

# Warranty Costs: An Age-Dependent Failure/Repair Model

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## Abstract

An age-dependent repair model is proposed. The notion of the “age” of the product and the degree of repair are used to define the virtual age of the product. The virtual failure rate function and the virtual hazard function related to the lifetime of the product are discussed. Under a non-homogeneous Poisson process scenario the expected warranty costs for repairable products associated with linear pro-rata, non-renewing free replacement and renewing free replacement warranties are evaluated. Illustration of the results is given by numerical and graphical examples.

**Key Words:** Warranty cost analysis, virtual age, virtual failure rate, virtual hazard function, pro-rata warranty, maintenance costs, Kijima Model I.

## 1 Introduction

In today’s market, product warranty plays an increasingly important role. The use of warranty is widespread and serves many purposes, including protection for manufacturers, sellers, insurance, buyers and users. It provides indirect information about the quality of the products and represents an important tool to influence the market. A detailed discussion of the various aspects of warranty problems can be found in the handbook of Blischke and Murthy (1996). One of the main problems for the companies offering warranties is the prediction of the warranty costs. For example, the pricing of the warranty should be balanced against the expenses throughout the warranty coverage. The evaluation of the warranty cost depends on the failure process, the degree of repairs and the prescribed maintenance of the item.

Repairable products are affected by the post-failure repairs in one of the following ways:

- A repair completely resets the failure rate of the product so that upon restart the product operates as a new one. This is known as a *complete repair*, and it is equivalent to a replacement of the faulty item by a new one.

- A repair has no impact on the failure rate. The failure rate remains the same as it was prior to the failure. The repair brings the product from a down to an up state without affecting its performance. This is known as a *minimal repair*.

- A repair contributes to some noticeable improvement of the product. This contribution is measured by an *age-reducing repair factor*. The repair sets back the clock of the repaired item. After the repair, the performance of the item is as it was at an earlier age. This is known as an *imperfect repair*.

- A repair may contribute to some noticeable degradation of the product. This contribution is measured by an *age-accelerating repair factor*. The repair sets forward the clock of the repaired item. After the repair, the performance of the item is as it will be at some later age. This type of repair is (called by Scarsini and Shaked 2000) a *sloppy repair*.

A repair process could be a mixture between minimal, imperfect, sloppy and complete repair.

Models of imperfect, sloppy or mixture of repairs specify the class of age-dependent repair models. The age dependent models of repairable products have been discussed in reliability literature. The model we consider is Kijima Model I, Kijima (1989). Scarsini and Shaked (2000) enriched this model with the concept of sloppy repair. Both papers focus on the application of the model on a finite horizon. Stochastic inequalities for various characteristics of the performance of the product are derived in terms of the inter-maintenance or inter-repair times. Guo and Love (1992, 1994), and Love and Guo (1994a, 1994b) provide an overview on statistical inferences of the age-dependent models. For warranty purposes such models have been considered in Chukova and Khalil (1990a, 1990b), and in Chukova (2000). To the best of our knowledge, models with imperfect or sloppy repairs have not been studied from warranty viewpoint.

In this paper we focus on the failure rate function of a product maintained under Kijima's Model I scenario. Following Scarsini & Shaked, we call it *virtual failure rate*. In Section 2, assuming that the *age-reducing, or age-accelerating repair factor* is a constant, we find a relationship between the virtual failure rate and the original failure rate. In Section 3 we apply the results from Section 2 to study warranty problems. In Section 4, assuming Weibull lifetime distribution, we illustrate the warranty problems studied in Section 3. We introduce the concept of an "age" of the products at the time the warranty is assigned. In addition, we refer to age-reducing or age-accelerating repair factor as an age-correcting repair factor. The degree of repair is measured by the value of the age-correcting repair factor. The length of the warranty period, the magnitude of the age-correcting factor and the "age" of the products are the parameters that will affect the warranty costs. In what follows, we reconsider the age-dependent models in Chukova and Khalil (1988), use some basic ideas of Nelson (1998) and Block et al. (1985), and give further development of the discussion in Chukova (2000). In addition, our results are an useful extension of the properties of Kijima's Model I and could be useful in other reliability contexts.

## 2 Model description

The most convenient description, from our point of view, of imperfect or sloppy repairs is given in terms of the failure rate function and corresponding hazard function of the lifetime of the product. Then, the intensity function approach (Kotz and Shanbhag, 1982) can be used to model the underlying random variables, and vice versa.

Let the initial lifetime,  $X$ , of a new product sold under warranty, be a continuous random variable (r.v.) with probability cumulative distribution function (c.d.f.)  $F(x) = P(X \leq x)$  and probability density function (p.d.f.)  $f(x) = \frac{d}{dx}F(x)$ . Then its hazard function is

$$\Lambda(x) = -\ln[1 - F(x)], \quad x \geq 0, \quad (1)$$

and its failure rate function is

$$\lambda(x) = \frac{d}{dx}\Lambda(x) = \frac{f(x)}{1 - F(x)}, \quad x \geq 0.$$

Block et al. (1985) and Beichelt (1991), have shown that under instantaneous minimal repairs the total number of failures within any time interval, say  $[u, u + v)$ ,  $u, v \geq 0$ , has the Poisson distribution with mean  $\Lambda(u + v) - \Lambda(u)$ ,  $u, v \geq 0$ . Let  $N_{[u, u+v)}$  be the number of failures during the interval  $[u, u + v)$ . Then the probability of no failure in  $[u, u + v)$  is

$$P(N_{[u, u+v)} = 0) = e^{-[\Lambda(u+v) - \Lambda(u)]}, \quad u, v \geq 0.$$

Moreover, the probability of first failure to occur after the moment  $u$  in the interval  $[u, u + v + \Delta v)$ ,  $\Delta v > 0$ , is

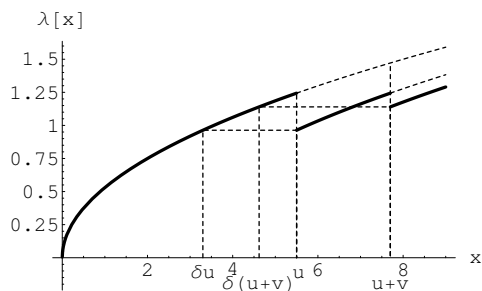
$$P(N_{[u, u+v)} = 0, N_{[u+v, u+v+\Delta v)} = 1) = \lambda(u + v) e^{-[\Lambda(u+v) - \Lambda(u)]} \cdot \Delta v + o(\Delta v). \quad (2)$$

The minimal repairs have no impact on the failure rate function. They switch the product from a down to an up state. At the same time they accumulate repair cost. The labor, time or money invested in the repair may have significant impact on the product improvement. The latter affects the warranty costs. We consider imperfect repairs that may affect the future performance of a repairable product, and are related to an age-reducing factor  $\delta$ . More specifically, following Scarsini and Shaked (2000), let  $X_i$  denote the inter-repair or inter-maintenance times of the product. Let  $\delta_i$  denote the lack of perfection of the  $i^{th}$  repair. Then

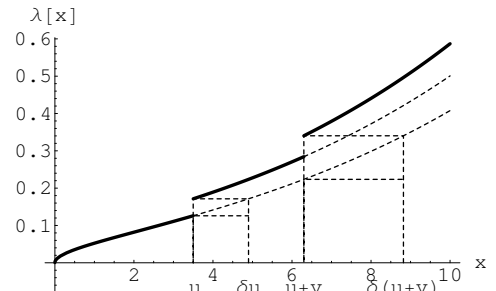
$$T_0 = 0, \quad T_i = T_{i-1} + \delta_i X_i, \quad i = 1, 2, \dots \quad (3)$$

are the values of the *virtual age* of the product immediately after the  $i^{th}$  repair. When  $\delta_i = 1$  then no improvement or deterioration of the product occurs at the  $i^{th}$  epoch of action. When  $\delta_i < (>)1$  then an improvement (deterioration) of the product occurs at that epoch. The extreme case of  $\delta_i = 0$  corresponds to the product being as good as new after the rectification. The model described in (3) is also known as Kijima's model I. In the sequel we consider this model with the assumption that  $\delta_i = \delta \neq 0$ , and call this  $\delta$  an *age-correcting factor*. If  $\delta < 1$  we call it *age-reducing factor*, and if  $\delta > 1$  we call it *age-accelerating factor*. In warranty, it is natural to assume that  $\delta < 1$ .

With no failures in  $[0, u)$ ,  $u > 0$  the product would have the original failure rate function  $\lambda(x)$  for  $x \in (0, u)$ . Referring to Fig. 1, the first age-reducing repair occurs at an instant  $u$ . After the repair, the product is improved and its performance is as it was earlier, when the age of the product was  $\delta u$ . At calendar age  $u$ , which is the time of the first repair, the virtual age of the product is  $\delta u$ . From time  $u$  onwards, until the next repair, the performance of the product is modeled by modified original failure rate function  $\lambda(x - (u - \delta u))$ . Assume that the next failure is at the calendar age  $u + v$ . The instantaneous repair improves the performance of the product and its virtual age is  $\delta u + \delta v$ . Physically, between the two consecutive failures, the product experiences age accumulation, say  $v$ , but due to the age-correcting repair, its virtual age accumulation is only  $\delta v$ . The failure rate function of the lifetime of the product maintained with age-correcting repairs is modified original one, as shown in Fig. 1.



**Fig.1.a** Original and individual virtual failure rates under age-reducing factor  $\delta = .6$ .



**Fig.1.b** Original and individual virtual failure rates under age-accelerating factor  $\delta = 1.2$ .

For any particular product in a homogeneous population this function will have its jumps whenever an age-correcting repair occurs. Therefore, future failures may reduce (or prolong) the increments in virtual age by factor of  $\delta$ . Hence, its virtual failure rate will be compressed (or stretched) compare to the original failure rate. For a population of products (with i.i.d. life times) maintained under age-correcting repairs with identical age-correcting factors, the population failure rate is obtained by averaging the possible individual failure rates of all products. The failure rate for this population is the virtual failure rate of one randomly selected product maintained under age-correcting repairs. We denote it by  $\lambda^*(x)$ , where  $x$  is a calendar age. It reflects the overall slow-down or acceleration of the aging process for an “average” product from the population. This concept is similar to Nelson (1998), used to evaluate the Mean Cumulative Cost Function of a population.

Since we assume instantaneous repairs, the *virtual hazard function*  $\Lambda^*(x)$  is a continuous nondecreasing function of the time  $t$ , and its right derivative is  $\lambda^*(x) = \left. \frac{d}{dx} \Lambda^*(x) \right|_{x+0}$ . The subscript here indicates that the value of  $\Lambda^*(x)$  is considered immediately after an age-correcting repair is completed.

In what follows, we will focus on products in a population where each individual item is maintained under age-correcting repairs of factor  $\delta$ . We derive a relationship between

the original failure rate and the virtual failure rate of the product from a population maintained under age-correcting instantaneous repairs of factor  $\delta$ .

Consider the sequence  $0 = T_0 \leq T_1 \leq T_2 \leq \dots \leq T_n \leq \dots$  of times representing the virtual age of the product after the  $n$ -th repair. To avoid technical difficulties, assume that the original life time of the product,  $X$ , has a c.d.f.  $F(x)$  with  $F(0) = 0$  and  $F(x) < 1$  for all  $x \geq 0$ . Denote the corresponding survival function by  $\bar{F}(t) = P\{X > t\} = 1 - F(t)$ . Taking into account the relationship between the calendar age of the item, and its virtual age after an age-correcting repair of factor  $\delta$ , given by (3), it follows that:

- The  $n$ -th step transition probability function is

$$\begin{aligned} P\{T_{n+1} > t \mid T_1, \dots, T_n\} &= P\{X > \frac{t}{\delta} \mid X > \frac{T_n}{\delta}\} \\ &= \frac{\bar{F}(\max(\frac{T_n}{\delta}, \frac{t}{\delta}))}{\bar{F}(\frac{T_n}{\delta})} \end{aligned} \quad (4)$$

for  $n = 1, 2, \dots$

- The initial distribution is

$$P\{T_1 > t\} = P\{X_1 > \frac{t}{\delta}\} = \bar{F}(\frac{t}{\delta}). \quad (5)$$

- From (4) and (5) it is easy to check by induction that for any non-negative measurable function  $g(t_1, \dots, t_n)$  and any  $n \geq 2$  it is true that:

$$\begin{aligned} E[g(T_1, \dots, T_n)] &= \int_0^\infty (\bar{F}(\frac{t_1}{\delta}))^{-1} \int_{t_1}^\infty (\bar{F}(\frac{t_2}{\delta}))^{-1} \dots \int_{t_{n-2}}^\infty (\bar{F}(\frac{t_{n-1}}{\delta}))^{-1} \times \\ &\quad \times \int_{t_{n-1}}^\infty g(t_1, \dots, t_n) dF(\frac{t_n}{\delta}) \dots dF(\frac{t_1}{\delta}). \end{aligned} \quad (6)$$

Let  $\{N_t^v, t \geq 0\}$  be the counting process corresponding to  $\{T_n\}_{n=0}^\infty$  defined by

$$N_t^v = \sum_n I_{[0,t)}(T_n),$$

where  $I_B(\cdot)$  is the indicator function of the set  $B$ .

**Theorem 1 (Age Transfer Theorem)**  $\{N_t^v, t \geq 0\}$  is a non-homogeneous Poisson process (NHPP) with a leading function

$$\Lambda^v(t) = E(N_t^v) = -\log(1 - F(\frac{t}{\delta})) = \Lambda(\frac{t}{\delta}), \quad (7)$$

where  $\Lambda(t) = -\log(1 - F(t))$  is the leading function of the NHPP associated to the process of instantaneous minimal repairs.

The proof of the theorem is in Appendix A.

Equation (7) shows that the transformation between the calendar and the virtual time scales is  $t^v \rightarrow t^v/\delta$ , i.e., if the virtual age is  $t^v$  corresponding calendar age is  $t^v/\delta$ . Therefore, when the product is at calendar age  $x$ , its virtual age measured at the calendar age scale is  $\delta x$ . Therefore, at calendar age  $x$ , an item maintained under age-correcting repairs of factor  $\delta$ , functions as an item at age  $\delta x$ . Thus

$$\lambda^*(x) dx = \lambda(\delta x) dx,$$

i.e., the probability to have a failure of the product from the original population within the interval  $[x, x + dx)$  is the same as the probability to have a failure of a product from the population of products, governed by age-reducing repairs, within the interval  $[\delta x, \delta x + dx)$ .

The relations

$$\Lambda^*(x) = \int_0^x \lambda^*(u) du, \text{ and } \Lambda(x) = \int_0^x \lambda(u) du,$$

will lead to

$$\Lambda^*(x) = \frac{1}{\delta} \Lambda(\delta x), \quad x \geq 0, \quad \delta > 0.$$

Hence, the following result holds:

**Theorem 2** The virtual failure rate  $\lambda^*(x)$  at calendar age  $x$ , and the original failure rate are related by the equality

$$\lambda^*(x) = \lambda(\delta x), \quad x \geq 0. \quad (8)$$

The virtual hazard rate  $\Lambda^*(t)$  and the original hazard rate  $\Lambda(t)$  are related by

$$\Lambda^*(x) = \frac{1}{\delta} \Lambda(\delta x), \quad x \geq 0, \quad \delta > 0. \quad (9)$$

□

Thus, we see that an age-reducing repair of factor  $\delta$  slows down the aging process of the product by  $100(1 - \delta)\%$ . The minimal repairs are particular case of age-reducing repairs with the age-reducing factor  $\delta = 1$ . However, the model with replacements of the faulty product by a new one is not analytically a particular case of the age-reducing repairs model, say with  $\delta = 0$ . For  $\delta = 0$ , Theorem 1 is not valid because it reflects the failure rate immediately after an age-reducing repair only. The values of  $0 < \delta < 1$  correspond to reliability improvement of the product.

Naturally there are certain costs associated with this improvement. Let us denote the cost of an age-reducing repair of factor  $\delta$  at calendar age  $u$  of the product by  $C_r(u, \delta)$ . A natural assumption is that  $C_r(u, \delta)$  satisfies the following inequalities

$$C_m(u) \leq C_r(u, \delta) \leq C_c(u), \quad (10)$$

where  $C_m(u)$  is the cost of a minimal repair of the product at calendar age  $u$ , and  $C_c(u)$  is the cost of replacement (complete repair) of the product at age  $u$ . The expected warranty costs associated with the product sold under warranty would depend on the warranty contract and the maintenance schedule associated to the product (Chukova, 2000). Different forms of warranty coverage with age dependent repairs will model large variety of practically feasible and important warranty policies. In this paper we confine ourselves to some of the typical warranty policies.

### 3 Warranty costs associated with a product during the assigned warranty time

Here and onwards the time scale is the calendar age time scale. In this section we discuss some of the typical warranty policies emphasizing on the age-dependent warranty, and show how the age of the product at the time of the sale, say  $t_0$ , can affect the warranty costs. We will refer to  $t_0$  as the initial age of the product. The warranty cost depends on the warranty policy and the degree of the warranty repairs (e.g. Blischke and Murthy, 1996, Chapter 10). The degree of the warranty repairs is measured by the age-correcting factor  $\delta$ . We assume that a sold item is covered by warranty for a calendar time of duration  $T$ , according to certain warranty agreement. The warranty coverage starts at the time the sale is completed. We consider warranty modelling from warranter's point of view. The warranter covers all, or a portion of the expenses associated with the failures and repairs of the product within the warranty coverage.

#### 3.1 Partial rebate warranty, non-repairable product

First we consider a warranty cost model for a non-repairable product with an initial age  $t_0$ . Assume that the initial life time  $X$  of the product is known and has failure rate function  $\lambda(t)$ . During the warranty period no repairs are possible. The initial age  $t_0$  of the product at the time of the sale, which is also the time when the warranty starts, is the parameter of interest.

If the failure is beyond the assigned warranty period, the warranter incurs no expenses. If the failure occurs at the moment  $t \in [t_0, t_0 + T)$ , the warranter refunds the user by

$$C(t) = C_0(t_0) \cdot \frac{T - t}{T},$$

where  $C_0(t_0)$  is the purchase price of the product at age  $t_0$ .

**Lemma 1** The expected warranty cost associated with an age-dependent product sold at age  $t_0$  with warranty of duration  $T$  is given by

$$C_W(t_0, T) = C_0(t_0) \int_0^T \lambda(t_0 + t) e^{-[\Lambda(t_0+t) - \Lambda(t_0)]} \cdot \frac{T - t}{T} dt, \quad (11)$$

where  $\lambda(t)$  and  $\Lambda(t)$  are the original failure rate and original hazard rate functions.

**Proof:** Based on NHPP concept for the failure rate of the product and using equation (2) for the probability of the first failure after  $t_0$ , we find that the probability to have a failure in  $[t_0 + t, t_0 + t + dt)$  is  $\lambda(t_0 + t)e^{-[\Lambda(t_0+t)-\Lambda(t_0)]}dt$ . Then the remaining time until the end of warranty is  $\max(T - t, 0)$ , and therefore the warranter refunds the user by  $C_0(t_0)\frac{T-t}{T}$ . The *total expectation rule* completes (11).  $\square$

**Note:** For a new product  $t_0 = 0$ . However, some used products are sold under this type of warranty. Thus, (11) represents the expected warranty cost under linear pro-rata warranty for non-repairable product which performance depends on the age it has been sold. Some illustrations of the dependence of the expected warranty cost on the initial age  $t_0$  are given in Section 4.

### 3.2 Age-reducing repairs under fixed warranty

In this section we consider repairable products sold under warranty of fixed duration  $T$ . We assume that during their usage, the products are maintained under age-reducing repairs, as in Theorem 1. We analyze a cost model for the *Free Replacement Warranty* (briefly FRW, Blischke and Murthy 1996, Chapter 10). The warranter agrees to repair or replace, at no cost to the consumer, any failed item up to the expiration of the warranty period of length  $T$ . This FRW is also known as *fixed free-replacement warranty*.

**Lemma 2** The expected warranty cost  $C_W(t_0, T, \delta)$  associated with a product sold at age  $t_0$  under FRW of duration  $T$  and maintained by age reducing repairs of factor  $\delta$ , satisfies the integral equation

$$C_W(t_0, T, \delta) = \int_{t_0}^{t_0+T} \lambda(\delta u)C_r(u, \delta) du, \quad (12)$$

where  $C_r(u, \delta)$  is the cost per an age-reducing repair of factor  $\delta$  for a product at calendar age  $u$ , and  $\lambda(t)$  is the original failure rate function of this product.

**Proof:** If there is no failure within the interval  $[t_0, t_0 + T]$ , the warranty cost equals to 0. It equals to the cost of one repair, namely  $C_r(u, \delta)$ , if a failure occurs at some calendar instant  $u \in [t_0, t_0 + T]$ . A failure occurs at an instant  $u$  after  $t_0$  with probability  $\lambda^*(u) du = \lambda(\delta u) du$ . Using the total expectation rule, we get (12).  $\square$

**Note:** For a new product  $t_0 = 0$ . Equation (12) represents the expected warranty cost for an item with an initial age  $t_0$  covered by non-renewing FRW of length  $T$ . In practice warranty coverage is assigned for some future calendar time, and consumers attention is focused mainly on the calendar age of the products. Consumers do not consider the virtual age of the product, even though it could be an important parameter of the performance of the product. From the warranter point of view, the virtual age of the product carries an important information and it is useful in evaluating the warranty expenses.

### 3.3 A mixture of minimal and age-reducing repairs

Most technical items have some prescribed maintenance schedule such that at certain (usually equi-distanced) moments of time preventive check ups are undertaken. For



example: the recommended oil change for motor vehicles after every 3000 miles; the seasonal (every 6 months) preventive check-ups for house heating/cooling systems, and so on. At any of these preventive check ups of the product some adjustments or rectifications are made. This usually contributes to some essential age-reduction like effects. We will assume that these check ups correspond to an age-reducing repair of factor  $\delta$ . Meanwhile, between the scheduled check-ups failures of the item may occur. We assume that the latter are fixed by minimal repairs. Repairs and check-ups are instantaneous.

Thus, we consider the following warranty-maintenance model. The moment of purchase  $t_0$  is a free of charge maintenance check-up. At the expiration of any  $b$  units of calendar time (we call  $b > 0$  the *inter-maintenance time*), a preventive maintenance must be performed (these are the planned check-ups). The age-dependent repair/maintenance check-up of factor  $\delta$  made at calendar age  $u$  will cost  $C_r(u, \delta)$ . All other failures are fixed by minimal repairs (each costs  $C_m$ ), and will not change the virtual failure rate of the product. A FRW is assumed over the time  $[t_0, t_0 + T]$ . We call this model *mixture of minimal and age reducing repairs*.

**Lemma 3** The expected warranty cost associated with a product sold at age  $t_0$  under FRW of duration  $T$  and maintained under a mixture of minimal and age-reducing repairs of factor  $\delta$  is given by the expression

$$\begin{aligned}
C_W(t_0, T, \delta) = & \sum_{k=1}^{\lfloor \frac{T}{b} \rfloor} C_r(t_0 + kb, \delta) \\
& + C_m \left\{ \sum_{k=1}^{\lfloor \frac{T}{b} \rfloor} [\Lambda((t_0 + (k-1)b)\delta + b) - \Lambda((t_0 + kb)\delta)] \right. \\
& \left. + \Lambda\left((t_0 + \lfloor \frac{T}{b} \rfloor b)\delta + T - \lfloor \frac{T}{b} \rfloor b\right) - \Lambda\left((t_0 + \lfloor \frac{T}{b} \rfloor b)\delta\right) \right\}, \tag{13}
\end{aligned}$$

where  $\lfloor \frac{T}{b} \rfloor$  is the smallest integer less than or equal to the number  $\frac{T}{b}$ .

**Proof:** For a fixed value of the inter-maintenance time ( $b > 0$ ), there will be exactly  $\lfloor \frac{T}{b} \rfloor$  check-ups during the warranty period of duration  $T$ . The  $k$ th repair/maintenance will contribute the amount  $C_r(t_0 + kb, \delta)$  to the expected warranty cost. This is the first sum of (13). Further, between the  $k$ th and  $(k+1)$ st preventive check-ups the virtual failure rate of the product is  $\lambda(t)$ ,  $t \in [(t_0 + kb)\delta, (t_0 + kb)\delta + b)$ . Thus, the expected number of failures within the corresponding interval is

$$\Lambda((t_0 + kb)\delta + b) - \Lambda((t_0 + kb)\delta) = \int_{(t_0 + kb)\delta}^{(t_0 + kb)\delta + b} \lambda(t) dt,$$

and each will cost  $C_m$ . This gives the second sum in (13). Similarly, the last (incomplete) period of time,  $\left[ t_0 + \lfloor \frac{T}{b} \rfloor b, t_0 + T \right)$  will contribute

$$\Lambda\left((t_0 + \lfloor \frac{T}{b} \rfloor b)\delta + T - \lfloor \frac{T}{b} \rfloor b\right) - \Lambda\left((t_0 + \lfloor \frac{T}{b} \rfloor b)\delta\right)$$

number of failures. □

This model is appropriate in many practical situations for calculating costs associated with service and maintenance policies of repairable products. Information about services prior to  $t_0$  may be helpful in refining the model.

### 3.4 Warranty costs under renewing warranty

Let us consider a cost model with renewing FRW. The warranter agrees to repair or replace any faulty item up to time  $T$  from the time of purchase, at no cost to the consumer. At the time of each warranty repair the item is warranted anew, i.e. the warranty period  $T$  is renewed. This *renewing FRW* is also known as *unlimited free-replacement warranty* (see Blischke and Murthy 1996, Chapter 10). We will assume, the items have been maintained under age-reducing repairs of factor  $\delta$  before the time of the sale, as in Section 3.2.

**Lemma 4** The expected warranty cost  $C_W(t_0, T)$  associated with a product sold at age  $t_0$  under unlimited FRW of duration  $T$  and maintained under age-reducing repairs of factor  $\delta$ , satisfies the integral equation

$$C_W(t_0, T) = \int_{t_0}^{t_0+T} \lambda^*(u) e^{-[\Lambda^*(u) - \Lambda^*(t_0)]} \cdot [C(u, \delta) + C_W(u, T)] du, \quad (14)$$

with the boundary conditions  $C_W(t_0, 0) = 0$ . Here  $C(u, \delta)$  is the cost per an age-reducing repair of factor  $\delta$  at age  $u$ , and  $\lambda^*(t)$ , and  $\Lambda^*(t)$  are defined in Theorem 2.

**Proof:** If there is no failure within the interval  $[t_0, t_0 + T]$ , the warranty cost equals to 0. It equals to the cost of one repair, namely  $C_r(u, \delta)$ , if the first failure occurs at an instant  $u \in [t_0, t_0 + T)$ , plus the remaining expected warranty cost,  $C_W(u, T, \delta)$ , due to the renewed warranty at age  $u$ . This first failure occurs at an instant  $u$  after  $t_0$  with probability  $\lambda^*(u) e^{-[\Lambda^*(u) - \Lambda^*(t_0)]} du$ . Using the total expectation rule, we obtain (14). □

**Note:** For a new item  $t_0 = 0$ . Equation (14) represents the expected warranty cost for an item with an initial age  $t_0$ , covered by renewing warranty.

## 4 Weibull life-time distribution with an age-reducing repair factor

As an illustration of the models in Section 3, we consider products with the Weibull life-time distribution, i.e.,

$$F(t) = P\{T_1 \leq t\} = 1 - e^{-\left(\frac{t}{\mu}\right)^\alpha}, \quad t \geq 0.$$

Then

$$f(t) = \frac{\alpha}{\mu} \left(\frac{t}{\mu}\right)^{\alpha-1} e^{-\left(\frac{t}{\mu}\right)^\alpha}, \quad t \geq 0,$$

and

$$\lambda(t) = \frac{\alpha}{\mu} \left( \frac{t}{\mu} \right)^{\alpha-1}, \quad t \geq 0.$$

The hazard rate, for this life time is

$$\Lambda(t) = \left( \frac{t}{\mu} \right)^{\alpha}, \quad t \geq 0.$$

Here  $\alpha$  is the shape parameter and  $\mu$  is the scale parameter of the Weibull distribution. For  $\alpha > 1$  the product will experience an increasing in time failure rate. For  $\alpha < 1$  the product has decreasing in time failure rate. If  $\alpha = 1$ , the product has a constant failure rate, i.e. the life-time has an exponential distribution.

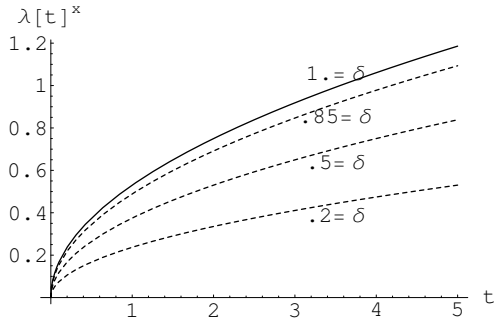
**Corollary 1** If a product has Weibull life-time distribution with parameters  $\alpha$  and  $\mu$  and it is maintained under age-reducing repairs of factor  $\delta > 0$ , then its virtual failure rate and virtual hazard rate are given by the equations

$$\lambda^*(t) = \frac{\alpha}{\mu} \left( \frac{\delta t}{\mu} \right)^{\alpha-1}, \quad t \geq 0, \quad 0 < \delta \leq 1, \quad (15)$$

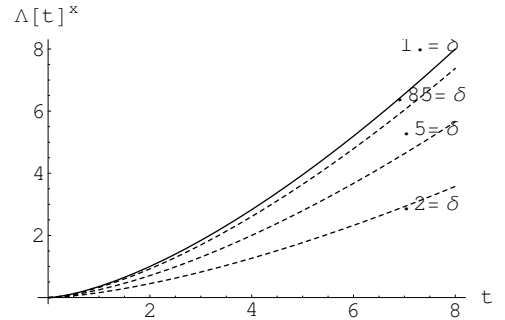
and

$$\Lambda^*(t) = \frac{1}{\delta} \left( \frac{\delta t}{\mu} \right)^{\alpha}, \quad t \geq 0, \quad 0 < \delta \leq 1. \quad (16)$$

**Proof:** Follows for **Theorem 2**.  $\square$



**Fig. 2.a.** Original and virtual failure rates under different age-reducing repair of factors.



**Fig. 2.b.** Original and virtual failure rates under different age-reducing repair of factors.

Fig. 2a illustrates the original failure rate and the virtual failure rates for the parameters  $\alpha = 1.5$ , and  $\mu = 2$ , and under age-reducing factors  $\delta = .2$ ,  $\delta = .5$ , and  $\delta = .85$ . Fig. 2b illustrates the original hazard rate and the virtual hazard rates for the parameters  $\alpha = 1.5$ , and  $\mu = 2$ , and under age-reducing factors  $\delta = .2$ ,  $\delta = .5$ , and  $\delta = .85$ . We see that higher depth of repair (which is equivalent to smaller values of  $\delta$ ), leads to a lower virtual failure rate, and lower virtual hazard rate.

Next, assuming Weibull distribution for the life time of the product, we derive the warranty costs for the models in Section 3 . Assume, the purchase price  $C_0(t_0)$  for a product, sold at age  $t_0$ , is given by the expression

$$C_0(t_0) = C_0 \frac{M - t_0}{M}. \quad (17)$$

This price can be justified in the following way: there is a limiting age,  $M$ , after which the product can not be sold, i.e. if  $t_0 \geq M$  no sales are possible. If the price of a new item is  $C_0$ , then a “fair sale price” at age  $t_0$  should be proportional to the residual time of the sale limit.

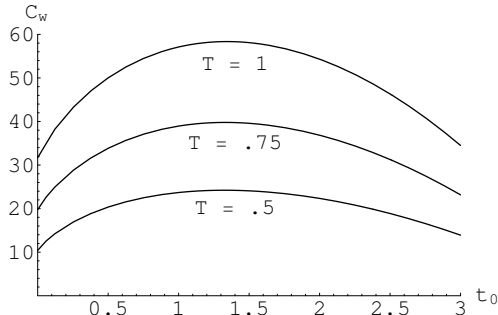
#### 4.1 Rebate warranty of non-repairable product

**Corollary 2** Under the conditions of Corollary 1 and Lemma 1, the expected warranty cost is given by the expression

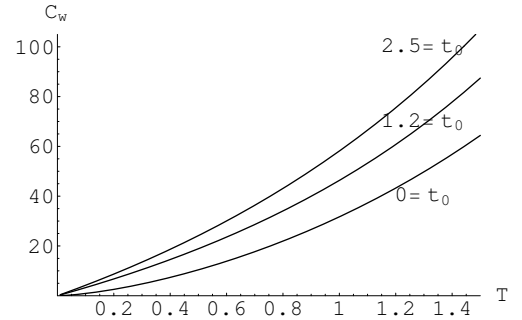
$$C_W(t_0, T) = C_0 \frac{M - t_0}{M} \int_0^T \frac{\alpha}{\mu} \left( \frac{t_0 + t}{\mu} \right)^{\alpha-1} e^{-[(\frac{t_0+t}{\mu})^\alpha - (\frac{t_0}{\mu})^\alpha]} \frac{T - t}{T} dt. \quad (18)$$

**Proof:** Substitute (17), (15), and (16) in (11), and obtain (18).  $\square$

Fig. 3.a illustrates the dependence of  $C_W(t_0, T)$  on the age at the time of purchase  $t_0 \in [0, 3]$ , for the Weibull distribution with parameters  $\alpha = 1.5$ ,  $\mu = 2$  and three different values of the warranty period  $T = .5$ ,  $T = .75$ , and  $T = 1$ . The limiting selling age of the product is  $M = 4$ . The price of a new product is assumed to be  $C_0 = 100$ . The warranty cost increases and after reaching certain threshold value of  $t_0$  decreases. This dependence is justified by the form of the sale price  $C_0(t_0)$  and its interaction with the increasing failure rate. There is an initial age where the warranty costs have a maximum value, and warranter should be aware of it.



**Fig. 3.a.** Dependence of  $C_W(t_0, T)$  on the age  $t_0$  at the time of purchase



**Fig. 3.b.** Dependence of  $C_W(t_0, T)$  on the length  $T$  of the warranty period.

For the same values of the sales/repairs related parameters, Fig. 3.b illustrates the dependence of  $C_W(t_0, T)$  on the length of the warranty period  $T \in [0, 3]$  for the same Weibull distribution and three initial ages  $t_0 = 0$ ,  $t_0 = 1.2$ , and  $t_0 = 2.5$ . We see that the warranty costs increase in both, the selling age  $t_0$ , and the duration  $T$  of the warranty

period. A comparison with the numerical results of Patankar and Mitra (Table 11.4 in Blischke and Murthy 1996), for the pro-rata warranty models with a Weibull distribution of same parameters, may be found useful and curious.

## 4.2 Fixed warranty

Now, let us consider repairable products. Assume that an age-reducing repair at age  $u$  by a factor  $\delta$  costs

$$C_r(u, \delta) = C_0(1 - \delta) \frac{M - u}{M} + C_m, \quad (19)$$

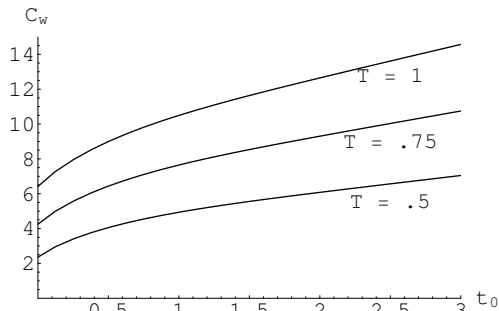
where  $C_0$  is the purchase cost of a new item,  $M$  is the age of the sale limit, and  $C_m$  is the average cost of a minimal repair.

**Corollary 3** Under the conditions of Corollary 1, and conditions of Lemma 2 the expected warranty costs are given by the expression

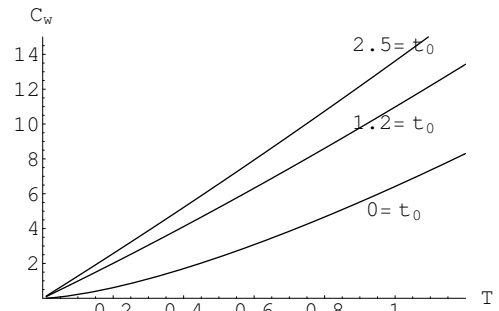
$$C_W(t_0, T, \delta) = \frac{C_0(1 - \delta)\alpha}{\mu} \int_{t_0}^{t_0+T} \frac{M - u}{M} \left(\frac{\delta u}{\mu}\right)^{\alpha-1} e^{-\left(\frac{\delta u}{\mu}\right)^{\alpha \frac{1}{\delta}}} du + \frac{C_m}{\delta} \left( \left(\frac{\delta(t_0 + T)}{\mu}\right)^{\alpha} - \left(\frac{\delta t_0}{\mu}\right)^{\alpha} \right). \quad (20)$$

**Proof:** Substitute (19), (15), and (16) in (12), solve an integral, and obtain (20).  $\square$

Fig. 4.a illustrates the dependence of  $C_W(t_0, T, \delta)$  on the age at the time of purchase  $t_0$ , ( $t_0 \in [0, 3]$ ), for the Weibull distribution with parameters  $\alpha = 1.5$  and  $\mu = 2$ . Three different lengths of the warranty period  $T = .5, .75$ , and  $1.0$  are considered. The limit of the selling age is equal to  $M = 4$ . The initial price of a new product is assumed to be  $C_0 = 100$  and a minimal repair costs  $C_m = 15$ . The age-reducing factor is  $\delta = .95$ , i.e. 5% age-reduction after an age-reducing repair. We see that double the warranty period  $T$  of a new product will lead to quite significant increase of the associated warranty costs. The difference between the warranty costs for the given warranty durations is an increasing function of the initial age of the product.



**Fig. 4.a.** Dependence of  $C_W(t_0, T, \delta)$  on the selling age  $t_0$ .



**Fig. 4.b.** Dependence of  $C_W(t_0, T, \delta)$  on the length  $T$  of the warranty period.

Fig. 4.b shows the dependence of  $C_W(t_0, T, \delta)$  on the length of the warranty period  $T \in [0, 3]$  for  $t_0 = 0, 1.2,$  and  $2.5$ . We observe that even for relatively short warranty periods, the initial age of the product has significant impact on the expected warranty costs. Similarly, based on the expression (20), the dependence of  $C_W(t_0, T, \delta)$  on some other parameters of the model (e.g. the selling age limit, the shape parameter  $\alpha$ , or on the ratio between warranty length  $T$  and the expected life time  $\mathbf{E}(X) = \mu\Gamma(1 + 1/\alpha)$ , where  $\Gamma(\cdot)$  is the Gamma function, etc.) can be studied.

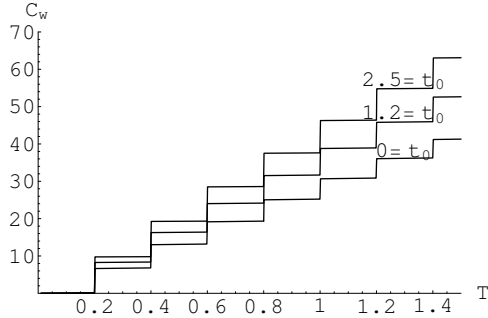
### 4.3 The mixture of minimal and age-reducing repairs

Let us assume that the cost of age-reducing repair at age  $u$  by a factor  $\delta$  is as in (19) and  $M$  is the age of the sale limit. Then:

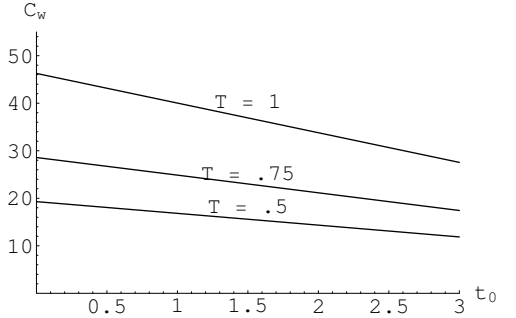
**Corollary 4** Under the conditions of Corollary 1 and Lemma 3, the expected warranty costs are given by the expression

$$\begin{aligned}
C_W(t_0, T, \delta) = & C_0(1 - \delta) \sum_{k=1}^{\lfloor \frac{T}{b} \rfloor} \frac{M - kb - t_0}{M} + \left\lfloor \frac{T}{b} \right\rfloor C_m \\
& + \left[ \sum_{k=1}^{\lfloor \frac{T}{b} \rfloor} \left( \frac{(t_0 + (k-1)b)\delta + b}{\mu} \right)^\alpha - \left( \frac{(t_0 + kb)\delta}{\mu} \right)^\alpha \right] C_m \\
& + \left[ \left( \frac{t_0 + \lfloor \frac{T}{b} \rfloor b}{\mu} \right)^\alpha - \left( \frac{(t_0 + \lfloor \frac{T}{b} \rfloor b)\delta}{\mu} \right)^\alpha \right] C_m.
\end{aligned} \tag{21}$$

**Proof:** Substitute (19), (15), and (16) in (13), and obtain (21).  $\square$



**Fig. 5.a** Dependence of  $C_W(t_0, T, \delta)$  on the warranty time  $T$ .



**Fig. 5.b** Dependence of  $C_W(t_0, T, \delta)$  on the selling age  $t_0$ .

Graphical illustration of the dependence of the expected warranty costs in (21) under mixture of minimal and age-reducing repairs for the Weibull distribution with parameters  $\alpha = 1.5$ ,  $\mu = 2$  is shown in Fig. 5. The age reduction factor is  $\delta = .95$ , the cost for a minimal repair is  $C_m = 15$ , the selling age limit is  $M = 4$ . The initial price of a new product is assumed to be  $C_0 = 100$ . The scheduled check ups are made in any  $b = .2$  time units. Dependence on the warranty period  $T$  and the initial selling age  $t_0$  are shown. Fig. 5.a shows how the expected warranty cost vary with the length of  $T$ . The stepwise-like

shape is due to the additional component to the cost at the time of the check-ups. Fig. 5b. illustrates the dependence of the cost on the initial age  $t_0$ , for three different warranty durations:  $T = .5, .75, \text{ and } 1.0$ . The decrease in the expected warranty costs is due to the impact of the selling price as a function of  $t_0$ . As expected, the expected warranty costs increase as  $T$  increases.

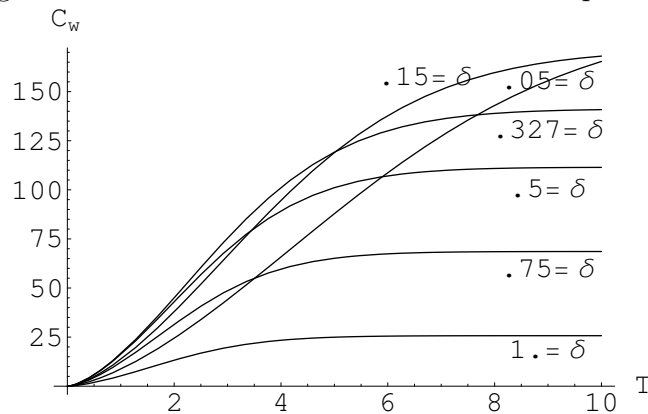
#### 4.4 Renewing warranty

**Corollary 5** Under the conditions of Corollary 1 and Lemma 4, the expected warranty costs for a product sold under unlimited FRW are given as a solution to the integral equation

$$C_W(t_0, T, \delta) = \int_{t_0}^{t_0+T} \frac{\alpha}{\mu} \left( \frac{\delta u}{\mu} \right)^{\alpha-1} e^{-\frac{1}{\delta} \left[ \left( \frac{\delta u}{\mu} \right)^\alpha - \left( \frac{\delta t_0}{\mu} \right)^\alpha \right]} \left[ C_0(1-\delta) \frac{M-u}{M} + C_m + C_W(u, T, \delta) \right] du. \quad (22)$$

**Proof:** Substitute (19), (15), and (16) in (14), and obtain (22).  $\square$

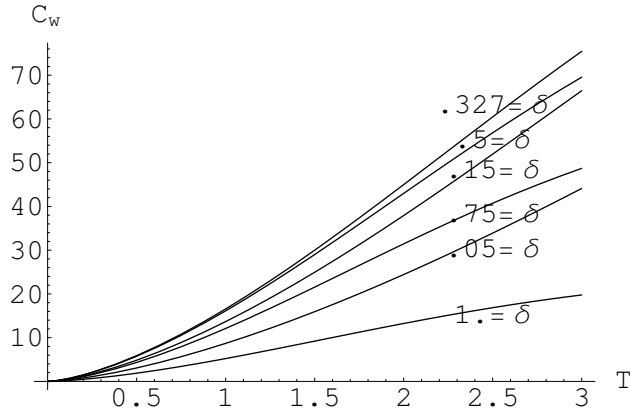
Fig. 6 illustrates the dependence of the expected warranty cost for a new item  $C_W(0, T, \delta)$  on the warranty period  $T$ , under age reducing repairs of factor  $\delta$ . Six values of  $\delta$  (.05, .15, .327, .5, .75 and 1.0) were selected to illustrate the dependence of  $C_W(0, T, \delta)$  on  $\delta$ . Product's life time is assumed to be Weibull with parameters  $\alpha = 1.5$  and  $\mu = 2$ . The age sale limit is  $M = 4$ . The initial price of a new product is assumed to be  $C_0 = 100$  and the cost of a minimal repair is  $C_m = 15$ . For fixed value of  $\delta$ , the expected warranty cost is an increasing function of  $T$ . On the other hand it depends on the value of  $\delta$ .



**Fig. 6.** Dependence of  $C_W(0, T, \delta)$  on the warranty duration  $T$ . and on the value of age-reducing factor  $\delta$ .

Using numerical optimization, for  $T = 3$  (and zooming in Fig.6.), the expected warranty costs as a function of  $\delta$  have a maximum at  $\delta_{max} = .327$ . The existence of  $\delta_{max}$  can be explained by the interruption of a renewing warranty coverage due to the improvement of the product after each age-reducing repair. The improvement of the product is a function of the degree of the repair  $\delta$ . Smaller values of  $\delta$  lead to shorter warranty

coverage and lower expected warranty costs. On the other hand the cost per age-reducing repair at an instant  $u$ ,  $C(u, \delta)$ , is a decreasing function of  $\delta$ .



**Fig. 7.** Dependence of  $C_W(0, T, \delta)$  on the warranty duration  $T$ . and on the value of age-reducing factor  $\delta$ .

The illustration is for a new product, i.e. for  $t_0 = 0$ . However, equation (22) allows study of the dependence of the warranty cost on any initial age  $t_0$ .

## 5 Conclusions

An age-reducing repair model is proposed, and the warranty cost for some typical warranty policies is analyzed. It is shown that the failure rate function and hazard function provide more convenient approach to age-dependent cost modeling than the approach based on the direct usage of the probability distributions of the product life time. Numerical examples illustrate the ideas of the proposed models for the Weibull distributed life time with an age-reducing factor.

## 6 Acknowledgments

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## 7 Appendix A

The proof of the theorem follows the ideas in Block et al. (1985).

**Theorem 3 (Age Transfer Theorem)**  $\{N_t^v, t \geq 0\}$  is a non-homogeneous Poisson process with a leading function

$$\Lambda^v(t) = E(N_t^v) = -\log(1 - F(\frac{t}{\delta})) = \Lambda(\frac{t}{\delta}), \quad (23)$$



where  $\Lambda(t) = -\log(1 - F(t))$  is the leading function of the NPP associated to the maintenance with minimal repairs.

**Proof:** It suffices to show that, (Cinlar (1975), p. 96)

$$\Lambda^v(T_1), \Lambda^v(T_2) - \Lambda^v(T_1), \dots, \Lambda^v(T_{n+1}) - \Lambda^v(T_n)$$

is a sequence of i.i.d's, exponentially distributed r.v. with parameter 1.

For any choice of  $t_1 > 0, \dots, t_{n+1} > 0$  define

$$g(x_1, \dots, x_n) = P\{\Lambda^v(T_1) > t_1, \Lambda^v(T_2) - \Lambda^v(T_1) > t_2, \dots, \\ \Lambda^v(T_{n+1}) - \Lambda^v(T_n) > t_{n+1} \mid T_1 = x_1, \dots, T_n = x_n\}.$$

Denoting the inverse function of  $\bar{F}(t)$  by  $G(t)$  and with reference to (5) we obtain

$$\begin{aligned} P\{\Lambda^v(T_1) > t_1\} &= P\{-\log \bar{F}(\frac{T_1}{\delta}) > t_1\} = P\{\bar{F}(\frac{T_1}{\delta}) < e^{-t_1}\} \\ &= P\{\frac{T_1}{\delta} > G(e^{-t_1})\} = P\{T_1 > \delta G(e^{-t_1})\} \\ &= P\{\frac{X_1}{\delta} > G^{-1}(e^{-t_1}) = \bar{F}(\frac{\delta G(e^{-t_1})}{\delta})\} = e^{-t_1}. \end{aligned}$$

Similarly, equivalent transformations give

$$g(x_1, \dots, x_n) = P\{T_1 > \delta G_\delta(e^{-t_1}), T_2 > \delta G_\delta(e^{-t_2} \bar{F}(\frac{x_1}{\delta})), \dots, T_{n+1} > \delta G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta}))\},$$

where  $G_\delta(t)$  denotes the left-continuous inverse of  $\bar{F}(\frac{t}{\delta})$ . From (4) we get

$$g(x_1, \dots, x_n) = \begin{cases} 0, & \text{if } x_i \leq G_\delta(e^{-t_i} \bar{F}(\frac{x_{i-1}}{\delta})) \\ & \text{for some } i = 2, \dots, n \\ & \text{or if } x_1 \leq G_\delta(e^{-t_1}) \\ \frac{\bar{F}(\max(\frac{x_n}{\delta}, G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta}))))}{\bar{F}(\frac{x_n}{\delta})} = e^{-t_{n+1}} & \text{if } x_i > G_\delta(e^{-t_i} \bar{F}(\frac{x_{i-1}}{\delta})) \\ & \text{for all } i = 2, \dots, n \\ & \text{and } x_1 > G_\delta(e^{-t_1}) \end{cases}$$

In the last equation we have used the following sequence of equivalent representations

$$\begin{aligned} &P\{\Lambda^v(T_{i+1}) - \Lambda^v(T_i) > t_{i+1} \mid T_1 = x_1, \dots, T_n = x_n\} \\ &= \begin{cases} 0, & \text{if } \Lambda^v(x_{i+1}) - \Lambda^v(x_i) \leq t_{i+1}; \\ 1, & \text{if } \Lambda^v(x_{i+1}) - \Lambda^v(x_i) > t_{i+1}. \end{cases} \end{aligned}$$

And also

$$\begin{aligned}
& P\{\Lambda^v(T_{n+1}) - \Lambda^v(T_n) > t_{n+1} \mid T_1 = x_1, \dots, T_n = x_n\} \\
&= P\{-\log \frac{\bar{F}(\frac{T_{n+1}}{\delta})}{\bar{F}(\frac{x_n}{\delta})} > t_{n+1} \mid T_n = x_n\} \\
&= P\{\bar{F}(\frac{T_{n+1}}{\delta}) < e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta}) \mid T_n = x_n\} \\
&= P\{T_{n+1} > \delta G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta})) \mid T_n = x_n\} \\
&= \frac{\bar{F}(\max(\frac{x_n}{\delta}, G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta}))))}{\bar{F}(\frac{x_n}{\delta})} = e^{-t_{n+1}}.
\end{aligned}$$

For the last equality we note that

$$\max(\frac{x_n}{\delta}, G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta}))) = G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta})),$$

since it is true that

$$G_\delta(e^{-t_{n+1}} \bar{F}(\frac{x_n}{\delta})) \geq G_\delta(\bar{F}(\frac{x_n}{\delta})) = \frac{x_n}{\delta}.$$

Finally, from (6) it follows that

$$\begin{aligned}
& P\{\Lambda^v(T_1) > t_1, \Lambda^v(T_2) - \Lambda^v(T_1) > t_2, \dots, \Lambda^v(T_{n+1}) - \Lambda^v(T_n) > t_{n+1}\} \\
&= E[g(T_1, \dots, T_n)] \\
&= e^{-t_{n+1}} \int_0^\infty (\bar{F}(\frac{x_1}{\delta}))^{-1} I_{\{G_\delta(e^{-t_1})\}} \times \\
&\times \int_{x_1}^\infty (\bar{F}(\frac{x_2}{\delta}))^{-1} I_{\{G_\delta(e^{-t_2} \bar{F}(\frac{x_1}{\delta}))\}} \times \dots \times \\
&\times \int_{x_{n-2}}^\infty (\bar{F}(\frac{x_{n-1}}{\delta}))^{-1} I_{\{G_\delta(e^{-t_{n-1}} \bar{F}(\frac{x_{n-2}}{\delta}))\}} \times \\
&\times \int_{x_{n-1}}^\infty I_{\{G_\delta(e^{-t_n} \bar{F}(\frac{x_{n-1}}{\delta}))\}} dF(\frac{x_n}{\delta}) \dots dF(\frac{x_1}{\delta}) \\
&= e^{-(t_1 + \dots + t_{n+1})}.
\end{aligned}$$

This completes the proof of the theorem. □

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