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Wave Character of the Time Dependent Ginzburg Landau Equation and the Fluctuating Pair Propagator in Superconductors*)

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The time dependent Ginzburg Landau equation and the fluctuation propagator of a superconductor in a static electromagnetic field are examined to the next order of the small parameters $1/\varepsilon_F\tau$ and T/ε_F, where ε_F is the Fermi energy and τ the relaxation time. The coefficient of the time derivative of the order parameter becomes a complex number, whose imaginary part is of order of T/ε_F smaller than the real part. This term is shown to be important for Hall effect due to the fluctuation near T_c, which has not been expected in the usual approximation. The arguments cover the cases of arbitrary mean free paths near T_c and of arbitrary temperatures except $T \ll T_c$.

§ 1. Introduction

In recent years effects of the fluctuations of the order parameter in superconductors on dynamical properties are studied extensively both near transition temperature, and in vortex states of type-II superconductors near the critical field H_{c2}. Aslamazov and Larkin[1] showed that there exist anomalously large contributions to the electrical conductivity, the ultrasonic attenuation and the specific heat by taking account of a process, now called the AL process, where the virtual cooper pairs represented by fluctuation propagators themselves respond to the external fields. Phenomenological descriptions[2] of such effects of fluctuation are based on the time dependent Ginzburg Landau (TDGL) equation, which is also used to examine the effects of dynamical order parameter in the vortex state near H_{c2}.[3)~5)]

In those arguments they used the fluctuation propagator $\mathscr{D}(\boldsymbol{q}, \omega)$ and the linearized TDGL equation for the order parameter $\Psi(\boldsymbol{r}, t)$, which are derived in the following assumptions and approximations. (i) The ladder approximation for the BCS effective interaction with a cutoff energy is adopted. (ii) The effects of randomly distributed impurities are taken into account in the Born approximation for the one-particle propagators and in the ladder approximation for $\mathscr{D}$. (iii) Electrons are assumed to have a dispersion similar to free electrons. (iv)

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Integration over a momentum variable is transformed into the energy integral multiplied by the density of states, N, at the Fermi energy.

By these approximations, we have

$$\mathscr{D}(\boldsymbol{q}, \omega+i\delta) = -\frac{1}{N}[\eta - i\lambda_0\omega + \lambda \boldsymbol{q}^2]^{-1}, \tag{1}$$

$$\gamma \frac{\partial}{\partial t}\Psi(\boldsymbol{r}, t) = \frac{1}{2m}\boldsymbol{\nabla}^2\Psi(\boldsymbol{r}, t) - a\Psi(\boldsymbol{r}, t), \tag{2}$$

with

$$\eta = (T - T_c)/T_c\,, \tag{3}$$

$$\lambda_0 = \frac{\pi}{8T}, \qquad \lambda = \frac{\pi D}{8T}, \qquad D = \frac{v^2\tau}{3}, \qquad a = \frac{\eta}{2m\lambda}, \qquad \gamma = \frac{\lambda_0}{2m\lambda}, \tag{4}$$

where v and τ are the Fermi velocity and relaxation time, respectively. Thus all coefficients are real positive numbers. In Eqs. (1) and (2), we neglect terms of the order of $1/\varepsilon_F\tau$ and T/ε_F, where ε_F is the Fermi energy. Using $\mathscr{D}$, Eq. (1), or the TDGL equation, Eq. (2), we can examine such dynamical properties as electrical resistivity and the ultrasonic attenuation. However as regards the Hall effect, Caroli and Maki[3] obtained a vanishing contribution in the vortex state, and Tsuzuki and the present authors[6] showed that the Hall conductivity does not have contributions from the AL process slightly above T_c. These vanishing results on Hall effects are intimately connected with the fact that λ_0 and λ in Eq. (1) or Eq. (2) are real quantities. The reality of λ_0 and λ are, however, true only if we neglect terms of the order of $1/\varepsilon_F\tau$ and T/ε_F in the derivation of $\mathscr{D}$. Thus we are in need to determine the fluctuation propagator $\mathscr{D}$ and the TDGL equation to this order to examine some dynamical processes. These determinations up to this order in the presence of external electric and magnetic fields are the purpose of the present article. We confine ourselves within the Born approximations and the ladder corrections to impurity scattering. That is, we work within (i) and (ii), but need not (iii) and (iv).

In § 1 the derivation is given for the nearly free electron system to clarify the expansion parameters. As is shown the corrections, which is of order of T/ε_F, depend on the energy dependence of the density of states function near the Fermi energy. Then we consider in § 3 the cases of arbitrary Bloch electrons. In § 4 we briefly discuss a new effect due to these corrections.

§ 2. Fluctuation propagator and TDGL equation for a nearly free electron system

For nearly free electrons, the structure of the propagator $\mathscr{D}$ is very simple. The model Hamiltonian is

$$\mathscr{H} = \sum_{\sigma}\int \psi^\dagger\left[\frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2 + \sum_j U(\boldsymbol{r}-\boldsymbol{R}_j)\right]\psi d\boldsymbol{r} - g\int \psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow d\boldsymbol{r}\,. \tag{5}$$

Here $\boldsymbol{A}$ is the vector potential and the impurity potential $U(\boldsymbol{r})$ is assumed of short range. The units are taken as $\hbar=c=k_B=1$ and the signs of g and e are positive. The one-particle Green's function, in a system without $\boldsymbol{A}$ or g, is given by

$$G^R(\boldsymbol{k}, x)=x-\xi_{\boldsymbol{k}}-\Sigma^R(x), \tag{6}$$

$$\xi_{\boldsymbol{k}}=\frac{1}{2m}\boldsymbol{k}^2-\varepsilon_F, \qquad \varepsilon_F=\frac{1}{2m}k_F^2, \tag{7}$$

$$\Sigma^R(x)=n_i u^2 \sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x)=-\frac{i}{2\tau}\varphi_1{}^R(x), \tag{8}$$

where

$$\varphi_1{}^R(x)\equiv\sqrt{1+\frac{x-\Sigma^R(x)}{\varepsilon_F}}\cong 1+\frac{x-(i/2\tau)(1+x/2\varepsilon_F)}{2\varepsilon_F}+\cdots$$

$$=1+\frac{x}{2\varepsilon_F}-\frac{i}{2}\gamma+\cdots, \tag{9}$$

$$\frac{1}{\tau}\equiv 2\pi n_i u^2 N, \qquad \gamma\equiv\frac{1}{2\varepsilon_F\tau}. \tag{10}$$

N is the density of states

$$N=\frac{mk_F}{2\pi^2}. \tag{11}$$

u is the Fourier transform of $U(\boldsymbol{r})$. The branch of square root is such as $\mathrm{Im}\,\varphi_1{}^R>0$. From now on, the approximation, $\mathrm{Re}\,\Sigma=0$, is adopted.

In the ladder approximation for the BCS coupling, the fluctuation propagator is given by

$$[\mathscr{D}^R(\boldsymbol{q}, \omega)]^{-1}=-[g^{-1}-\Pi^R(\boldsymbol{q}, \omega)], \tag{12}$$

$$\Pi(\boldsymbol{q}, i\omega_\lambda)=-\frac{1}{\beta}\int_0^\beta\int_0^\beta d\tau d\tau' \exp[i\omega_\lambda(\tau-\tau')]\langle T_\tau \Psi^\dagger(\boldsymbol{q}, \tau)\Psi(-\boldsymbol{q}, \tau')\rangle, \tag{13}$$

where

$$\Psi^\dagger(\boldsymbol{q})=\int d\boldsymbol{r}\exp[-i\boldsymbol{q}\cdot\boldsymbol{r}]\psi_\uparrow{}^\dagger(\boldsymbol{r})\psi_\downarrow{}^\dagger(\boldsymbol{r}). \tag{14}$$

Here Π^R is an analytic continuation of $\Pi(\boldsymbol{q}, i\omega_\lambda)$ from the upper plane of ω and the bracket means the thermal average. Adopting the ladder approximation for impurity scattering, which is consistent with Eq. (8), one obtains

$$\Pi^R(\boldsymbol{q}, \omega)=\frac{1}{4\pi i}\int_{-\infty}^{\infty}dx \tanh\frac{x}{2T}[\tilde{\Pi}^{RR}(x, x+\omega)-\tilde{\Pi}^{AR}(x, x+\omega)$$

$$+\tilde{\Pi}^{AR}(x-\omega, x)-\tilde{\Pi}^{AA}(x-\omega, x)], \tag{15}$$

where

$$\tilde{\Pi}^{BB'}(x, x') = g^{BB'}(\boldsymbol{q}, x, x')[1 - n_i u^2 g^{BB'}(\boldsymbol{q}, x, x')]^{-1}, \tag{16}$$

$$g(\boldsymbol{q}, i\varepsilon_n, i\varepsilon_n') = \sum_k G(\boldsymbol{k}, i\varepsilon_n) G(-\boldsymbol{k}-\boldsymbol{q}, -i\varepsilon_n'), \tag{17}$$

B and B' take A or R, which means that the function is analytically continued to real axis from below or above, respectively.

Explicit calculations lead to

$$g^{BB'}(\boldsymbol{q}, x, x') = \frac{i\pi}{\varepsilon_F} N \frac{1}{\varphi_1{}^B(x) + \varphi_2{}^{B'}(x')} \left\{1 + \frac{\boldsymbol{q}^2}{3k_F{}^2} \frac{1}{[\varphi_1{}^B(x) + \varphi_2{}^{B'}(x')]^2}\right\}, \tag{18}$$

in the small q limit, where

$$\varphi_2{}^R(x') \equiv \sqrt{1 + \frac{-x' - \Sigma^A(-x')}{\varepsilon_F}} \simeq -1 + \frac{x'}{2\varepsilon_F} + \frac{i}{2}\gamma + \cdots. \tag{19}$$

One can immediately see that the fourth term in Eq. (15) gives the same contribution as the first. In the dirty limit Eq. (15) is calculated as

$$\begin{aligned}
\Pi^R(\boldsymbol{q}, \omega) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx \tanh\frac{x}{2T} \frac{i\pi N}{\varepsilon_F}\left\{\varphi_1{}^R(x) + \varphi_2{}^R(x+\omega) - i\gamma - \frac{\boldsymbol{q}^2}{3k_F{}^2}\frac{1}{(\varphi_1{}^R + \varphi_2{}^R)}\right\}^{-1} \\
&\quad + \frac{\omega}{4\pi i}\int_{-\infty}^{\infty} dx \tanh\frac{x}{2T}\frac{d}{dx}\left\{\varphi_1{}^A(x) + \varphi_2{}^R(x) - i\gamma - \frac{\boldsymbol{q}^2}{3k_F{}^2}\frac{1}{(\varphi_1{}^A + \varphi_2{}^R)}\right\}^{-1} \\
&= \frac{N}{2}\left(1 + \frac{\omega}{4\varepsilon_F}\right)\int_{-\infty}^{\infty} dx \tanh\frac{x}{2T} \\
&\quad \times \left\{\left[\left\{x + \frac{\omega + iD\boldsymbol{q}^2}{2}\left(1 + \frac{\omega}{4\varepsilon_F}\right)\right\}^{-1} - (x + i\delta)^{-1}\right] + \frac{1}{x + i\delta}\right\},
\end{aligned} \tag{20}$$

which is valid to the linear order of ω. Here we have performed the expansion of φ's with respect to $1/\varepsilon_F\tau$ and $x/\varepsilon_F \sim T/\varepsilon_F$ (or ω_D/ε_F) to the order following the one that leads to the ordinary $\mathscr{D}$. The expansion parameter with $\boldsymbol{q}^2$ is $\tau D q^2$. Introducing the BCS cutoff, $|x| \leqq \omega_D$, in the second term, and using the relation

$$\frac{1}{g} - \Pi^R(0, 0) = \frac{1}{g} - N\ln\frac{2\gamma\omega_D}{\pi T} = \ln\frac{T}{T_{c0}}, \tag{21}$$

one obtains Π^R and then $\mathscr{D}^R$ as follows:

$$\begin{aligned}
[\mathscr{D}^R(\boldsymbol{q}, \omega)]^{-1} = -N\Bigg\{&\ln\frac{T}{T_{c0}} + \psi\left(\frac{1}{2} + \zeta\right) - \psi\left(\frac{1}{2}\right) \\
&+ \frac{1}{4\varepsilon_F}\left[\frac{1}{gN} - \ln\frac{T}{T_{c0}} - \psi\left(\frac{1}{2} + \zeta\right) + \psi\left(\frac{1}{2}\right)\right]\Bigg\},
\end{aligned} \tag{22}$$

where ψ is the di-gamma function and

$$\zeta = \frac{-i\omega + D\boldsymbol{q}^2}{4\pi T}\left(1 + \frac{\omega}{4\varepsilon_F}\right). \tag{23}$$

In the presence of a magnetic field, $\boldsymbol{q}$ should be replaced by $\boldsymbol{Q}=\boldsymbol{q}-2e\boldsymbol{A}$.[7] This result, Eq. (23), has new terms proportional to ω, which is smaller than the ordinary terms by a factor T/ε_F.

If $D\boldsymbol{q}^2/T \ll 1$, we get

$$[\mathscr{D}^R(\boldsymbol{q},\omega)]^{-1} = -N\left[\ln\frac{T}{T_{c0}} + \lambda q^2 - i\lambda_0\omega + \frac{\omega}{4\varepsilon_F}\frac{1}{gN}\right]. \tag{24}$$

This equation is valid for cases with arbitrary values of mean free path, if we set

$$\lambda = \frac{7\zeta(3)v^2}{48(\pi T)^2}\chi(\rho), \tag{25}$$

where $\chi(\rho)$ is the Gor'kov function,[8] given by

$$\chi(\rho) = \frac{8}{7\zeta(3)}\frac{1}{\rho}\left\{\frac{\pi^2}{8} + \frac{1}{2\rho}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2}+\frac{1}{2}\rho\right)\right]\right\}, \tag{26}$$

$$\rho = \frac{1}{2\pi\tau T}.$$

So far we have neglected the effects of electric fields. For the derivation of the TDGL equation which includes a scalar potential of an electric field, careful treatments are necessary concerning the ordering of the operator $\partial/\partial\boldsymbol{r}$ and the potential. Introducing an external field by

$$\mathscr{H}_{\text{ext}} = -e\int\psi^\dagger V(\boldsymbol{r})\psi d\boldsymbol{r}\,, \tag{27}$$

one has the equation for the order parameter $\Delta^\dagger$,[9]

$$\Delta^\dagger(\boldsymbol{q},\omega) = g\{\Pi^R(\boldsymbol{q},\omega)\Delta^\dagger(\boldsymbol{q},\omega) + P^R(\boldsymbol{q}_1,\boldsymbol{q}_2,\omega)\Delta_0^\dagger(\boldsymbol{q}_1)2eV(\boldsymbol{q}_2,\omega)\}, \tag{28}$$

to the linear order of V. The function P is defined by

$$P(\boldsymbol{q}_1,\boldsymbol{q}_2,i\omega_\lambda) = -\frac{1}{\beta}\int_0^\beta\int_0^\beta d\tau d\tau' \exp[i\omega_\lambda(\tau-\tau')]$$
$$\times\langle T_\tau\Psi^\dagger(\boldsymbol{q}_1+\boldsymbol{q}_2,\tau)\Psi(-\boldsymbol{q}_1,\tau)n(-\boldsymbol{q}_2,\tau')\rangle \tag{29}$$

where

$$n(\boldsymbol{q}) = \int d\boldsymbol{r}\exp[-i\boldsymbol{q}\cdot\boldsymbol{r}]\psi_\uparrow^\dagger(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r}).$$

In the dirty limit, a similar calculation to those for Π^R yield $P^R(\boldsymbol{q}_1,\boldsymbol{q}_2,\omega)$ for the limit, $\omega\to 0$ (Appendix A).

$$P^R(\boldsymbol{q}_1,\boldsymbol{q}_2) = -\frac{iN}{4\pi T}\psi^{(1)}\left(\frac{1}{2}+\frac{D\boldsymbol{q}^2}{4\pi T}\right)$$
$$+\frac{N}{4\varepsilon_F}\left[\frac{1}{gN} - \ln\frac{T}{T_{c0}} - \psi\left(\frac{1}{2}+\frac{D\boldsymbol{q}_1^2}{4\pi T}\right) + \psi\left(\frac{1}{2}\right)\right.$$

$$-4\pi T\sum_n \frac{D(\boldsymbol{q}_1+\boldsymbol{q}_2)^2}{(2\varepsilon_n+D(\boldsymbol{q}_1+\boldsymbol{q}_2)^2)(2\varepsilon_n+D\boldsymbol{q}_1^2)}+4\pi T\sum_n \frac{D(\boldsymbol{q}_1+\boldsymbol{q}_2)^2}{(2\varepsilon_n+D\boldsymbol{q}_1^2)^2}\Bigg], \tag{30}$$

where $\psi^{(1)}$ is the tri-gamma function. In the $\boldsymbol{r}$-representation, $\boldsymbol{q}_1$ or $\boldsymbol{q}_2$ is an operator $\partial/\partial\boldsymbol{r}$ which operates on $\Delta^\dagger(\boldsymbol{r},t)$ or $V(\boldsymbol{r})$, respectively. Thus we obtain

$$\begin{aligned}\Bigg\{&\psi\left(\frac{1}{2}+\frac{D\boldsymbol{Q}^2}{4\pi T}\right)-\psi\left(\frac{1}{2}\right)+\ln\frac{T}{T_{c0}}+\frac{-i\omega+2ieV(\boldsymbol{r})}{4\pi T}\psi^{(1)}\left(\frac{1}{2}+\frac{D\boldsymbol{Q}^2}{4\pi T}\right)\\&+\frac{\omega-2eV(\boldsymbol{r})}{4\varepsilon_F}\left[\frac{1}{gN}-\psi\left(\frac{1}{2}+\frac{D\boldsymbol{Q}^2}{4\pi T}\right)+\psi\left(\frac{1}{2}\right)-\ln\frac{T}{T_{c0}}-\frac{D\boldsymbol{Q}^2}{4\pi T}\psi^{(1)}\left(\frac{1}{2}+\frac{D\boldsymbol{Q}^2}{4\pi T}\right)\right.\\&-4\pi T\sum_n D\boldsymbol{Q}^2\left[2eV(\boldsymbol{r}),\frac{1}{2\varepsilon_n+D\boldsymbol{Q}^2}\right]\frac{1}{2\varepsilon_n+D\boldsymbol{Q}^2}\Bigg\}\Delta^\dagger(\boldsymbol{r},t)=0\,,\end{aligned} \tag{31}$$

where [,] is a commutator and

$$\boldsymbol{Q}=\boldsymbol{q}-2e\boldsymbol{A}\,.$$

§ 3. Fluctuation propagator in the case of Bloch electrons

We assume an arbitrary $\varepsilon_{\boldsymbol{k}}-\boldsymbol{k}$ relation in a single band dispersion, restricting ourselves only to systems with the cubic symmetry. In this section we confine ourselves only in the dirty limit, and in the systems with a Fermi energy such that $1/\varepsilon_F\tau\ll1$, $T/\varepsilon_F\ll1$ and $\omega_D/\varepsilon_F\ll1$. Equation (15) is still valid if one use the correct expressions for $\tilde{\Pi}$, i.e., for g. The self-energy is calculated as

$$\Sigma^R(x)=n_iu^2\sum_{\boldsymbol{k}}G^R(\boldsymbol{k},x)=-\frac{i}{2\tau}\frac{N(x)}{N}\,, \tag{32}$$

where $N(x)$ is the density of states at ε_F+x. Similarly one obtains

$$g^{AR}(x,x)=\int d\varepsilon_{\boldsymbol{k}}N(\varepsilon_{\boldsymbol{k}})\frac{1}{x-\varepsilon_{\boldsymbol{k}}+\varepsilon_F-i/2\tau}\frac{1}{-x-\varepsilon_{\boldsymbol{k}}+\varepsilon_F-i/2\tau}\cong-i\pi N', \tag{33}$$

where N' is the derivative of N with respect to x at $x=0$. Using the relation

$$\frac{d}{dx}g^{AR}(x,x)\cong-i\pi N''\sim\mathcal{O}(N/\varepsilon_F^2)$$

and neglecting the quantity of the order of ω_D/ε_F, one gets

$$\Pi^{R,2+3}(\boldsymbol{q},\omega)\equiv\frac{\omega}{4\pi i}\int_{-\omega_D}^{\omega_D}dx\tanh\frac{x}{2T}\frac{d}{dx}[g^{AR}(x,x)^{-1}-n_iu^2]^{-1}=0\,. \tag{34}$$

Another branch of g is expressed as

$$\begin{aligned}g^{RR}(x,x+\omega)=&\sum_{\boldsymbol{k}}G^R(\boldsymbol{k},x)G^A(\boldsymbol{k},-x-\omega)\\&+\frac{1}{2}\sum_\mu q_\mu^2\sum_{\boldsymbol{k}}G^R(\boldsymbol{k},x)\frac{\partial^2}{\partial k_\mu^2}G^A(\boldsymbol{k},-x-\omega),\end{aligned} \tag{35}$$

where the first term is expanded in terms of ω as

$$\sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x) G^A(\boldsymbol{k}, -x) + \omega\left(1 - \frac{N'}{2\tau N}\right) \sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x) G^A(\boldsymbol{k}, -x)^2. \quad (36)$$

The $\boldsymbol{k}$ summation can be performed if one notes that one can expand the denominator of G in terms of $\tau x \sim \tau T$.

$$\begin{aligned}\sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x) G^A(\boldsymbol{k}, -x) &= \int d\varepsilon N(\varepsilon) \left[\varepsilon_F - \varepsilon + \frac{i}{2\tau} - x\left(1 + \frac{i}{2\tau}\frac{N'}{N}\right)\right]^{-1} \\ &\quad \times \left[\varepsilon_F - \varepsilon - \frac{i}{2\tau} + x\left(1 - \frac{i}{2\tau}\frac{N'}{N}\right)\right]^{-1} \\ &= 2\pi N \tau (1 - 2\tau x), \end{aligned} \quad (37)$$

$$\sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x) G^A(\boldsymbol{k}, -x)^2 = 2\pi i N \tau^2 \left(1 + \frac{i}{2\tau}\frac{N'}{N}\right) - 8\pi\tau^3 x N \left(1 + \frac{iN'}{4\tau N}\right), \quad (38)$$

$$\begin{aligned}\sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x) \frac{\partial^2}{\partial k_\mu^2} G^A(\boldsymbol{k}, -x-\omega) &= -\sum_{\boldsymbol{k}} \frac{\partial}{\partial k_\mu} G^R(\boldsymbol{k}, x) \frac{\partial}{\partial k_\mu} G^A(\boldsymbol{k}, -x-\omega) \\ &= -\sum_{\boldsymbol{k}} G^R(\boldsymbol{k}, x) G^A(\boldsymbol{k}, -x-\omega)^2 \left(\frac{\partial}{\partial k_\mu}\varepsilon(\boldsymbol{k})\right)^2 \\ &= -\sum_{\boldsymbol{k}} \left(\frac{\partial}{\partial k_\mu}\varepsilon(\boldsymbol{k})\right)^2 G^R(\boldsymbol{k}, x)^2 G^A(\boldsymbol{k}, -x)^2 \\ &\quad -2\omega\left(1 - \frac{i}{2\tau}\frac{N'}{N}\right) \sum_{\boldsymbol{k}} \left(\frac{\partial}{\partial k_\mu}\varepsilon(\boldsymbol{k})\right)^2 G^R(\boldsymbol{k}, x)^2 G^A(\boldsymbol{k}, -x)^3. \end{aligned} \quad (39)$$

The first of Eq. (39) is evaluated as follows.

$$\begin{aligned} -\sum_{\boldsymbol{k}} \left(\frac{\partial}{\partial k_\mu}\varepsilon(\boldsymbol{k})\right)^2 G^R(\boldsymbol{k}, x)^2 G^A(\boldsymbol{k}, -x)^2 \\ = -\int d\varepsilon a(\varepsilon) \left[\varepsilon_F - \varepsilon + \frac{i}{2\tau} - x\left(1 + \frac{i}{2\tau}\frac{N'}{N}\right)\right]^{-2} \\ \times \left[\varepsilon_F - \varepsilon - \frac{i}{2\tau} + x\left(1 - \frac{i}{2\tau}\frac{N'}{N}\right)\right]^{-2} \\ = 4\pi\tau^3 a(\varepsilon_F)(1 + 3i\tau x), \end{aligned} \quad (40)$$

where

$$a(\varepsilon_F) = \sum_{\boldsymbol{k}} \delta(\varepsilon(\boldsymbol{k}) - \varepsilon_F) \left(\frac{\partial}{\partial k_\mu}\varepsilon(\boldsymbol{k})\right)^2. \quad (41)$$

In an isotropic system, $a(\varepsilon_F)$ is independent of μ. By similar procedures to Eq. (40) one obtains

$$\sum_{\boldsymbol{k}} \left(\frac{\partial}{\partial k_\mu}\varepsilon(\boldsymbol{k})\right)^2 G^R(\boldsymbol{k}, x)^2 G^A(\boldsymbol{k}, -x)^3$$

$$=6\pi i\tau^4 a\left[1+\frac{ia'}{6\tau a}+8i\tau x\left(1+\frac{ia'}{8\tau a}\right)\right]. \tag{42}$$

By Eqs. (37), (38), (39), (40) and (42), Eq. (35) is expressed as

$$g^{RR}(x, x+\omega)=2\pi N\left\{1+i\omega\tau+2i\tau x\left[1+i\omega\tau\left(1-\frac{i}{4\tau}\frac{N'}{N}\right)\right]\right\}-2\pi\tau^3 a\boldsymbol{q}^2. \tag{43}$$

Thus we get

$$\Pi^{1+4}(\boldsymbol{q},\omega)=\frac{1}{2\pi i}\int dx\tanh\frac{x}{2T}[g^{RR}(x,x+\omega)^{-1}-n_i u^2]$$

$$=\frac{N}{2}\left[1+2i\omega\tau\left(1-\frac{i}{4\tau}\frac{N'}{N}\right)\right]\int dx\tanh\frac{x}{2T}\frac{1}{x+i\zeta}, \tag{44}$$

where

$$\zeta=\frac{1}{4\pi T}(-i\omega+D\boldsymbol{q}^2)\left[1+2i\omega\tau\left(1-\frac{i}{4\tau}\frac{N'}{N}\right)\right],\tag{45}$$

$$D=a\tau^2/N.$$

The same procedures of the integration over x as used in § 2 yield

$$[\mathscr{D}^R(\boldsymbol{q},\omega)]^{-1}=-N\left\{\ln\frac{T}{T_{c0}}+\psi\left(\frac{1}{2}+\zeta\right)-\psi\left(\frac{1}{2}\right)\right.$$

$$\left.+\frac{\omega}{2}\frac{N'}{N}\left[\frac{1}{gN}-\psi\left(\frac{1}{2}+\zeta\right)+\psi\left(\frac{1}{2}\right)-\ln\frac{T}{T_c}\right]\right\}. \tag{46}$$

Again $\boldsymbol{q}$ should be replaced by $\boldsymbol{Q}=\boldsymbol{q}-2e\boldsymbol{A}$ in the presence of a magnetic field.

§ 4. Discussion

We have derived the fluctuation propagator and the TDGL equation for the order parameter and found the new terms proportional to ω of order of T/ε_F. Though it is small it is important in some situation. One can show that there is a contribution to Hall conductivity due to fluctuation by use of the newly derived TDGL equation, Eq. (31), instead of Eq. (2). The arguments are restricted in the range near T_c. The basic equations are

$$\hbar(\gamma+i\varepsilon)\left(\frac{\partial}{\partial t}+2ie\phi\right)\Delta^\dagger(\boldsymbol{r},t)=-\left[\frac{1}{2m}\left(-i\boldsymbol{\nabla}-\frac{2e\boldsymbol{A}}{c}\right)^2+a\right]\Delta^\dagger(\boldsymbol{r},t), \tag{47}$$

$$j(\boldsymbol{r},t)=-\frac{ie}{2m}\left(\boldsymbol{\nabla}-\boldsymbol{\nabla}'+4i\frac{e}{c}\boldsymbol{A}\right)\Delta^\dagger(\boldsymbol{r}',t)\Delta(\boldsymbol{r},t)\Big|_{\boldsymbol{r}'\to\boldsymbol{r}}, \tag{48}$$

where

$$\varepsilon=\frac{2\alpha}{\pi}\gamma. \tag{49}$$

As we are concerned only with weak magnetic field cases, we apply the Wigner representation[10] to treat the magnetic field for the system of fluctuating order parameters. Using the formal solution $\Delta^\dagger(t)$, one gets the density matrix of the system as

$$\begin{aligned}\hat{\rho} &= \langle \Delta(t)\Delta^\dagger(t)\rangle \\ &= \frac{2k_BT}{\hbar\gamma}\int_{-\infty}^{t} dt' \exp\left[\left(-\frac{\mathcal{H}}{\hbar(\gamma-i\varepsilon)}-\frac{i\mathcal{H}_{\text{ext}}}{\hbar}\right)(t-t')\right]\cdot\mathbf{1} \\ &\qquad\times\exp\left[\left(-\frac{\mathcal{H}}{\hbar(\gamma+i\varepsilon)}+\frac{i\mathcal{H}_{\text{ext}}}{\hbar}\right)(t-t')\right],\end{aligned} \tag{50}$$

where

$$\left.\begin{aligned}\mathcal{H} &= \frac{1}{2m}\boldsymbol{\pi}^2 + a\,, \\ \boldsymbol{\pi} &= \boldsymbol{p} + 2\frac{e}{c}\boldsymbol{A}\,, \\ \mathcal{H}_{\text{ext}} &= 2eEx\,.\end{aligned}\right\} \tag{51}$$

The bracket means ensemble average with respect to stochastic variables, and **1** means the unit matrix. The factor in front of the integral in Eq. (50) is adjusted such that in the absence of external fields $\hat{\rho}$ may be consistent with GL free energy.

Constructing the Liouville operator for the Wigner distribution function f corresponding to $\hat{\rho}$ from the equation of motion, one obtains

$$\frac{\partial f}{\partial t} = i(\mathcal{L}_0 + \mathcal{L}' + \mathcal{L}_{\text{ext}})f + \frac{2k_BT}{\hbar\gamma}\,, \tag{52}$$

$$\left.\begin{aligned} i\mathcal{L} &= -\frac{2}{\hbar\gamma}\left(\frac{1}{2m}\pi^2 + a\right) - \frac{\varepsilon}{\gamma^2}\frac{\boldsymbol{\pi}}{m}\frac{\partial}{\partial \boldsymbol{x}}\,, \\ i\mathcal{L}' &= -\frac{\varepsilon 2\omega_c}{\gamma^2}\left(\pi_x\frac{\partial}{\partial\pi_y} - \pi_y\frac{\partial}{\partial\pi_x}\right), \\ i\mathcal{L}_{\text{ext}} &= 2eE\frac{\partial}{\partial\pi_x}\,.\end{aligned}\right\} \tag{53}$$

The equilibrium distribution function is

$$f^0 = \frac{k_BT}{(1/2m)\boldsymbol{\pi}^2 + a}\,. \tag{54}$$

Following Kubo's derivation of the expression for the conductivity tensor to the linear order of H, one gets

$$\sigma'_{xy} = \int_0^\infty dt\,\phi_{xy}(t)\,, \tag{55}$$

$$\phi_{xy}(t) = -\frac{4e^2}{m^2}\int d\Gamma\int_0^t dt' e^{i\mathcal{L}_0(t-t')} i\mathcal{L}' e^{i\mathcal{L}_0 t'} \frac{\partial}{\partial \pi_x} f^0$$

$$= \frac{8e^2\omega_c t}{m^2}\frac{\varepsilon k_B T}{\gamma}\frac{1}{2\pi d}\int d\pi_x d\pi_y \pi_y{}^2 \frac{\exp[-(2/\hbar\gamma)(1/2m\cdot\boldsymbol{\pi}^2+a)]}{(1/2m\cdot\boldsymbol{\pi}^2+a)^2}, \tag{56}$$

where the restriction of phase integral is used. Explicit calculations lead to

$$\sigma'_{xy} = \frac{e^2}{6\pi d}\omega_c \varepsilon k_B T \frac{1}{a^2}, \tag{57}$$

i.e.

$$\sigma'_{xy} = \frac{\omega_c \tau}{9}\frac{1}{gN}\frac{e^2}{16\hbar d}\frac{1}{\eta^2}. \tag{58}$$

The temperature dependence is thus found more singular than σ'_{xx} by Aslamazov-Larkin.[1)]

The full discussion of this large contribution to the Hall conductivity due to fluctuations will be given in a following paper, based on the microscopic theory, including contributions form the Maki process.[11)]

A note is added concerning the difference of the new term in the fluctuation propagator or the TDGL equation form the corresponding one by Abrahams-Tsuneto[12)] and Maki.[13)] Theirs include the $\boldsymbol{q}^2$ term as

$$\left(D+\frac{i}{4m}\right)\boldsymbol{q}^2. \tag{59}$$

As is shown in Appendix B, $\mathcal{D}^R(\boldsymbol{q}, \omega)$ is a real number for $\omega=0$ in any approximation for the impurity scattering other than the Born and the ladder one as far as we work in the ladder approximation for the BCS coupling.

In such approximations we may expect the corrections of order of $1/\varepsilon_F\tau$ to the coefficient of ω, which do not appear in the ladder approximation for the impurity scattering. The consistent treatments over Σ and the vertex corrections must be done. One more open problem now is the validity of neglecting the real part of Σ, which is closely related to the model potential due to impurities.

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Appendix A

Calculation of $P^R(\boldsymbol{q}, \boldsymbol{q}')$

In the ladder approximation Eq. (29) becomes

$$P(\boldsymbol{q}_1, \boldsymbol{q}_2, i\omega_\lambda) = -T\sum_n \sum_{\boldsymbol{k}} G(\boldsymbol{k}-\boldsymbol{q}, i\varepsilon_n) G(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2, i\varepsilon_n - i\omega_\lambda) G(-\boldsymbol{k}, -i\varepsilon_n)$$
$$\times [1-n_i u^2 g(\boldsymbol{q}_1, i\varepsilon_n, i\varepsilon_n)]^{-1} [1-n_i u^2 g(\boldsymbol{q}_2, i\varepsilon_n, -i\varepsilon_n + i\omega_\lambda)]^{-1}$$
$$\times [1-n_i u^2 g(\boldsymbol{q}_1+\boldsymbol{q}_2, i\varepsilon_n, i\varepsilon_n - i\omega_\lambda)]^{-1}. \tag{A·1}$$

The analytically continued function of the summand is calculated similarly to $\tilde{\Pi}^{BB'}$, as

$$\tilde{P}^{BB'}(\boldsymbol{q}_1, \boldsymbol{q}_2, x, x') = \frac{i\pi N}{\varepsilon_F^2} \frac{1}{\varphi_1+\varphi_1'-i\gamma} \frac{1}{\varphi_1+\varphi_2-i\gamma-(\boldsymbol{q}^2/3k_F^2)\cdot i\gamma/(\varphi_1+\varphi_2)^2}$$
$$\times \frac{1}{\varphi_1'+\varphi_2-i\gamma-(\boldsymbol{q}_1+\boldsymbol{q}_2)^2/3k_F^2\cdot i\gamma/(\varphi_1'+\varphi_2)}$$
$$\times \left\{1+\frac{\boldsymbol{q}_1^2}{3k_F^2}\left[\frac{1}{(\varphi_1+\varphi_2)^2}+\frac{1}{2(\varphi_1+\varphi_1')(\varphi_1+\varphi_2)}\right.\right.$$
$$\left.+\frac{1}{2(\varphi_1+\varphi_2)(\varphi_1'+\varphi_2)}-\frac{1}{2(\varphi_1+\varphi_1')(\varphi_1'+\varphi_2)}\right]$$
$$\left.+\frac{(\boldsymbol{q}_1+\boldsymbol{q}_2)^2}{3k_F^2}(\text{terms obtained by exchange of } \varphi_1 \text{ and } \varphi_1')\right\},$$

where φ_1, φ_2 or φ_1' means the abbreviation of $\varphi_1^B(x)$, $\varphi_2^B(x)$ or $\varphi_1^{B'}(x')$, respectively. Here $\boldsymbol{q}'^2$ is put to zero becouse the space charge, $\boldsymbol{\nabla}^2 V$, is vanishing. Expanding φ's in terms of T/ε_F and $1/\varepsilon_F\tau$, one obtains, to the lowest order of ω,

$$P^R(\boldsymbol{q}_1, \boldsymbol{q}_2) = \frac{N}{4}\int dx \tanh\frac{x}{2T}\frac{x}{-2\varepsilon_F}\frac{1}{x+iD/2\cdot\boldsymbol{q}_1^2}\frac{1}{x+iD/2\cdot(\boldsymbol{q}_1+\boldsymbol{q}_2)^2}$$
$$+\frac{\omega N}{4}\int dx \tanh\frac{x}{2T}\frac{d}{dx}\left[\frac{2}{\omega(1-x/2\varepsilon_F)}\frac{1}{x-iD/2\cdot\boldsymbol{q}_1^2}\frac{1}{-2}\left(1+\frac{i}{4\varepsilon_F}D(\boldsymbol{q}_1+\boldsymbol{q}_2)^2\right)\right], \tag{A·2}$$

which leads to Eq. (30).

Appendix B

Reality of $\mathcal{D}^R(\boldsymbol{q}, 0)$

By definition, we have

$$\Pi^R(\boldsymbol{q}, 0) = -\frac{1}{4\pi i}\int dx \int d\boldsymbol{r} \exp[i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')]$$
$$\times \{\overline{G^R(\boldsymbol{r}, \boldsymbol{r}', x) G^A(\boldsymbol{r}, \boldsymbol{r}', -x)} - \overline{G^A(\boldsymbol{r}, \boldsymbol{r}', x) G^R(\boldsymbol{r}, \boldsymbol{r}', -x)}\}, \tag{B·1}$$

where bar means the ensemble average over the random distribution of impurities. In Eq. (B·1) we have put $\omega=0$, for no singularity appears if $\omega \to 0$. Impurity potential is real and the scattering is elastic. We have, then,

$$[G^R(\boldsymbol{r}, \boldsymbol{r}', x)]^* = G^A(\boldsymbol{r}, \boldsymbol{r}', x).$$

Thus $\Pi^R(\boldsymbol{q}, 0)$ is real.

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