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# WARPED PRODUCT CR-SUBMANIFOLDS OF LP-COSYMPLECTIC MANIFOLDS

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**Abstract**

In this paper, we study warped product CR-submanifolds of LP-cosymplectic manifolds. We have shown that the warped product of the type $M = N_T \times {}_f N_\perp$ does not exist, where $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds of an LP-cosymplectic manifold $\bar{M}$, respectively. Also, we have obtained a characterization result for a CR-submanifold to be locally a CR-warped product.

## 1 Introduction

The geometry of warped product was introduced by Bishop and O'Neill [1]. These manifolds appear in differential geometric studies in natural way and these are generalization of Riemannian product manifolds and then it was studied by many geometers in different known spaces [2, 5]. Recently, B.Y. Chen has introduced the notion of CR-warped product in Kaehler manifolds and showed that there exist no proper warped product CR-submanifolds in the form $M = N_\perp \times {}_f N_T$ in a Kaehler manifold [3]. Later on, Hasegawa and Mihai proved that warped product CR-submanifolds $N_\perp \times {}_f N_T$ in Sasakian manifolds are trivial where $N_T$ and $N_\perp$ are $\phi-$invariant and anti-invariant submanifolds of Sasakian manifold respectively [5].

Matsumoto [7] introduced the notion of a Lorentzian almost paracontact manifold. Then Mihai and Rosca [8] introduced the same notion and obtained several results in this manifold. Submanifolds of a Lorentzian almost paracontact manifold have been studied by Prasad and Ojha and defined a class of Lorentzian almost paracontact manifold as an LP-cosymplectic manifold in [9].

In view of the physical applications of these manifolds, the question of existence or non existence of warped product submanifolds assumes significance. In the

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present paper, we have shown that the warped product in the form $M = N_T \times_f N_\perp$ is trivial where $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. On the other hand we have obtained a characterization result for the warped product of the type $M = N_\perp \times_f N_T$ when $\xi$ is tangent to $N_\perp$. Also, we have shown that there is no warped product $M = N_1 \times_f N_2$ when $\xi$ is tangent to $N_2$, where $N_1$ and $N_2$ are submanifolds of an LP-cosymplectic manifold.

# 2 Preliminaries

Let $\bar{M}$ be a $n$-dimensional Lorentzian almost paracontact manifold with the almost paracontact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is a $(1,1)$ tensor field, $\xi$ is a contravariant vector field, $\eta$ is a $1-$form and $g$ is a Lorentzian metric with signature $(-,+,+,\cdots,+)$ on $\bar{M}$, satisfying [7]:

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \mathrm{rank}(\phi) = n-1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi), \tag{2.2}$$

$$\Phi(X,Y) = g(\phi X, Y) = g(X, \phi Y) = \Phi(Y,X), \tag{2.3}$$

for all $X, Y \in T\bar{M}$, where $\Phi$ is the fundamental $2-$form defined as above.

A Lorentzian almost contact metric structure on $\bar{M}$ is called a *Lorentzian para-cosymplectic structure* if $\bar{\nabla}\phi = 0$, where $\bar{\nabla}$ denotes the Riemannian connection with respect to $g$. The manifold $\bar{M}$ in this case is called a *Lorentzian para-cosymplectic* (in brief, an *LP-cosymplectic*) manifold. From formula $\bar{\nabla}\phi = 0$, it follows that $\bar{\nabla}_X \xi = 0$.

Let $M$ be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$. Let the induced metric on $M$ also be denoted by $g$. Then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.5}$$

for any $X$, $Y$ in $TM$ and $N$ in $T^\perp M$, where $TM$ is the Lie algebra of vector field in $M$ and $T^\perp M$ is the set of all vector fields normal to $M$. $\nabla^\perp$ is the connection in the normal bundle, $h$ the second fundamental form and $A_N$ is the Weingarten endomorphism associated with $N$. It is easy to see that

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.6}$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.7}$$

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. Similarly for $N \in T^\perp M$, we write

$$\phi N = BN + CN, \tag{2.8}$$

where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$.

The covariant derivatives of the tensor fields $\phi$, $P$ and $F$ are defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \qquad \forall\ X, Y \in T\bar{M} \tag{2.9}$$

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \qquad \forall\ X, Y \in TM \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \qquad \forall\ X, Y \in TM. \tag{2.11}$$

Moreover, for an LP-cosymplectic manifold we have

$$(\bar{\nabla}_X P)Y = A_{FY}X + Bh(X, Y), \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = Ch(X, Y) - h(X, PY). \tag{2.13}$$

For submanifolds tangent to the structure vector field $\xi$, there are different classes of submanifolds. We mention the following.

(i) A submanifold $M$ tangent to $\xi$ is called an *invariant* submanifold if $F$ is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand $M$ is said to be an *anti-invariant* submanifold if $P$ is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

(ii) A submanifold $M$ tangent to $\xi$ is called a *contact CR-submanifold* if it admits a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}^{\perp}_x) \subset T^{\perp}_x M$, for every $x \in M$.

Let $M$ be an $m-$dimensional CR-submanifold of an LP-cosymplectic manifold $\bar{M}$. Then, $F(T_xM)$ is a subspace of $T^{\perp}_x M$. Thus it follows that $T_xM \oplus F(T_xM)$ is invariant with respect to $\phi$. Then for every $x \in M$, there exists an invariant subspace $\nu_x$ of $T_x\bar{M}$ such that

$$T_x\bar{M} = T_xM \oplus F(T_xM) \oplus \nu_x.$$

# 3 Warped and Doubly Warped Product Submanifolds

Let $(N_1, g_1)$ and $(N_2, g_2)$ be two semi-Riemannian manifolds and $f$, a positive differentiable function on $N_1$. The warped product of $N_1$ and $N_2$ is the manifold $N_1 \times_f N_2 = (N_1 \times N_2,\ g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

We recall the following general formula on a warped product [1].

$$\nabla_X V = \nabla_V X = (X \ln f)V, \tag{3.2}$$

where $X$ is tangent to $N_1$ and $V$ is tangent to $N_2$.

Let $M = N_1 \times {}_f N_2$ be a warped product manifold, this means that $N_1$ is totally geodesic and $N_2$ is totally umbilical submanifold of $M$, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by B. Ünal [10]. A *doubly warped product manifold* of $N_1$ and $N_2$, denoted as ${}_{f_2}N_1 \times {}_{f_1}N_2$ is endowed with a metric $g$ defined as

$$g = f_2^2 g_1 + f_1^2 g_2 \tag{3.3}$$

where $f_1$ and $f_2$ are positive differentiable functions on $N_1$ and $N_2$ respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1)Z + (Z \ln f_2)X \tag{3.4}$$

for each $X \in TN_1$ and $Z \in TN_2$ [10].

If neither $f_1$ nor $f_2$ is constant we have a non trivial doubly warped product $M = {}_{f_2}N_1 \times {}_{f_1}N_2$. Obviously in this case both $N_1$ and $N_2$ are totally umbilical submanifolds of $M$.

We now consider a doubly warped product of two semi-Riemannian manifolds $N_1$ and $N_2$ embedded into an LP-cosymplectic manifold $\bar{M}$ such that the structure vector field $\xi$ is tangential to the submanifold $M = {}_{f_2}N_1 \times {}_{f_1}N_2$.

**Theorem 3.1.** *There does not exist a proper doubly warped product submanifold in LP-cosymplectic manifolds.*

*Proof.* Let $M = {}_{f_2}N_1 \times {}_{f_1}N_2$ be a doubly warped product submanifold of an LP-cosymplectic manifold $\bar{M}$, where $N_1$ and $N_2$ are submanifolds of $\bar{M}$. We have using Gauss formula and the fact that $\bar{M}$ is LP-cosymplectic, for any $U \in TM$

$$\nabla_U \xi = 0. \tag{3.5}$$

Thus in case $\xi \in TN_1$ and $U \in TN_2$ equation (3.4) and (3.5) imply that $(\xi \ln f_1)U + (U \ln f_2)\xi = 0$, which shows that $f_2$ is constant. Similarly, for $\xi \in TN_2$ and $U \in TN_1$, we have $(\xi \ln f_2)U + (U \ln f_1)\xi = 0$, showing that $f_1$ is constant. This completes the proof. □

In above theorem we see that $f_2$ is constant if the structure vector field $\xi$ is tangent to $N_1$ and $f_1$ is constant if the structure vector field $\xi$ is tangent to $N_2$. The following corollary is an immediate consequence of the above theorem.

**Corollary 3.1.** *There does not exist a warped product submanifold $N_1 \times {}_f N_2$ of an LP-cosymplectic manifold $\bar{M}$ such that $\xi$ is tangent to $N_2$.*

Thus the only remaining case to study is the warped product submanifold $N_1 \times {}_f N_2$ with structure vector field $\xi$ tangential to $N_1$, we first obtain some useful formulae for later use.

**Lemma 3.1.** *Let $M = N_1 \times {}_f N_2$ be a proper warped product submanifold of an LP-cosymplectic manifold $\bar{M}$ such that $\xi$ is tangent to $N_1$, where $N_1$ and $N_2$ are submanifolds of $\bar{M}$. Then*

(i) $\xi \ln f = 0$,

(ii) $A_{FZ}X = -Bh(X, Z)$,

(iii) $g(h(X, Y), FZ) = -g(h(X, Z), FY)$,

(iv) $g(h(X, Z), FW) = -g(h(X, W), FZ)$

*for any* $X,\ Y \in TN_1$ *and* $Z,\ W \in TN_2$.

*Proof.* The first part of the lemma is an immediate consequence of the fact that $\bar{\nabla}_U \xi = 0$, for $U \in TM$ and using formula (2.4) and separating the tangential and normal parts. Now, for any $X \in TN_1$ and $Z \in TN_2$, then formula (2.12) gives

$$(\bar{\nabla}_X P)Z = A_{FZ}X + Bh(X, Z). \tag{3.6}$$

Also, we have

$$(\bar{\nabla}_X P)Z = \nabla_X PZ - P\nabla_X Z = (X \ln f)PZ - P(X \ln f)Z = 0, \tag{3.7}$$

for any $X \in TN_1$ and $Z \in TN_2$. Part (ii) follows by equations (3.6) and (3.7). Parts (iii) and (iv) follow by taking the product in (ii) by $Y$ and $W$ respectively. □

# 4 CR-Warped Product Submanifolds

Throughout this section the structure vector field $\xi$ is either tangent to the invariant submanifold $N_T$ or tangent to the anti-invariant submanifold $N_\perp$. There are two types of warped product in an LP-cosymplectic manifold $\bar{M}$, namely $N_T \times_f N_\perp$ and $N_\perp \times_f N_T$ are called *CR-warped product* submanifolds with $\xi$ tangent to $N_T$ and $N_\perp$, respectively. The following theorem is dealt with the case when $\xi$ is tangent to $N_T$.

**Theorem 4.1.** *There does not exist a proper warped product submanifold* $N_T \times_f N_\perp$ *where* $N_T$ *is an invariant and* $N_\perp$ *is an anti-invariant submanifolds of an LP-cosymplectic manifold* $\bar{M}$ *such that* $\xi$ *is tangent to* $N_T$.

*Proof.* Let $M = N_T \times_f N_\perp$ be a warped product CR-submanifold of an LP-cosymplectic manifold $\bar{M}$ with $\xi \in TN_T$ then from equations (2.2), (2.4) and the fact that $\bar{M}$ is an LP-cosymplectic, we have

$$g(\nabla_X Z, W) = g(\nabla_Z X, W) = g(\bar{\nabla}_Z X, W) = g(\phi\bar{\nabla}_Z X, \phi W)$$

for any $X \in TN_T$ and $Z \in TN_\perp$. Using (3.2), we get

$$(X \ln f)g(Z, W) = g(\bar{\nabla}_Z \phi X, \phi W) = g(\nabla_Z \phi X + h(Z, \phi X), \phi W),$$

or

$$(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + (\phi X \ln f)g(Z, \phi W) = g(h(Z, \phi X), \phi W).$$

That is,

$$(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W). \tag{4.1}$$

Again, we have

$$g(h(Z, \phi X), \phi W) = g(\bar{\nabla}_{\phi X} Z, \phi W). \tag{4.2}$$

Making use of equations (2.3), (2.5), (2.6) and (2.10) we deduce from (4.2) that

$$g(h(Z, \phi X), \phi W) = -g(h(\phi X, W), \phi Z). \tag{4.3}$$

Interchanging $Z$ and $W$ in (4.1) and then adding the resulting equation in (4.1), we get

$$2(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + g(h(\phi X, W), \phi Z).$$

Using (4.3), we obtain

$$(X \ln f)g(Z, W) = 0, \tag{4.4}$$

for all $X \in TN_T$ and $Z,\ W \in TN_\perp$. As $N_\perp \neq \{0\}$ anti-invariant submanifold then equation (4.4) and Lemma 3.1 (i) imply that $f$ is constant on $N_T$, proving the result. □

Now, the other case i.e., $N_\perp \times {}_f N_T$ with $\xi$ is tangent to $N_\perp$.

**Lemma 4.1.** *Let $M = N_\perp \times {}_f N_T$ be a warped product submanifold of an LP-cosymplectic manifold $\bar{M}$. Then*

$$g(h(X, \phi Y), \phi Z) = -(Z \ln f)g(X, Y), \tag{4.5}$$

*for any $X,\ Y \in TN_T$ and $Z \in TN_\perp$.*

*Proof.* For any $X, Y \in TN_T$ and $Z \in TN_\perp$, by formula (3.2) we have

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X Y, Z) = -g(\nabla_X Z, Y) = -(Z \ln f)g(X, Y). \tag{4.6}$$

Now, for any $X, Y \in TN_T$ and $Z \in TN_\perp$, consider

$$\begin{aligned} g(\bar{\nabla}_X Y, Z) &= g(\phi\bar{\nabla}_X Y, \phi Z) \\ &= g(\bar{\nabla}_X \phi Y, \phi Z) \\ &= g(h(X, \phi Y), \phi Z), \end{aligned}$$

i.e,

$$g(\bar{\nabla}_X Y, Z) = g(h(X, \phi Y), \phi Z). \tag{4.7}$$

Thus equation (4.5) follows by (4.6) and (4.7). This completes the proof of the lemma. □

**Theorem 4.2.** *Let $M$ be a CR-submanifold of an LP-cosymplectic manifold $\bar{M}$. Then $M$ is locally a contact CR-warped product if and only if*

$$A_{\phi Z} X = -Z(\mu)\phi X, \quad X \in \mathcal{D}, \ \ Z \in \mathcal{D}^\perp \oplus \langle\xi\rangle \tag{4.8}$$

*for some function* $\mu$ *on* $M$ *satisfying* $W'(\mu) = 0$ *for each* $W' \in \mathcal{D}$.

*Proof.* If $M = N_{\perp} \times {}_f N_T$ is CR-warped product submanifold, then on applying Lemma 4.1, we obtain (4.8). In this case $\mu = \ln f$.

Conversely, suppose $M$ is CR-submanifold of $\bar{M}$ and satisfying

$$A_{\phi Z} X = -Z(\mu)\phi X,$$

then

$$g(h(X,X), \phi Z) = g(A_{\phi Z} X, X) = -Z(\mu) g(\phi X, X) = 0$$

i.e., $h(X,Y) \in \nu$ the orthogonal complementary distribution of $\phi(\mathcal{D}^{\perp} \oplus \langle \xi \rangle)$. On the other hand, for any $X \in TN_T$ and $Z,\ W \in TN_{\perp}$ we have

$$g(\nabla_W Z, \phi X) = g(\bar{\nabla}_W Z, \phi X).$$

As $g$ is Lorentzian and $\bar{M}$ is LP-cosymplectic, the above equation takes the form

$$g(\nabla_W Z, \phi X) = -g(\bar{\nabla}_W \phi Z,\ X).$$

Thus, on using (2.5) and (2.6) we get

$$g(\nabla_W Z, \phi X) = g(A_{\phi Z} W,\ X) = g(h(X,\ W),\ \phi Z).$$

Also, by (2.4) we have

$$\begin{aligned} g(h(X,\ W),\ \phi Z) &= g(\bar{\nabla}_X W, \phi Z) \\ &= -g(\bar{\nabla}_X \phi Z,\ W) \\ &= g(A_{\phi Z} X, W). \end{aligned}$$

Using (4.8) in above, we get

$$g(\nabla_W Z, \phi X) = -(Z\mu) g(\phi X,\ W) = 0.$$

This means that $\mathcal{D}^{\perp} \oplus \langle \xi \rangle$ is integrable and its leaves are totally geodesic in $M$. Also, we have

$$\begin{aligned} g(\nabla_X Y, Z) = g(\bar{\nabla}_X Y, Z) &= -g(\bar{\nabla}_X Z, Y) = -g(\bar{\nabla}_X \phi Z, \phi Y) \\ &= g(A_{\phi Z} X, \phi Y) = -Z(\mu) g(\phi X, \phi Y) = -Z(\mu) g(X,Y) \end{aligned}$$

i.e.,

$$g(\nabla_X Y, Z) = -Z(\mu) g(X,Y) \tag{4.9}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp} \oplus \langle \xi \rangle$. Now, by Gauss formula

$$g(h'(X,Y), Z) = g(\nabla_X Y, Z)$$

where $h'$ denotes the second fundamental form of the immersion of $N_T$ into $M$. On using (4.9), the last equation gives

$$g(h'(X,Y),Z) = -Z(\mu)g(X,Y)$$

which shows that each leaf of $N_T$ of $\mathcal{D}$ is totally umbilical in $M$. Moreover the fact that $W'\mu = 0$ for all $W' \in \mathcal{D}$, implies that the mean curvature vector on $N_T$ is parallel along $N_T$ i.e., each leaf of $\mathcal{D}$ is an extrinsic sphere in $M$. Hence by virtue of a result in [6] which states that -*"If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum* $TM = E_0 \oplus E_1$ *of non trivial vector sub bundles such that* $E_1$ *is spherical and its orthogonal complement* $E_0$ *is auto parallel, then the manifold M is locally isometric to a warped product* $M_0 \times_f M_1$", we get that, $M$ is locally a warped $N_\perp \times_f N_T$ of a holomorphic submanifold $N_T$ and a totally real submanifold $N_\perp$ of $M$. Here $N_T$ is a leaf of $\mathcal{D}$ and $N_\perp$ is a leaf of $\mathcal{D}^\perp \oplus \langle\xi\rangle$ and $f$ is a warping function. □

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## References

[1] B. O'Neill, *Semi-Riemannian geometry,* Academic Press, New York, 1983.

[2] B.Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifold*, Monatsh. Math. 133 (2001), 177-195.

[3] B.Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II*, Monatsh. Math. 134 (2001), 103-119.

[4] U.C. De and K. Arslan, *Certain curvature conditions on an LP-Sasakian manifold with a coefficient* $\alpha$, Bull. Korean Math. Soc. 46 (2009), 401-408.

[5] I. Hasegawa and I. Mihai, *Contact CR-warped product submanifolds in Sasakian manifolds*, Geom. Dedicata 102 (2003), 143-150.

[6] S. Hiepko, *Eine Inner Kennzeichungder verzerrten Produkte*, Math. Anal. 241 (1979), 209-215.

[7] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. 12 (1989), 151-156.

[8] I. Mihai and R. Rosca *On Lorentzian P-Sasakian manifolds*, Classical analysis (Kazimierz Donly, 1991), 155-169, World Sci. Publ., River Edge, NJ, 1992.

[9] S. Prasad and R.H. Ojha, *Lorentzian paracontact submanifolds*, Publ. Math Debrecen 44 (1994), 215-223.

[10] B. Ünal, *Doubly warped products*, Diff. Geom. Appl. 15 (2001), 253-263.

[11] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.

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