CHAPTER 13

Water-wave modulation

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Abstract

Accurate numerical computations of wave trains with small modulations that grow are described. The range of wave steepness and modulation lengths that give breaking waves are determined. The number of wave crests is reduced for a time at greatest modulation but frequency downshifting is not observed. Near-breaking wave-group structure is always similar, with energy concentrated into one or two wavelengths.

1. Introduction

The ocean surface rarely, if ever, has the form of a uniform periodic travelling wave train. In part, this is due to the instabilities of such waves. The first instability to be found was the Benjamin-Feir (1967) modulational instability. Unlike other instabilities it occurs for wave trains of steepness well below the maximum steepness of steady waves at which crests approach a 120° angle, and so may be especially relevant to the evolution of ocean waves. For more recent theoretical studies of the instability itself see McLean (1982 a,b).

Experiments demonstrate that the instability leads to the formation of strongly modulated wave groups, and further evolution can lead to a recurrence of an approximately uniform wave train. One of the most striking features of experiments is that for sufficiently steep initial waves the recurrence of a uniform wave is accompanied by a decrease in frequency. (Lake et al, 1977, Melville 1982, Su et al 1982). Lake and Yuen (1978) conjecture that this frequency downshifting is related to the decrease in the dominant frequency of wind waves with fetch and duration.

Previous theoretical studies have been made with approximate equations, see Lake et al (1977), Stiassnie and Kroszynski (1982), and Lo and Mei (1985). They show strong modulation and recurrence of uniform waves, but no frequency downshifting. Since these approximations are weakly nonlinear they cannot describe the wave breaking that occurs at strongest modulation.

This work describes numerical computations of the evolution of small amplitude modulations on a uniform wave train. These computations use an efficient program solving for irrotational flow with the exact

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inviscid boundary conditions. The computations can continue to describe the initial overturning stage of wave breaking, and hence clearly indicate when breaking occurs.

2. Computation

The method of computation is based on a boundary-integral solution of Laplace's equation for the complex velocity $\phi_x - i\phi_y$ in two dimensions. It is based on Cauchy's integral theorem for functions of a complex variable $(x + iy)$. The method is briefly described in Dold and Peregrine (1984, 1985): a more detailed account is in preparation.

Initial conditions are chosen for the surface elevation, ζ, and velocity potential, ϕ, and were made up as follows:

(i) an initial uniform wave train on deep water; $\zeta = a \cos x$ is the relevant first approximation, but an accurate steady wave form and potential were used in units with gravity = 1, wavelength = 2π.

(ii) the computational region contained n waves, where $3 \leqslant n \leqslant 10$.

(iii) a perturbation of the form

$$\varepsilon a\left[\cos\left(\frac{n+m}{n}x - \tfrac{1}{4}\pi\right) + \cos\left(\frac{n-m}{n}x - \tfrac{1}{4}\pi\right)\right]$$

was added to ζ together with a corresponding perturbation to ϕ. This gave a modulation length of n/m waves; m was 1 or 2. The phase shift was chosen to be $\tfrac{1}{4}\pi$ since Stiassnie (private communication) indicated this gave the most rapid growth. The value of ε was usually 0.1 and sometimes 0.05. We did not intend to study the growth of infinitesimal perturbations.

Periodic boundary conditions were imposed on the computational region, and the effect of the above initial conditions is to give a weak periodic modulation on the wave train. The time evolution of this modulation was then followed through its initial growth until either:

(a) the computation failed because the wave crest had become too sharply curved for satisfactory resolution, which is a clear indication of the proximity of breaking. By rearrangement of computational points wave overturning could be modelled.

or (b) a near recurrence of the uniform wave state occurred. In some cases more than one recurrence was computed.

Most previous calculations with this general type of program had been for one or two periods involving no more than two waves. The present examples were for up to 10 waves followed for a hundred or more periods. Accuracy was checked in this case by running the same initial conditions with various numbers of initial points and varying the explicit tolerances for iterations and time-stepping within the computation. We aimed to find the most economical values which gave sufficient accuracy. The high-order approximations used in the method permitted as little as 8 computational points per wave and time-steps as large as 0.1 period for the gentler waves. Figure 1 shows part of one such computation just after it has lost accuracy, together with a more accurate profile using 12 points per wavelength from the initial time.

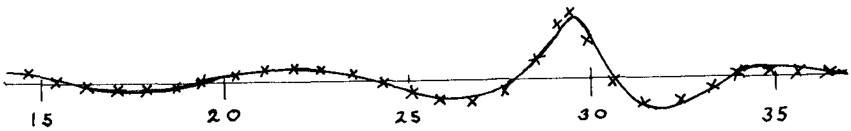


Figure 1. A partial comparison of computations of surface elevation with 8 and 12 points per wave length after 60 linear periods, i.e. $t = 120\pi$. The more accurate profile is shown with a continuous line, the computational discretization is shown for the 8 point profile. The initial steepness is 0.12, and the whole computation includes 2 modulations over 9 wavelengths. The vertical exaggeration in the figure is 5 times. The actual maximum wave slope is 23°.

3. Some examples related to wave breaking

A summary of the calculations is given in figure 2 which indicates whether or not waves grow to breaking. Some of the shorter modulations of 2½ or 3 waves are too short to be unstable. For the instability of just two waves and its evolution see Longuet-Higgins (1978) and Longuet-Higgins and Cokelet (1978).

One important feature of all the computations is that except for a fraction of a wave period before wave breaking the linear-wave concepts of phase velocity and group velocity give a good approximation for any time interval of two or three wave periods. Hence, should anyone wish to compare any of the spatial wave profiles with wave measurements at a fixed point this is possible. The most significant point to be noted is that since for deep water waves group velocity, c_g, is half the phase velocity, c, the number of waves in a modulation on a temporal record is twice the spatial number used here; see figure 6.

Figure 3 is a representative example of cases where the small initial modulation evolved into a steep, short wave group and back to a recurrence of a near uniform wave-train. Slightly more than one modulation length of 5 waves is shown. The wave profiles are given at multiples of 10 or 20 periods since by choosing an interval of $2n$ or $4n$ periods the effects on the wave profile of both phase velocity and linear group velocity lead to a recurrence of the wave and modulation positions.

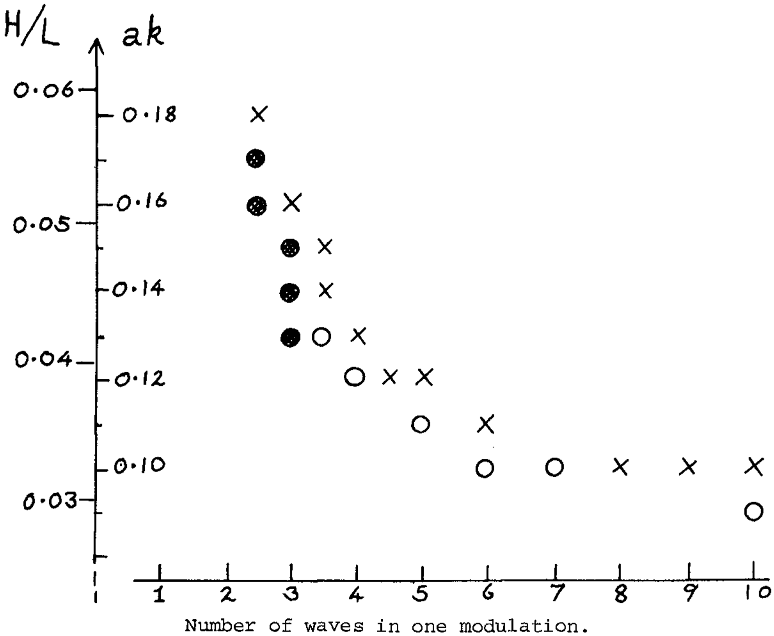


Figure 2. Summary of wave modulation computations. Crosses indicate waves grew to breaking. Open circles indicate modulation grows and uniform waves recur. Filled circles indicate no growth of modulation.

Typical features shown in figure 3 include:

(a) a short group of steep waves at maximum modulation.

(b) waves of close to zero amplitude each side of the steep waves. The modulation does have two precise zeros in the evolution.

(c) a time interval around the time of maximum modulation during which one wave crest disappears.

(d) an advance in the relative position of the peak of the modulation over that predicted by linear group velocity.

(e) a change of relative phase of waves at recurrence, but no lasting frequency downshift which requires the loss of a wave.

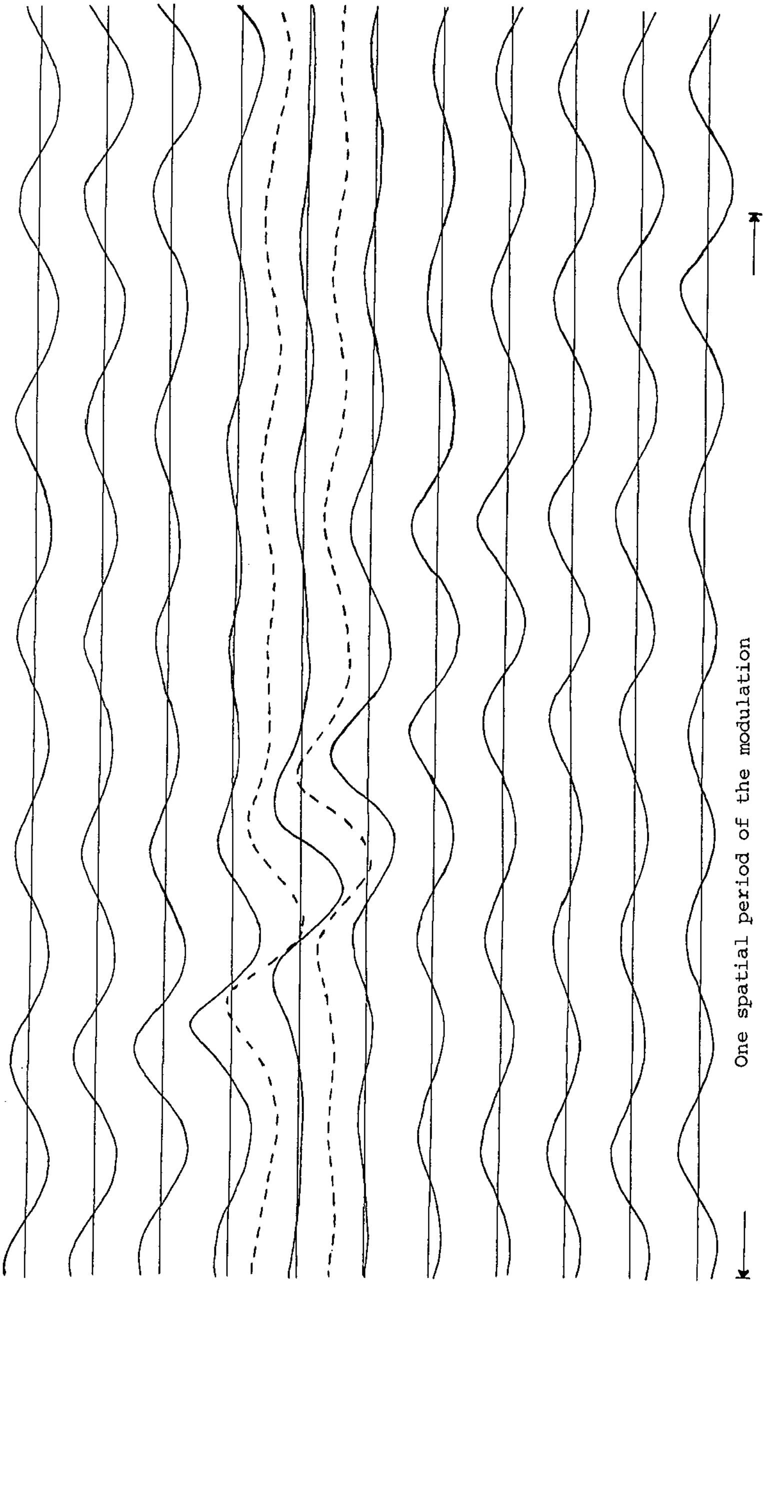


Figure 3. Evolution of the surface profile of a 5 wave modulation shown at intervals of 20 periods with supplementary profiles at 70 and 90 periods. The initial wave steepness is $ak = 0.11$, and the perturbation has $\varepsilon = 0.1$. The vertical exaggeration is 5 times.

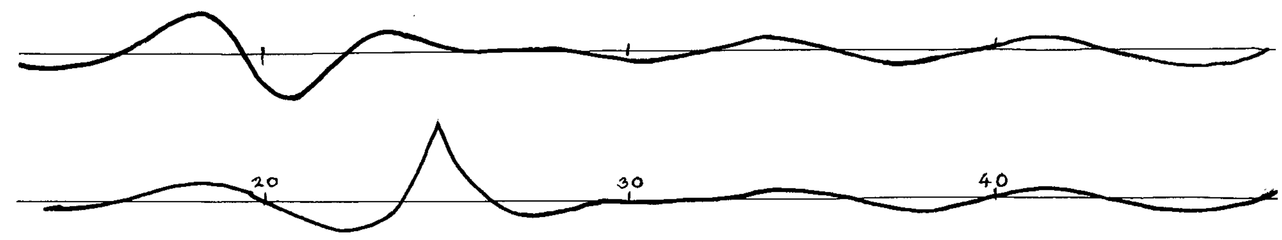


Figure 4. The surface profile of a 5 wave modulation at 60 periods and 61 periods for initial wave steepness $ak = 0.12$ with perturbation $\varepsilon = 0.1$. The vertical exaggeration is 5 times.

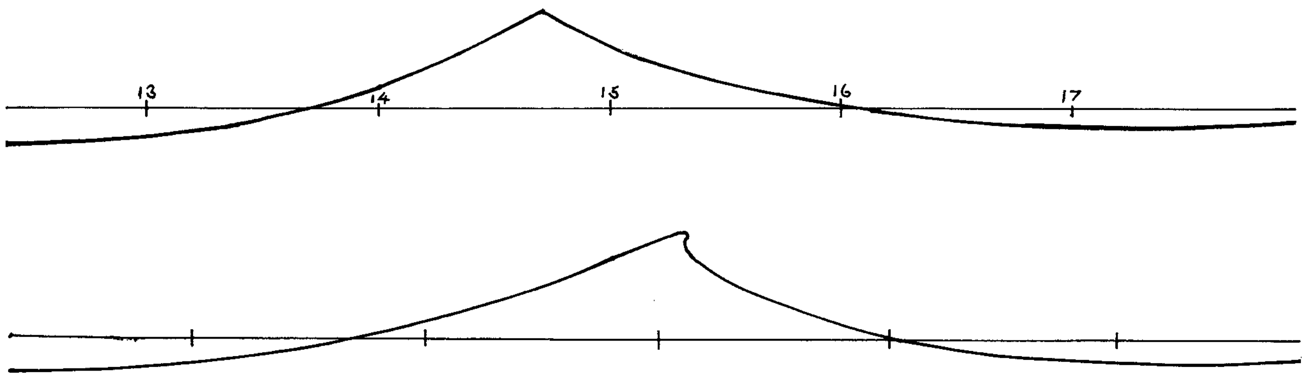


Figure 5. Computed wave profile at "breaking" for a 5 wave modulation with initial steepness $ak = 0.12$. (a) with initial modulation $\varepsilon = 0.1$ (b) with $\varepsilon = 0.05$. No vertical exaggeration.

Those modulations that break differ little from the first part of figure 3. This is illustrated in figure 4 which shows two profiles for the slightly greater initial wave steepness of $ak = 0.12$, compared with $ak = 0.11$ in figure 3. The first profile, at 60 periods, fits almost precisely on the supplementary profile at 70 periods in figure 3. It is significant that the ratio 70:60 is close to $(0.12)^2:(0.11)^2$ which is the relative scaling one would expect from the nonlinear Schrödinger equation. A wave trough is near the maximum point of the modulation.

The second profile in figure 4 is just one period later: the crest of a wave is now at a point close to the peak of the modulation. A very short time later the computation ceased because the wave crest was too sharp. Improved numerical resolution met the same problem. This wave crest is very close to the form of the limiting travelling wave with a corner of 120^o at the crest. The maximum slopes of the waves are close to 30^o. This crest is shown without vertical exaggeration in figure 5.

Also in figure 5 is a breaking wave derived from the same wave train but with the initial perturbation $\varepsilon = 0.05$ instead of $\varepsilon = 0.10$. This produced a slightly more energetic breaker at a later time.

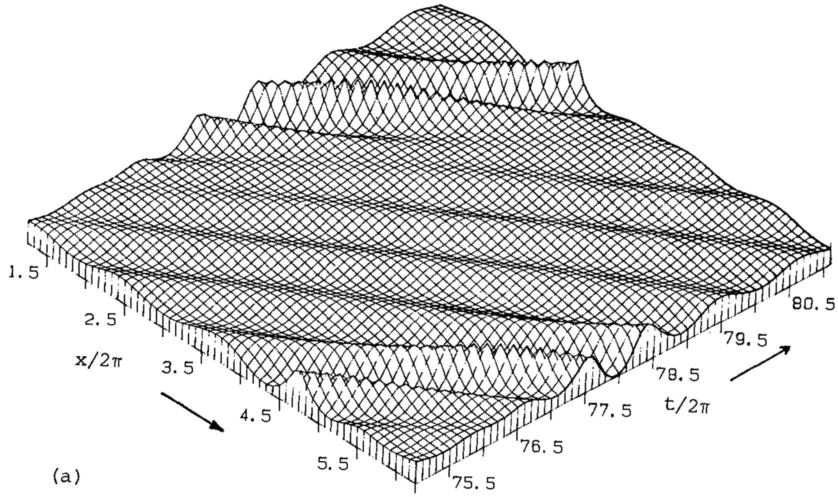
Two isometric plots of the later stages of the computation with $\varepsilon = 0.05$ are chosen to show profiles in the time direction cutting a wave group when the peak modulation is in a trough, figure 6(a) and when it is at a crest, figure 6(b). They also show how each successive wave passes through the energy maximum.

The energy of the modulation continues to be focussed near the peak so that there is eventually one wave which either just breaks or breaks more strongly depending on the relationship between the energy focussing and wave phase. In reality each successive wave passing through such a group can break. This is reported from ocean observations by Donelan et al (1982). In a real sea the relationship between energy focussing and wave phase is likely to be even more variable, so this aspect of wave breaking has not been studied further.

The detailed shape and evolution of the short steep wave group that occurs for some periods before breaking, as illustrated in figure 6, is remarkably similar for a range of initial wave steepnesses and modulations. Profiles chosen at comparable times of wave passage through the groups can be superposed with only slight differences detectable.

4. Frequency downshifting

The problem of frequency downshifting has been examined carefully. In these calculations there is <u>no</u> frequency downshift at recurrence. On the other hand, at maximum modulation there is every sign of downshifting. There is normally one wave crest lost for each modulation: see figure 7 where at the end of a computation with 9 waves and two modulations only seven crests are visible. Similarly the spatial Fourier coefficients show a strong spectral shift to the lower perturbation wavenumber: figure 8 shows the time evolution of the coefficients



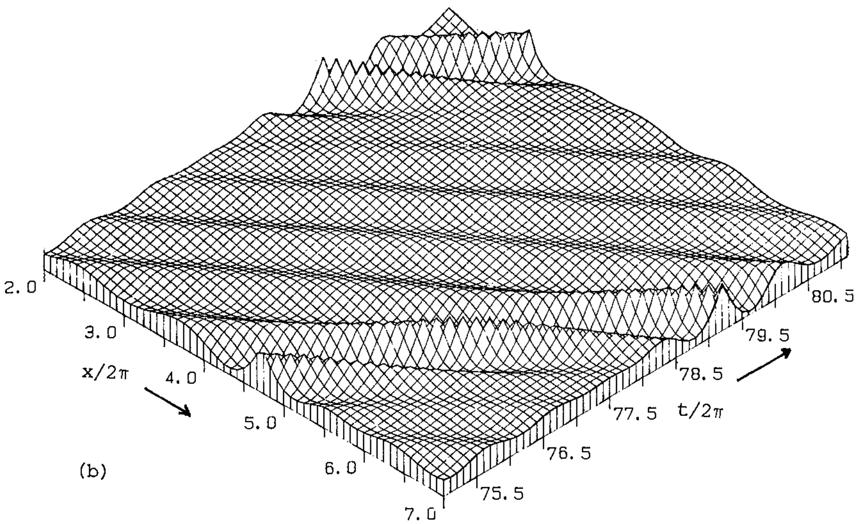


Figure 6. Isometric plots of the last 6 periods before breaking for the 5 wave modulation with initial steepness ak = 0.12, and initial modulation $\epsilon = 0.05$. The "exposed" time profile has a trough at peak modulation in (a), and a crest in (b).

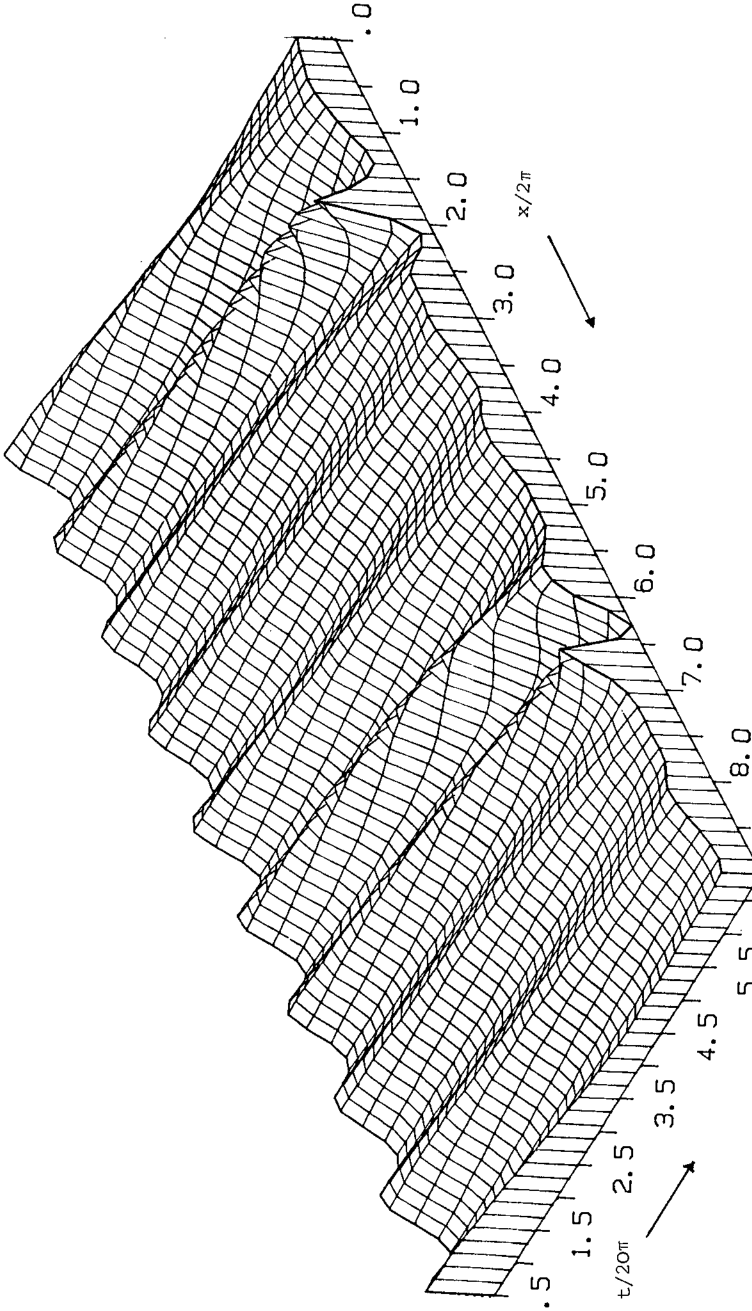


Figure 7. Plot of the time development for 64 periods of two $4\frac{1}{2}$ wave modulations on an initial steepness of ak = 0.12. Phase and group velocity effects are minimised by taking data only every 2 periods. (The apparent roughness of the profiles shown is due to limitations of the plotting routine). Note the transition to 7 crests and 7 troughs.

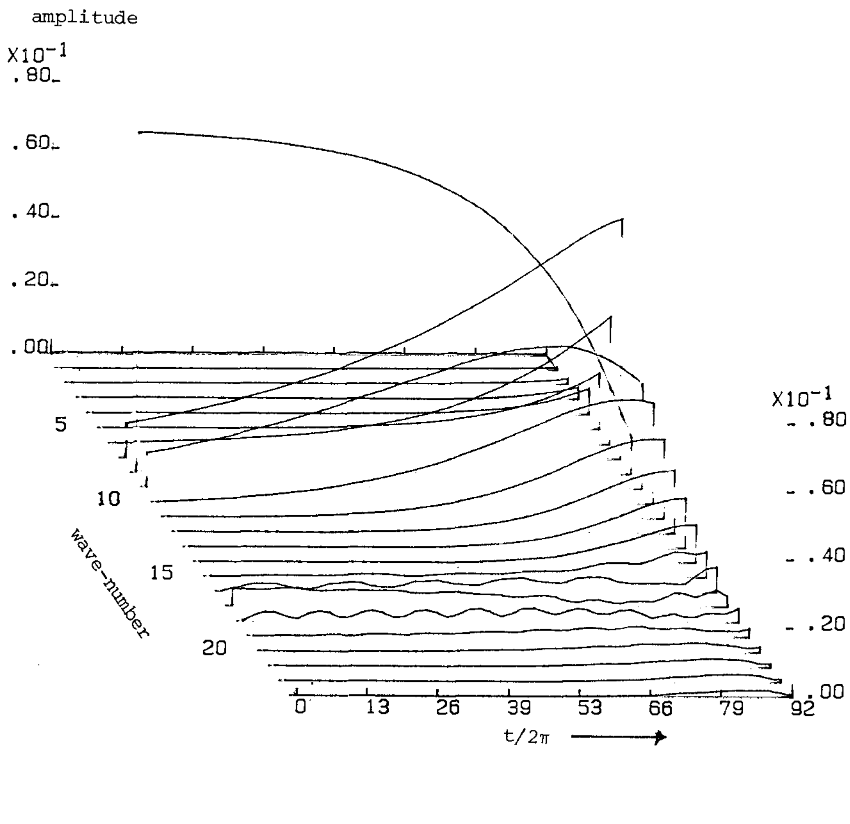


Figure 8. Time evolution of the spatial Fourier coefficients for a 9 wave modulation on waves with initial steepness 0.10. Breaking occurs at 92 periods with most energy in the "8" wave number component.

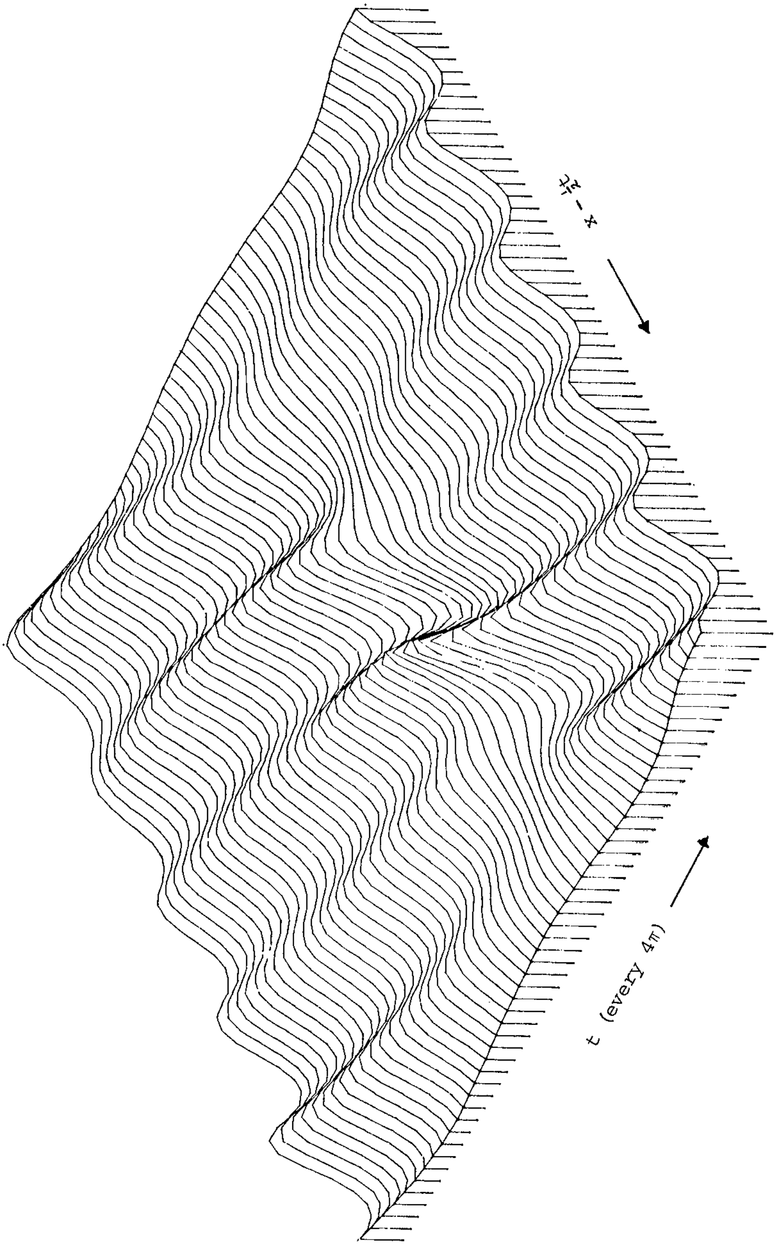


Figure 9. Evolution to recurrence, with linear phase and group velocity effects minimised, of a 5 wave modulation on wave steepness 0.11.

for a single 9-wave modulation. However, when recurrence occurs, as in figures 3 and 9 the original wavelength and period also return.

These computations support the hypothesis that breaking is essential for frequency downshift. This is consistent with experiments. However, waves actually disappear, or reappear, in the vicinity of zeros of wave amplitude. In each case of recurrence two such zeros were found at different times and one less wave can be seen during the time interval between zeros.

5. Conclusions

Computations describe many details of the evolution of weak modulations on a uniform wave train. The boundary of modulations that grow to breaking is outlined in terms of initial wave steepness and modulation length. Wave trains as gentle as $ak = 0.10$, $H/L \doteq 0.03$, can develop into breaking waves. For comparison, the energy density of a wind driven sea is equivalent to a uniform wave train, at the frequency of the spectral peak, of around $ak = 0.13$, and the limiting wave train of maximum steepness has $ak = 0.44$.

The steepest waves all occur in similar short groups a little more than one wavelength long. Such groups endure for around ten or more wave periods.

Although there are indications of frequency downshifting in the computations, the origin of the phenomenon is still obscure.

Comparisons with theory suggest:

(a) for times of a few periods linear theory gives a fair indication of wave behaviour, even for the steepest nonbreaking waves, but wave shape is nonlinear.

(b) for long time intervals, the weakly nonlinear NLS equation gives a qualitatively correct evolution, with quantitative discrepancies which are consistent with Dysthe's (1979) higher order approximation. These comparisons will be reported elsewhere.

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