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Warped Brane Worlds in Six Dimensional Supergravity

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Abstract: We present warped compactification solutions of six-dimensional supergravity, which are generalizations of the Randall-Sundrum (RS) warped brane world to codimension two and to a supersymmetric context. In these solutions the dilaton varies over the extra dimensions, and this makes the electroweak hierarchy only power-law sensitive to the proper radius of the extra dimensions (as opposed to being exponentially sensitive as in the RS model). Warping changes the phenomenology of these models because the Kaluza-Klein gap can be much larger than the internal space's inverse proper radius. We provide examples both for Romans' nonchiral supergravity and Salam-Sezgin chiral supergravity, and in both cases the solutions break all of the supersymmetries of the models. We interpret the solution as describing the fields sourced by a 3-brane and a boundary 4-brane (Romans' supergravity) or by one or two 3-branes (Salam-Sezgin supergravity), and we identify the topological constraints which are required by this interpretation. For both types of solutions the 3-branes are flat for all topologically-allowed values of the brane tensions. We identify the general mechanism for and limitations of the self-tuning of the effective 4D cosmological constant in higher-dimensional supergravity which these models illustrate.

Keywords: brane-world, supergravity, warped geometries.

Contents

1. Introduction

Although higher-dimensional models have a long history within supersymmetric theories, there has been less exploration within supergravity theories of the low-energy implications of warped compactifications [1, 2]. This is by contrast with nonsupersymmetric models, for which warped compactifications have been explored in some detail in 5 spacetime dimensions [3], and more recently in 6 dimensions [4, 5, 6]. The reason for this difference is partly due to the point of view taken by workers on the 5D models, for whom part of the basic motivation was to provide an approach to the hierarchy problem which is an alternative to supersymmetric models.

In the end Nature may not feel the need to choose to solve the hierarchy problem using only supersymmetry or only warping. Warping may play a role in the hierarchy problem in addition to supersymmetry, rather than in competition with it. In order to decide whether or not it does requires a better theoretical exploration of what is possible. Certainly if string theory proves to be the correct theory of very short distances warping is only likely to play a role at low-energies within a supergravity framework.

One of the main difficulties to constructing warped brane-world models in general is the absence of explicit solutions describing branes within compact spaces including the back-reaction on the space due to the branes. The Randall-Sundrum [3] construction provides such solutions for 3-branes in five dimensions, with the transverse dimension described by a line segment. (Solutions to the corresponding 5D supergravity equations are also known having a warped geometry [1].) What makes these solutions possible is the fact that the branes have codimension 1, and so their gravitational back-reaction may be summarized by the Israel junction conditions.

Six-dimensions are also attractive for constructing brane world models with compact internal spaces since the gravitational back-reaction problem for 3-branes (codimension two objects) is also soluble in terms of δ-function curvature singularities [7]. Warped examples of this type have been constructed [5, 6], based on the AdS soliton solution [4] to the Einstein equations with negative cosmological constant. Unwarped brane-world solutions have also been constructed, both for nonsupersymmetric [8] and for supersymmetric [9] systems [1].

In this paper we describe the first examples of warped brane-world compactifications of six-dimensional supergravity. We do so by explicitly solving the 6D coupled Einstein-Maxwell-dilaton equations in both their Romans' [10, 11] and Salam-Sezgin [12, 13, 14] variants [2]. In all of these solutions the warping of the 4D metric goes

[1] Codimension two warped solutions of type IIB string theory have also been considered [2], with supersymmetry broken by a global cosmic brane of finite extent.

[2] These solutions are analytical continuations of the solutions recently found in [16]. While writing this article a general solution of the Salam-Sezgin supergravity was discussed in [17] having similar properties as the ones discussed in section 3.

hand in hand with a nontrivial dilaton configuration, and so these solutions generalize the simpler product-space spherical compactifications of the Salam-Sezgin model [14, 15, 18, 9, 19]. Unlike the spacetime curvature, the dilaton and electromagnetic fields in our solution *ansatz* are nonsingular at the positions of the 3-branes, and so the solutions can only describe the fields due to 3-branes which do not couple to these fields.

Typically we find that the warped solutions for Romans' supergravity resemble and generalize the nonsupersymmetric AdS soliton solutions, with the internal dimensions being bounded by a 4-brane and containing 'our' 3-brane at an interior point. By contrast, the solutions for Salam-Sezgin supergravity generalize the unwarped solutions, for which a pair of 3-branes sit at opposite poles of an internal 2-sphere. In neither case does the electroweak hierarchy depend exponentially on the size of the internal dimensions, because the solutions are not asymptotically anti-de Sitter. For Salam-Sezgin supergravity they are not because the scalar potential is positive. For Romans' supergravity it is because the dilaton field varies in such a way as to run asymptotically to a zero of the potential. Because it is not exponential we find that some dimensionless combinations of brane tensions and couplings must be chosen to be very large if the hierarchy is to be sufficiently big.

In all of our solutions the internal geometry of the 3-branes is flat for all values of the brane tensions. This provides a final motivation for exploring their properties: to explore further the nature of the self-tuning of the effective 4D cosmological constant in 6D supergravity theories, as was discussed for unwarped compactifications in ref. [9]. We find the result that self-tuning also occurs for these warped compactifications provided the branes are assumed to have specific kinds of charges, but without requiring any adjustments of the bulk coupling constants.

We organize our presentation as follows. The next section describes the warped solution to Romans' supergravity, and in particular examines how its low-energy features (such as the electroweak hierarchy) depend on the physical properties of the branes involved. Section 3 repeats this discussion for warped compactifications of Salam-Sezgin supergravity. The nature of the cosmological-constant self-tuning is then described for both models in section 4. Finally, our conclusions are summarized in section 5.

2. Romans Supergravity in 6D

We now describe nonchiral six-dimensional supergravity, which is the first example for which we present warped compactifications. The solutions which we find in this case are supersymmetric generalizations of the well-known AdS soliton [4]. The 6D supergravities described in this section are the $N = 4^g$ and $N = \tilde{4}^g$ models of ref. [10]. It is known how to obtain these theories from 10D supergravity, and we summarize in an appendix how the solutions we find may be lifted, or oxidized, to higher dimensions than six.

2.1 The Model

The bosonic field content of the theory consists of a metric (g_{MN}), antisymmetric gauge field (B_{MN}), dilaton (ϕ) plus 6D gauge potentials (A^α_M) for the gauge group $G = SU(2) \times U(1)$. The fermionic field content comprises 4 gravitini (ψ^i_M) and four spin-$\frac{1}{2}$ fields (χ^i), where $i = 1, 2$ are indices on which the $SU(2)$ gauge-group factor acts with generators $(T_\alpha)^i{}_j$. (We use the index $I = 1, \dots, 4$ to denote G generators and $\alpha = 1, 2, 3$ to label the $SU(2)$ subgroup.) The fermions all satisfy an $SU(2)$ symplectic Majorana condition. Although this condition is compatible with a simultaneous Weyl condition in six dimensions, we do *not* impose this additional condition, and so the theory is trivially anomaly free. For instance, the fermion covariant derivative is

$$D_M \chi^i = \left[\left(\partial_M + \frac{1}{4} \omega_M{}^{AB} \Gamma_{AB} \right) \delta^i{}_j + g_2 A^\alpha_M \, (T_\alpha)^i{}_j \right] \chi^j_N, \tag{2.1}$$

where $\omega_M{}^{AB}$ denotes the spin connection and g_2 denotes the 6D $SU(2)$ gauge coupling. The field strengths for B_{MN} and the $U(1)$ gauge potential, $\mathcal{A}_M$, are the usual abelian expressions $G = \mathrm{d}B$ and $\mathcal{F} = \mathrm{d}\mathcal{A}$, while F^α_{MN} denotes the usual $SU(2)$ nonabelian field strength.

The bosonic part of the classical 6D supergravity action is:[3]

$$\begin{aligned} e_6^{-1} \mathcal{L}_B = &- \frac{1}{2} R - \frac{1}{2} \partial_M \phi \, \partial^M \phi - \frac{1}{12} e^{-2\zeta\phi} \, G_{MNP} G^{MNP} \\ &+ \frac{1}{2} g_2^2 \, e^{\phi} - \frac{1}{4} e^{-\phi} \left(F^\alpha_{MN} F^{MN}_\alpha + \mathcal{F}_{MN} \mathcal{F}^{MN} \right) \\ &- \frac{1}{8\sqrt{2}} \epsilon^{MNPQRS} B_{MN} \Big(F^\alpha_{PQ} F_{\alpha RS} + \mathcal{F}_{PQ} \mathcal{F}_{RS} \Big), \end{aligned} \tag{2.2}$$

where as usual $e_6 = |\det e_M{}^A| = \sqrt{-\det g_{MN}}$. The parameter ζ which defines the dilaton coupling to G_{MNP} takes values $\zeta = -1$ for Romans' $N = 4^g$ theory. By contrast it is $\zeta = +1$ for the $N = \tilde{4}^g$ theory, which is obtained from $N = 4^g$ by dualising $G_{MNP} \to \widetilde{G}_{MNP} = \frac{1}{3!} \, e^{-2\phi} \, \epsilon_{MNPQRS} \, G^{QRS}$.

For later purposes it is useful to record here the supersymmetry transformation rules for the fermions of the model. For the $N = 4^g$ theory these are

$$\delta\chi^i = \frac{1}{\sqrt{2}} \Gamma^M \epsilon^i \, \partial_M \phi + \frac{g_2 \, e^{\phi/2}}{2\sqrt{2}} \, \Gamma_7 \epsilon^i - \frac{e^\phi}{12} \Gamma_7 \Gamma^{MNP} \epsilon^i G_{MNP}$$

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[3]Our conventions differ from ref. [10] in that we use Weinberg's curvature conventions [21] and we set $\kappa_6^2 = 8\pi G_6 = 1$ rather than 2, and with this choice the canonical normalization of the dilaton requires $\phi_R \to \phi = \sqrt{2} \, \phi_R$.

$$+\frac{e^{-\phi/2}}{4\sqrt{2}}\,\Gamma^{MN}\Big(\mathcal{F}_{MN}\delta^i_j + 2\,\Gamma_7 F^{\alpha}_{MN}(T_\alpha)^i_j\Big)\,\epsilon^j \tag{2.3}$$

$$\begin{aligned}\delta\psi^i_M &= \sqrt{2}\,D_M\epsilon^i - \frac{g_2\,e^{\phi/2}}{4\sqrt{2}}\,\Gamma_M\Gamma_7\epsilon^i - \frac{e^{\phi}}{24}\,\Gamma_7\Gamma^{PQR}\epsilon^i\,G_{PQR}\\ &\quad -\frac{e^{-\phi/2}}{8\sqrt{2}}\Big(\Gamma_M{}^{PQ} - 6\delta^P_M\Gamma^Q\Big)\Big(\mathcal{F}_{PQ}\delta^i_j + 2\,\Gamma_7 F^{\alpha}_{PQ}(T_\alpha)^i_j\Big)\,\epsilon^j\,, \end{aligned}\tag{2.4}$$

where ϵ is the supersymmetry parameter. As usual

$$\Gamma^{A_1\ldots A_n} = \frac{1}{n!}\,\left[\Gamma^{A_1}\cdots\Gamma^{A_n} \pm \text{permutations}\right] \tag{2.5}$$

denotes the completely antisymmetric product. Letters from the beginning of the alphabet denote tangent-frame indices, those from the middle of the alphabet denote world indices, and these are related to one another by the vielbein by $\Gamma^M = e_A{}^M\,\Gamma^A$. The connection appearing in the covariant derivative of the SUSY parameter ϵ^i here is given by the spin connection, according to

$$\nabla_M\epsilon^i \equiv \left(\partial_M + \frac{1}{4}\omega^{AB}_M\Gamma_{AB}\right)\epsilon^i\,, \tag{2.6}$$

and in our conventions $\Gamma_7^2 = 1$.

2.2 Warped Compactifications

We now turn to the 4D compactifications of the model, for which the internal two dimensions are rotationally invariant about the position of a centrally-placed 3-brane. In order to make the resulting two dimensions compact we also take the outer edge of this two-dimensional space to be bounded by a 4-brane.

The field equations which follow from the action, eq. (2.2), are

$$\begin{aligned}&\Box\,\phi + \frac{\zeta}{6}\,e^{-2\zeta\phi}\,G^2 + \frac{1}{4}\,e^{-\phi}\left(F^2 + \mathcal{F}^2\right) + \frac{g_2^2}{2}\,e^{\phi} = 0\\ &D_P\Big(e^{-2\zeta\phi}\,G^{PMN}\Big) - \frac{1}{4\sqrt{2}}\,\epsilon^{MNPQRS}\Big(F^{\alpha}_{PQ}F_{\alpha RS} + \mathcal{F}_{PQ}\mathcal{F}_{RS}\Big) = 0\\ &D_M\Big(e^{-\phi}\,F^{MN}_{\alpha}\Big) - \frac{1}{6\sqrt{2}}\,\epsilon^{RSPQMN}G_{MRS}\,F_{\alpha PQ} = 0\\ &D_M\Big(e^{-\phi}\,\mathcal{F}^{MN}\Big) - \frac{1}{6\sqrt{2}}\,\epsilon^{RSPQMN}G_{MRS}\,\mathcal{F}_{PQ} = 0\\ &R_{MN} + \partial_M\phi\,\partial_N\phi + \frac{1}{2}\,e^{-2\zeta\phi}\,G_{MPQ}\,G_N{}^{PQ} + e^{-\phi}\left(F^{\alpha}_{MP}F_{\alpha N}{}^{P} + \mathcal{F}_{MP}\mathcal{F}_N{}^{P}\right)\\ &-\left[\frac{1}{12}\,e^{-2\zeta\phi}\,G^2 + \frac{1}{8}\,e^{-\phi}\left(F^2 + \mathcal{F}^2\right) + \frac{g_2^2}{4}\,e^{\phi}\right]\,g_{MN} = 0,\end{aligned}\tag{2.7}$$

Notice that when $B_{MN} = 0$ these equations are invariant under the rescaling $g_{MN} \to \Omega g_{MN}$, $e^{\phi} \to \Omega^{-1}e^{\phi}$, for constant Ω.

To obtain solutions we adopt the following *ansätze*:

$$\begin{aligned} ds_6^2 &= a(r)\Big[h_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + b(r)\,\mathrm{d}\theta^2\Big] + \frac{\mathrm{d}r^2}{a(r)\,b(r)} \\ F^{\hat\alpha}_{r\theta} &= f(r)\,\epsilon_{r\theta} \\ \phi &= \phi(r)\,, \end{aligned} \tag{2.8}$$

where $\epsilon_{r\theta} = \pm e_2$ is the volume form in the internal two dimensions, and $\hat\alpha$ denotes the gauge-group element whose background field strength is nonzero. All other fields vanish. The intrinsic 4D metric, $h_{\mu\nu}$ is assumed to be maximally symmetric but warped, with warp factor $a(r)$. This is the most general form which is consistent with the product of maximal symmetry in the four noncompact dimensions and rotational invariance in the two internal dimensions. In practice our interest is particularly in solutions for which the intrinsic four dimensions are flat: $h_{\mu\nu} = \eta_{\mu\nu}$, and in whether this requires a fine-tuning of the theory's couplings.

With these choices the generalized Maxwell equation becomes

$$\Big(e_6\, e^{-\phi}\, F^{r\theta}_{\hat\alpha}\Big)' = \Big(a^2\, e^{-\phi}\, f\Big)' = 0\,, \tag{2.9}$$

where the prime denotes differentiation with respect to the coordinate r. This has as solution

$$f(r) = \frac{A\, g_2\, e^{\phi}}{a^2}\,, \tag{2.10}$$

where A is an arbitrary constant of integration and the factor g_2 is included for later convenience.

This expression for f is sufficient to exclude the possibility of obtaining solutions with constant ϕ, as may be seen by using the above *ansatz* with $\phi' = 0$ in the dilaton field equation, leading to

$$0 = f^2 + g_2^2\, e^{2\phi} = g_2^2\, e^{2\phi}\left(1 + \frac{A^2}{a^2}\right)\,. \tag{2.11}$$

Clearly this cannot be satisfied for real fields and nonzero g_2 and A unless $\phi \to -\infty$.

Using eq. (2.10), the Einstein equations reduce to the system

$$\begin{aligned} \frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2 + \frac{a'\,b'}{ab} &= \frac{g_2^2\, e^{\phi}}{2ab}\left(1 + \frac{A^2}{a^4}\right) \\ \frac{a''}{a} + \frac{b''}{b} + \frac{4\,a'b'}{ab} + 2\left(\frac{a'}{a}\right)^2 &= \frac{g_2^2\, e^{\phi}}{2ab}\left(1 - \frac{3A^2}{a^4}\right) \\ \frac{5\,a''}{a} + \frac{b''}{b} + \frac{4\,a'b'}{ab} &= -2\,\phi'^2 + \frac{g_2^2\, e^{\phi}}{2ab}\left(1 - \frac{3A^2}{a^4}\right)\,. \end{aligned} \tag{2.12}$$

We need not explicitly write the dilaton equation, as this is not independent of the ones already written so long as $\phi' \neq 0$.

To find solutions it is useful to eliminate b by taking the difference of the last two of eqs. (2.12), to obtain

$$2\left(\frac{a''}{a}\right)-\left(\frac{a'}{a}\right)^2=-\,\phi'^{\,2}\,. \tag{2.13}$$

It is also useful to eliminate all second derivatives by subtracting half of this last equation from the first of eqs. (2.12), giving

$$5\left(\frac{a'}{a}\right)^2+2\left(\frac{a'b'}{ab}\right)=\phi'^{\,2}+\frac{g_2^2\,e^{\phi}}{ab}\left(1+\frac{A^2}{a^4}\right)\,. \tag{2.14}$$

The invariance of eq. (2.13) with respect to rescaling r suggests a power-law solution,

$$\phi(r)=\phi_0-\sqrt{p\,(2-p)}\,\ln r\qquad a(r)=a_0\,r^p\,, \tag{2.15}$$

where $0<p<2$. Notice the cases $p=0$ and $p=2$ may be excluded because these would imply $\phi'=0$, which we have seen is inconsistent with the dilaton field equation.

Using this in eq. (2.14) gives a linear first-order equation for $b(r)$, whose general solution is

$$b(r)=b_1\,r^{\beta_1}-b_2r^{\beta_2}-B\,r^{\beta_h}\,, \tag{2.16}$$

where B is an integration constant, while

$$\beta_1=2-p-\sqrt{p(2-p)}\,,\qquad \beta_2=2-5p-\sqrt{p(2-p)}\,,\qquad \beta_h=1-3p\,, \tag{2.17}$$

and

$$b_1=\frac{g_2^2\,e^{\phi_0}}{2p\,a_0\,(\beta_1+3p-1)}\,,\qquad b_2=\frac{A^2\,g_2^2\,e^{\phi_0}}{2p\,a_0^5(1-3p-\beta_2)}\,. \tag{2.18}$$

Finally, substitution of this solution back into eqs. (2.12) (or into the dilaton equation) shows that we must further require $p=1$. This leaves the final expression

$$b(r)=b_1-\frac{B}{r^2}-\frac{b_2}{r^4}\,, \tag{2.19}$$

with

$$b_1=\frac{g_2^2\,e^{\phi_0}}{4\,a_0}\,,\qquad b_2=\frac{A^2\,g_2^2\,e^{\phi_0}}{4\,a_0^5}\,, \tag{2.20}$$

both positive.

In summary, we obtain in this way as solutions the explicit field configurations

$$\phi(r)=\phi_0-\ln r\qquad f(r)=\frac{A\,g_2\,e^{\phi_0}}{a_0^2\,r^3}\qquad a(r)=a_0\,r\qquad b(r)=b_1-\frac{B}{r^2}-\frac{b_2}{r^4}\,. \tag{2.21}$$

This solution is also obtainable by appropriately continuing the solutions of ref. [16].

For future reference we also note that the nonvanishing component of the gauge potential itself is given locally by

$$A_\theta^{\hat{\alpha}} = C - \frac{A\, g_2\, e^{\phi_0}}{2a_0^2 r^2} \tag{2.22}$$

where, as before, $\hat{\alpha}$ is the $SU(2) \times U(1)$ index of the background field and C is a constant of integration.

At first sight there appear to be 5 constants of integration to be determined: A, B, C, a_0, ϕ_0. Physically, A corresponds to the strength of the gauge-field flux, and B is analogous to the black hole mass in the Schwarzschild solution. Locally, C is an irrelevant gauge degree of freedom, but we keep it here since it can encode gauge-invariant information in spaces with nontrivial topology.

In fact, only 4 of these 5 are set by boundary conditions because the rescaling symmetry of the field equations for $B_{MN} = 0$ allow one combination to be set by an appropriate choice of units in the 4 noncompact dimensions. To see this explicitly, perform the following rescaling of the integration constants A, B, ϕ_0, a_0:

$$\begin{aligned} e^{\phi_0} &\to e^{\phi_0} \qquad & a_0 &\to c^{-2} a_0 \\ A &\to c^{-4} A \qquad & B &\to c^2 B\,, \end{aligned} \tag{2.23}$$

where c is an arbitrary constant parameter. This rescaling does not alter the form of the solution obtained above, because its effects can be compensated by performing the coordinate transformation $x^\mu \to c x^\mu$, with r and θ held fixed. Below we will use this rescaling to fix a_0, and so to reduce the number of integration constants to 4. For the moment, however, we keep all 5 parameters.

Notice also that even once a_0 is fixed in this way, the rescalings

$$r \to \Omega\, r, \qquad B \to \Omega^2\, B, \qquad A \to \Omega^2\, A\,, \tag{2.24}$$

have the effect of rescaling the solution according to $g_{MN} \to \Omega\, g_{MN}$, $e^\phi \to \Omega^{-1}\, e^\phi$ and $F_{MN} \to F_{MN}$, which we have seen is a symmetry of the classical equations. To the extent that the boundary conditions also respect this symmetry we should not expect to be able to determine the combination of integration constants which corresponds to this rescaling.

Finally, notice that the solution only depends on the gauge coupling, g_2, and ϕ_0 through the combination $g_2^2\, e^{\phi_0}$. As such, one can — although we shall not — set $g_2 = 1$ during all manipulations, secure in the knowledge that the appropriate factors of g_2 can be restored easily.

2.2.1 Supersymmetry of the Solution

We examine the supersymmetry of this solution, and show that it is supersymmetric only if $A = B = 0$. To do so we evaluate the supersymmetry transformations, (2.3), at the bosonic solution with vanishing fermion fields, $G_3 = 0$, and with a single $U(1)$ gauge field. With these choices we are left with the equation[4]

$$\delta\chi^i = \frac{1}{\sqrt{2}}\,\Gamma^M\epsilon^i\,\partial_M\phi + \frac{g_2\,e^{\phi/2}}{2\sqrt{2}}\,\Gamma_7\epsilon^i + \frac{e^{-\phi/2}}{4\sqrt{2}}\,\Gamma^{MN}\,\epsilon^i\,F_{MN}\,, \tag{2.25}$$

while for the gravitino we have

$$\delta\psi^i_M = \sqrt{2}\,D_M\epsilon^i - \frac{g_2\,e^{\phi/2}}{4\sqrt{2}}\,\Gamma_M\Gamma_7\epsilon^i - \frac{e^{-\phi/2}}{8\sqrt{2}}\Big(\Gamma_M{}^{PQ} - 6\delta^P_M\Gamma^Q\Big)\;\epsilon^i\,F_{PQ}\,. \tag{2.26}$$

Let us concentrate on the condition $\delta\chi_i = 0$ since the condition found using the gravitino transformation can be worked out in a similar manner. Specializing as above we have

$$\Gamma^r\phi'\,\epsilon + \frac{g_2}{2}e^{\phi/2}\Gamma_7\,\epsilon + \frac{1}{2}e^{-\phi/2}\Gamma^r\Gamma^\theta\,F_{r\theta}\,\epsilon = 0\,. \tag{2.27}$$

Multiplying both sides of this equation by the tangent-frame Dirac matrix in the r direction, $\overline{\Gamma}^r = \Gamma^r/\sqrt{g^{rr}}$, and considering the two possible eigenvalues

$$\overline{\Gamma}^r\Gamma_7\,\epsilon_\pm = \pm\epsilon_\pm\,, \tag{2.28}$$

the equation (2.27) becomes

$$\left[\sqrt{g^{rr}}\phi' \pm \frac{g_2}{2}e^{\phi/2}\right]\epsilon_\pm = -\frac{1}{2}e^{-\phi/2}\sqrt{g^{rr}}\Gamma^\theta\,F_{r\theta}\,\epsilon_\pm\,. \tag{2.29}$$

Squaring both sides of this equality then allows us to remove all Dirac matrices, leading to the equation

$$\left[\sqrt{g^{rr}}\phi' \pm \frac{g_2}{2}e^{\phi/2}\right]^2\epsilon_\pm = \frac{1}{4}e^{-\phi}g^{rr}g^{\theta\theta}F^2_{r\theta}\,\epsilon_\pm \tag{2.30}$$

which is satisfied for our solution only if both sides vanish, requiring $A = B = 0$.

2.2.2 Conical Singularities

The metric which results from these functions describes a geometry which is singular for $r \to 0$, where there are curvature invariants which diverge. Because we regard our field equations as valid only in the limit of small curvatures, we cannot trust our solution in this region. Furthermore, since both b_1 and b_2 are nonnegative the function $b(r)$ is negative for $r \to 0$ and positive for $r \to \infty$, passing through zero at the point $r = r_3$, where[5]

$$\frac{2b_2}{r_3^2} = -B + \sqrt{B^2 + 4b_1b_2}\,. \tag{2.31}$$

[4]We are outlining the calculation for the case where the vacuum expectation value of the gauge field lies in the $U(1)$ subgroup, but the result is the same in the case that it lies in $SU(2)$.

[5]If $b_2 = 0$, such as when $A = 0$, then instead one finds $r_3^2 = B/b_1$.

The metric therefore has Lorentzian signature for $r > r_3$, while for $r < r_3$ its signature is $(3,3)$.

Since the proper circumference of a circle at fixed r goes to zero as $r \to r_3$, the Lorentzian-signature space ($r \geq r_3$) pinches off there, and although all curvature invariants have smooth limits as $r \to r_3$ the metric acquires a conical singularity at this point. The conical singularity is exhibited by writing $r = r_3 + \delta$ (with $\delta \ll 1$), in which case the two-dimensional metric becomes

$$\begin{aligned} ds_2^2 &= a(r)\, b(r)\, \mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{a(r)\, b(r)} \\ &\approx a_3\, b_3' \delta\, \mathrm{d}\theta^2 + \frac{\mathrm{d}\delta^2}{a_3\, b_3' \delta} \\ &= \left(\frac{1}{a_3\, b_3'}\right) \left[\mathrm{d}\rho^2 + \left(\frac{a_3\, b_3'}{2}\right)^2 \rho^2 \mathrm{d}\theta^2\right], \end{aligned} \tag{2.32}$$

where $\rho = 2\sqrt{\delta}$, $a_3 = a(r_3) = a_0\, r_3$ and

$$b_3' = \left(\frac{db}{dr}\right)_{r=r_3} = \frac{2B}{r_3^3} + \frac{4b_2}{r_3^5}. \tag{2.33}$$

The geometry is therefore locally a cone, with a delta-function singularity in the curvature which is proportional to the defect angle $\Delta\theta = 2\pi\varepsilon_3$, with

$$\varepsilon_3 = 1 - \frac{a_3 b_3'}{2} = 1 - a_3 \left(\frac{B}{r_3^3} + \frac{2b_2}{r_3^5}\right). \tag{2.34}$$

If $a_3 b_3' = 2$ the solution is nonsingular. This can be achieved by appropriately restricting the parameters A, B, a_0, ϕ_0 (such as by choosing $A = 0$ and $g_2^2\, e^{\phi_0} = 4$). Otherwise the solution has a conical singularity at $r = r_3$, and the nonvanishing defect angle can be interpreted as the response of the geometry to the presence of a 3-brane located there.

At this point we use the freedom described earlier to rescale the 4D coordinates to set $a_0 = 1/r_3$, and so to ensure $a_3 = a_0\, r_3 = 1$. (This is accomplished by choosing $c^2 = 1/r_3$ in the scaling transformations (2.23).) It is also convenient to define a new parameter $\alpha_3 = e^{\phi_0}/r_3$ so that $e^{\phi(r_3)} = \alpha_3$ denotes the effective 4D bulk gauge coupling at $r = r_3$. The solution and its dependent parameters then take the form

$$\begin{aligned} e^{\phi(r)} &= \alpha_3 \left(\frac{r_3}{r}\right); \quad A_\theta^{\hat{\alpha}}(r) = C - \frac{A\, g_2\, \alpha_3\, r_3^3}{2\, r^2}; \\ a(r) &= \frac{r}{r_3}; \quad b_1 = \frac{g_2^2\, \alpha_3\, r_3^2}{4}; \quad b_2 = \frac{A^2\, g_2^2\, \alpha_3\, r_3^6}{4}, \end{aligned} \tag{2.35}$$

with four independent integration constants, A, B, C and α_3. Since these equations imply that b_1 and b_2 are themselves functions of r_3, eq. (2.31) can itself be solved to give the more explicit result

$$r_3^4 = \left(\frac{4B}{g_2^2\, \alpha_3} \right) \frac{1}{1 - A^2} \,, \tag{2.36}$$

from which we see that $B > 0$ implies $|A| < 1$, while $B < 0$ requires $|A| > 1$.

Many of our later expressions simplify considerably in the limits $A^2 \gg 1$, $A^2 \ll 1$ and $A^2 = 1 - \epsilon$ with $|\epsilon| \ll 1$, so we list some helpful approximate expressions here. For this purpose the relation $4b_1 b_2/B^2 = 4A^2/(1 - A^2)^2$ is very useful. We have:

- *The case $A^2 \ll 1$:*

  In this limit we require $B > 0$ and have $r_3^4 \approx 4B/(\alpha_3 g_2^2)$, and so $4b_1 b_2/B^2 \approx 4A^2 \ll 1$. Consequently $r_3^2 \approx B/b_1$ and so $b_2/r^2 < b_2/r_3^2 \approx b_1 b_2/B \ll B$ for all $r > r_3$. This allows the simplification $b(r) \approx b_1 - B/r^2 = \frac{1}{4}\, g_2^2\, \alpha_3\, r_3^2 - B/r^2$ for all $r > r_3$.

- *The case $A^2 \gg 1$:*

  In this limit we must have $B < 0$ and so $r_3^4 \approx -4B/(A^2\, g_2^2\, \alpha_3)$. This implies $4b_1 b_2/B^2 \approx 4/A^2 \ll 1$. Consequently $r_3^2 \approx -B/b_1$ and so $b_2/r^2 < b_2/r_3^2 \approx b_1 b_2/|B| \ll |B|$ for all $r > r_3$. This again allows the simplification $b(r) \approx b_1 - B/r^2 = \frac{1}{4}\, g_2^2\, \alpha_3\, r_3^2 - B/r^2$ for all $r > r_3$.

- *The case $A^2 = 1 - \epsilon$ with $|\epsilon| \ll 1$:*

  In this limit we have $r_3^4 \approx 4B/(\alpha_3 \epsilon)$, and so $\mathrm{sign}\, B = \mathrm{sign}\, \epsilon$. This implies $4b_1 b_2/B^2 \approx 4/\epsilon^2 \gg 1$. Consequently $r_3^4 \approx b_2/b_1$ and so $b_2/r_3^2 \approx (b_1 b_2)^{1/2} \gg |B|$. This allows the simplification $b(r) \approx b_1 - b_2/r^4 = \frac{1}{4}\, g_2^2\, \alpha_3\, r_3^2 (1 - r_3^4/r^4)$ for $r_3 < r \lesssim (b_2/|B|)^{1/2}$, while $b(r) \approx b_1 - B/r^2 \approx \frac{1}{4}\, g_2^2\, \alpha_3\, r_3^2 - B/r^2$ for $r \gtrsim (b_2/|B|)^{1/2}$.

We end this section with some relevant expressions characterizing the geometry of the warped cone. For this metric the circle with coordinate radius r_c has circumference

$$\rho(r_c) = 2\pi \sqrt{a(r_c)\, b(r_c)} = 2\pi\, r_3^{-1/2} \Big[b_1 r_c - (B/r_c) - (b_2/r_c^3) \Big]^{1/2} \,, \tag{2.37}$$

and the proper radius of such a circle, measured from the conical defect, is similarly

$$\ell(r_c) = \int_{r_3}^{r_c} \frac{dr}{\sqrt{a(r)\, b(r)}} = {r_3}^{1/2} \int_{r_3}^{r_c} \frac{dr}{\sqrt{b_1 r - (B/r) - (b_2/r^3)}} \,. \tag{2.38}$$

If $r_c \gg |B|^{1/2}, b_2^{1/4}$ then $\rho(r_c) \approx 2\pi \sqrt{b_1\, r_c/r_3} = \pi g_2 \sqrt{\alpha_3 r_3 r_c}$ and $\ell(r_c) \approx 2\sqrt{r_c r_3/b_1} = (4/g_2)\sqrt{r_c/(r_3 \alpha_3)}$. Notice that these imply the ratio $\rho(r)/\ell(r)$ is independent of r for large enough r.

Using the condition $a_3 = 1$, the deficit angle at $r = r_3$, eq. (2.34), becomes

$$\varepsilon_3 = 1 - \frac{b_3'}{2} = 1 - \frac{B}{r_3^3}\left(\frac{1+A^2}{1-A^2}\right). \tag{2.39}$$

Clearly this expression always satisfies $\varepsilon_3 \le 1$, and $\varepsilon_3 > 0$ implies a lower limit to the ratio $|(A^2-1)/B|$. In both of the limits $A^2 \ll 1$ and $A^2 \gg 1$ the expression for ε_3 simplifies to $\varepsilon_3 \approx 1 - |B|/r_3^3$.

For large r the 2D metric becomes

$$ds_2^2 = a(r)\,b(r)\,d\theta^2 + \frac{dr^2}{a(r)\,b(r)} \approx \frac{b_1\,r}{r_3}\,d\theta^2 + \frac{r_3\,dr^2}{b_1\,r}\,, \tag{2.40}$$

which the coordinate transformation $\rho = 2\sqrt{r}$ shows to be locally flat, but with a conical deficit angle given by

$$\varepsilon_\infty = 1 - \frac{b_1}{2\,r_3} = 1 - \frac{g_2^2\,\alpha_3\,r_3}{8}\,. \tag{2.41}$$

In general the geometry is one of a cone which is curved near its apex, and so whose deficit angle differs when measured at infinity and near the apex: $\varepsilon_\infty \neq \varepsilon_3$.

2.3 Brane Worlds

We obtain the desired brane world by placing ourselves on a 3-brane which is located at the position of the conical defect, $r = r_3$. In this way the conical defect can be ascribed to the response of the gravitational field to the brane's tension. In order to obtain a finite extra-dimensional volume the space will also be terminated at a 4-brane located at $r = r_4$. In this section we determine how the bulk fields respond to the presence of these branes, and in so doing relate the integration constants of the solution just described to the physical properties of the branes.

2.3.1 The Electroweak Hierarchy

The first issue to settle for a brane-world application of these solutions is how large the space must be in order to properly describe the electroweak hierarchy $M_w/M_p \sim 10^{-15}$. In the present case the effective 4D Planck mass may be read off from the dimensional reduction of the Einstein-Hilbert lagrangian:

$$\int d^2x\,\sqrt{-g_6}\,g^{\mu\nu}R_{\mu\nu}(g) = M_p^2\sqrt{-h}\,\left[h^{\mu\nu}R_{\mu\nu}(h) + \cdots\right], \tag{2.42}$$

and so

$$M_p^2 = 2\pi\int_{r_3}^{r_4} dr\,a(r) = \frac{\pi}{r_3}(r_4^2 - r_3^2)\,. \tag{2.43}$$

(Recall our units, for which $\kappa_6 = M_6^{-2} = 1$ and $a(r_3) = 1$.)

This must be compared with the mass scales for particles located on the 3-brane at $r = r_3$. Taking for example a scalar field, χ, with mass parameter μ_3, we see that the 3-brane action is

$$\mathcal{L}_3 = -\frac{1}{2}\sqrt{-g_4}\Big(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \mu_3^2\,\chi^2\Big) = -\frac{1}{2}\sqrt{-h}\,a^2(r_3)\left[\frac{h^{\mu\nu}}{a(r_3)}\,\partial_\mu\chi\partial_\nu\chi + \mu_3^2\,\chi^2\right], \tag{2.44}$$

and so the particle's physical mass is $m_3 = \mu_3\sqrt{a(r_3)} = \mu_3$. For $\mu_3 \sim \kappa_6^{-1/2} = M_6$, $r_3 \ll r_4$, and dropping $O(1)$ factors we find (after temporarily restoring powers of M_6)

$$\frac{m_3}{M_p} \sim \left(\frac{r_3}{M_6 r_4^2}\right)^{1/2}. \tag{2.45}$$

Performing the same exercise for the mass, m_4, of particles confined to the 4-brane at $r = r_4$ gives

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{2}\sqrt{-g_5}\Big(g^{pq}\partial_p\chi\partial_q\chi + \mu_4^2\,\chi^2\Big)\\ &= -\frac{1}{2}\sqrt{-h}\,a^{5/2}b^{1/2}\left[\frac{h^{\mu\nu}}{a}\,\partial_\mu\chi\partial_\nu\chi + \frac{1}{ab}\,(\partial_\theta\chi)^2 + \mu_4^2\,\chi^2\right].\end{aligned} \tag{2.46}$$

This shows that the Kaluza-Klein (KK) zero mode has a mass of order $m_4 = \mu_4\sqrt{a(r_4)} = \mu_4\sqrt{r_4/r_3}$, and so if $\mu_4 \sim \mu_3$, the physical masses satisfy $m_4/m_3 \sim \sqrt{r_4/r_3}$. Consequently $m_4/M_p \sim (M_6 r_4)^{-1/2}$.

Eq. (2.46) also implies that the KK masses associated with the circular direction on the 4-brane have a mass gap $M_4 = 1/\sqrt{b(r_4)}$ — which becomes $M_4 \approx 1/\sqrt{b_1} = 4/(\hat{g}_2 r_3\sqrt{\alpha_3})$ for sufficiently large r_4. (Here $\hat{g}_2 = g_2 M_6$ is the dimensionless six-dimensional gauge coupling.) Notice in particular that the KK mass M_4 does *not* vanish in the limit of large r_4, although the relative spacing of KK masses to bare masses as measured purely on the 4-brane, M_4/m_4, does vanish as $r_4 \to \infty$.

In the simplest scenario we choose all parameters except for r_4 to be of the same size, and we take the fundamental scale on the 3 brane to be M_w: $\mu_3 \sim \mu_4 \sim M_6 \sim 1/g_2 \sim 1/r_3 \sim M_w \sim 1$ TeV. With this choice we have $m_4 \sim \sqrt{M_w M_p} \sim 3\times 10^{10}$ GeV, making the scale of the 4-brane the intermediate scale. By contrast, the KK spacing of the 4-brane modes is much smaller (if $g_2^2\,\alpha_3 \sim 1$), being of order $1/r_3 \sim M_w$. This implies the intriguing possibility that massive particles on the 4-brane are naturally extremely heavy, while the nominally massless modes there form a KK tower which remains at the electroweak scale.

Since $e^{-\phi}$ pre-multiplies the gauge kinetic terms, e^{ϕ} can be interpreted as a position-dependent modulation of the gauge coupling. From the form of the dilaton solution we see that the couplings on the 4-brane are much weaker than those on the 3-brane, by an amount: $e^{\phi(r_4)}/e^{\phi(r_3)} = r_3/r_4 \sim 10^{-15}$. Thus the TeV-mass 4-brane modes are naturally extremely weakly coupled amongst themselves relative to the couplings of those TeV modes at $r = r_3$.

With these choices we have $r_4/r_3 \sim (m_4/m_3)^2 \sim 10^{15}$, and so $r_3 \sim (1\,\text{TeV})^{-1} \sim 10^{-19}\,\text{m}$ implies $r_4 \sim 0.1\,\text{mm}$. This corresponds to proper distance $\ell(r_4)/r_3 \sim \sqrt{r_4/r_3} \sim 3 \times 10^7$, or $\ell(r_4) \sim 3 \times 10^{-12}\,\text{m} \sim 0.03$ Angstroms. Hence $\ell(r_4)$ is well below the current limits on short-distance deviations from Newton's Gravitational Law [20], although this limit is more properly compared with the scale of KK masses in the bulk.

2.3.2 Bulk KK Modes

Decomposing the bulk KK modes as $\Psi(x, r, \theta) = \sum_{nl} \psi_{nl}(x)\, u_{nl}(r)\, e^{in\theta}$, the bulk action becomes

$$\begin{aligned} \mathcal{L}_4 &= \int dr\, d\theta\, \sqrt{-g_6} \Big(\Psi^* g^{MN} \nabla_M \nabla_n \Psi \Big) \\ &= \sum_{nl} \sqrt{-h}\, \psi^*_{nl} \Big(h^{\mu\nu}\, \nabla_\mu \nabla_\nu - \lambda_{nl} \Big) \psi_{nl} \,, \end{aligned} \tag{2.47}$$

which follows from the orthogonality relation $\int dr\, d\theta\, a\, u^*_{nl}\, u_{mk} e^{i(m-n)\theta} = \delta_{mn}\, \delta_{lk}$ satisfied by the mode functions, and the eigenvalue condition

$$\Delta_2\, u_{nl} = -\frac{1}{a} \Big(a^3\, b\, u'_{nl} \Big)' + \frac{n^2}{b}\, u_{nl} = \lambda_{nl}\, u_{nl} \,, \tag{2.48}$$

where primes denote differentiation with respect to r. The eigenvalues, λ_{nl}, then give the squares of the bulk KK masses.

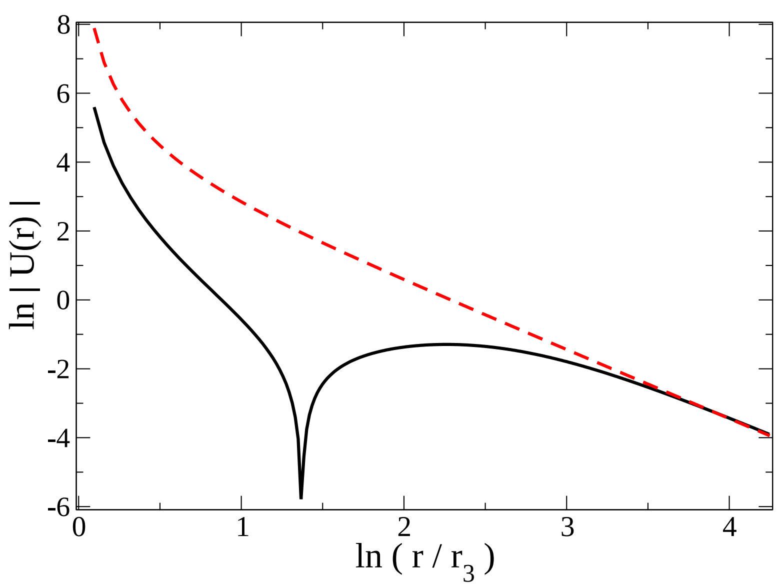

Figure 1: $\ln |U(r)|$ vs. $\ln r/r_3$ for two illustrative cases. For the solid curve, $U(r) < 0$ near $r = r_3$, but changes sign as r gets larger. $U(r)$ is always positive for the dashed curve.

We may estimate how the KK masses scale for large r_4 by looking for WKB solutions to eq. (2.48). For these purposes it is convenient to put this equation into a Schrödinger-like form, $-v''_{nl} + U(r)\, v_{nl} = 0$, by rescaling $u_{nl} = v_{nl}/(a^3\, b)^{1/2}$, in which case the 'potential' $U(r)$ takes the form

$$U(r) = \frac{1}{a^2(r)b(r)} \left(\frac{n^2}{b(r)} - \lambda_{nl} \right) + \frac{1}{2} \frac{d^2}{dr^2} \Big[\ln(a^3\, b) \Big] + \frac{1}{4} \left[\frac{d}{dr} \ln(a^3\, b) \right]^2 . \qquad (2.49)$$

A plot of this potential is given in figure (1). Near $r = r_3$ we have $b(r) \approx b'_3(r - r_3)$ and $a(r) \approx 1$ and so in this region $U(r) \approx U_3/(r - r_3)^2$ with $U_3 = (n/b'_3)^2 - \frac{1}{4}$. We shall see later that positive 3-brane tension requires the defect angle $\varepsilon_3 = 1 - b'_3/2$ to be positive, implying $b'_3 < 2$. This in turn ensures that $U_3 > 0$ for any nonzero n. For $r \to \infty$, on the other hand, we have $a(r) = r/r_3$ and $b(r) \approx b_1$ and so $U(r) \approx U_\infty/r^2$ with $U_\infty = \frac{3}{4} + (n^2/b_1 - \lambda_{nl})(r_3^2/b_1)$. Clearly $U_\infty \geq 0$ for $\lambda_{nl} \leq (3b_1/4r_3^2) + (n^2/b_1)$.

We seek the eigenstates of this potential having zero energy, and for these the region around $r = r_3$ is classically allowed provided $U_3 < 0$. For $n \neq 0$ we see that both U_3 and U_∞ are positive for small enough λ_{nl}, and for these choices there is typically no classically-allowed region for which $U \leq 0$. This shows that the least-massive $n \neq 0$ bulk states have masses which are of order $m^2 \sim \lambda_{min} = (3b_1/4r_3^2) + n^2/b_1$, as expected.

The least massive bulk KK states must therefore have $n = 0$, in which case $U_3 = -\frac{1}{4}$ and $U_\infty = \frac{3}{4} - \lambda_{0l} r_3^2/b_1$. If $\lambda_{0l} < 3b_1/4r_3^2$, then $U(r)$ is negative near $r = r_3$ and positive at large r, implying U must pass through zero at least once for finite r. Denoting the smallest zero of U by r_z, we see the existence of zero-energy states localized near the 3-brane in the classically-allowed region $r_3 < r < r_z$.

The eigenstates for this system in the WKB approximation are

$$u^\pm_{nl} \approx \frac{A^\pm_{nl}}{[a^3(r)\, b(r)]^{1/2}} \exp \left[\pm i \int_{r_3}^{r} dr' \sqrt{-U(r')} \right] , \qquad (2.50)$$

where $A^\pm_{nl}$ are constants. These behave like $u^\pm_{nl} \sim (r - r_3)^{\alpha_\pm}$ as $r \to r_3$, where $\alpha_\pm = -\frac{1}{2} \pm i\sqrt{-U_3}$. These states are therefore only marginally normalizable at $r = r_3$, using the required norm: $2\pi \int_{r_3}^{r_4} a\, |u^\pm_{nl}|^2\, dr$. Since they are localized within $r < r_z$, there is a discrete spectrum of eigenvalues for these lowest-energy $n = 0$ states, and the spacing of these eigenvalues should be independent of r_4 in the limit $r_4 \gg r_z$. This indicates that the bulk KK modes are generically independent of r_4 as r_4 is made large, indicating that the KK gap remains fixed in this limit. If all scales other than r_4 are chosen at the TeV scale, we therefore expect the spectrum of massive bulk KK modes to also start in the TeV region. In addition to these modes there will generally also be a few bulk (and possibly 4-brane) massless modes (like the graviton), which can appear in the low-energy, sub-TeV 4D effective theory.

In summary, we have been led to a warped relatively-large extra-dimensional scenario [22], with TeV physics on our brane coupled to bulk modes which are generically at the TeV scale, and to very weakly coupled physics at both TeV and intermediate scales on the 4-brane. Although we have found that the electroweak hierarchy requires the proper size of these extra dimensions to be quite large compared to microscopic scales, the large values we find for masses of the bulk KK modes makes this theory safe with respect to tests of Newton's Gravitational Law on submillimeter scales. The large KK masses also allow these models to evade the serious astrophysical problems [23] which large-extra-dimensional models generically have, and which are typically much worse within a supersymmetric context [24].

2.3.3 Brane Boundary Conditions

Clearly in these models an understanding of the hierarchy problem involves an understanding of why r_4 should be so much larger than r_3. Since this is determined by the brane properties, we now turn to a more detailed description of how the branes couple.

Before plunging into the details, it is worth considering the broader picture by first counting equations and unknowns. There is a boundary condition at each brane for each field in the problem. Given the symmetries of our solution this gives rise to 3 conditions at the 3-brane (one each for the dilaton, metric and Maxwell fields) plus 4 more at the 4-brane (keeping in mind that the $(\mu\nu)$ and $(\theta\theta)$ metric conditions on the 4-brane are independent). Thus there is a total of 7 conditions which must be solved for the various integration constants of the solution.

Since there are 5 independent integration constants (A, B, C, α_3 and r_4), the system is overconstrained and thus requires 2 independent conditions on the brane couplings of the model. We show in this section that these 2 constraints may be satisfied by choosing the dilaton coupling on the 3-brane to vanish, plus a topological condition that relates the coupling g_2 to g (the coupling corresponding to the gauge generator whose background field is nonzero).

With the above choices we therefore fix all of the integration constants, showing that our ansatz has no moduli and hence no classically massless dilaton or metric breathing modes. In contrast, if the dilaton brane couplings are chosen to preserve the classical scale invariance of the bulk action, one combination of the integration constants is a modulus which remains unfixed at the classical level. The counting of constraints also changes, but leads to the same conclusion as before. In this case the existence of the undetermined modulus shows that the equations are not all independent, so we must solve one fewer equation (*i.e.* 6) for one fewer (*i.e.* 4) combination of parameters. This leaves the same two required adjustments among the coupling constants as before.

Besides identifying how the couplings must be chosen in order to interpret our solutions as being sourced by 3- and 4-branes, we also explicitly solve for the dynamically-determined position of the 4-brane, r_4, and in the process find what properties the branes must have in order to obtain a large hierarchy $r_4 \gg r_3$.

The 3 Brane

We start with the 3-brane, for which the dilaton and metric couplings in the brane action are taken to be

$$S_3 = -T_3 \int d^4\xi \; e^{\lambda_3 \phi} \sqrt{-\det \gamma_{\mu\nu}} \,, \tag{2.51}$$

Here the induced metric is related to the 6D metric, g_{MN}, and the 3-brane position, $x^M(\xi)$, by $\gamma_{\mu\nu} = g_{MN}\, \partial_\mu x^M \partial_\nu x^N$. For coordinates $\xi^\mu = x^\mu$, this becomes $\gamma_{\mu\nu} = g_{\mu\nu} + g_{mn}\partial_\mu x^m \partial_\nu x^n$, where $\mu, \nu = 0, ..., 3$ and $m, n = 4, 5$. For a brane at rest at $r = r_3$ we also have $x^m = 0$. The quantities T_3 and λ_3 are the physical 3-brane properties which we wish to relate to the bulk geometry.

This action adds source terms to the dilaton and Einstein equations, eqs. (2.7). If the three brane is located at position x_3^m, the source terms are of the form

$$\begin{aligned} \Box\phi + (\cdots) &= \lambda_3 \, T_3 \, \frac{e^{\lambda_3\phi}}{e_2} \, \delta^2(x - x_3) \\ R_{MN} + (\cdots)_{MN} &= T_3 \, \frac{e^{\lambda_3\phi}}{e_2} \left(g_{\mu\nu}\delta^\mu_M \delta^\nu_n - g_{MN} \right) \delta^2(x - x_3) \,, \end{aligned} \tag{2.52}$$

where $e_2 = \sqrt{\det g_{mn}}$. These δ-function sources imply nontrivial boundary conditions for the bulk fields at the brane position, as may be determined by integrating the field equations over a small volume of infinitesimal proper radius about the 3-brane position. Assuming the metric, dilaton and Maxwell fields to be continuous at the brane position, we learn how the dilaton derivative and the curvature behave there.

The dilaton derivative at the 3-brane position becomes:

$$\lambda_3 T_3 \, e^{\lambda_3 \phi} \Big|_{r=r_3} = \sqrt{ab}\, \phi' \Big|_{r=r_3} \tag{2.53}$$

which should be read as a condition relating ϕ and ϕ' at the brane position, given the known couplings T_3 and λ_3. Since ϕ' is bounded as $r \to r_3$ in the solution of interest, using $a(r_3) = 1$, $e^{\phi(r_3)} = \alpha_3$ and $b(r_3) = 0$ shows that the right-hand side vanishes, and so

$$\lambda_3 T_3 \alpha_3^{\lambda_3} = 0 \,. \tag{2.54}$$

Since we do not wish to allow either T_3 or α_3 to vanish, we take this last condition to require $\lambda_3 = 0$.

A similar argument applied to the curvature singularity implies the standard relation between the conical defect angle and the 3-brane tension [7]:

$$T_3 = \Delta\theta = 2\pi \left[1 - \frac{b'(r_3)}{2} \right] = 2\pi \left[1 - \frac{B}{r_3^3} \left(\frac{1 + A^2}{1 - A^2} \right) \right] \,. \tag{2.55}$$

(recall that our units satisfy $\kappa_6^2 = 8\pi G_6 = 1$). We regard this solution as fixing the value of B once T_3 and A are given. Notice that if $T_3 > 0$ then we must require $b'(r_3) < 2$, and so A and B must satisfy $|1 - A^2| \, r_3^3 > |B|(1 + A^2)$.

The 3-brane condition satisfied by the Maxwell field is found by expressing the field strength in terms of a gauge potential as in eq. (2.22). The integration constant C in this expression is fixed by the requirement that there be vanishing magnetic flux through an infinitesimal surface enclosing the 3-brane position. This is equivalent to demanding that the gauge potential of eq. (2.22) vanish as $r \to r_3$, and so

$$A^{\hat{\alpha}} = \frac{A\, g_2 \alpha_3 r_3}{2} \left(1 - \frac{r_3^2}{r^2}\right) \mathrm{d}\theta \,. \tag{2.56}$$

In summary, the above equations determine how the 3-brane properties T_3 and λ_3 are related to properties of the bulk field configuration. In particular, the two integration constants B and C are fixed by eqs (2.55) and (2.56). On the other hand, the dilaton condition, eq. (2.53), in general implies that the dilaton should be singular at the 3-brane position, and so cannot be satisfied for the smooth dilaton configuration considered here unless the 3-brane does not couple to the dilaton field at all — *i.e.* $\lambda_3 = 0$. Both of the integration constants α_3 and A, as well as the 4-brane position r_4, then remain undetermined by the 3-brane properties, and so are arbitrary at this point. They are ultimately determined by the physics of the 4-brane, which is also what determines the volume of the internal two dimensions.

The 4 Brane

We next ask what properties the bulk solution implies for the 4-brane which we assume terminates the extra dimensions at $r = r_4 > r_3$. The precise nature of the conditions we obtain depends on the kinds of 4-brane couplings we are prepared to entertain, but in addition to the usual Nambu action it must also contain the physics whose currents are generated by the electromagnetic fields in the bulk. We start by considering the simplest case, corresponding to the Stückelberg action for a superconducting 4-brane

$$S_4 = - \int d^5\xi \, \sqrt{- \det \gamma} \, \Big[T_4 \, e^{\lambda_4 \, \phi} + \tfrac{1}{2} \, e^{\zeta_4 \phi} \gamma^{ps} (\partial_p \, \sigma - q A_p)(\partial_s \, \sigma - q A_s) \Big] , \tag{2.57}$$

where γ_{pq} is the 4-brane's induced metric, T_4 is its tension, λ_4 and ζ_4 are dilaton couplings, and σ is a Goldstone mode living on the brane which arises due to an assumed spontaneous breaking of the electromagnetic gauge invariance. q is a dimensionful quantity describing the energy scale of this symmetry breaking. The derivative $D_p \, \sigma = \partial_p \, \sigma - q A_p$ is gauge invariant given the transformation rules $\delta A_p = \partial_p \, \omega$ and $\delta\sigma = q\omega$.

With these choices the field equations for σ, ϕ and A_M at the position of the boundary 4-brane become

$$\frac{\delta S_4}{\delta \sigma} = \partial_p \Big[e_5 \, e^{\zeta_4 \phi} D^p \sigma \Big]_{r = r_4} = 0$$

$$\Big[n_M \partial^M \phi\Big]_{r=r_4} = \frac{1}{e_5}\,\frac{\delta S_4}{\delta \phi} = -\Big[\lambda_4 T_4 e^{\lambda_4 \phi} + \tfrac{1}{2}\,\zeta_4\, e^{\zeta_4 \phi}(D\sigma)^2\Big]_{r=r_4} \qquad (2.58)$$

$$\Big[n_M\, e^{-\phi}\, F^{Mp}\Big]_{r=r_4} = \frac{1}{e_5}\,\frac{\delta S_4}{\delta A_p} = q\, e^{\zeta_4 \phi(r_4)}\, D^p \sigma\,,$$

where $n_M\, dx^M = dr/\sqrt{g^{rr}}$ is the outward-pointing unit normal to the surface $r = r_4$, and $e_5 = \sqrt{-\det\gamma}$ is the volume element on the 4-brane. In what follows we further simplify these expressions by choosing the gauge $\sigma = 0$ on the 4-brane, in which case $D_p\,\sigma = -qA_p$. Using (2.22), we see that the first condition of (2.58) is trivially satisfied since A_θ is independent of θ.

The last of these equations is the electromagnetic boundary condition at the 4-brane position which expresses how much surface current must flow in order to maintain the given magnetic flux, A. We find

$$n_M g^{MN}\, e^{-\phi}\, F_{Np}\Big|_{r=r_4} = \sqrt{g^{rr}}\, e^{-\phi}\, F_{rp}\Big|_{r=r_4} = -q^2\, e^{\zeta_4 \phi}\, A_p\Big|_{r=r_4}\,, \qquad (2.59)$$

which takes the more explicit form

$$g_2\, A\sqrt{[ab]}_{r_4}\, \left(\frac{r_3}{r_4}\right)^2 = -\,q^2\, \left(\alpha_3\,\frac{r_3}{r_4}\right)^{\zeta_4}\, A_\theta(r_4)\,. \qquad (2.60)$$

We regard this as an equation for C_4, where the gauge potential at $r = r_4$ is written near the 4-brane as $A_\theta(r) = \big[C_4 - \frac{1}{2}\, A\, g_2 \alpha_3 r_3^3/r^2\big]$, leading to

$$C_4 = \left(\frac{A\, g_2 r_3^2}{r_4^2}\right)\left[\frac{\alpha_3 r_3}{2} - \frac{\sqrt{[ab]}_{r_4}}{q^2}\,\left(\alpha_3\,\frac{r_3}{r_4}\right)^{-\zeta_4}\right]. \qquad (2.61)$$

This can only differ from our 3-brane determination, $C_3 = \frac{1}{2}\, A\, g_2 \alpha_3 r_3$, by at most a periodic gauge transformation: $C_3 - C_4 = N/g$, where N is an integer and g is the gauge coupling for the gauge generator $\hat\alpha$ whose background field is turned on (and so $g = g_2$ if $\hat\alpha \in SU(2)$). This implies the constraint

$$\frac{A\, g_2 \alpha_3 r_3}{2}\left(1 - \frac{r_3^2}{r_4^2}\right) + \frac{\sqrt{[ab]}_{r_4}}{q^2}\,\left(\frac{A\, g_2 r_3^2}{r_4^2}\right)\left(\alpha_3\,\frac{r_3}{r_4}\right)^{-\zeta_4} = \frac{N}{g}\,. \qquad (2.62)$$

which is a flux-quantization condition, restricting A to take discrete values, labeled by an integer N.

The second equation of (2.58), the dilaton equation of motion near the 4-brane, determines the radial derivative of ϕ at $r = r_4$.

$$n_M g^{MN} \partial_N \phi\Big|_{r=r_4} = \sqrt{g^{rr}}\, \partial_r \phi\Big|_{r=r_4} = -\Big[\lambda_4 T_4 e^{\lambda_4 \phi} + \tfrac{1}{2}\, q^2\, \zeta_4 e^{\zeta_4 \phi} g^{\theta\theta} A_\theta^2\Big]_{r=r_4}\,. \qquad (2.63)$$

From this equation it follows that the 4-brane dilaton boundary condition is

$$-\sqrt{ab}\,\phi'\Big|_{r=r_4} = \frac{\sqrt{ab}}{r}\Bigg|_{r=r_4} = \left[\lambda_4 T_4\, e^{\lambda_4 \phi} + \frac{q^2 \zeta_4 e^{\zeta_4 \phi} A_\theta^2}{2\, ab}\right]_{r=r_4}\,. \qquad (2.64)$$

Finally we consider the junction conditions for the metric elements at the 4-brane, which express the response of the bulk metric to the brane tension. The boundary couplings to the metric are obtained from the 4-brane action, plus the Gibbons-Hawking extrinsic-curvature term [32] (which is proportional to integral over the boundary of the trace of the boundary's extrinsic curvature). The result relates the extrinsic curvature K_{pq} of the 4-brane to the 4-brane stress energy, S_{pq}, according to

$$S_{pq} = -K_{pq} + g_{pq} g^{rs} K_{rs} \,. \tag{2.65}$$

The brane stress energy to be used in this condition is

$$S_{pq} = -\Big[T_4 e^{\lambda_4 \phi} + \tfrac{1}{2}\, e^{\zeta_4 \phi} (D\sigma)^2\Big] g_{pq} + e^{\zeta_4 \phi} D_p\, \sigma D_q\, \sigma \,. \tag{2.66}$$

The extrinsic curvature of the surface $r = r_4$ is given by

$$K_{pq} = -\left(\Gamma^M_{pq}\, \hat{n}_M\right)_{r=r_4} = -\left(\frac{1}{2}\sqrt{g^{rr}}\, g'_{pq}\right)_{r=r_4} = \begin{pmatrix} K_{\mu\nu} & 0 \\ 0 & K_{\theta\theta} \end{pmatrix} , \tag{2.67}$$

where $\hat{n} = \hat{n}_M \, \mathrm{d}x^M = -\mathrm{d}r/\sqrt{g^{rr}}$ is the unit normal pointing into the bulk. The indices $p, q = 0, ..., 4$ include the four maximally-symmetric coordinates, $\mu, \nu = 0, ..., 3$, and θ. The components of K_{pq} so obtained are

$$K_{\mu\nu} = -\frac{\sqrt{ab}}{2}\left(\frac{a'}{a}\right) g_{\mu\nu} \qquad \text{and} \qquad K_{\theta\theta} = -\frac{\sqrt{ab}}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) g_{\theta\theta} \,, \tag{2.68}$$

implying that the combination which appears in the jump conditions, $\mathcal{K}_{pq} = K_{pq} - g_{pq} g^{rs} K_{rs}$, is given by

$$\mathcal{K}_{\mu\nu} = \frac{\sqrt{ab}}{2}\left(\frac{4a'}{a} + \frac{b'}{b}\right) g_{\mu\nu} \qquad \text{and} \qquad \mathcal{K}_{\theta\theta} = \frac{\sqrt{ab}}{2}\left(\frac{4a'}{a}\right) g_{\theta\theta} \,. \tag{2.69}$$

Using this extrinsic curvature with the stress-energy of eq. (2.66) implies the two conditions

$$\begin{aligned} \Big[T_4\, e^{\lambda_4 \phi} + \tfrac{1}{2}\, q^2 e^{\zeta_4 \phi}\, g^{\theta\theta} A_\theta^2\Big]_{r=r_4} &= \frac{\sqrt{ab}}{2}\left(\frac{4a'}{a} + \frac{b'}{b}\right)_{r=r_4} \\ \text{and} \qquad q^2 e^{\zeta_4 \phi} A_\theta^2\, g^{\theta\theta}\Big|_{r=r_4} &= \frac{\sqrt{ab}}{2}\left(\frac{b'}{b}\right)_{r=r_4} \,. \end{aligned} \tag{2.70}$$

Two conditions arise in this case because the electromagnetic currents which support the bulk magnetic field generate an asymmetric stress in the θ direction, ensuring that the stress energy is not proportional to the metric, g_{pq}. We may use one of these two conditions to determine the one remaining undetermined integration constant, r_4. This leaves the other as a redundant condition, which in general has a solution only for specific choices for the couplings T_3 or $\mathcal{Q}$. (The latter coupling will be introduced in section 5.2.)

Notice also that the second of eqs. (2.70) only has a solution if $b'(r_4) > 0$, and inspection of the figure shows that this is only possible if either: (*i*) $B > 0$ (and hence $|A| < 1$); or (*ii*) $B < 0$ (implying $|A| > 1$) and $r_4 < r_*$, where r_* is the position at which $b'(r_*) = 0$.

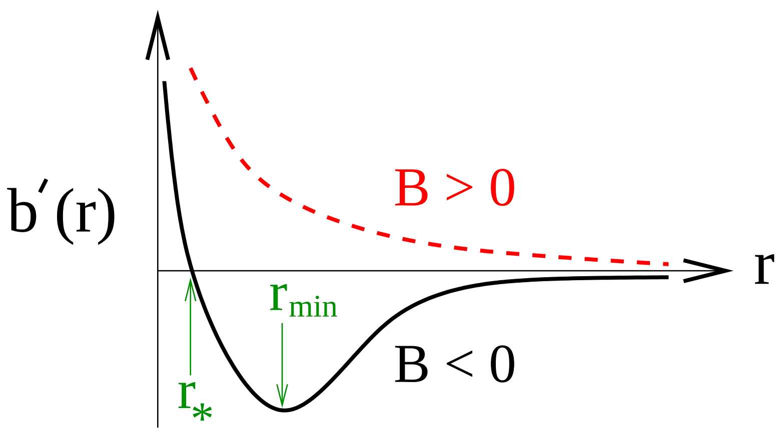

Figure 2: $b'(r)$ vs. r in the two cases $B < 0$ and $B > 0$.

We now turn to a more explicit solution of these last three 4-brane boundary equations (*i.e.* dilaton and metric conditions) to see what brane properties are required in order to obtain a large hierarchy, $r_4 \gg r_3$.

Conditions for a Solution

Our goal now is to solve eqs. (2.64) and (2.70) to determine r_4, α_3 and A. We treat these three variables as independent, although in so doing we redundantly determine A, which must also satisfy eq. (2.62). It is the reconciliation of these two conditions which requires us to adjust the coupling g of eq. (2.62) to g_2.

To solve these conditions it is useful to define the following auxiliary quantities,

$$D = \left[\frac{q^2}{2}\, e^{\zeta_4\phi} g^{\theta\theta} A_\theta^2\right]_{r_4}, \qquad E = \left[T_4 e^{\lambda_4\phi}\right]_{r_4}, \tag{2.71}$$

and first solve for E, D and r_4. The relevant three 4-brane conditions can be written as

$$E + D = \frac{\sqrt{[ab]_{r_4}}}{2}\left(\frac{4a'}{a} + \frac{b'}{b}\right)_{r_4} \tag{2.72}$$

$$D = \frac{\sqrt{[ab]_{r_4}}}{2}\left(\frac{b'}{2b}\right)_{r_4} \tag{2.73}$$

$$\lambda_4 E + \zeta_4 D = \frac{\sqrt{[ab]}_{r_4}}{r_4} \qquad (2.74)$$

Let us first concentrate on r_4, by eliminating E and D from these equations, leading to the result $4(1-2\lambda_4)\, b(r_4) = (\lambda_4+\zeta_4)\, r_4\, b'(r_4)$, or:

$$\frac{2b_2[\zeta_4 - \lambda_4 + 1]}{r_4^4} + \frac{B[\zeta_4 - 3\lambda_4 + 2]}{r_4^2} - 2(1-2\lambda_4)\, b_1 = 0\,. \qquad (2.75)$$

This may be solved to give

$$\frac{2b_2}{r_4^2} = \frac{|B|}{2\,(\zeta_4 - \lambda_4 + 1)} \left[(3\lambda_4 - \zeta_4 - 2)\eta_B \pm \Delta\right]\,, \qquad (2.76)$$

with

$$\Delta = \left[(3\lambda_4 - \zeta_4 - 2)^2 + \frac{16A^2(\zeta_4 - \lambda_4 + 1)(1-2\lambda_4)}{(1-A^2)^2}\right]^{1/2}\,, \qquad (2.77)$$

where $\eta_B = B/|B|$ and the sign in front of Δ in eq. (2.76) is the choice which makes the overall result positive (and so is equal to the sign of the product $(\zeta_4 - \lambda_4 + 1)(1 - 2\lambda_4)$). We also use here the result $b_1 b_2 = B^2 A^2/(1-A^2)^2$.

Since the hierarchy is determined by the ratio r_3/r_4, it is useful to divide eq. (2.76) by the earlier result, eq. (2.31), in the form $2b_2/r_3^2 = -B + \sqrt{B^2 + 4b_1 b_2} = 2BA^2/(1-A^2)$. This gives

$$\left(\frac{r_3}{r_4}\right)^2 = \frac{|1-A^2|}{4A^2}\left[\frac{(3\lambda_4 - \zeta_4 - 2)\eta_B \pm \Delta}{(\zeta_4 - \lambda_4 + 1)}\right]\,. \qquad (2.78)$$

Several conclusions may be drawn from these expressions.

- Eq. (2.78) clearly shows that the hierarchy is completely determined by the magnetic flux, A, and the 4-brane dilaton couplings, λ_4 and ζ_4.

- In the limit $A = 1-\epsilon$ with $|\epsilon| \ll 1$ we have $\Delta \approx (2/|\epsilon|)\Big[(\zeta_4 - \lambda_4 + 1)(1-2\lambda_4)\Big]^{1/2}$ and so
$$\left(\frac{r_3}{r_4}\right)^2 \approx \sqrt{\frac{1-2\lambda_4}{\zeta_4 - \lambda_4 + 1}}\,. \qquad (2.79)$$
This limit only makes sense (for real r_4) if $1-2\lambda_4$ and $\zeta_4 - \lambda_4 + 1$ share the same sign. A hierarchy is in this case ensured if λ_4 is chosen close to $\frac{1}{2}$ but with ζ_4 not close to $-\frac{1}{2}$.

- If $|A| \ll 1$, then $B > 0$ and real solutions for r_4 exist provided $(\zeta_4 - 3\lambda_4 + 2)$ and $(1-2\lambda_4)$ share the same sign. In this case
$$\Delta \approx |3\lambda_4 - \zeta_4 - 2| + \frac{8A^2(\zeta_4 - \lambda_4 + 1)(1-2\lambda_4)}{|3\lambda_4 - \zeta_4 - 2|}\,, \qquad (2.80)$$

and so the hierarchy is found to be

$$\left(\frac{r_3}{r_4}\right)^2 \approx \frac{2(1-2\lambda_4)}{\zeta_4 - 3\lambda_4 + 2}\,. \tag{2.81}$$

Again a large hierarchy is obtained if λ_4 is adjusted to be close to $\frac{1}{2}$, keeping ζ_4 not too close to $-\frac{1}{2}$.

- In the special case $\lambda_4 = -\zeta_4 = \frac{1}{2}$, the radius r_4 remains completely undetermined by the metric/dilaton conditions for any A. This acts as a check on our calculations, as we shall see in subsequent sections that in this limit the 4-brane action preserves the classical bulk scale invariance. In this case one combination of integration constants cannot be determined from the boundary conditions, and we expect to have a massless modulus in the 4D spectrum. If the modulus is taken to be r_4, then a large hierarchy may always be chosen by moving along this flat direction out to large values of r_4/r_3.

- If $\lambda_4 = \frac{1}{2}$ but ζ_4 is kept general, then there is no solution. This can be seen because there are contradictory conditions for A. On one hand, the relation fixing r_4 reduces in this case to $b'(r_4) = 0$, which is only possible if $B < 0$ and so $|A| > 1$, since $r_4^2 = r_*^2 = -2b_2/B$. On the other hand, the condition $b'(r_4) = 0$ in the metric matching conditions implies $D = 0$ and so also $qA_\theta(r_4) = 0$. Consequently the 4-brane stress energy is $SO(4,1)$ invariant, and must therefore be pure tension. In this limit the 4-brane electromagnetic boundary condition, eq. (2.60), implies $A = 0$, contradicting the earlier condition $|A| > 1$.

- Similarly, if $\zeta_4 = -\frac{1}{2}$ with λ_4 kept general, then r_4 becomes fixed by the condition $b(r_4) = 0$. We discard this degenerate case since it corresponds to a bulk with one less dimension, where $r_4 = r_3$.

In the generic case, with r_4 determined, we can solve eqs. (2.72) and (2.73) for E and D. E immediately determines the value of the dilaton at the 4-brane, which can be taken to α_3. As stated above, eq. (2.73) then provides a second determination of $A_\theta(r_4)$, and so also A. The result obtained in general need not be consistent with eq. (2.62), and so requires an adjustment of the coupling constant g relative to g_2. Alternatively, we can adjust the 4-brane symmetry-breaking scale, q.

The scale-invariant case ($\lambda_4 = \frac{1}{2}$ and $\zeta_4 = -\frac{1}{2}$) is similar. Here we may solve eqs. (2.72), (2.73) and (2.59) for $x = r_3/r_4$ and A in terms of g_2, T_4 and q. We then use the 3-brane tension condition to fix B/r_3^3 and this, with the definition of r_3 (*i.e.* eq. (2.36)), gives $\alpha_3 r_3$ purely in terms of couplings and tensions. Since the flux-quantization condition itself is a function only of the combinations $\alpha_3 r_3$ and B/r_3^3, it provides a redundant constraint whose satisfaction requires an adjustment of g or q. We find that a large hierarchy, $r_3 \ll r_4$, may be obtained in this case, for example by choosing q to be small. The details of this solution are provided in an appendix (section 7).

In summary, we see that our solution in general only describes the back reaction of the bulk fields to a 3- and 4-brane for specific choices of brane coupling, $\lambda_3 = 0$, and subject to a constraint which relates g and g_2.

3. Salam-Sezgin Supergravity in 6D

In this section we present a warped compactification which is a solution of the Salam-Sezgin chiral six-dimensional supergravity-supermatter system. We begin by recapitulating the relevant features of this model [12, 13, 14].

3.1 The Model

The field content of Salam-Sezgin supergravity consists of a supergravity-tensor multiplet consisting of a metric (g_{MN}), antisymmetric Kalb-Ramond field (B_{MN}, with field strength G_{MNP}), dilaton (ϕ), gravitino (ψ^i_M) and dilatino (χ^i). The fermions are all real Weyl spinors, satisfying $\Gamma_7 \psi_M = \psi_M$ and $\Gamma_7 \chi = -\chi$ and so the model is anomalous unless it is coupled to an appropriate matter content [25]. The appropriate chiral 6D matter consists of a combination of gauge multiplets, containing gauge potentials (A_M) and gauginos (λ^i), and n_H hyper-multiplets, with scalars Φ^a and fermions $\Psi^{\hat{a}}$. The index $i = 1, 2$ is an $Sp(1)$ index, $\hat{a} = 1, \ldots, 2n_H$ and $a = 1, \ldots, 4n_H$. The gauge multiplets transform in the adjoint representation of a gauge group, G. The $Sp(1)$ symmetry is broken explicitly to a $U(1)$ subgroup, which is gauged.

The matter fermions are also chiral, $\Gamma_7 \lambda = \lambda$ and $\Gamma_7 \Psi^{\hat{a}} = -\Psi^{\hat{a}}$, but the anomalies can be cancelled *via* the Green-Schwarz mechanism [26], for specific gauge groups and hypermultiplets [15, 27]. An explicit example [15] of an anomaly-free choice is $G = E_6 \times E_7 \times U(1)$, with the hyper-multiplet scalars living on the noncompact quaternionic Kähler manifold $\mathcal{M} = Sp(456, 1)/(Sp(456) \times Sp(1))$.

The bosonic part of the classical 6D supergravity action is:

$$e^{-1}\mathcal{L}_B = -\frac{1}{2}\,R - \frac{1}{2}\,\partial_M\phi\,\partial^M\phi - \frac{1}{2}\,G_{ab}(\Phi)\,D_M\Phi^a\,D^M\Phi^b - \frac{1}{12}\,e^{-2\phi}\,G_{MNP}G^{MNP} - \frac{1}{4}\,e^{-\phi}\,F^\alpha_{MN}F^{MN}_\alpha - e^\phi\,v(\Phi)\,. \qquad (3.1)$$

Here the index $\alpha = 1, \ldots, \dim(G)$ runs over the gauge-group generators, $G_{ab}(\Phi)$ is the metric on $\mathcal{M}$ and D_m are gauge and Kähler covariant derivatives whose details are not important for our purposes. We only require the dependence on ϕ of the scalar potential for $\Phi^a = 0$, which is $V(\phi, \Phi) = 2\,g_1^2\,e^\phi$. The coupling g_1 denotes the $U(1)$ gauge coupling.

When the hypermultiplets and all but one of the gauge multiplets are set to zero then the supersymmetry transformations reduce to

$$
\begin{aligned}
\delta e_M^A &= \frac{1}{\sqrt{2}} \left(\bar{\epsilon}\Gamma^A\psi_M - \bar{\psi}_M\Gamma^A\epsilon\right) \\
\delta\phi &= -\frac{1}{\sqrt{2}}\left(\bar{\epsilon}\chi + \bar{\chi}\epsilon\right) \\
\delta B_{MN} &= \sqrt{2}\, A_{[M}\delta A_{N]} + \frac{e^{\phi}}{2} \Big(\bar{\epsilon}\Gamma_M\psi_N - \bar{\psi}_N\Gamma_M\epsilon \\
&\qquad -\bar{\epsilon}\Gamma_N\psi_M + \bar{\psi}_M\Gamma_N\epsilon - \bar{\epsilon}\Gamma_{MN}\chi + \bar{\chi}\Gamma_{MN}\epsilon\Big) \qquad (3.2)\\
\delta\chi &= \frac{1}{\sqrt{2}}\, \partial_M\phi\, \Gamma^M\epsilon + \frac{e^{-\phi}}{12} G_{MNP}\, \Gamma^{MNP}\epsilon \\
\delta\psi_M &= \sqrt{2}\, D_M\epsilon + \frac{e^{-\phi}}{24}\, G_{PQR}\, \Gamma^{PQR}\Gamma_M\epsilon \\
\delta A_M &= \frac{1}{\sqrt{2}}\left(\bar{\epsilon}\Gamma_M\lambda - \bar{\lambda}\Gamma_M\epsilon\right)e^{\phi/2} \\
\delta\lambda &= \frac{e^{-\phi/2}}{4}\, F_{MN}\, \Gamma^{MN}\epsilon - \frac{i}{\sqrt{2}}\, g_1\, e^{\phi/2}\, \epsilon\,,
\end{aligned}
$$

where the supersymmetry parameter is complex and Weyl: $\Gamma_7\epsilon = \epsilon$.

3.2 Compactification

For our purposes we may set all gauge fields to zero except for a single gauge potential, A, and we also set $\Phi^a = 0$. In this section we derive a warped brane-world solution by continuing a related nontrivial solution for the same system which was found in ref. [16]. The solution in [16] is given by

$$
\begin{aligned}
ds_6^2 &= -h(\rho)\, \mathrm{d}\tau^2 + \frac{\rho^2}{h(\rho)}\mathrm{d}\rho^2 + \rho^2\mathrm{d}x_{0,4}^2\,, \\
\phi(\rho) &= -2\, \ln\rho\,, \qquad (3.3)\\
F_{\tau\rho} &= \frac{\hat{\mathcal{A}}}{\rho^5}\, \epsilon_{\tau\rho}\,,
\end{aligned}
$$

where $dx_{0,4}$ denotes a flat 4-dimensional spatial slice, and

$$
h(\rho) = -\frac{2\mathcal{M}}{\rho^2} - \frac{g_1^2\,\rho^2}{4} + \frac{\hat{\mathcal{A}}^2}{16\,\rho^6}\,. \qquad (3.4)
$$

This function has only a single zero for real positive ρ, and $\mathcal{M}$ and $\hat{\mathcal{A}}$ are integration constants which can be positive or negative. This is not a brane-world solution since the point where h vanishes corresponds to a null Cauchy horizon of the geometry.

A warped brane-world solution may be obtained from this one by performing a suitable analytic continuation, in which we first redefine the coordinate $r = \frac{1}{2}\,\rho^2$ so that the previous solution takes the form

$$
\begin{aligned}
ds_6^2 &= -h(r)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{h(r)} + 2\,r[\mathrm{d}x_1^2 + \mathrm{d}x_{0,3}^2]\,, \\
\phi(r) &= -\ln(2r)\,, \\
F_{\tau r} &= \frac{\hat{\mathcal{A}}}{8\,r^3}\,\epsilon_{\tau r}\,,
\end{aligned}
\tag{3.5}
$$

with

$$
h(r) = \frac{2M}{r} - \frac{g_1^2\,r}{2} + \frac{\hat{\mathcal{A}}^2}{128\,r^3}\,. \tag{3.6}
$$

Here we redefine the integration constant according to $\mathcal{M} = -2M$, in anticipation of our later choice $\mathcal{M} < 0$. The new solution is obtained by performing the analytic continuation

$$
\tau \to i\,\theta\,, \qquad x_1 \to it \qquad \frac{\hat{\mathcal{A}}}{8} \to i\mathcal{A}\,, \tag{3.7}
$$

in which case the it becomes:

$$
\begin{aligned}
ds_6^2 &= 2\,r[-\mathrm{d}t^2 + \mathrm{d}x_3^2] + h(r)\,\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{h(r)}\,, \\
\phi(r) &= -\ln(2r)\,, \\
F_{\theta r} &= -\frac{\mathcal{A}}{r^3}\,\epsilon_{\theta r}\,,
\end{aligned}
\tag{3.8}
$$

with

$$
h(r) = \frac{2M}{r} - \frac{g_1^2\,r}{2} - \frac{\mathcal{A}^2}{2\,r^3}\,. \tag{3.9}
$$

This is the desired solution whose properties we now explore.

3.2.1 Singularities and Supersymmetry

Eq. (3.8) describes a Lorentzian-signature solution provided $h(r) > 0$, and so it is useful to enumerate the zeroes of $h(r)$, which occur at

$$
r_\pm^2 = \frac{2\,M}{g_1^2}\left[1 \pm \sqrt{1 - \left(\frac{g_1\,\mathcal{A}}{2M}\right)^2}\,\right]\,. \tag{3.10}
$$

Since $h(r) < 0$ when $r \to \infty$ and $r \to 0$, the regime of interest for a brane-world solution is the interval $r_- < r < r_+$. This interval is not empty provided $M > \frac{1}{2}|g_1\mathcal{A}| > 0$, a condition which we henceforth assume.

The geometry pinches off at the points $r = r_\pm$, at each of which it generically has conical singularities. We therefore place a 3-brane at each of these points when constructing a brane-world model. Repeating the discussion of the previous sections shows that the conical defect at $r = r_\pm$ is given by

$$
\varepsilon_\pm = 1 - \frac{|h'(r_\pm)|}{2} = 1 - \frac{g_1^2}{2\,r_\pm^2}\left(r_+^2 - r_-^2\right)\,. \tag{3.11}
$$

This last equality is obtained by writing $h(r) = -\frac{1}{2}\,(g_1^2/r^3)(r^2 - r_+^2)(r^2 - r_-^2)$.

These conditions show that the defect angles are completely determined by the two quantities r_-/r_+ and g_1. In particular, one of the conical defects can be smoothed over if r_-/r_+ is chosen appropriately. We find

$$\varepsilon_+ = 0 \quad \Rightarrow \quad \frac{r_-^2}{r_+^2} = 1 - \frac{2}{g_1^2} \qquad \text{and} \qquad \varepsilon_- = 0 \quad \Rightarrow \quad \frac{r_+^2}{r_-^2} = 1 + \frac{2}{g_1^2}\,. \tag{3.12}$$

Notice that the condition for the removal of the singularity at r_+ requires a large coupling $g_1 > \sqrt{2}$, and so is only of doubtful validity in a perturbative calculation such as ours.

Supersymmetry

This solution generically breaks supersymmetry, as is most easily seen by specializing the χ supersymmetry transformation to it, with the result

$$\delta\chi = \frac{1}{\sqrt{2}}\,\partial_M\phi\,\Gamma^M\epsilon\,. \tag{3.13}$$

This clearly cannot vanish because $\partial_M\phi \neq 0$.

3.3 Brane Worlds

In this section we examine the properties of the brane-world scenario constructed from the warped solution given above. In this case the construction requires two 3-branes, respectively located at the conical singularities $r = r_\pm$, allowing us to interpret these singularities as the gravitational back-reaction due to the presence of the branes.

3.3.1 Electroweak Hierarchy

In the present instance the warp factor is $w(r) = 2\,r$ and so the expression for the effective 4D Planck mass becomes

$$M_p^2 = 2\pi \int_{r_-}^{r_+} dr\, w(r) = 2\pi(r_+^2 - r_-^2) = \frac{8\pi\, M}{g_1^2}\sqrt{1-x^2}\,, \tag{3.14}$$

where $x = g_1\mathcal{A}/(2M)$. For comparison, the physical mass of a particle localized on the 3-brane located at $r = r_\pm$ is

$$m_\pm = \mu_\pm\sqrt{w(r_\pm)}\,, \tag{3.15}$$

where the particle action is assumed to be proportional to $g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \mu_\pm^2\chi^2$.

The hierarchy between these scales is therefore

$$\begin{aligned} \frac{M_p^2}{m_\pm^2} &= \frac{4\pi M}{g_1^2\mu_\pm^2 r_\pm}\sqrt{1-x^2} = \frac{2\pi\,\mathcal{A}}{g_1\mu_\pm^2 r_\pm}\left(\frac{\sqrt{1-x^2}}{x}\right) \\ \frac{m_+^2}{m_-^2} &= \frac{\mu_+^2\, r_+}{\mu_-^2\, r_-} = \frac{\mu_+^2}{\mu_-^2}\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)^{1/2}, \end{aligned} \tag{3.16}$$

so a large hierarchy can be achieved, for example, if all dimensionful quantities are the same order of magnitude, except, say, M, which we take to be much larger. The hierarchy is then controlled by $x \ll 1$, or $g_1\mathcal{A} \ll 2M$, and in this case the previous formulae for $r_\pm$ reduce to

$$r_+^2 \approx \frac{4M}{g_1^2} \qquad \text{and} \qquad r_-^2 \approx \frac{\mathcal{A}^2}{4M}\,, \tag{3.17}$$

and so $r_-/r_+ \approx x/2$. Clearly this does not really provide a satisfactory explanation for the electroweak hierarchy, since the desired scales are simply inserted into the higher-dimensional solution.

If the gauge coupling $e^{\phi(r_-)}$ is assumed small, then the solution guarantees the gauge coupling to be even smaller at $r = r_+$ by an amount $e^{\phi(r_+)}/e^{\phi(r_-)} = r_-/r_+$.

3.3.2 Brane Boundary Conditions

To understand what the previous choices for $\mathcal{A}$ and M mean physically it is necessary to connect these integration constants to brane properties.

The counting of boundary conditions proceeds as follows. As before, the smoothness of the dilaton field at the 3-brane positions precludes these branes from directly coupling to the dilaton. Because this is also the choice which preserves the bulk scale invariance, the metric condition at each 3-brane only involves the scale-invariant ratio $\mathcal{A}/M$, implying a topological constraint which relates the two tensions to one another. The Maxwell boundary conditions at each 3-brane then lead to contradictory conditions on the gauge potentials, which imply a final topological restriction, also involving only the ratio $\mathcal{A}/M$.

We are therefore led in this case to three kinds of constraints. Two of these (vanishing 3-brane/dilaton charge, and the Maxwell flux-quantization condition) are similar to those found earlier for Romans' supergravity. Flux-quantization can be satisfied by adjusting the background gauge coupling, g, in terms of the coupling, g_1, appearing in the scalar potential. The third restriction, relating the 3-brane tensions, has no counterpart for Romans' supergravity and arises in the Salam-Sezgin case because of the compactness of the internal two dimensions. (This constraint is the analog of the condition of equal tensions which arises in the unwarped case [9].) In summary, we are led in this model to a picture which is very similar to what was encountered elsewhere for the unwarped solutions to Salam-Sezgin supergravity.

Dilaton and Metric

Following the reasoning of previous sections we see that 3-branes having the actions

$$S_\pm = -T_\pm \int_{r_\pm} d^4\xi \, e^{\lambda_\pm \phi} \sqrt{-\det \gamma}\,, \tag{3.18}$$

implies the dilaton couplings must satisfy

$$\lambda_\pm = 0\,, \tag{3.19}$$

in order for our assumed solution to describe correctly the fields generated by branes having the assumed action. This condition follows from the smoothness of the dilaton at the 3-brane positions.

Similarly, the brane tensions are related to the corresponding conical defect angles by the conditions

$$T_\pm = 2\pi \left[1 - \frac{|h'(r_\pm)|}{2} \right] = 2\pi \left[1 - \frac{g_1^2}{2\, r_\pm^2} \left(r_+^2 - r_-^2 \right) \right] , \tag{3.20}$$

from which we see that positive tensions imply that the radii $r_\pm$ must satisfy $(r_+/r_-)^2 < 1 + 2/g_1^2$, or in terms of $x = g_1 \mathcal{A}/(2M)$: $x^2 > 1 - (g_1^2 + 1)^{-2}$. Notice that large r_+, $r_+ \gg r_-$, therefore clearly requires $g_1 \ll 1$.

These last two brane boundary conditions determine only one of the two integration constants, M and $\mathcal{A}$, (or equivalently of r_+ and r_-) because they depend on the ratio r_-/r_+, and so can only determine the combination $x = g_1 \mathcal{A}/(2M)$. The fact that the two tensions are both determined by the single variable x implies the existence of a constraint relating these tensions. Eliminating r_-/r_+ from eq. (3.20) gives

$$\frac{T_+ - T_-}{2\pi} - \frac{2}{g_1^2} \left(1 - \frac{T_+}{2\pi} \right) \left(1 - \frac{T_-}{2\pi} \right) = 0 \, . \tag{3.21}$$

This is the analogue of the condition that the two 3-brane tensions be equal, which obtains for the unwarped 2-sphere solution [9].

Gauge Fields

A similar condition applies at the position of each brane, which follows from the nature of the brane coupling to the background Maxwell field. For the action of eq. (3.18), the brane carries no flux, and so the flux through a small patch of infinitesimal radius ϵ about each brane position must vanish in the limit $\epsilon \to 0$. This condition applied to both branes leads to a topological constraint which the parameters of our solution must satisfy.

To see this, notice that the gauge potential for the magnetic field strength, $F = (\mathcal{A}/r^3)\, \mathrm{d}r \wedge \mathrm{d}\theta$ may be written

$$A = \left(c - \frac{\mathcal{A}}{2r^2} \right) \mathrm{d}\theta \, , \tag{3.22}$$

where c is an integration constant. The condition that F not contain delta-function contributions at $r = r_\pm$ requires A to vanish at these two positions, and this imposes contradictory constraints on c: $c = c_\pm = \mathcal{A}/(2\, r_\pm^2)$. Consequently F can only be non-singular at *both* $r = r_+$ and $r = r_-$ if eq. (3.22) holds separately for two overlapping patches, $P_\pm$, each of which includes only one of r_+ or r_-.

Although the gauge potential can take different values ($A = A_\pm$ distinguished by constants $c_\pm$) on each of these patches, $A_+ - A_-$ must be a gauge transformation.

Periodicity of the coordinate θ on the overlap then requires $c_+ - c_- = n/g$, where g is the gauge coupling appropriate for the background gauge field which has been turned on. Combined with the expressions for $c_\pm$ we find the requirement

$$\frac{\mathcal{A}}{2}\left(\frac{1}{r_-^2} - \frac{1}{r_+^2}\right) = \frac{2M}{\mathcal{A}}\sqrt{1 - \left(\frac{g_1 \mathcal{A}}{2M}\right)^2} = \frac{n}{g}\,. \tag{3.23}$$

For the case of large, $r_+ \gg r_-$, this condition simplifies to $2M/\mathcal{A} \approx n/g$, and so $r_-/r_+ \approx g_1\mathcal{A}/(4M) \approx g\, g_1/(2n) \ll 1$. Since the ratio r_-/r_+ is already fixed given T_+ or T_-, we instead read eq. (3.23) as a condition relating g to g_1.

Since all of these conditions only fix the ratio $\mathcal{A}/M$ and none separately determine $\mathcal{A}$ or M, the overall scale of the extra dimensions (say, its volume) remains undetermined. As described in detail in the next section, this is consistent with the scale invariance of the bulk equations which is not broken by the 3-brane. Consequently $\mathcal{A}$ parameterizes a flat direction, for which we expect a classically massless modulus in the low-energy 4D theory. This behavior is in contrast to that of non-supersymmetric versions of this model, lacking the dilaton, where the volume of the extra dimensions is automatically stabilized in the presence of nonvanishing gauge flux [8].

4. Self-Tuning in Six Dimensions

The solutions we have found have flat 4D slices for all values of the various integration constants they involve. On the other hand, we have found that regarding these solutions as being sourced by simple 3- or 4-branes requires nontrivial relations amongst the couplings of the model. It is natural to then ask whether these choices were also required in order to adjust the 4D cosmological constant to vanish, or if they are choices which are only forced on us by our inability to find the general solution corresponding to the fields set up by a generic brane configuration.

In this section we partially address this question by identifying the source of the vanishing of the 4D cosmological constant in as much generality as possible. In particular, we avoid use of the detailed properties of the solutions, to see which features are important (and which are not) for ensuring flatness in 4D. By generalizing the argument of ref. [9] we show here that 4D flatness turns on the classical scale invariance of the bulk equations, and so hinges on whether the dilaton/brane couplings are chosen in a scale-invariant way. By contrast flatness does not appear to depend in an important way on the various topological conditions we have found.

To show this, in this section we explicitly integrate out the bulk massive KK modes, which at the classical level amounts to setting the fermionic modes to zero and eliminating the bosonic modes from the action using their classical equations of motion. We are therefore interested in the value of the action when evaluated at the solution to the classical equations of motion.

Before performing this integration for the specific models described above, we first recast the argument in its most general form which exposes the connection to the classical bulk scale invariance. Our arguments show that this self tuning property need not be specific to six dimensions, and may rather be generic to higher-dimensional supergravities. Making the connection to scale invariance also allows a more precise comparison of these models with the general self-tuning formulation given in ref. [28] (for discussions on the 5D case see for instance [30, 31]). Making this connection explicit makes it possible to ask whether there are loopholes to the no-go arguments that the general four-dimensional analysis suggests. We reserve our remarks concerning possible loopholes for the discussion, Section 5.

4.1 General Arguments

The classical self-tuning properties of the 6D (and higher-dimensional) supergravity equations follows from the classical scale invariance which they enjoy. In this section we review the general argument as to how this ensures a self-tuning of the cosmological argument.

Consider therefore the interactions of generic matter fields ϕ and the metric g_{MN} in an n-dimensional bulk, coupled to various brane modes on a set of $(n-1)$-dimensional boundary branes. Our later application is to a 6-dimensional bulk bounded by 5-dimensional 4-branes. 3-branes may also be included, in which case the boundary contribution consists of a small circle of infinitesimal radius which surrounds the 3-branes. We take the action for the theory to be

$$\begin{aligned} S &= \int_M d^n x \; \mathcal{L}_B(\phi, g_{MN}, \cdots) + \int_{\partial M} d^{n-1}x \; \mathcal{L}_b(\phi, g_{MN}, \cdots) \\ &= \int_M d^n x \; \mathcal{L}_B + \int_{\partial M} d^{n-1}x \left[\mathcal{L}_b^0(\phi, g_{MN}, \cdots) + \widehat{\mathcal{L}}_b(\phi, g_{MN}, \cdots) \right] \end{aligned} \tag{4.1}$$

where $\mathcal{L}_B$ is the bulk lagrangian density, and $\mathcal{L}_b$ is the brane action, which the second line splits into two pieces, $\mathcal{L}_b^0$ and $\widehat{\mathcal{L}}_b$. $\mathcal{L}_b^0$ here consists of the boundary pieces (such as the Gibbons-Hawking extrinsic-curvature term [32]) which are required by the bulk action, while $\widehat{\mathcal{L}}_b$ denotes the explicit brane action (such as the Nambu action used above).

Higher-dimensional supergravity theories typically have the following rescaling property for constant c:

$$\begin{aligned} \mathcal{L}_B(\phi, g_{MN}, \cdots) &= e^{-\omega_B c} \mathcal{L}_B(\phi - c, e^{sc} g_{MN}, \cdots) \\ \mathcal{L}_b^0(\phi, g_{MN}, \cdots) &= e^{-\omega_B c} \mathcal{L}_b^0(\phi - c, e^{sc} g_{MN}, \cdots) \\ \widehat{\mathcal{L}}_b(\phi, g_{MN}, \cdots) &= e^{-\omega_b c} \widehat{\mathcal{L}}_b(\phi - c, e^{sc} g_{MN}, \cdots). \end{aligned} \tag{4.2}$$

When regarded as low-energy vacua of string theory this invariance can be traced to the dependence on the dilaton, for which the classical scale invariance is manifest in the string frame. The ellipses in these equations denote any other fields, some of which may transform under the rescaling symmetry.

Now comes the main point. We rewrite the action, (4.1) using the above relations, and take the derivative with respect to c, setting $c = 0$ afterwards. Since S is independent of c, we have

$$0 = \frac{\partial S}{\partial c} = -\int_M d^n x \left[\omega_B \mathcal{L}_B(\phi, g_{MN}, \cdots) + \frac{\partial \mathcal{L}_B}{\partial \phi}\right] - \int_{\partial M} d^{n-1}x \, \frac{\partial \mathcal{L}_b}{\partial \phi}$$
$$- \omega_B \int_{\partial M} d^{n-1}x \, \mathcal{L}_b^0(g_{MN}, \cdots) - \omega_b \int_{\partial M} d^{n-1}x \, \widehat{\mathcal{L}}_b(\phi, g_{MN}, \cdots)$$
$$+ \text{(terms vanishing on use of all but the } \phi \text{ equations of motion)} \,. \tag{4.3}$$

The key observation is that the terms involving differentiation with respect to ϕ cancel when evaluated at the solutions to the dilaton equation of motion. This cancellation arises between the bulk and boundary terms, as may be seen from the bulk equations of motion,

$$\frac{\partial \mathcal{L}_B}{\partial \phi} = \partial_M \left(\frac{\partial \mathcal{L}_B}{\partial \partial_M \phi}\right) \equiv \partial_M V^M \,, \tag{4.4}$$

together with their counterparts on the boundary:

$$\frac{\partial \mathcal{L}_b}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}_b}{\partial \partial_\mu \phi}\right) - n_M V^M \,. \tag{4.5}$$

Here n_M denotes the outward pointing unit normal on the boundary brane, and the last term in this equation arises due to an integration by parts in the bulk. Keeping in mind that total derivatives may be dropped on the boundary, we are finally left with

$$0 = \omega_B \left[\int_M d^n x \, \mathcal{L}_B + \int_{\partial M} d^{n-1}x \, \mathcal{L}_b^0\right] + \omega_b \int_{\partial M} d^{n-1}x \, \widehat{\mathcal{L}}_b \,. \tag{4.6}$$

We now use (4.6) to eliminate all pieces of the bulk action in favor of the brane action to get

$$S = \left(1 - \frac{\omega_b}{\omega_B}\right) \int_{\partial M} d^{n-1}x \, \widehat{\mathcal{L}}_b \,. \tag{4.7}$$

For example, for the dilaton gravity such as arises in D-dimensional supergravity we have $s = 2\omega_B/(D-2)$ and $\omega_B = 2$. Consequently $s = 1$ in $D = 6$ and $s = \frac{1}{2}$ for $D = 10$. Furthermore, $\widehat{\mathcal{L}}_b = e^{\lambda\phi}\sqrt{-g}$ implies $\omega_b = -\lambda + ds/2$ where $d = n - 1 = p + 1$ is the dimension of the world-volume for a p-brane. Therefore, in the 6D case of present interest we have, using $\omega_B = 2$ and $s = 1$,

$$1 - \frac{\omega_b}{\omega_B} = 1 + \frac{\lambda}{2} - \frac{d}{4} \,. \tag{4.8}$$

Applied to a 4-brane we have $d = 5$, and so $S = 0$ when evaluated at the classical equations if $\lambda_4 = 1/2$. For a 3-brane we instead use $d = 4$ to get $S = 0$ if $\lambda_3 = 0$.

We expect from these arguments that scale invariance should ensure a classical self-tuning of the 4D cosmological constant for configurations built using $\lambda_3 = 0$ 3-branes and $\lambda_4 = \frac{1}{2}$ 4-branes. A similar argument applied to the 4-brane action for the σ fields implies self-tuning should occur if the dilaton coupling of eq. (2.57) satisfies

$$\zeta_4 = -1 + \frac{d-4}{2}\,, \tag{4.9}$$

which reduces to $\zeta_4 = -\frac{1}{2}$ for a 4-brane ($d = 5$).

We now investigate more explicitly how these general arguments work for the two 6D supergravity models considered in previous sections.

4.2 Romans' Supergravity

To explicitly see how self-tuning works for Romans' supergravity, consider the following expression for the effective 4D cosmological constant, obtained by evaluating the classical action with the bulk Kaluza-Klein modes integrated out at tree level. Since this is equivalent to their elimination using their classical equations of motion, we have

$$\begin{aligned}\rho_{\rm eff} \;=\; & T_3 a^2(r_3)\, e^{\lambda_3\phi}\Big|_{r=r_3} + T_4\, a^2(r_4) \int_0^{2\pi} d\theta\, e^{\lambda_4\phi}\sqrt{g_{\theta\theta}}\bigg|_{r=r_4} \\ & + \int_M d^2y\; e_2\, a^2 \left[\frac{1}{2}R_6 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{12}\, e^{-2\zeta\phi}\, G^2 + \frac{1}{4}\, e^{-\phi}\left(F^2 + \mathcal{F}^2\right)\right. \\ & \qquad \left. -\frac{1}{2}\, g_2^2\, e^{\phi} + \frac{1}{8\sqrt{2}}\, \epsilon^{MNPQRS} B_{MN}\Big(F^{\alpha}_{PQ}F_{\alpha RS} + \mathcal{F}_{PQ}\mathcal{F}_{RS}\Big)\right]_{cl}\,,\end{aligned} \tag{4.10}$$

where the subscript 'cl' indicates the evaluation of the result at the solution to the classical equations of motion. For simplicity we choose $q = 0$ and so neglect to the contributions to $\rho_{\rm eff}$ of the superconducting currents.

Here we adopt a procedure for which the branes are represented as delta-function contributions to the bulk equations of motion, and so if $\widehat{M}$ is the two-dimensional bulk having the 4-brane as a boundary, M denotes the two-dimensional bulk manifold obtained by gluing two copies of $\widehat{M}$ together at the 4-brane position. For future purposes it is important to recognize that whereas $\widehat{M}$ has a boundary, M does not unless we introduce boundaries by excising small circles about the positions of any 3-branes. This choice is purely a matter of convenience, and we have verified that our conclusions are unchanged if we instead work directly with $\widehat{M}$, keeping an explicit boundary at the position of the 4-brane.

Eliminating the metric using the Einstein equation (2.7) allows the 6D curvature scalar to be replaced by

$$\begin{aligned}R_6 \;=\; & -(\partial\phi)^2 + \frac{3}{2}\, g_2^2\, e^{\phi} - \frac{1}{4}\, e^{-\phi}\left(F^2 + \mathcal{F}^2\right) \\ & -\frac{2}{e_2}\, T_3\, e^{\lambda_3\phi}\, \delta^2(x - x_3) - \frac{5}{2}\, T_4\, e^{\lambda_4\phi}\frac{1}{\sqrt{g_{rr}}}\, \delta(r - r_4)\,.\end{aligned} \tag{4.11}$$

Substituting this into eq. (4.10) we find

$$\rho_{\rm eff} = -\frac{1}{4}\,T_4\,a^2(r_4)\int_0^{2\pi} d\theta\, e^{\lambda_4\phi}\sqrt{g_{\theta\theta}}\Big|_{r=r_4} + \int_M d^2y\; a^2\, e_2\;\Big[\frac{1}{12}\,e^{-2\zeta\phi}\,G^2 + \frac{1}{8}\,e^{-\phi}\Big(F^2+\mathcal{F}^2\Big) + \frac{1}{4}\,g_2^2\,e^{\phi} + \frac{1}{8\sqrt{2}}\,\epsilon^{MNPQRS}B_{MN}\Big(F^{\alpha}_{PQ}F_{\alpha RS}+\mathcal{F}_{PQ}\mathcal{F}_{RS}\Big)\Big]_{cl}\,. \tag{4.12}$$

Notice that the 3-brane tension cancels, just as in ref. [9].

Repeating this process, to integrate out the dilaton classically, allows us to use

$$\frac{1}{2}\,g_2^2\,e^{\phi} + \frac{1}{4}\,e^{-\phi}\,\Big(F^2+\mathcal{F}^2\Big) + \frac{\zeta}{6}\,e^{-2\zeta\phi}\,G^2 = -\Box\phi + \lambda_3\,T_3\,e^{\lambda_3\phi}\frac{1}{e_2}\,\delta^2(x-x_3) + \lambda_4\,T_4\,e^{\lambda_4\phi}\frac{1}{\sqrt{g_{rr}}}\,\delta(r-r_4)\,, \tag{4.13}$$

and further simplify eq. (4.12) to become

$$\rho_{\rm eff} = \frac{1}{2}\left(\lambda_4-\frac{1}{2}\right)\,a^2(r_4)\,T_4\int_0^{2\pi} d\theta\,e^{\lambda_4\phi}\sqrt{g_{\theta\theta}}\Big|_{r=r_4} + \frac{1}{2}\,\lambda_3\,a^2(r_3)\,T_3\,e^{\lambda_3\phi}\Big|_{r=r_3} + \int_M d^2y\;a^2\,e_2\;\Big[\frac{1}{12}(1-\zeta)\,e^{-2\zeta\phi}\,G^2 - \frac{1}{2}\,\Box\phi + \frac{1}{8\sqrt{2}}\,\epsilon^{MNPQRS}B_{MN}\Big(F^{\alpha}_{PQ}F_{\alpha RS}+\mathcal{F}_{PQ}\mathcal{F}_{RS}\Big)\Big]_{cl}\,. \tag{4.14}$$

If we now integrate out B_{MN} using its equation of motion, we may write

$$\frac{1}{8\sqrt{2}}\,\epsilon^{MNPQRS}B_{MN}\Big(F^{\alpha}_{PQ}F_{\alpha RS}+\mathcal{F}_{PQ}\mathcal{F}_{RS}\Big) = \frac{1}{2}\,B_{MN}D_P\Big(e^{-2\zeta\phi}\,G^{PMN}\Big)\,, \tag{4.15}$$

leading to

$$\begin{aligned}\rho_{\rm eff} &= \frac{1}{2}\left(\lambda_4-\frac{1}{2}\right)\,a^2(r_4)\,T_4\int_0^{2\pi} d\theta\,e^{\lambda_4\phi}\sqrt{g_{\theta\theta}}\Big|_{r=r_4} + \frac{1}{2}\,\lambda_3\,a^2(r_3)\,T_3\,e^{\lambda_3\phi}\Big|_{r=r_3} \\ &\quad - \int_M d^2y\;a^2\,e_2\;\Big[\frac{1}{12}(1+\zeta)\,e^{-2\zeta\phi}\,G^2 + \frac{1}{2}\,\Box\phi - \frac{1}{2}\,D_P\Big(e^{-2\zeta\phi}\,B_{MN}\,G^{PMN}\Big)\Big]_{cl} \\ &= -\frac{1}{12}(1+\zeta)\int_M d^2y\;a^2\,e_2\,e^{-2\zeta\phi}\,G^2 + a^2(r_4)\left[\pi\left(\lambda_4-\frac{1}{2}\right)\,T_4\,e^{\lambda_4\phi}\sqrt{g_{\theta\theta}}\right]_{r=r_4} \\ &\quad + a^2(r_3)\left[\frac{1}{2}\,\lambda_3\,T_3\,e^{\lambda_3\phi} - \pi\,e_2 n_P\Big(\partial^P\phi - e^{-2\zeta\phi}\,B_{MN}\,G^{PMN}\Big)\right]_{r=r_3}\,.\end{aligned} \tag{4.16}$$

Here the surface integral for the 3-brane is evaluated on an infinitesimal 5 dimensional surface at $r = r_3 + \delta$, with the limit $\delta \to 0$ taken at the end. Notice that total derivatives in the bulk, such as $\Box\phi$, do *not* give boundary contributions at the position of the 4-brane because the 4-brane does not represent a boundary of the manifold M. (Alternatively, the boundary contributions cancel for each of the two copies of $\widehat{M}$ of which M is composed.)

We see that for the $N = 4^g$ theory, where $\zeta = -1$, the bulk contribution to the result cancels, leaving only terms evaluated at the positions of the two branes.

It is straightforward to see that the 3-brane contributions to eq. (4.16) vanish for the warped solution considered earlier. The first term does so straightforwardly because of the dilaton 3-brane boundary condition, which required us to choose $\lambda_3 = 0$. The surface term also vanishes in this case because $B^{MN} = 0$ and because the outward-pointing unit normal, $n_M dy^M = -dr/\sqrt{g^{rr}}$, contributes an amount $n_M \partial^M \phi = -\sqrt{g^{rr}}\,\phi' = \sqrt{ab}\,\phi'$, which vanishes as $r \to r_3$ by virtue of the vanishing of $b(r_3)$. This agrees with the general scaling argument given above, which indicated self-tuning in the case $\lambda_3 = 0$.

We see that the 4-brane contribution to $\rho_{\rm eff}$ also vanishes provided that $\lambda_4 = \frac{1}{2}$, again in agreement with the general scaling argument. Since our explicit warped solution of earlier sections does not require a specific value for λ_4, we are free to make this choice and so to ensure the vanishing of $\rho_{\rm eff}$. It is a straightforward exercise to verify that the choice $\zeta_4 = -\frac{1}{2}$ would also be required to ensure $\rho_{\rm eff} = 0$ if we had taken $q \neq 0$ and followed the 4-brane fields σ.

4.3 Salam-Sezgin Supergravity

We here repeat the above exercise for the solution to Salam-Sezgin supergravity. We keep the presentation concise since the arguments largely follow the discussion in ref. [9].

For two parallel 3-branes positioned at $y = y^m_\pm$ in the internal dimensions the effective 4D vacuum energy in Salam-Sezgin supergravity is

$$\rho_{\rm eff} = \sum_{i=\pm} w^2(r_i)\, T_i + \int_M d^2y\; e_2\, w^2 \left[\frac{1}{2} R_6 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2} G_{ab}(D\Phi^a)(D\Phi^b) + \frac{1}{12}\, e^{-2\phi}\, G^2 + \frac{1}{4}\, e^{-\phi}\, F^2 + v(\Phi)\, e^{\phi} \right]_{cl} \Bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} \qquad (4.17)$$

where $w(r) = 2\,r$ is the warp factor, and M denotes the internal two-dimensional bulk manifold. As before the subscript '*cl*' indicates the evaluation of the result at the solution to the classical equations of motion.

Using the Einstein equation to eliminate the metric gives

$$R_6 = -(\partial\phi)^2 - G_{ab} D\Phi^a D\Phi^b - 3 v(\Phi)\, e^{\phi} - \frac{1}{4}\, e^{-\phi}\, F^2 - \frac{2}{e_2} \sum_{i=\pm} T_i\, \delta^2(y - y_i)\,, \qquad (4.18)$$

and using this in $\rho_{\rm eff}$ gives

$$\rho_{\rm eff} = \int_M d^2y\; e_2\, w^2 \left[\frac{1}{12}\, e^{-2\phi}\, G^2 + \frac{1}{8}\, e^{-\phi}\, F^2 - \frac{1}{2}\, v(\Phi)\, e^{\phi} \right]_{cl} \Bigg|_{cl} . \qquad (4.19)$$

The dilaton equation of motion now reads

$$v(\Phi)\, e^{\phi} - \tfrac{1}{4}\, e^{-\phi}\, F^2 - \tfrac{1}{6}\, e^{-2\phi}\, G^2 = \Box\phi - \sum_{i=\pm} \lambda_i T_i e^{\lambda_i \phi}\, \frac{1}{e_2}\, \delta^2(y - y_\pm)\,, \tag{4.20}$$

which gives when inserted into eq. (4.19)

$$\begin{aligned} \rho_{\rm eff} &= -\frac{1}{2}\int_M d^2y\; e_2\, w^2\, \Box\phi_{cl} + \frac{1}{2}\sum_{i=\pm} \lambda_i T_i\, w^2(r_i) \\ &= \sum_{i=\pm} w^2(r_i) \left[\frac{1}{2}\lambda_i T_i - \pi\, e_2\, n_M\, \partial^M \phi\right]_{r=r_i}\,, \end{aligned} \tag{4.21}$$

where we evaluate total derivative using the boundary surface ∂M_i, consisting of an infinitesimal region surrounding the 3-brane positions. For the solution considered above this consists of an infinitesimal circle surrounding the brane positions at $r = r_\pm$.

The two contributions to $\rho_{\rm eff}$ therefore vanish when evaluated at the solutions derived in earlier sections. The first term vanishes because we have already seen that the solution described above requires $\lambda_\pm = 0$, and the second likewise vanishes because ϕ' is bounded but $n_M\, \partial^M \phi = \sqrt{g^{rr}}\, \phi' = \sqrt{h}\, \phi'$ vanishes at the brane positions, $r = r_\pm$.

5. Discussion

In this paper we constructed explicit warped, axisymmetric solutions to the dilaton-Einstein-Maxwell field equations arising from both Romans' and Salam-Sezgin supergravity in six dimensions. We identified the circumstances under which they may be interpreted as being generated by simple 3- and 4-brane sources, and what geometrical features are required in order for the resulting brane systems to be used as brane-world models having a realistic electroweak hierarchy. Since all of the solutions have flat 4-dimensional sections regardless of the values of the tensions and couplings on the various branes, they resemble the unwarped solution of ref. [9]. We therefore also examine more generally how self-tuning of the 4D cosmological constant arises in these models. This allows us to identify some of the issues which must be addressed in order to promote these features into a real solution to the cosmological constant problem.

Our results, in more detail, are as follows.

5.1 Brane World Solutions

We considered two kinds of supergravities — Romans' and Salam-Sezgin — whose bosonic parts mainly differ in the overall sign of the exponential potential they predict for the 6D dilaton, and we found warped solutions for both theories.

Romans' Supergravity

The warped solution to Romans' supergravity can have a conical defect at its origin, which we interpreted as the position of a 3-brane. It is also terminated by a boundary 4-brane in order to ensure the transverse dimensions to have finite volume. By making simple assumptions about the physics of the 3- and 4-brane, we investigated how the parameters of the bulk solution are related to the physical properties of the branes. We found that in the generic case the number of boundary conditions is larger than the number of integration constants, implying that the bulk solutions we find can only be interpreted as being generated by the assumed brane sources if some of the brane couplings are adjusted.

In detail, our assumption that the dilaton remains nonsingular at the 3-brane position required us to choose the dilaton 3-brane coupling to vanish: $\lambda_3 = 0$. In addition, we found that the magnetic flux is determined both by the metric/dilaton boundary conditions and by a topological condition. These conditions are generically not consistent with one another, but can be made consistent by adjusting the background gauge coupling, g. Alternatively, since we assumed for simplicity the 4-brane to be superconducting, we could satisfy this equation by adjusting the 4-brane symmetry breaking scale (or 'penetration depth'), q.

In the generic case, the 4-brane couplings break the classical scale invariance of the bulk theory and so we were able to determine all parameters of the solution using the boundary conditions. In this sense our ansatz has no moduli, and so does not have a classically-massless dilaton or breathing mode. The scale invariance is not broken for the special case $\lambda_4 = \frac{1}{2}$ and $\zeta_4 = -\frac{1}{2}$, and in this case there is at least one flat direction.

We briefly examined brane-world models based on this solution and saw that an electroweak hierarchy could be obtained, but only by choosing a hierarchy in the underlying 6D theory or by adjusting dilaton couplings to be near to their scale-invariant values. In the precisely scale-invariant case the overall hierarchy could be simply set by the position chosen along the flat direction, and so would have to be explained by whatever physics stabilizes this direction.

Our inability to account for the electroweak hierarchy as cleanly as was possible for nonsupersymmetric systems follows from the presence of the dilaton, since the dilaton is free to roll to its potential minimum asymptotically, at which point the spacetime curvature also vanishes. Consequently the solution we found is asymptotically locally flat (conical), rather than being asymptotically anti-de Sitter, as is the case for the 5D Randall Sundrum [3] and 6D ADS soliton [4, 5, 6] solutions; in the latter the dilaton is replaced by a negative cosmological constant.

The warping of the metric ensures that the KK spectrum of the model need not involve many states lighter than the weak scale even if the proper radius of the internal space is comparatively large. This is because the lightest bulk KK modes tend to be localized near the 3-brane, and so do not 'see' the entire extent of the extra dimensions. In particular, many of the attractive relationships between scales which occur in the unwarped case (such as that relating the electroweak hierarchy to the effective cosmological constant) do not appear to also hold for these warped solutions.

Salam-Sezgin Supergravity

The warped solution to Salam-Sezgin supergravity can have either one or two conical defects, which we interpreted as the position of one or two 3-branes. Again the number of boundary conditions is larger than the number of integration constants, and so the bulk solutions are only produced by the assumed branes if their couplings are adjusted in particular ways.

As for the Romans' case, nonsingularity of the dilaton requires vanishing dilaton 3-brane couplings: $\lambda_\pm = 0$. Furthermore, the 3-brane tensions are subject to a topological condition which generalizes the condition found in the unwarped case (for which the tensions must be equal). Finally, we found a topological condition on the total magnetic flux through the space, whose satisfaction requires the adjustment of one of the couplings, such as the background gauge coupling, g.

Since the required dilaton couplings preserve the classical scale invariance of the bulk theory there is at least one classically flat direction corresponding to the overall volume of the internal dimensions.

Brane-world models based on this solution can have acceptable electroweak hierarchies, but apparently only by inserting the required hierarchies by hand into the 6D theory. Again, this can be chosen to be along the flat direction, pending an understanding of modulus stabilization in this direction. Unlike the unwarped example, there does not seem to be any compelling numerology which relates the required extra-dimensional sizes to the observed electroweak or cosmological constant hierarchies.

5.2 Self-Tuning Issues

All of the solutions which we considered have flat 4D slices for any values of the various couplings, suggesting these share the self-tuning properties of the unwarped example. In section 4 we traced the origin of self-tuning to the classical bulk scale invariance, and so made an explicit connection between our higher-dimensional self-tuning and Weinberg's general formulation of self-tuning in four dimensions. This connection allowed us to clarify how the usual objections to self-tuning arise in the 6D context.

Tuning of Couplings

Since special adjustments of couplings are required to interpret our solutions as the fields set up by simple brane sources, one worries that these adjustments may also be responsible for tuning the 4D cosmological constant. But given that classical scale invariance is the central property required, we believe the dilaton/brane coupling conditions ($\lambda_3 = 0$ for 3-branes and $\lambda_4 = -\zeta_4 = \frac{1}{2}$ for 4-branes) are the ones which are important for the self-tuning mechanism (as was also true in the original self-tuning solutions of [30]). Phrased in this way, the self-tuning is not seen as an exclusively 6D property, and is likely to apply more generally to brane configurations in compactified spaces.

On the other hand, since our self-tuning calculations of section 4 do not use any of the detailed properties of the solution, we believe the topological conditions which our models satisfy do not play a similarly important role. For instance, for 3-branes we find that the necessary condition for a solution to be self-tuning is that it have a nonsingular dilaton at the brane positions. Given only this, the bulk curvature automatically cancels the brane tensions regardless of the values these tensions take. In particular, the cancellation occurs for *any* value of the tensions, and does not depend on whether the tensions are related to one another by topological conditions.

We believe the same to be true for the magnetic flux-quantization conditions, since these conditions are actually very similar in form to the tension constraints. To see this, imagine including the following direct 3-brane coupling to the magnetic flux, obtained by integrating the Hodge dual ${}^*F$ over the four-dimensional brane world volume

$$\Delta S_3 = -\frac{\mathcal{Q}}{2}\int_{r_\pm} d^4\xi \, e_4 \, e^{-\phi} \, \epsilon^{mn} F_{mn} \,. \tag{5.1}$$

This term causes the 3-brane itself to carry magnetic flux, since it causes the flux through an infinitesimal surface surrounding the brane to be nonvanishing. This may be seen from the Maxwell field equation, which is modified to become (when $B_{MN} = 0$)

$$\partial_M \Big(e_6 \, e^{-\phi} \, F^{MN} \Big) = \delta^N_n \, \mathcal{Q} \, \partial_m \Big(e_4 \, e^{-\phi} \, \epsilon^{mn} \delta^2(\vec{y} - \vec{y}_3) \Big) \,, \tag{5.2}$$

thus showing that the bulk magnetic flux acquires delta-function contributions at either of the two brane positions.

Given this choice, for the Salam-Sezgin model the gauge potential again has the form of eq. (3.22), but now with the condition that $A \to (\mathcal{Q}_\pm/2\pi)\,\mathrm{d}\theta$ as $r \to r_\pm$, leading to

$$A_\pm = \left[\frac{\mathcal{Q}_\pm}{2\pi} + \frac{\mathcal{A}}{2\, r_\pm^2} \left(1 - \frac{r_\pm^2}{r^2} \right) \right] \mathrm{d}\theta \,, \tag{5.3}$$

in the patch centered at $r = r_\pm$. Requiring, as before, the two patches to be related by a periodic gauge transformation in this case replaces eq. (3.23) with the condition

$$\frac{\mathcal{Q}_+ - \mathcal{Q}_-}{2\pi} + \frac{\mathcal{A}}{2} \left(\frac{1}{r_+^2} - \frac{1}{r_-^2} \right) = \frac{n}{g} \,. \tag{5.4}$$

This is the direct analogue of the tension constraint, eq. (3.21), which relates the 3-brane tensions for compact extra dimensions. (No integer appears in the tension constraint because the integer has already been chosen, since the internal space has Euler number 2.)

Seen in this light, the flux-quantization constraint may be viewed as a condition on the charges carried by various branes, rather than as a tuning of gauge couplings which are parameters of the bulk action. (Constraints on the gauge couplings, like eq. (3.23), are seen in this light as being forced by the specialization to two branes having identical flux: $\mathcal{Q}_+ = \mathcal{Q}_-$.) In this sense, the tuning simply expresses global constraints on what combinations of brane charges it makes sense to include within extra dimensions of the assumed topology, in a similar manner to the well-known Gauss' Law requirement that the total charge for a collection of particles in a compact space must vanish.

Just like the Gauss' Law constraint (or the quantization condition for a magnetic monopole, or the flux-quantization condition for annular superconductors) we expect these constraints to be stable under UV-sensitive radiative corrections in the 6D theory; hence they are not fine-tunings in the sense of the cosmological constant problem. This radiative stability relies on the fact that short-distance quantum corrections must be local, and so are unlikely to affect long-distance topological effects.

Of course, an explicit demonstration of this stability is more persuasive than a hand-waving argument in its favor, and work on this is in progress.

Quantum Corrections and the No-Go Theorem

Because models of this class obtain a zero 4D cosmological constant by virtue of their classical scale invariance, they fall directly into the category of self-tuning models, and so also into Weinberg's related no-go theorem, described in ref. [28]. This suggests that there are *two* ways in which quantum corrections can ruin the self-tuning, rather than one.

The simplest problem which quantum effects raise is that they need not respect the scale invariance. This is certainly true in the theories studied here, for which the scale transformations do not leave the action invariant, but rather transform it into a multiple of itself. Although this suffices to ensure a symmetry of the classical equations of motion, it does not guarantee invariance for the full path integral and so the low-energy quantum-corrected action need not be scale invariant.

Even if quantum corrections were to respect scale invariance, the no-go theorem of ref. [28] raises another problem. This is because within any phenomenologically-successful scale-invariant theory the symmetry must be spontaneously broken in order to allow nonzero particle masses. It must therefore contain an effective 4D dilaton, φ, which is the 4D Goldstone boson for the scale invariance, and which therefore shifts under a scaling transformation: $\varphi \to \varphi + c$. (All other fields can then be made invariant by performing appropriate field redefinitions [29].) This transformation law ensures that the dilaton equations of motion suffice to ensure that flat space solves Einstein's equations. In the 6D models studied here, this dilaton is a linear combination of the 6D dilaton, ϕ, and the internal metric's 'breathing' mode.

The difficulty with these models is that scale invariance cannot forbid a term in the 4D scalar potential of the form $V_{\rm eff} = v\, e^{a\varphi}$, where v and a are dilaton-independent quantities. Since these are not constrained at all by scale invariance, scale invariance by itself cannot ensure $v = 0$, and so typically quantum corrections make $v \neq 0$ even if they are scale invariant. This lifts the degeneracy along the φ direction, making the ground state unique, and thereby leads to a vacuum which does not spontaneously break scale invariance at all. Although the 4D cosmological constant vanishes, it does so by driving the theory to a scale-invariant vacuum. The resulting theory cannot be said to solve the cosmological constant problem, because it is not a great achievement to obtain a vanishing cosmological constant in a theory for which all masses are also zero.

There are clearly two problems, and supersymmetry may be able to help with both of them. Although quantum corrections do break scale invariance, and can lift flat directions, supersymmetry typically ensures the scale for doing so is the supersymmetry-breaking scale. In the 6D models of interest here the self tuning mechanism can handle any quantum corrections on the brane, but cannot do so at the quantum level for the bulk modes. However the scale of supersymmetry breaking for the dilaton sector in these models is of order the bulk KK mass scale, m_{KK}, which can be much smaller than the usually-assumed TeV scale without running into observational difficulties. This is particularly striking for the unwarped solutions, for which m_{KK} can be as small as 10^{-3} eV.

Although no general proof exists that quantum corrections to the dilaton potential must be as small as $O(m_{KK}^4)$, there are encouraging indications. Explicit calculations in (unwarped) supersymmetric string and field theories with supersymmetry broken on branes indicate that the effective 4D cosmological constant generated at one loop *are* of order m_{KK}^4, rather than being set by the scale of the brane tension [33]. In six dimensions self-tuning itself has been argued for unwarped geometries to ensure that quantum corrections to the dilaton potential are at most of order $M_w^2 m_{KK}^2$ [9].

Notice that m_{KK} is typically much smaller in the unwarped compactifications than is found for the solutions examined here. Consequently it is the unwarped, large-extra-dimensional scenario which is the most attractive for potentially addressing the cosmological-constant problem in the low-energy 4D theory. Furthermore, the choice of the unwarped vacuum is likely to be stable against quantum corrections because (unlike the warped solutions) in the absence of branes it preserves an unbroken $N = 1$ supersymmetry in four dimensions.

Although none of these lines of argument are yet conclusive, we believe it is sufficiently encouraging to warrant more fully exploring how quantum corrections arise in the low-energy sector of these theories.

5.3 Open Issues

Our discussion suggests several directions for further exploration. Most notable among these is the solution to the general problem of finding the back reaction of simple 3-brane configurations in six dimensions without the neglect of dilaton or electromagnetic couplings. Given the general configuration it would be possible to identify whether the brane-coupling choices we make play an important role in the low-energy properties and with the self-tuning of the 4D cosmological constant.

An equally important issue to be addressed is the extent to which bulk radiative corrections change our results. In particular one would like to address the extent to which supersymmetry helps protect the electroweak hierarchy and 4D cosmological constant, given that these are chosen to be acceptably small at the classical level.

Given that the field equations we examine are supersymmetric, it would be useful to know how our solutions may be embedded into a still-higher-dimensional theory like 10D supergravity or string theory. At present this connection can be made more explicit for Romans' supergravity — such as for the explicit lift to ten dimensions described in the appendix (section 6) — because it is known how to obtain this theory by consistent truncation from higher dimensions. Similar constructions for Salam-Sezgin supergravity are presently being developed, [34], [35].[6]

We believe that further explorations in these directions is warranted by the preliminary features we have been able to identify here.

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6. Appendix: Romans with 10D Lifts

Much of the motivation for studying compactifications of higher-dimensional supergravities comes from their interpretation as low-energy vacua of string theory. This allows the identification of any phenomenologically-attractive low-energy features of these models to be taken as guidelines when searching for realistic string vacua. For

[6] Ref. [35] obtains an embedding of Salam-Sezgin supergravity by performing a consistent Pauli reduction of 11D/10D supergravity on the non-compact hyperboloid $\mathcal{H}^{2,2}$ times S^1.

these purposes it is necessary to know how a lower-dimensional supergravity arises from its higher-dimensional counterparts in order to be able to properly identify how the lower-dimensional fields correspond to explicit string modes. Since this matching has been partially performed for the supergravities we use, we pause here to record how it works. Our discussion, given for the $N = \tilde{4}^g$ theory (but similar to what happens for $N = 4^g$), follows that given in ref. [11].

Given a solution to Romans' $N = \tilde{4}^g$ six-dimensional supergravity involving the fields we consider here, one may always generate a solution to the bosonic field equations of a 10-dimensional supergravity, in which the metric, dilaton and Ramond-Ramond (RR) 4-form, F_4, take nontrivial values. We will consider, as internal manifold in the uplifting procedure, the space $S^3 \times T^1$.

If we adopt a notation for which 10D and 6D quantities are distinguished by marking the 10D fields using tildes, then the same uplifting procedure gives us the relevant part of the bosonic action for the 10D theory, that may be written as

$$\mathcal{L}_{10} = -\tilde{R} - \frac{1}{2}(\partial\tilde{\phi})^2 - \frac{1}{2}e^{-\tilde{\phi}/2}\tilde{F}_{\mu\nu\rho\lambda}\tilde{F}^{\mu\nu\rho\lambda}\,. \tag{6.1}$$

We adopt the standard convention that $e^{\phi} \to 0$ corresponds to weak string coupling, so our results differ from the conventional form for the (truncated) bosonic action of 10D type IIA supergravity by simply re-defining the scalar field according to $\tilde{\phi} \to -\tilde{\phi}$ [11].

The ten dimensional field configuration corresponding to a solution for the equations relative to (6.1) is then given in terms of the six-dimensional one by the following expressions:

$$\begin{aligned} d\tilde{s}^2_{10} &= \frac{1}{2}e^{\phi/4}ds^2_6 + \frac{1}{2\,g_2^2}e^{-3\phi/4}\sum_{\alpha=1}^{3}\left(\sigma^\alpha - \frac{g_2\,A_1^\alpha}{\sqrt{2}}\right)^2 + e^{5\phi/4}dZ^2\,,\\ \tilde{F}_4 &= \left(\tilde{G}_3 - \frac{1}{\sqrt{2}\,g_2^2}\,h^1\wedge h^2\wedge h^3 + \frac{1}{2\,g_2}F_2^\alpha\wedge h^\alpha\right)\wedge \mathrm{d}Z\,,\\ \tilde{\phi} &= \frac{1}{2}\,\phi\,, \end{aligned} \tag{6.2}$$

where θ_1, ψ, φ and Z are coordinates on the 4 new dimensions, $h^\alpha = \sigma^\alpha - \frac{g_2}{\sqrt{2}}\,A_1^\alpha$, and the σ^α are left-invariant 1-forms for $SU(2)$ given by

$$\begin{aligned} \sigma_1 &= \cos\psi\,\mathrm{d}\theta_1 + \sin\psi\sin\theta_1\,\mathrm{d}\varphi\,,\\ \sigma_2 &= \sin\psi\,\mathrm{d}\theta_1 - \cos\psi\sin\theta_1\,\mathrm{d}\varphi\,,\\ \sigma_3 &= \mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\varphi\,. \end{aligned} \tag{6.3}$$

The 3-form, $\tilde{G}_3$, appearing within the expression for the 4-form in eq. (6.2) is the 6-dimensional dual

$$\tilde{G}_{\mu\nu\rho} = \frac{1}{6}\,e^{2\phi}\,\epsilon_{\mu\nu\rho\lambda\beta\gamma}G^{\lambda\beta\gamma}\,, \tag{6.4}$$

of the 3-form field strength of the field B appearing in the 6D Romans' Lagrangian.

The obtained uplifted solution is quite interesting, since, in general, it can geometrically be interpreted as a configuration of intersecting D-branes. We see in the following how this works in a specific example.

We now specialize these general formulae to specific 6D solution considered in previous sections into a ten dimensional solution to the equations obtained from the Lagrangian (6.1). We get

$$\begin{aligned} d\tilde{s}_{10}^2 &= \frac{e^{\phi_0}\,r^{1/4}}{2}\left(a_0\,r\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + a_0\,r\,b(r)\,\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{a_0\,r\,b(r)}\right) \\ &\quad + \frac{e^{-3\phi_0/4}}{2\,g_2^2\,r^{3/4}}\left((\sigma^1)^2 + (\sigma^2)^2 + \left(\sigma^3 + \frac{g_2\,A\,e^{\phi_0}}{2\sqrt{2}\,a_0^2\,r^2}\right)^2\right) + e^{5\phi_0/4}r^{5/4}\mathrm{d}Z^2\,, \\ \tilde{\phi} &= \frac{\phi_0}{2} + \frac{1}{2}\ln r\,, \qquad (6.5) \\ \tilde{F}_4 &= \left(-\frac{1}{g_2^2\sqrt{2}}\sin\theta_1\,\mathrm{d}\theta_1\wedge\mathrm{d}\varphi\wedge\mathrm{d}\psi - \frac{Ae^{\phi_0}}{4\,g_2\,a_0^2\,r^2}\sin\theta_1\,\mathrm{d}\varphi\wedge\mathrm{d}\theta_1\wedge\mathrm{d}\theta \right. \\ &\quad \left. + \frac{A\,e^{\phi_0}}{2\,a_0^2\,g_2\,r^3}\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge(\mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\varphi)\right)\wedge\mathrm{d}Z\,. \end{aligned}$$

This configuration represents three D4 branes that intersect over three spatial directions. Two of the D4 branes, moreover, wrap the three sphere, which is part of the internal manifold. As one might expect, in the supersymmetric limit discussed in the previous section, the angle of intersection vanishes.

7. Appendix: Explicit Solution with $\lambda_4 = -\zeta_4 = \frac{1}{2}$

Here we will show explicitly how the counting of free parameters works in the Romans case. In general we have 5 parameters A, B, C, α_3, r_4. The parameter C can be eliminated right away using the matching of the gauge potentials as in equation (2.60). So we will concentrate on the 4 parameters A, B, α_3 and $x = r_3/r_4$. To determine them we have one condition coming from the 3-brane, namely, eq. (2.55), plus 4 conditions coming from the 4-brane, which are eqs. (2.72, 2.73, 2.74) and the flux quantisation condition (2.62). So in the general case we have one more equation than parameters and therefore there has to be at least one constraint involving g_2 and the brane parameters $T_4, T_3, q, \lambda_4, \zeta_4$.

In the conformal invariant case $\lambda_4 = -\zeta_4 = 1/2$, equation (2.74) is automatically satisfied. We could have then concluded that with one less equation we have the same number of equations and paramters and no constraint may be needed. However this is not the case. The reason is precisely because in this case we have the extra scaling symmetry (2.24) which implies that one of the parameters is actually redundant. We can see this explicitly by trying to solve the 5 equations mentioned above. This is what we will do now.

First we need to recall that $e^{\phi(r_4)} = \alpha_3 x$ and $a(r_4) = 1/x$ as well as the relation between A_θ and A given in (2.60). Also since r_3 appears explicitly in most of the equations we need to use often the expression (2.36). To simplify the calculations we work in the limit of large r_4 meaning that $b(r_4) \sim b_1 = g_2^2 \alpha_3 r_3^2/4$ and $b'(r_4) \sim 2B/r_4^3$.

From this we can see that equations (2.72) and (2.73) can be solved for x and A giving:

$$x^2 \;=\; \rho \left[-1 \pm \sqrt{1 - \frac{\mu}{\rho}} \right] \tag{7.1}$$

where $\rho \equiv 2(T_4 - g_2)$ and $\mu = q^2/g_2^2$. For A we find:

$$A^2(x) \;=\; \left(1 + \frac{2g_2^2}{q^2} x^2 \right)^{-1} \tag{7.2}$$

For the remaining parameters α_3 and B, we can easily see that equation (2.62) implies:

$$\alpha_3 r_3 \;=\; \frac{2N}{g g_2 A} \left[1 - x^2 \left(1 - \frac{g_2}{q^2} \right) \right]^{-1} \;\equiv F_1(x) \tag{7.3}$$

and the 3-brane condition (2.55), implies

$$\frac{B}{r_3^3} \;=\; \frac{\left(1 - \frac{T_3}{2\pi}\right)(1 - A^2)}{1 + A^2} \;\equiv F_2(x). \tag{7.4}$$

Since r_3 appears on both equations we can eliminate it by taking their ratio. But precisely the ratio in the left hand side is what appears in the expression for r_3 in (2.36). This then implies that:

$$\frac{F_1(x)}{F_2(x)} \;=\; \frac{4}{g_2^2} \frac{1}{1 - A^2(x)} \tag{7.5}$$

This is a constraint that involves *only* the external parameters: T_3, T_4, q as well as g_2 (since we have the explicit solutions for x and A) but not α_3, B. This also implies that one combination of the parameters α_3, B remains unfixed. Therefore we have shown explicitly that in this case there is still one free parameter, unlike the generic nonconformal, cases and that, similar to those cases, there is still one consistency constraint to be satisfied. This illustrates the general arguments given in the text.

Finally we can see from the expression for x above, which amounts to fixing the size of the extra dimensions, that in order to obtain a hierarchy $x \ll 1$ we may have to have either $\rho \ll 1$ or $q^2 \ll g_2^2$.

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