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WATSON'S METHOD OF SOLVING A QUINTIC EQUATION

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Abstract

Watson's method for determining the roots of a solvable quintic equation in radical form is examined in complete detail. New methods in the spirit of Watson are constructed to cover those exceptional cases to which Watson's original method does not apply, thereby making Watson's method completely general. Examples illustrating the various cases that arise are presented.

1. Introduction

In the 1930's the English mathematician George Neville Watson (1886-1965) devoted considerable effort to the evaluation of singular moduli and class invariants arising in the theory of elliptic functions

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[6]-[11]. These evaluations were given in terms of the roots of polynomial equations whose roots are expressible in terms of radicals. In order to solve those equations of degree 5, Watson developed a method of finding the roots of a solvable quintic equation in radical form. He described his method in a lecture given at Cambridge University in 1948. A commentary on this lecture was given recently by Berndt, Spearman and Williams [1]. This commentary included a general description of Watson's method. However it was not noted by Watson (nor in [1]) that there are solvable quintic equations to which his method does not apply. In this paper we describe Watson's method in complete detail treating the exceptional cases separately, thus making Watson's method applicable to any solvable quintic equation. Several examples illustrating Watson's method are given. Another method of solving the quintic has been given by Dummit [4].

2. Watson's Method

Let $f(x)$ be a monic solvable irreducible quintic polynomial in $\mathbb{Q}[x]$. By means of a linear change of variable we may suppose that the coefficient of x^4 is 0 so that

$$f(x) = x^5 + 10Cx^3 + 10Dx^2 + 5Ex + F \tag{2.1}$$

for some $C, D, E, F \in \mathbb{Q}$. Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{C}$ be the five roots of $f(x)$. The discriminant δ of $f(x)$ is the quantity

$$\delta = \prod_{1 \le j < k \le 5} (x_j - x_k)^2. \tag{2.2}$$

In terms of the coefficients of $f(x)$, δ is given by

$$\begin{aligned}\delta = {} & 8250000C^2D^2F^2 + 500000CE^2F^2 - 375000CDF^3 \\ & - 6750000D^4E^2 - 10000000C^3D^2E^2 - 4500000C^3EF^2 \\ & + 10800000D^5F + 18000000CD^2E^3 + 16000000C^3D^3F \\ & + 14000000C^2DE^2F - 36000000C^4DEF - 31500000CD^3EF\end{aligned}$$

$$+ 1125000D^2EF^2 - 2000000DE^3F + 20000000C^4E^3$$

$$- 8000000C^2E^4 + 800000E^5 + 3125F^4 + 10800000C^5F^2. \quad (2.3)$$

As $f(x)$ is solvable and irreducible, we have [4, p. 390]

$$\delta > 0. \quad (2.4)$$

We set

$$K = E + 3C^2, \quad (2.5)$$

$$L = -2DF + 3E^2 - 2C^2E + 8CD^2 + 15C^4, \quad (2.6)$$

$$M = CF^2 - 2DEF + E^3 - 2C^2DF - 11C^2E^2$$

$$+ 28CD^2E - 16D^4 + 35C^4E - 40C^3D^2 - 25C^6. \quad (2.7)$$

Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{C}$ be the five roots of $f(x)$. Cayley [2] has shown that

$$\phi_1 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 - x_1x_3 - x_3x_5 - x_5x_2 - x_2x_4 - x_4x_1,$$

$$\phi_2 = x_1x_3 + x_3x_4 + x_4x_2 + x_2x_5 + x_5x_1 - x_1x_4 - x_4x_5 - x_5x_3 - x_3x_2 - x_2x_1,$$

$$\phi_3 = x_1x_4 + x_4x_2 + x_2x_3 + x_3x_5 + x_5x_1 - x_1x_2 - x_2x_5 - x_5x_4 - x_4x_3 - x_3x_1,$$

$$\phi_4 = x_1x_2 + x_2x_5 + x_5x_3 + x_3x_4 + x_4x_1 - x_1x_5 - x_5x_4 - x_4x_2 - x_2x_3 - x_3x_1,$$

$$\phi_5 = x_1x_3 + x_3x_5 + x_5x_4 + x_4x_2 + x_2x_1 - x_1x_5 - x_5x_2 - x_2x_3 - x_3x_4 - x_4x_1,$$

$$\phi_6 = x_1x_4 + x_4x_5 + x_5x_2 + x_2x_3 + x_3x_1 - x_1x_5 - x_5x_3 - x_3x_4 - x_4x_2 - x_2x_1,$$

(2.8)

are the roots of

$$g(x) = x^6 - 100Kx^4 + 2000Lx^2 - 32\sqrt{\delta}x + 40000M \in \mathbb{Q}(\sqrt{\delta})[x]. \quad (2.9)$$

Watson [1] has observed as $f(x)$ is solvable and irreducible that $g(x)$ has a root of the form $\phi = \rho\sqrt{\delta}$, where $\rho \in \mathbb{Q}$, so that $\phi \in \mathbb{Q}(\sqrt{\delta})$. We set

$$\theta = \frac{\phi\sqrt{5}}{50} \in \mathbb{Q}(\sqrt{5\delta}) \subseteq \mathbb{R}. \quad (2.10)$$

Clearly θ is a root of

$$h(x) = x^6 - \frac{K}{5}x^4 + \frac{L}{125}x^2 - \frac{\sqrt{5}\delta}{390625}x + \frac{M}{3125}. \tag{2.11}$$

The following simple lemma enables us to determine the solutions of a quintic equation.

Lemma. *Let* $C, D, E, F \in \mathbb{Q}$. *If* $u_1, u_2, u_3, u_4 \in \mathbb{C}$ *are such that*

$$u_1u_4 + u_2u_3 = -2C, \tag{2.12}$$

$$u_1u_2^2 + u_2u_4^2 + u_3u_1^2 + u_4u_3^2 = -2D, \tag{2.13}$$

$$u_1^2u_4^2 + u_2^2u_3^2 - u_1^3u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3 - u_1u_2u_3u_4 = E, \tag{2.14}$$

$$u_1^5 + u_2^5 + u_3^5 + u_4^5 - 5(u_1u_4 - u_2u_3)(u_1^2u_3 - u_2^2u_1 - u_3^2u_4 + u_4^2u_2) = -F, \tag{2.15}$$

then the five roots of $f(x) = 0$ *are*

$$x = \omega u_1 + \omega^2 u_2 + \omega^3 u_3 + \omega^4 u_4, \tag{2.16}$$

where ω *runs through the fifth roots of unity.*

Proof. This follows from the identity

$$(\omega u_1 + \omega^2 u_2 + \omega^3 u_3 + \omega^4 u_4)^5 - 5U(\omega u_1 + \omega^2 u_2 + \omega^3 u_3 + \omega^4 u_4)^3$$

$$- 5V(\omega u_1 + \omega^2 u_2 + \omega^3 u_3 + \omega^4 u_4)^2 + 5W(\omega u_1 + \omega^2 u_2 + \omega^3 u_3 + \omega^4 u_4)$$

$$+ 5(X - Y) - Z = 0,$$

where

$$U = u_1u_4 + u_2u_3,$$

$$V = u_1u_2^2 + u_2u_4^2 + u_3u_1^2 + u_4u_3^2,$$

$$W = u_1^2u_4^2 + u_2^2u_3^2 - u_1^3u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3 - u_1u_2u_3u_4,$$

$$X = u_1^3u_3u_4 + u_2^3u_1u_3 + u_3^3u_2u_4 + u_4^3u_1u_2,$$

$$Y = u_1u_3^2u_4^2 + u_2u_1^2u_3^2 + u_3u_2^2u_4^2 + u_4u_1^2u_2^2,$$

$$Z = u_1^5 + u_2^5 + u_3^5 + u_4^5,$$

see for example [5, p. 987].

If $\theta \neq 0, \pm C$ Watson's method of determining the roots of $f(x) = 0$ in radical form is given in the next theorem.

Theorem 1. *Let $f(x)$ be the solvable irreducible quintic polynomial* (2.1). *Suppose that $\theta \neq 0, \pm C$. Set*

$$\begin{aligned} p(T) = T^4 &+ (-14C\theta^2 - 2D^2 + 2CE - 2C^3)T^2 + 16D\theta^3 T \\ &+ (-25\theta^6 + (35C^2 + 6E)\theta^4 + (-11C^4 - 2CD^2 - 4C^2E - E^2)\theta^2 \\ &+ (C^6 + 2C^3D^2 - 2CD^2E - 2C^4E + C^2E^2 + D^4)) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} q(T) = -CT^3 - D\theta T^2 &+ (25\theta^4 - (10C^2 + E)\theta^2 + C^4 + CD^2 - C^2E)T \\ &+ (-F\theta^3 + (-2CDE + C^2F + D^3)\theta). \end{aligned} \tag{2.18}$$

Then the pair of equations

$$p(T) = q(T) = 0 \tag{2.19}$$

has at least one solution $T \in \mathbb{C}$, which is expressible by radicals. Set

$$R_1 = \sqrt{(D - T)^2 + 4(C - \theta)^2(C + \theta)} \tag{2.20}$$

and

$$R_2 = \begin{cases} \dfrac{C(D^2 - T^2) + (C^2 - \theta^2)(C^2 + 3\theta^2 - E)}{R_1\theta}, & \text{if } R_1 \neq 0, \\ \sqrt{(D + T)^2 + 4(C + \theta)^2(C - \theta)}. & \text{if } R_1 = 0. \end{cases} \tag{2.21}$$

If $R_1 \neq 0$, then we have

$$R_2 = \pm\sqrt{(D + T)^2 + 4(C + \theta)^2(C - \theta)}. \tag{2.22}$$

Set

$$X = \frac{1}{2}(-D + T + R_1), \quad \overline{X} = \frac{1}{2}(-D + T - R_1), \tag{2.23}$$

$$Y = \frac{1}{2}(-D - T + R_2), \quad \overline{Y} = \frac{1}{2}(-D - T - R_2), \tag{2.24}$$

$$Z = -C - \theta \neq 0, \quad \overline{Z} = -C + \theta \neq 0. \tag{2.25}$$

Let u_1 *be any fifth root of* X^2Y/Z^2. *Set*

$$u_2 = \frac{\overline{X}}{\overline{Z}^2}u_1^2, \quad u_3 = \frac{\overline{X}\overline{Y}}{Z\overline{Z}^3}u_1^3, \quad u_4 = \frac{\overline{X}^2\overline{Y}}{Z^2\overline{Z}^4}u_1^4. \tag{2.26}$$

Then the five roots of $f(x) = 0$ *are given in radical form by* (2.16).

The proof of Theorem 1 is given in Section 3. Theorem 1 does not apply when (i) $\theta = 0$, since in this case R_2 is not always defined; when (ii) $\theta = C$, since $\overline{Z} = 0$ and u_2, u_3, u_4 are not defined; and when (iii) $\theta = -C$, since $Z = 0$ and u_1, u_3, u_4 are not defined. These excluded cases were not covered by Watson [1] and are given in Theorems 2, 3, 4. Theorem 2 covers $\theta = \pm C \neq 0$, Theorem 3 covers $\theta = C = 0$, and Theorem 4 covers $\theta = 0, C \neq 0$.

Theorem 2. *Let* $f(x)$ *be the solvable irreducible quintic polynomial* (2.1). *Suppose that* $\theta = \pm C \neq 0$. *Set*

$$r(T) = T^3 + DT^2 + (-16C^3 + 2CE - D^2)T + (2CDE - D^3) \tag{2.27}$$

and

$$\begin{aligned} s(T) = {} & T^6 + (88C^3 - 3D^2)T^4 + (112C^3D - 4C^2F)T^3 \\ & + (-64C^6 - 32C^3D^2 - 4C^2DF + 3D^4)T^2 \\ & + (128C^6D - 48C^3D^3 + 4C^2D^2F)T \\ & + (-64C^6D^2 + 8C^3D^4 + 4C^2D^3F - D^6). \end{aligned} \tag{2.28}$$

Then the pair of equations

$$r(T) = s(T) = 0 \tag{2.29}$$

has at least one solution $T \in \mathbb{C}$ *expressible by radicals.*

(a) *If* $D \neq \pm T$, *we let* u_1 *be any fifth root of*

$$\begin{cases} \dfrac{-(D-T)^2(D+T)}{4C^2}, & \text{if } \theta = C, \\ \dfrac{-16C^4(D-T)}{(D+T)^2}, & \text{if } \theta = -C, \end{cases} \tag{2.30}$$

and set

$$u_2 = \begin{cases} \dfrac{2C}{(D-T)} u_1^2, & \text{if } \theta = C, \\ \dfrac{-(D+T)}{4C^2} u_1^2, & \text{if } \theta = -C, \end{cases} \tag{2.31}$$

$$u_3 = \begin{cases} \dfrac{4C^2}{D^2 - T^2} u_1^3, & \text{if } \theta = C, \\ 0, & \text{if } \theta = -C, \end{cases} \tag{2.32}$$

$$u_4 = \begin{cases} 0, & \text{if } \theta = C, \\ \dfrac{(D+T)^2}{8(D-T)C^3} u_1^4, & \text{if } \theta = -C. \end{cases} \tag{2.33}$$

(b) *If* $D = \pm T$, *then*

$$D = 0 \tag{2.34}$$

and either

$$\text{(i) } E = 4C^2 \text{ or (ii) } E = \frac{16C^3 - Z^2}{2C}, \ F = \frac{64C^6 - 88C^3Z^2 - Z^4}{4C^2Z} \tag{2.35}$$

for some $Z \in \mathbb{Q}$, $Z \neq 0$. *Let* u_1 *be any fifth root of*

$$\begin{cases} \dfrac{-F + \sqrt{128C^5 + F^2}}{2}, & \text{(i)}, \\ 2CZ, & \text{(ii)}, \end{cases} \tag{2.36}$$

and set

$$u_2 = 0, \tag{2.37}$$

$$u_3 = \begin{cases} 0, & \text{(i)}, \\ \dfrac{1}{2C} u_1^3, & \text{(ii)}, \end{cases} \tag{2.38}$$

$$u_4 = \begin{cases} \dfrac{1}{16C^4}\left(\dfrac{-F-\sqrt{128C^5+F^2}}{2}\right)u_1^4, & \text{(i)}, \\ -\dfrac{1}{Z}u_1^4, & \text{(ii)}. \end{cases} \tag{2.39}$$

Then in both cases (a) *and* (b) *the five roots of* $f(x) = 0$ *are given by* (2.16).

Theorem 3. *Let* $f(x)$ *be the solvable irreducible quintic polynomial* (2.1). *Suppose that* $\theta = C = 0$. *In this case*

$$\text{(i) } D = E = 0 \ \ \textit{or} \ \ \text{(ii) } D \neq 0,\ E \neq 0. \tag{2.40}$$

In case (ii) *we have*

$$F = \frac{E^3 - 16D^4}{2DE}. \tag{2.41}$$

Let u_1 *be any fifth root of*

$$\begin{cases} -F, & \text{(i)}, \\ \dfrac{8D^3}{E}, & \text{(ii)}. \end{cases} \tag{2.42}$$

Set

$$u_2 = u_4 = 0 \tag{2.43}$$

and

$$u_3 = \begin{cases} 0, & \text{(i)}, \\ -\dfrac{E}{4D^2}u_1^3, & \text{(ii)}. \end{cases} \tag{2.44}$$

Then the five roots of $f(x) = 0$ *are given by* (2.16).

Theorem 4. *Let* $f(x)$ *be the solvable irreducible quintic polynomial* (2.1). *Suppose that* $\theta = 0$, $C \neq 0$. *Set*

$$T = \sqrt{C^3 - CE + D^2}. \tag{2.45}$$

Then

$$T^2((D-T)^2 + 4C^3)((D+T)^2 + 4C^3)$$

$$= (2C^3D - C^2F + D^3 - DT^2)^2. \tag{2.46}$$

Let $R_1, R_2 \in \mathbb{C}$ *be such that*

$$R_1^2 = (D - T)^2 + 4C^3, \tag{2.47}$$

$$R_2^2 = (D + T)^2 + 4C^3, \tag{2.48}$$

$$TR_1R_2 = 2C^3D - C^2F + D^3 - DT^2. \tag{2.49}$$

Set

$$X = \frac{1}{2}(-D + T + R_1), \quad \overline{X} = \frac{1}{2}(-D + T - R_1), \tag{2.50}$$

$$Y = \frac{1}{2}(-D - T + R_2), \quad \overline{Y} = \frac{1}{2}(-D - T - R_2). \tag{2.51}$$

Let u_1 *be any fifth root of* $\frac{X^2Y}{C^2}$. *Set*

$$u_2 = \frac{\overline{X}}{C^2}u_1^2, \quad u_3 = \frac{\overline{X}\overline{Y}}{C^4}u_1^3, \quad u_4 = \frac{\overline{X}^2\overline{Y}}{C^6}u_1^4. \tag{2.52}$$

Then the five roots of $f(x) = 0$ *are given by* (2.16).

3. Proof of Theorem 1

If $C \neq 0$ or $D \neq 0$ or $25\theta^4 - (10C^2 + E)\theta^2 + (C^4 + CD^2 - C^2E) \neq 0$, then the polynomial $q(T)$ is non-constant and the resultant $R(p, q)$ of p and q is

$$R(p, q) = 5^{10}\theta^4(C^2 - \theta^2)^3h(\theta)h(-\theta). \tag{3.1}$$

As $h(\theta) = 0$, we have $R(p, q) = 0$. Thus (2.19) has at least one solution $T \in \mathbb{C}$, which is expressible by radicals as it is a root of the quartic polynomial $p(T)$.

On the other hand if $C = D = 25\theta^4 - (10C^2 + E)\theta^2 + (C^4 + CD^2 - C^2E) = 0$, then we show that $q(T)$ is identically zero and the assertion remains valid. In this case $25\theta^4 - E\theta^2 = 0$. As $\theta \neq 0$ we have $\theta^2 = E/25$. Thus (2.3), (2.5), (2.6) and (2.7) give

$$\delta = 2^8 5^5 E^5 + 5^5 F^4, \quad K = E, \quad L = 3E^2, \quad M = E^3.$$

Hence

$$h(x) = x^6 - \frac{E}{5}x^4 + \frac{3E^2}{5^3}x^2 - \frac{(2^8E^5 + F^4)^{1/2}}{5^5}x + \frac{E^3}{5^5}.$$

Thus

$$0 = h(\theta) = \frac{2^4E^3}{5^6} \pm \frac{(2^8E^6 + EF^4)^{1/2}}{5^6}.$$

As $E \neq 0$ we deduce that $F = 0$ proving that $q(T) \equiv 0$.

Multiplying $p(T) = 0$ by $C^2 - \theta^2$ ($\neq 0$) and rearranging, we obtain

$$(C(D^2 - T^2) + (C^2 - \theta^2)(C^2 + 3\theta^2 - E))^2$$
$$= \theta^2((D - T)^2 + 4(C - \theta)^2(C + \theta))((D + T)^2 + 4(C + \theta)^2(C - \theta)). \qquad (3.2)$$

Define R_1 and R_2 as in (2.20) and (2.21). If $R_1 \neq 0$, as $\theta \neq 0$, then we deduce from (3.2) that

$$R_2^2 = (D + T)^2 + 4(C + \theta)^2(C - \theta),$$

which is (2.22). Define $X, \overline{X}, Y, \overline{Y}, Z, \overline{Z}$ as in (2.23), (2.24) and (2.25). Clearly

$$X + \overline{X} + Y + \overline{Y} = -2D, \quad X + \overline{X} - Y - \overline{Y} = 2T. \qquad (3.3)$$

From (2.20), (2.23) and (2.25) we deduce that

$$X\overline{X} = Z\overline{Z}^2. \qquad (3.4)$$

From (2.21), (2.22), (2.24) and (2.25) we deduce that

$$Y\overline{Y} = Z^2\overline{Z}. \qquad (3.5)$$

From (2.23), (2.24) and (2.25) we obtain

$$Z^2 - Z\overline{Z} + \overline{Z}^2 - \left(\frac{XY}{Z} + \frac{\overline{X}Y}{\overline{Z}} + \frac{X\overline{Y}}{\overline{Z}} + \frac{\overline{X}\overline{Y}}{Z}\right)$$
$$= \frac{-C^4 + 3\theta^4 - 2C^2\theta^2 + R_1R_2\theta - CD^2 + CT^2}{-(C^2 - \theta^2)}.$$

Appealing to (2.20) and (3.2) if $R_1 = 0$ and to (2.21) if $R_1 \neq 0$, then we deduce that

$$Z^2 - Z\overline{Z} + \overline{Z}^2 - \left(\frac{XY}{Z} + \frac{\overline{X}Y}{\overline{Z}} + \frac{X\overline{Y}}{\overline{Z}} + \frac{\overline{X}\overline{Y}}{Z}\right) = E. \tag{3.6}$$

From (2.20)-(2.25) we obtain

$$\frac{X^2Y}{Z^2} + \frac{\overline{X}Y^2}{\overline{Z}^2} + \frac{X\overline{Y}^2}{\overline{Z}^2} + \frac{\overline{X}^2\overline{Y}}{Z^2} - 20T\theta$$

$$= \frac{q(T) + F\theta^3 - C^2F\theta}{(C^2 - \theta^2)\theta} = -F. \tag{3.7}$$

Now define u_1, u_2, u_3, u_4 by (2.26). As $Z, \overline{Z} \neq 0$ we deduce from (3.4) and (3.5) that $X, \overline{X}, Y, \overline{Y} \neq 0$. Further, by (3.4) and (3.5), we have

$$\frac{\overline{X}\overline{Y}}{Z\overline{Z}^3} = \frac{Z^2}{XY}, \quad \frac{\overline{X}^2\overline{Y}}{Z^2\overline{Z}^4} = \frac{\overline{Y}}{X^2},$$

so that

$$u_3 = \frac{Z^2}{XY}u_1^3, \quad u_4 = \frac{\overline{Y}}{X^2}u_1^4.$$

Then

$$u_1u_4 = \frac{Y\overline{Y}}{Z^2} = \overline{Z}, \quad u_2u_3 = \frac{X\overline{X}}{\overline{Z}^2} = Z \tag{3.8}$$

and

$$u_2^5 = \frac{\overline{X}Y^2}{\overline{Z}^2}, \quad u_3^5 = \frac{X\overline{Y}^2}{\overline{Z}^2}, \quad u_4^5 = \frac{\overline{X}^2\overline{Y}}{Z^2}. \tag{3.9}$$

Hence

$$u_1u_4 + u_2u_3 = Z + \overline{Z} = -2C,$$

which is (2.12). Next

$$u_1^2u_3 \pm u_2^2u_1 \pm u_3^2u_4 + u_4^2u_2$$

$$= u_1^5 \frac{Z^2}{XY} \pm u_1^5 \frac{\overline{X}^2}{\overline{Z}^4} \pm u_1^{10} \frac{Z^4\overline{Y}}{X^4Y^2} + u_1^{10} \frac{\overline{Y}^2\overline{X}}{X^4\overline{Z}^2}$$

$$= X \pm Y \pm \overline{Y} + \overline{X},$$

that is,

$$u_1^2u_3 \pm u_2^2u_1 \pm u_3^2u_4 + u_4^2u_2 = \begin{cases} -2D, & \text{with + signs,} \\ 2T, & \text{with − signs.} \end{cases} \tag{3.10}$$

The first of these is (2.13).

Further

$$u_1^2u_4^2 + u_2^2u_3^2 - u_1u_2u_3u_4 - u_1^3u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3$$

$$= u_1^{10}\left(\frac{\overline{Y}^2}{X^4} + \frac{Z^6}{X^4Y^2} - \frac{Z^3\overline{Y}}{X^4Y}\right)$$

$$-\left(u_1^5 \frac{Z}{X} + u_1^{10} \frac{Z^3\overline{Y}}{X^5} + u_1^{10} \frac{Z^6}{X^3Y^3} + u_1^{15} \frac{\overline{Y}^3Z^2}{X^7Y}\right)$$

$$= \overline{Z}^2 + Z^2 - Z\overline{Z} - \left(\frac{XY}{Z} + \frac{\overline{X}Y}{\overline{Z}} + \frac{X\overline{Y}}{\overline{Z}} + \frac{\overline{X}\overline{Y}}{Z}\right)$$

$$= E,$$

which is (2.14).

Finally, from (3.7), (3.8), (3.9) and (3.10), we obtain

$$u_1^5 + u_2^5 + u_3^5 + u_4^5 - 5(u_1u_4 - u_2u_3)(u_1^2u_3 - u_2^2u_1 - u_3^2u_4 + u_4^2u_2)$$

$$= \frac{X^2Y}{Z^2} + \frac{\overline{X}Y^2}{\overline{Z}^2} + \frac{X\overline{Y}^2}{\overline{Z}^2} + \frac{\overline{X}^2\overline{Y}}{Z^2} - 20T\theta$$

$$= -F,$$

which is (2.15). By the Lemma the roots of $f(x) = 0$ are given by (2.16). As θ and T are expressible by radicals so are R_1, R_2, X, $\overline{X}$, Y, $\overline{Y}$, Z, $\overline{Z}$. Hence u_1, u_2, u_3, u_4 are expressible by radicals. Thus the roots x_1, x_2, x_3, x_4, x_5 of $f(x) = 0$ are expressible by radicals.

4. Proof of Theorem 2

Using MAPLE we find that

$$R(r, s) = -\frac{5^{15}}{C^2} h(C)h(-C) = 0, \tag{4.1}$$

as $\theta = \pm C$, so that there is at least one solution $T \in \mathbb{C}$ of (2.29). As T is a root of a cubic equation, T is expressible in terms of radicals.

If $D \neq \pm T$, then we define u_1, u_2, u_3, u_4 as in (2.30)-(2.33). Thus

$$u_1u_4 + u_2u_3 = \begin{cases} \dfrac{8C^3u_1^5}{(D-T)^2(D+T)}, & \text{if } \theta = C, \\ \dfrac{(D+T)^2u_1^5}{8C^3(D-T)}, & \text{if } \theta = -C, \end{cases}$$

$$= -2C,$$

which is (2.12).

Next

$$u_1u_2^2 + u_2u_4^2 + u_3u_1^2 + u_4u_3^2 = \begin{cases} \dfrac{8C^2Du_1^5}{(D-T)^2(D+T)}, & \text{if } \theta = C, \\ \dfrac{4C^2u_1^5}{(D-T)^2} + \dfrac{(D+T)^4u_1^{10}}{32C^5(D-T)^3}, & \text{if } \theta = -C, \end{cases}$$

$$= -2D,$$

which is (2.13).

Further, for both $\theta = C$ and $\theta = -C$, we have

$$u_1^2u_4^2 + u_2^2u_3^2 - u_1^3u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3 - u_1u_2u_3u_4$$

$$= 4C^2 + \frac{(D^2 - T^2)}{2C} - 4C^2\frac{(D-T)}{(D+T)} = E,$$

by (2.27) and (2.29), which is (2.14).

Finally, using (2.30)-(2.33), we obtain

$$u_1^5 + u_2^5 + u_3^5 + u_4^5 - 5(u_1u_4 - u_2u_3)(u_1^2u_3 - u_2^2u_1 - u_3^2u_4 + u_4^2u_2)$$

$$= \begin{cases} u_1^5 + u_2^5 + u_3^5 - 20CT, & \text{if } \theta = C, \\ u_1^5 + u_2^5 + u_4^5 - 20CT, & \text{if } \theta = -C, \end{cases}$$

$$= \begin{cases} -\dfrac{(D-T)^2(D+T)}{4C^2} + 2C\dfrac{(D+T)^2}{(D-T)} - \dfrac{16C^4(D-T)}{(D+T)^2} - 20CT, & \text{if } \theta = C, \\ -\dfrac{16C^4(D-T)}{(D+T)^2} - \dfrac{(D+T)(D-T)^2}{4C^2} + \dfrac{2C(D+T)^2}{(D-T)} - 20CT, & \text{if } \theta = -C, \end{cases}$$

$$= -F,$$

by (2.28) and (2.29), which is (2.15).

If $D = \pm T$, then from $r(T) = r(\pm D) = 0$, we obtain

$$\begin{cases} 128C^3D^4 = 0, & \text{if } T = D, \\ -256C^6D^2 = 0, & \text{if } T = -D. \end{cases}$$

As $C \neq 0$ we deduce that $D = 0$. Then (2.5)-(2.7) become

$$\begin{cases} K = 3C^2 + E, \\ L = 15C^4 - 2C^2E + 3E^2, \\ M = -25C^6 + 35C^4E - 11C^2E^2 + CF^2 + E^3. \end{cases}$$

From

$$h(C)h(-C) = h(\theta)h(-\theta) = 0$$

and (2.3) with $D = 0$, we obtain

$$(4C^2 - E)^2(-160000C^8 + 48000C^6E - 4400C^4E^2$$
$$+ 16C^3F^2 + 120C^2E^3 - 2CEF^2 - E^4) = 0.$$

If $E = 8C^2$, then this equation becomes $-2^{12}C^{12} = 0$, contradicting $C \neq 0$. Thus $E \neq 8C^2$. Hence either

$$\text{(i) } E = 4C^2$$

or

$$\text{(ii) } F^2 = \frac{(400C^4 - 60C^2E + E^2)^2}{2C(8C^2 - E)}.$$

Since $f(x)$ is irreducible, $F \neq 0$, and in case (ii) we have

$$2C(8C^2 - E) = Z^2$$

for some $Z \in \mathbb{Q}$ with $Z \neq 0$. Then

$$E = \frac{16C^3 - Z^2}{2C}$$

and

$$F = \frac{400C^4 - 60C^2E + E^2}{Z} = \frac{64C^6 - 88C^3Z^2 - Z^4}{4C^2Z}.$$

Now define u_1, u_2, u_3, u_4 as in (2.36)-(2.39). Then

$$u_1u_4 + u_2u_3 = u_1u_4 = -2C,$$

which is (2.12).

Next

$$u_1u_2^2 + u_2u_4^2 + u_3u_1^2 + u_4u_3^2 = u_3u_1^2 + u_4u_3^2 = 0 = -2D,$$

which is (2.13).

Further

$$u_1^2u_4^2 + u_2^2u_3^2 - u_1^3u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3 - u_1u_2u_3u_4$$

$$= u_1^2u_4^2 - u_3^3u_1 - u_4^3u_3$$

$$= \begin{cases} 4C^2, & \text{(i)}, \\ \dfrac{16C^3 - Z^2}{2C}, & \text{(ii)}, \end{cases}$$

$$= E,$$

which is (2.14).

Finally

$$u_1^5 + u_2^5 + u_3^5 + u_4^5 - 5(u_1u_4 - u_2u_3)(u_1^2u_3 - u_2^2u_1 - u_3^2u_4 + u_4^2u_2)$$

$$= \begin{cases} u_1^5 + u_4^5, & \text{(i)}, \\ u_1^5 + u_3^5 + u_4^5 + 20CZ, & \text{(ii)}, \end{cases}$$

$$= \begin{cases} -F, & \text{(i)}, \\ 22CZ + \dfrac{Z^3}{4C^2} - \dfrac{16C^4}{Z}, & \text{(ii)}, \end{cases}$$

$$= -F,$$

which is (2.15).

In both cases (i) and (ii), by the Lemma the roots of $f(x) = 0$ are given by (2.16). As T is expressible in terms of radicals, so are u_1, u_2, u_3, u_4, and thus x_1, x_2, x_3, x_4, x_5 are expressible in radicals.

5. Proof of Theorem 3

As $\theta = C = 0$ we deduce from $h(\theta) = 0$ that

$$M = -16D^4 - 2DEF + E^3 = 0.$$

If $E = 0$, then $D = 0$ and conversely. Thus

$$\text{(i) } D = E = 0 \text{ or (ii) } D \neq 0, E \neq 0.$$

In case (ii) we have

$$F = \frac{E^3 - 16D^4}{2DE}.$$

Define u_1, u_2, u_3, u_4 as in (2.42)-(2.44). Then

$$u_1u_4 + u_2u_3 = 0 = -2C,$$

which is (2.12). Also

$$u_1u_2^2 + u_2u_4^2 + u_3u_1^2 + u_4u_3^2 = u_1^2u_3 = \left. \begin{cases} 0, & \text{(i)} \\ -2D, & \text{(ii)} \end{cases} \right\} = -2D,$$

which is (2.13). Further

$$u_1^2u_4^2 + u_2^2u_3^2 - u_1^3u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3 - u_1u_2u_3u_4$$

$$= -u_1u_3^3 = \begin{cases} 0, & \text{(i)} \\ E, & \text{(ii)} \end{cases} = E,$$

which is (2.14). Finally

$$u_1^5 + u_2^5 + u_3^5 + u_4^5 - 5(u_1u_4 - u_2u_3)(u_1^2u_3 - u_2^2u_1 - u_3^2u_4 + u_4^2u_2)$$

$$= u_1^5 + u_3^5 = \begin{cases} -F, & \text{(i)} \\ \dfrac{16D^4 - E^3}{2DE}, & \text{(ii)} \end{cases} = -F,$$

which is (2.15). Hence by the Lemma the roots of $f(x) = 0$ are given by (2.16). Clearly u_1, u_2, u_3, u_4 can be expressed in radical form so that x_1, x_2, x_3, x_4, x_5 are expressible by radicals.

6. Proof of Theorem 4

We define T by (2.45). As $\theta = 0$ we have $h(\theta) = h(0) = \dfrac{M}{3125} = 0$ so that

$$\begin{aligned} & 25C^6 + 35C^4E - 40C^3D^2 - 2C^2DF - 11C^2E^2 \\ & + 28CDE + CF^2 - 16D^4 - 2DEF + E^3 = 0. \end{aligned} \tag{6.1}$$

Replacing E by $\dfrac{(C^3 + D^2 - T^2)}{C}$ in (6.1), we obtain (2.46). Define R_1 and R_2 as in (2.47)-(2.49). Define $X, \overline{X}, Y, \overline{Y}$ as in (2.50) and (2.51). Clearly

$$X, \overline{X}, Y, \overline{Y} \neq 0$$

and

$$X\overline{X} = Y\overline{Y} = -C^3, \quad X + \overline{X} + Y + \overline{Y} = -2D. \tag{6.2}$$

Next

$$C^2 + \frac{(X + \overline{X})(Y + \overline{Y})}{C} = \frac{C^3 - T^2 + D^2}{C} = E. \tag{6.3}$$

Also

$$\frac{X^2Y + \overline{X}Y^2 + X\overline{Y}^2 + \overline{X}^2\overline{Y}}{C^2}$$

$$= \frac{TR_1R_2 - \frac{1}{4}DR_1^2 - \frac{1}{4}TR_1^2 - \frac{1}{4}DR_2^2 + \frac{1}{4}TR_2^2 - \frac{1}{2}D^3 + \frac{1}{2}DT^2}{C^2}$$

$$= -F. \tag{6.4}$$

Now define u_1, u_2, u_3, u_4 by (2.52). Then

$$u_1u_4 = u_2u_3 = -C.$$

Also

$$u_2^5 = \frac{\overline{X}Y^2}{C^2}, \quad u_3^5 = \frac{X\overline{Y}^2}{C^2}, \quad u_4^5 = \frac{\overline{X}^2\overline{Y}}{C^2}.$$

Then

$$u_1u_4 + u_2u_3 = -2C,$$

which is (2.12). Also

$$u_1u_2^2 + u_2u_4^2 + u_3u_1^2 + u_4u_3^2 = Y + X + \overline{Y} + \overline{X} = -2D,$$

which is (2.13). Further

$$u_1^2u_4^2 + u_2^2u_3^2 - u_1u_2u_3u_4 - (u_1^3u_2 + u_2^3u_4 + u_3^3u_1 + u_4^3u_3)$$

$$= C^2 + \left(\frac{XY}{C} + \frac{\overline{X}Y}{C} + \frac{X\overline{Y}}{C} + \frac{\overline{X}\overline{Y}}{C}\right)$$

$$= E,$$

by (6.3), which is (2.14). Finally

$$u_1^5 + u_2^5 + u_3^5 + u_4^5 - 5(u_1u_4 - u_2u_3)(u_1^2u_2 - u_2^2u_1 - u_3^2u_4 + u_4^2u_2)$$

$$= u_1^5 + u_2^5 + u_3^5 + u_4^5$$

$$= \frac{X^2Y}{C^2} + \frac{\overline{X}Y^2}{C^2} + \frac{X\overline{Y}^2}{C^2} + \frac{\overline{X}^2\overline{Y}}{C^2}$$

$$= -F,$$

by (6.4), which is (2.15). By the Lemma the roots of $f(x) = 0$ are given by

(2.16). As T, R_1, R_2, X, $\overline{X}$, Y, $\overline{Y}$ are expressible by radicals, so are u_1, u_2, u_3, u_4, and thus x_1, x_2, x_3, x_4, x_5 are expressible by radicals.

7. Examples

We present eight examples.

Example 1. This is Example 3 from [1] with typos corrected

$$f(x) = x^5 - 25x^3 + 50x^2 - 25, \quad \mathrm{Gal}(f) = \mathbb{Z}/5\mathbb{Z} \text{ [MAPLE]}$$

$$C = -\frac{5}{2}, \quad D = 5, \quad E = 0, \quad F = -25$$

$$K = \frac{75}{4}, \quad L = \frac{5375}{16}, \quad M = \frac{-30625}{64}$$

$$\delta = 5^{12}7^2$$

$$h(x) = x^6 - \frac{15}{4}x^4 + \frac{43}{16}x^2 - \frac{7}{25}\sqrt{5}x - \frac{49}{320}$$

$$= \left(x + \frac{\sqrt{5}}{2}\right)\left(x^5 - \frac{\sqrt{5}}{2}x^4 - \frac{5}{2}x^3 + \frac{5\sqrt{5}}{4}x^2 - \frac{7}{16}x - \frac{49\sqrt{5}}{800}\right)$$

$$\theta = -\frac{\sqrt{5}}{2}, \quad T = 0$$

Theorem 1 and (3.9) give

$$R_1 = \sqrt{-25 + 10\sqrt{5}}, \quad R_2 = -\sqrt{-25 - 10\sqrt{5}}$$

$$X = \frac{-5 + \sqrt{-25 + 10\sqrt{5}}}{2}, \quad Y = \frac{-5 - \sqrt{-25 - 10\sqrt{5}}}{2}, \quad Z = \frac{5 + \sqrt{5}}{2}$$

$$u_1 = \left(\frac{X^2Y}{Z^2}\right)^{1/5} = \left(\frac{25 + 15\sqrt{5} + 5\sqrt{-130 - 58\sqrt{5}}}{4}\right)^{1/5}$$

$$u_2 = \left(\frac{\overline{X}Y^2}{\overline{Z}^2}\right)^{1/5} = \left(\frac{25 - 15\sqrt{5} + 5\sqrt{-130 + 58\sqrt{5}}}{4}\right)^{1/5}$$

$$u_3 = \left(\frac{X\overline{Y}^2}{\overline{Z}^2}\right)^{1/5} = \left(\frac{25 - 15\sqrt{5} - 5\sqrt{-130 + 58\sqrt{5}}}{4}\right)^{1/5}$$

$$u_4 = \left(\frac{\overline{X}^2\overline{Y}}{Z^2}\right)^{1/5} = \left(\frac{25 + 15\sqrt{5} - 5\sqrt{-130 - 58\sqrt{5}}}{4}\right)^{1/5}.$$

Example 2.

$f(x) = x^5 + 10x^3 + 10x^2 + 10x + 78$, $\mathrm{Gal}(f) = F_{20}$ [MAPLE]

$C = 1$, $D = 1$, $E = 2$, $F = 78$

$K = 5$, $L = -125$, $M = 5625$, $\delta = 2^4 5^{13}$

$$h(x) = x^6 - x^4 - x^2 - \frac{4}{5}x + \frac{9}{5} = (x-1)\left(x^5 + x^4 - x - \frac{9}{5}\right)$$

$\theta = 1$, $T = 3$.

Theorem 2(a) $(\theta = C)$ gives

$$u_1^5 = -4,\ u_2 = -u_1^2,\ u_3 = -\frac{1}{2}u_1^3,\ u_4 = 0$$

$$x = -\omega 2^{2/5} - \omega^2 2^{4/5} + \omega^3 2^{1/5}.$$

Example 3.

$f(x) = x^5 + 10x^3 + 20x + 1$, $\mathrm{Gal}(f) = F_{20}$ [MAPLE]

$C = 1$, $D = 0$, $E = 4$, $F = 1$

$K = 7$, $L = 55$, $M = 4$, $\delta = 3^2 5^5 43^2$

$$h(x) = x^6 - \frac{7}{5}x^4 + \frac{11}{25}x^2 - \frac{129}{3125}x + \frac{4}{3125}$$

$$= (x-1)\left(x^5 + x^4 - \frac{2}{5}x^3 - \frac{2}{5}x^2 + \frac{1}{25}x - \frac{4}{3125}\right)$$

$\theta = 1$, $T = 0$.

Theorem 2(b)(i) gives

$$u_1 = \left(\frac{-1 + \sqrt{129}}{2}\right)^{1/5},\ u_2 = u_3 = 0,\ u_4 = \left(\frac{-1 - \sqrt{129}}{2}\right)^{1/5}$$

$$x = \omega\left(\frac{-1+\sqrt{129}}{2}\right)^{1/5} + \omega^4\left(\frac{-1-\sqrt{129}}{2}\right)^{1/5}.$$

Example 4.

$$f(x) = x^5 + 10x^3 + 30x - 38, \quad \mathrm{Gal}(f) = F_{20} \text{ [MAPLE]}$$

$$C = 1, \quad D = 0, \quad E = 6, \quad F = -38$$

$$K = 9, \quad L = 111, \quad M = 1449, \quad \delta = 2^4 5^5 431^2$$

$$h(x) = x^6 - \frac{9}{5}x^4 + \frac{111}{125}x^2 - \frac{1724}{3125}x + \frac{1449}{3125}$$

$$= (x-1)\left(x^5 + x^4 - \frac{4}{5}x^3 - \frac{4}{5}x^2 + \frac{11}{125}x - \frac{1449}{3125}\right)$$

$$\theta = 1, \quad T = 0.$$

Theorem 2(b)(ii) gives

$$Z = 2$$

$$u_1 = 2^{2/5}, \quad u_2 = 0, \quad u_3 = 2^{1/5}, \quad u_4 = -2^{3/5}$$

$$x = \omega 2^{2/5} + \omega^3 2^{1/5} - \omega^4 2^{3/5}.$$

Example 5.

$$f(x) = x^5 - 20x^3 + 180x - 236, \quad \mathrm{Gal}(f) = F_{20} \text{ [MAPLE]}$$

$$C = -2, \quad D = 0, \quad E = 36, \quad F = -236$$

$$K = 48, \quad L = 3840, \quad M = -103200, \quad \delta = 2^8 5^9$$

$$h(x) = x^6 - \frac{48}{5}x^4 + \frac{768}{25}x^2 - \frac{16}{125}x - \frac{4128}{125}$$

$$= (x-2)\left(x^5 + 2x^4 - \frac{28}{5}x^3 - \frac{56}{5}x^2 + \frac{208}{25}x + \frac{2064}{125}\right)$$

$$\theta = 2$$

$$T = 0, 4.$$

With $T = 0$ Theorem 2(b)(ii) gives

$$Z = -4$$

$$u_1 = 2^{4/5}, \ u_2 = 0, \ u_3 = -2^{2/5}, \ u_4 = 2^{6/5}$$

$$x = \omega 2^{4/5} - \omega^3 2^{2/5} + \omega^4 2^{6/5}.$$

With $T = 4$ Theorem 2(a) ($\theta = -C$) gives

$$u_1 = 2^{6/5}, \ u_2 = -2^{2/5}, \ u_3 = 0, \ u_4 = 2^{4/5}$$

$$x = \omega 2^{6/5} - \omega^2 2^{2/5} + \omega^4 2^{4/5}$$

in agreement with the solutions given with $T = 0$.

Example 6.

$$f(x) = x^5 + 10x^2 + 10x - 2, \quad \mathrm{Gal}(f) = F_{20} \text{ [MAPLE]}$$

$$C = 0, \ D = 1, \ E = 2, \ F = -2$$

$$K = 2, \ L = 16, \ M = 0, \ \delta = 2^4 5^5 19^2$$

$$h(x) = x^6 - \frac{2}{5}x^4 + \frac{16}{125}x^2 - \frac{76}{3125}x$$

$$\theta = 0.$$

Theorem 3(ii) gives

$$u_1 = 2^{2/5}, \ u_2 = 0, \ u_3 = -2^{1/5}, \ u_4 = 0$$

$$x = \omega 2^{2/5} - \omega^3 2^{1/5}.$$

Example 7.

$$f(x) = x^5 - 20x^3 + 8x^2 + 76x - \frac{96}{25}, \quad \mathrm{Gal}(f) = F_{20} \text{ [MAPLE]}$$

$$C = -2, \ D = \frac{4}{5}, \ E = \frac{76}{5}, \ F = -\frac{96}{25}$$

$$K = \frac{-136}{5}, \ L = \frac{100928}{125}, \ M = 0, \ \delta = \frac{2^{16}1999^2}{5^3}$$

$$h(x) = x^6 - \frac{136}{25}x^4 + \frac{100928}{15625}x^2 - \frac{511744}{1953125}x$$

$$\theta = 0.$$

Theorem 4 gives

$$T = \frac{24}{5}, \quad R_1 = 4i, \quad R_2 = \frac{4}{5}i$$

$$X = 2 + 2i, \quad \overline{X} = 2 - 2i, \quad Y = \frac{-14 + 2i}{5}, \quad \overline{Y} = \frac{-14 - 2i}{5}$$

$$u_1 = \left(\frac{-4 - 28i}{5}\right)^{1/5}, \quad u_2 = \left(\frac{1 - i}{2}\right)u_1^2$$

$$u_3 = \left(\frac{-4 + 3i}{10}\right)u_1^3, \quad u_4 = \left(\frac{-1 + 7i}{20}\right)u_1^4$$

$$x = \omega u_1 + \omega^2\left(\frac{1 - i}{2}\right)u_1^2 + \omega^3\left(\frac{-4 + 3i}{10}\right)u_1^3 + \omega^4\left(\frac{-1 + 7i}{20}\right)u_1^4.$$

Example 8.

$$f(x) = x^5 + 40x^3 - 120x^2 + 160x + 96, \quad \mathrm{Gal}(f) = D_5 \text{ [MAPLE]}$$

$$C = 4, \quad D = -12, \quad E = 32, \quad F = 96$$

$$K = 80, \quad L = 12800, \quad M = 0, \quad \delta = 2^{20}3^4 5^{10}$$

$$h(x) = x^6 - 16x^4 + \frac{512}{5}x^2 - \frac{288}{125}\sqrt{5}x$$

$$\theta = 0.$$

Theorem 4 gives

$$T = 4\sqrt{5}$$

$$R_1 = 4\sqrt{30 + 6\sqrt{5}}, \quad R_2 = -4\sqrt{30 - 6\sqrt{5}} \quad \text{(by (2.49))}$$

$$X = 6 + 2\sqrt{5} + 2\sqrt{30 + 6\sqrt{5}}, \quad \overline{X} = 6 + 2\sqrt{5} - 2\sqrt{30 + 6\sqrt{5}}$$

$$Y = 6 - 2\sqrt{5} - 2\sqrt{30 - 6\sqrt{5}}, \quad \overline{Y} = 6 - 2\sqrt{5} + 2\sqrt{30 - 6\sqrt{5}}$$

$$u_1 = (-24 - 40\sqrt{5} - 8\sqrt{150 + 30\sqrt{5}})^{1/5}$$

$$u_2 = \left(\frac{3 + \sqrt{5} - \sqrt{30 + 6\sqrt{5}}}{8}\right)u_1^2$$

$$u_3 = \left(\frac{1 - 3\sqrt{5} + \sqrt{30 - 6\sqrt{5}}}{16}\right)u_1^3$$

$$u_4 = \left(\frac{-3 - 5\sqrt{5} + \sqrt{150 + 30\sqrt{5}}}{32}\right)u_1^4$$

$$x = \omega u_1 + \omega^2\left(\frac{3 + \sqrt{5} - \sqrt{30 + 6\sqrt{5}}}{8}\right)u_1^2 + \omega^3\left(\frac{1 - 3\sqrt{5} + \sqrt{30 - 6\sqrt{5}}}{16}\right)u_1^3 + \omega^4\left(\frac{-3 - 5\sqrt{5} + \sqrt{150 + 30\sqrt{5}}}{32}\right)u_1^4.$$

8. Concluding Remarks

We take this opportunity to note some corrections to [1].

On page 20 in the expression for Δ the terms

$$-20a^3bcf^3,\quad 4080a^2bd^2c^2f,\quad -180ab^3c^3f,\quad -3375b^4c^4$$

should be replaced by

$$-20a^3bef^3,\quad 4080a^2bd^2e^2f,\quad -180ab^3e^3f,\quad -3375b^4e^4.$$

On page 24 in the second column S_2, S_4 and S_6 should be replaced by E_2, E_4 and E_6.

On page 27 in the first column in the equation given by MAPLE the term $3CD\theta^2 - T\theta^3$ should be replaced by $C^3D - 3C^2T\theta + 3CD\theta^2 - T\theta^3$.

On page 28 in Step 3 equation (7) should be replaced by (8).

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