\left|W^{\prime}\right|$ is equal to the Hilbert polynomial of the quotient $T / I\left(W^{\prime}\right)$, which is a constant;
(2) Since $Z$ is not a zero-divisor on $T / I\left(W^{\prime}\right)$, the above Hilbert polynomial is just the $\mathbb{C}$-dimension of the Artinian algebra $T /\left(I\left(W^{\prime}\right)+(Z)\right.$;
(3) Since $I(W) \subset A n n(F)$, we have

$$
I\left(W^{\prime}\right)=I(W):(Z) \subset \operatorname{Ann}(F):(Z)=\left(g_{1}, g_{2}, Z\right)
$$

It follows that

$$
\left|W^{\prime}\right|=\operatorname{dim} \frac{T}{I\left(W^{\prime}\right)+(Z)} \geq \operatorname{dim} \frac{T}{(\operatorname{Ann}(F):(Z))+(Z)}=\operatorname{dim} \frac{T}{\left(g_{1}, g_{2}, Z\right)} .
$$

On the other hand, we have

$$
\operatorname{dim} \frac{T}{\left(g_{1}, g_{2}, Z\right)}=\operatorname{dim} \frac{Q}{\left(g_{1}, g_{2}\right)}=d_{1} d_{2},
$$

and this proves our claim.
Example 5.2. In the previous sections, we have given examples of quartic binary forms $f$ (respectively quintic binary forms $f$ ) without multiple factors and such that $\left(d_{1}, d_{2}\right)=(2,4)$ and $\left(d_{1}, d_{2}\right)=(3,3)$ for quartic forms, and respectively $\left(d_{1}, d_{2}\right)=(2,5)$ and $\left(d_{1}, d_{2}\right)=(3,4)$ for quintic forms. Note that the associated line arrangements in $\mathbb{P}^{2}$, namely

$$
\mathcal{A}(F): z f(x, y)=0,
$$

have a very simple combinatorics, namely a pencil of 4 or 5 lines through a common point, plus a transversal line. However, the Waring rank of $F$ can be 8 or 9 for a quartic form $f$, and 10 or 12 for a quintic form $f$. In particular, the combinatorics of the line arrangement $\mathcal{A}(F)$ cannot determine the Waring rank of the defining equation $F=z f$ of the line arrangement.

Remark 5.3. Note the following analog of Lemma 2.2 (3) above. A line arrangement $\mathcal{A}: F(x, y, z)=0$ in $\mathbb{P}^{2}$ satisfies $\operatorname{Ann}(F)_{1}=0$ and $\operatorname{Ann}(F)=\mathbb{C} \ell^{2}$ for some linear form $\ell \in R_{1}$ if and only if $\mathcal{A}$ has the same combinatorics as the line arrangements considered in Example 5.2, namely a pencil of $d$ lines through a point of $\mathbb{P}^{2}$ plus a transversal line. Note that any such arrangement, regarded as a central arrangement in $\mathbb{C}^{3}$ is free with exponents $\left(e_{1}, e_{2}, e_{3}\right)=(1,1, d-1)$, which are the degrees of a basis for the free $R$-module of derivations $D(\mathcal{A})$, see for instance [13, Chapter 8 ] for generalities on free arrangements. On the other hand, the generators of the ideal $\operatorname{Ann}(F) \subset T$, as we have seen above, have degrees $\left(2, d_{1}, d_{2}\right)$ with $d_{1}+d_{2}=d+2$. It does not seem to be a simple relation between the module of derivations $D(\mathcal{A})$ and the ideal $\operatorname{Ann}(F)$, even in this simple situation.

## References

[1] A. Abdesselam, A computational solution to a question by Beauville on the invariants of the binary quintic, J. Algebra 303 (2006), 771-788. 3, 4
[2] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, J. Algebraic Geom. 4 (1995), 201-222. 2
[3] A. Bernardi, E. Carlini, M. V. Catalisano, A. Gimigliano, A. Oneto, The Hitchhiker guide to: Secant Varieties and Tensor Decomposition, arXiv:1812.10267. 1
[4] L. Brustenga i Moncusí, S. K. Masuti, On the Waring rank of binary forms: the binomial formula and a dihedral cover of rank two forms, arXiv:1901.08320. [1, 4.2, 4.4
[5] J. Buczyński, Z. Teitler, Some examples of forms of high rank, Collectanea Mathematica 67(2016), 431-441. 2
[6] E. Carlini, M.V. Catalisano, A. Oneto, Waring loci and the Strassen conjecture. Adv. Math. 314(2017), 630-662. I $^{1}$
[7] E. Carlini, M. V. Catalisano, L. Chiantini, A. V. Geramita, Y. Woo, Symmetric tensors: rank, Strassens conjecture and e-computability, Ann. Scuola Normale Sup. Pisa. 18 (2018), 363-390. 15
[8] C. Ciliberto, Geometric aspects of polynomial interpolation in more variables and of Warings problem. In: European Congress of Mathematics, Barcelona 2000, pages 289-316. Springer, 2001. 1
[9] CoCoA-5 (15 Sept 2014): a system for doing Computations in Commutative Algebra, available at http://cocoa.dima.unige.it
[10] G. Comas and M. Seiguer, On the rank of a binary form. Foundations of Computational Mathematics, 11(1)52011), 65-78. [1. 2
[11] P. Comon, G. Golub, L.H. Lim, B. Mourrain, Symmetric tensors and symmetric tensor rank, SIAM Journal on Matrix Analysis and Applications, 30(2008),1254-1279. [1
[12] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann. Singular 4-0-1 - A computer algebra system for polynomial computations, available at http://www.singular.uni-kl.de (2014). 1
[13] A. Dimca, Hyperplane Arrangements: An Introduction, Universitext, Springer, 20175.3
[14] I. Dolgachev, Lectures on Invariant Theory, London Mathematical Society Lecture Note Series, Cambridge: Cambridge University Press, 2003. 3
[15] R. Fröberg, G. Ottaviani, B. Shapiro, On the Waring problem for polynomial rings Proceedings of the National Academy of Sciences, 109 (2012), 5600-5602. [
[16] R. Fröberg, S. Lundqvist, A. Oneto, B. Shapiro, Algebraic stories from one and from the other pockets, Arnold Math. J. 4 (2018), 137-160. 1
[17] G. B. Gurevich, Foundations of the Theory of Algebraic Invariants, P. Noordhoff, 1964. 3.3
[18] A. Iarrobino, V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Springer Lecture Notes 1721, 1999. 1
[19] J.M. Landsberg, Tensors: Geometry and Applications, Graduate Studies in Mathematics vol.128, American Mathematical Soc., 2012. 1
[20] J.M. Landsberg, Z. Teitler, On the ranks and border ranks of symmetric tensors, Found. Comput. Math. 10(3) (2010) 339-366. [
[21] B. Mourrain, A. Oneto, On minimal decompositions of low rank symmetric tensors, arXiv:1805.11940. [
[22] P. Olver, Classical Invariant Theory, London Mathematical Society Student Texts 44, Cambridge Univ. Press, 1999. 3.3
[23] A. Oneto, Waring type problems for polynomials, Doctoral Thesis in Mathematics at Stockholm University, Sweden, 2016. 1
[24] J.J. Sylvester. Lx. on a remarkable discovery in the theory of canonical forms and of hyperdeterminants. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 2(12)(1851), 391-410. 2
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