J. Fluid Mech. (1993), vol. 249, pp. 241–259

Wave-drift damping of floating bodies

By J. N. NEWMAN

Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(Received 14 April 1992 and in revised form 26 October 1992)

Wave-drift damping results from low-frequency oscillatory motions of a floating body, in the presence of an incident wave field. Previous works have analysed this effect in a quasi-steady manner, based on the rate of change of the added resistance in waves, with respect to a small steady forward velocity. In this paper the wave-drift damping coefficient is derived more directly, from a perturbation analysis where the low-frequency body oscillations are superposed on the diffraction field. Unlike the case of body oscillations in calm water, where the damping due to wave radiation is asymptotically small for low frequencies, the superposition of oscillatory motions on the diffraction field results in an order-one damping coefficient. All three degrees of freedom are considered in the horizontal plane. The resulting matrix of damping coefficients is derived from pressure integration on the body, and transformed in special cases to a far-field control surface.

1. Introduction

The linear exciting force exerted by water waves on a floating body is proportional to the wave amplitude A, and acts with the same frequency ω. Quadratic nonlinear interactions at second order result in a steady 'mean drift force' of order A^2 which is independent of time. More generally, in a realistic spectrum of ocean waves, slowly varying second-order forces occur at the differences $\sigma = |\omega_i - \omega_j|$ between the frequencies (ω_i, ω_j) of each pair of spectral components. (Second-order interactions also cause high-frequency forces which are important for certain types of offshore platforms, cf. Lee *et al.* 1991, but these are quite different in their character, and are not considered in the present work.)

Vessels moored in deep water can experience resonant low-frequency motions in the horizontal plane, due to excitation from the slowly varying second-order wave forces. Important practical examples are moored ships, where the mooring system has a relatively small linear restoring force, and tension-leg platforms which are similar dynamically to an inverted pendulum with the buoyancy force directed upwards. The resonant response of these vessels is limited only by the relevant hydrodynamic damping mechanisms.

At low frequencies the conventional linear damping of body motions due to wave radiation is negligible. For example, the horizontal exciting force acting on a fixed three-dimensional body in long wavelengths is proportional to the pressure gradient of the incident waves, or $O(\sigma^2)$ for waves of unit amplitude and low frequency σ. It follows from the Haskind relations (cf. Newman 1977) that the horizontal damping coefficients are $O(\sigma^7)$. On the other hand, the second-order wave force acting on the body tends to a finite limit equal to the mean drift force, as the difference-frequency tends to zero. Thus, in the absence of more significant damping effects, resonant second-order motions would occur with velocity proportional to $O(\sigma^{-7})$ and amplitude proportional to $O(\sigma^{-8})$.

An obvious alternative source of damping is viscous drag. However, the quasi-steady drag force is proportional to the square of the relative velocity between the body and surrounding fluid. From an equivalent-damping synthesis (cf. Faltinsen 1991) the resultant damping coefficient is formally of order σ.

The relevance of 'wave-drift damping' was suggested in an experimental study by Wichers & van Sluijs (1979), where the oscillatory surge motions were measured on two ship models restrained by spring moorings. Comparisons of the extinction rate in calm water and in waves of differing heights clearly indicated the presence of a damping force proportional to the square of the wave height. (See also Faltinsen 1991, figure 5.20, and Chakrabarti & Cotter 1992, figure 11.)

These experimental observations have been explained in a quasi-steady manner, by considering the added resistance in waves due to steady translation of the body with small velocity U. This force is proportional to the square of the incident-wave amplitude, tending to the zero-speed mean drift force as $U \rightarrow 0$, with the leading-order correction proportional to U. The derivative with respect to U, evaluated at $U = 0$, represents a force proportional to the velocity, which is interpreted as a damping coefficient.

This quasi-steady explanation has been used as the basis for several theoretical and computational studies where the diffraction problem is solved for a body moving with a steady forward velocity U, in the presence of incident waves. The derivative of the resulting mean force with respect to U is derived analytically, using pertinent asymptotic analysis for $U \ll 1$ (Nossen, Grue & Palm 1991; Emmerhoff & Sclavounos 1992). Alternatively, in the approach followed by Zhao & Faltinsen (1989), the damping coefficient is evaluated by numerical differentiation from computations with small non-zero velocity.

In the present paper the phenomenon of wave-drift damping is considered in a more direct manner, without introducing a steady forward velocity. Our approach is motivated by the conditions of the experimental observations. Whereas the damping due to wave radiation is asymptotically small with respect to the frequency of body oscillations in calm water, a more significant damping force occurs in the presence of an incident wave field. This suggests the use of perturbation methods to analyse the higher-order interaction between low-frequency body motions with frequency σ, and the diffraction problem for the fixed body in incident waves of frequency ω. The analysis is simplified by assuming that $\sigma \ll \omega$. In this respect the formulation is similar to that of Agnon & Mei (1985), who employ the method of multiple scales to analyse the corresponding two-dimensional problem for a rectangular body in shallow water.

One feature of the present approach is that, like the conventional linear analysis of floating-body motions, it is possible to consider not only the force due to longitudinal surge motions, but also the more general case of arbitrary motions with multiple degrees of freedom. In practice the most important modes are translations in the horizontal plane (surge and sway), and rotation about the vertical axis (yaw). Our principal objective is to evaluate the three-by-three matrix, of the components of the wave-drift damping horizontal force and vertical moment, due to low-frequency oscillatory motions in the corresponding modes.

The perturbation expansion for the velocity potential is postulated in §2, and appropriate boundary conditions are derived. In §3 the low-frequency approximation is introduced to simplify the free-surface boundary conditions for the required

higher-order potentials. In §4 the hydrodynamic force acting on the body is expressed in an analogous perturbation expansion, and the wave-drift damping coefficient is derived from pressure integration as the component of the force in phase with the body's velocity which is proportional to the square of the incident wave amplitude. Integral relations are used in §5 to replace local integration over the body and free surface by integrals over a control surface in the far field. Comparisons are made with the results of the quasi-steady analyses by Emmerhoff & Sclavounos (1992) and by Grue & Palm (1993). In §6 the present results are discussed from the standpoints of physical interpretation and computational implementation. Various integral relations used in the analysis are derived in the Appendix.

2. Expansion of the velocity potential and boundary conditions

Consider the diffraction problem, resulting from the interaction of monochromatic incident waves with frequency ω and amplitude $|A|$, and also the radiation problem resulting from oscillatory body motions $\xi(t)$ in the horizontal plane with frequency σ. Three separate modes of motion are included: surge (parallel to the x-axis), sway (parallel to the y-axis), and yaw (rotation about the vertical z-axis). An indicial notation ($j = 1, 2, 6$) is used to denote each of these three modes of motion, respectively, with the corresponding oscillatory displacement $\xi_j \sin(\sigma t)$ and velocity $\sigma\xi_j \cos(\sigma t)$. Since the wave-drift damping force and moment are linear in these displacements, it is sufficient to consider a single degree of motion without regard for nonlinear interactions between different modes. The phase of the incident wave is not restricted, hence there is no loss of generality in defining the body motions to be in phase with $\sin(\sigma t)$; similarly, when a complex representation is adopted for the oscillatory time dependence, it will be assumed that ξ_j is a real coefficient. Later it will be assumed that the frequency of the body motions is much less than that of the incident waves, i.e. $\sigma \ll \omega$.

The fluid is considered to be infinitely deep, and the flow is assumed to be irrotational. For the above inputs the appropriate perturbation expansion for the velocity potential can be expressed in the form

$$\begin{aligned}\Phi(x,t) = \mathrm{Re}\{&\phi_1 \mathrm{e}^{\mathrm{i}\omega t} + \phi_2^{(0)} + \phi_2^{(2)} \mathrm{e}^{2\mathrm{i}\omega t} + \ldots \\ &+ \xi_j[\phi_{0j} \mathrm{e}^{\mathrm{i}\sigma t} + \phi_{1j}^{(+)} \mathrm{e}^{\mathrm{i}(\omega+\sigma)t} + \phi_{1j}^{(-)} \mathrm{e}^{\mathrm{i}(\omega-\sigma)t} \\ &+ \phi_{2j}^{(0)} \mathrm{e}^{\mathrm{i}\sigma t} + \phi_{2j}^{(2+)} \mathrm{e}^{\mathrm{i}(2\omega+\sigma)t} + \phi_{2j}^{(2-)} \mathrm{e}^{\mathrm{i}(2\omega-\sigma)t} + \ldots]\}. \qquad (2.1)\end{aligned}$$

Here the potentials ϕ_m and ϕ_{mj} depend on the space coordinates x. The first subscript refers to the order of magnitude in A, and the second subscript refers to the mode of motion. Thus $\phi_m = O(A^m)$ are the components of the diffraction solution, and ϕ_{mj} are potentials of the same order in A, due to the body motions. Superscripts are used when necessary to denote harmonic time dependence in the respective frequencies. The symbol Re denotes the real part of the complex expression. Without loss of generality the potential $\phi_2^{(0)}$, the component of the second-order diffraction solution which is independent of time, is assumed to be real. The remaining potentials displayed on the right-hand side of (2.1) are complex. Terms which are conjugate to those in (2.1) can be neglected, hence it is permissible to include only the complex exponentials which have a positive imaginary argument when $\omega \gg \sigma > 0$.

The functions ϕ_j and ϕ_{mj} in (2.1) are governed by Laplace's equation in the fluid domain, with appropriate boundary conditions specified on the body and free surface. The boundary conditions are completed by requiring each potential to vanish at large depths below the free surface. Except for the incident-wave potential

$$\phi_I = (gA/\omega)\,\mathrm{e}^{Kz-\mathrm{i}K(x\cos\beta+y\sin\beta)}, \qquad (2.2)$$

which is a specified component of the first-order diffraction solution ϕ_1, each potential in (2.1) must satisfy the radiation condition of outgoing waves in the far field. In (2.2) g is the gravitational acceleration, $K = \omega^2/g$ is the wavenumber, and β denotes the angle of incidence relative to the positive x-axis. Since the phase is unrestricted, the amplitude A is complex.

The first-order diffraction potential ϕ_1 is subject to the boundary conditions

$$\phi_{1n} = 0 \quad \text{on } S_b, \qquad (2.3)$$

and

$$g\phi_{1z} - \omega^2\phi_1 = 0 \quad \text{on } z = 0. \qquad (2.4)$$

Here S_b is the submerged portion of the body surface, in its mean position. The subscript n denotes the normal derivative, with the unit normal vector $\boldsymbol{n}$ defined in the positive sense to point out of the fluid domain, and hence into the interior of the body. Subscripts (x, y, z, t) denote partial differentiation with respect to the corresponding variables.

The first-order radiation potential ϕ_{0j} satisfies the boundary condition

$$\phi_{0jn} = \sigma n_j \quad \text{on } S_b, \qquad (2.5)$$

where the three components of the vector $\{n_j\}$ are defined by

$$n_1 = n_x, \quad n_2 = n_y, \quad n_6 = xn_y - yn_x. \qquad (2.6)$$

The appropriate free-surface boundary condition is

$$g\phi_{0jz} - \sigma^2\phi_{0j} = 0 \quad \text{on } z = 0. \qquad (2.7)$$

The higher-order potentials in (2.1) satisfy inhomogeneous boundary conditions on the body and/or the free surface. In the analysis to follow it will be necessary to consider various products of the time-dependent potential (2.1) and its derivatives. These products can be expressed in a similar form, with appropriate coefficients. Thus, if two functions $A(t)$ and $B(t)$ are represented as in (2.1) with corresponding coefficients a_{mn} and b_{mn}, the product $C = AB$ can be represented in the same form. The relevant low-frequency components of C are as follows:

$$c_2^{(0)} = \tfrac{1}{2}\,\mathrm{Re}\{a_1 b_1^*\}, \qquad (2.8)$$

$$c_{1j}^{(+)} = \tfrac{1}{2}(a_1 b_{0j} + a_{0j} b_1), \qquad (2.9)$$

$$c_{1j}^{(-)} = \tfrac{1}{2}(a_1 b_{0j}^* + a_{0j}^* b_1), \qquad (2.10)$$

$$c_{2j}^{(0)} = a_2^{(0)} b_{0j} + a_{0j} b_2^{(0)} + \tfrac{1}{2}(a_{1j}^{(+)} b_1^* + a_1^* b_{1j}^{(+)} + a_{1j}^{(-)*} b_1 + a_1 b_{1j}^{(-)*}). \qquad (2.11)$$

The coefficients of triple products can be derived by repeated application of the same relations.

On the exact oscillatory position $\tilde{S}_b$ of the body surface the kinematic boundary condition is

$$\boldsymbol{\Phi}_n(\boldsymbol{x}, t) = \dot{\boldsymbol{\xi}}(t)\cdot\boldsymbol{n}. \qquad (2.12)$$

Boundary conditions for the potentials in (2.1) on the mean body surface are derived by Taylor series expansion of the left-hand side of (2.12) to the mean position S_b, and accounting for the rotation of the body-fixed normal vector $\boldsymbol{n}$. The appropriate modification of (2.12) follows from the analysis outlined by Newman (1978, equation 3.28). Neglecting terms of order ξ^2 yields the following boundary conditions for the three separate modes of motion:

$$\boldsymbol{\Phi}_{1n} = \dot{\xi}_1(t)\, n_x - \xi_1(t)\,\boldsymbol{\Phi}_{1xn}, \qquad (2.13)$$

$$\boldsymbol{\Phi}_{2n} = \dot{\xi}_2(t)\, n_y - \xi_2(t)\,\boldsymbol{\Phi}_{2yn}, \qquad (2.14)$$

$$\boldsymbol{\Phi}_{6n} = \dot{\xi}_6(t)(xn_y - yn_x) - \xi_6(t)(x\boldsymbol{\Phi}_{6yn} - y\boldsymbol{\Phi}_{6xn}) + \xi_6(t)(n_x\boldsymbol{\Phi}_{6y} - n_y\boldsymbol{\Phi}_{6x}) \quad \text{on } S_b. \qquad (2.15)$$

The last pair of terms in (2.15) accounts for the rotation of the coordinate system. The other terms in (2.13)–(2.15) are the results of Taylor expansion between the oscillatory and mean body surfaces.

The boundary conditions (2.13)–(2.15) can be expressed on the unified form

$$\boldsymbol{\Phi}_{jn} = \dot{\xi}_j(t)\, n_j - \xi_j(t)\,\mathscr{D}_{jn}(\boldsymbol{\Phi}_j) \quad \text{on } S_b, \qquad (2.16)$$

where

$$\mathscr{D}_1(\phi) = \phi_x, \quad \mathscr{D}_2(\phi) = \phi_y, \quad \mathscr{D}_6(\phi) = x\phi_y - y\phi_x. \qquad (2.17)$$

Since $\nabla^2\phi = 0$, each of the three functions defined by (2.17) is harmonic. Normal derivatives of the same functions are denoted by

$$\mathscr{D}_{1n}(\phi) = \phi_{xn}, \quad \mathscr{D}_{2n}(\phi) = \phi_{yn}, \quad \mathscr{D}_{6n}(\phi) = x\phi_{yn} - y\phi_{xn} + n_x\phi_y - n_y\phi_x. \qquad (2.18)$$

When applied to the potential for steady-state translation of the body, the normal derivatives (2.18) are equivalent to the so-called 'm-terms' which appear in the quasi-steady analyses (Nossen *et al.* 1991; Emmerhoff & Sclavounos 1992). In the present work the operators (2.17) are applied in a different manner, to the diffraction solution with the body fixed.

Collecting the terms of the same order in (2.16) and using (2.8)–(2.11), the following boundary conditions are derived on the mean position of the body:

$$\phi_{2n}^{(0)} = 0, \qquad (2.19)$$

$$\phi_{1jn}^{(\pm)} = \pm\tfrac{1}{2}\mathrm{i}\,\mathscr{D}_{jn}(\phi_1), \qquad (2.20)$$

$$\phi_{2jn}^{(0)} = \mathrm{i}\,\mathscr{D}_{jn}(\phi_2^{(0)}). \qquad (2.21)$$

Next we consider the free-surface condition, which is expressed in exact form as

$$\boldsymbol{\Phi}_{tt} + g\boldsymbol{\Phi}_z = -\frac{\partial}{\partial t}V^2 - \tfrac{1}{2}\boldsymbol{V}\cdot\nabla(V^2) \quad \text{on } z = \zeta, \qquad (2.22)$$

where $\boldsymbol{V} = \nabla\boldsymbol{\Phi}$ is the fluid velocity vector. This boundary condition is transferred to the mean free surface $z = 0$ using the following expansion for ζ:

$$\zeta = -\frac{1}{g}(\boldsymbol{\Phi}_t + \tfrac{1}{2}V^2)_{z=\zeta} = -\frac{1}{g}(\boldsymbol{\Phi}_t + \tfrac{1}{2}V^2 + \zeta\boldsymbol{\Phi}_{tz})_{z=0} + O(\boldsymbol{\Phi}^3)$$

$$= -\frac{1}{g}\left(\boldsymbol{\Phi}_t + \tfrac{1}{2}V^2 - \frac{1}{g}\boldsymbol{\Phi}_t\boldsymbol{\Phi}_{tz}\right)_{z=0} + O(\boldsymbol{\Phi}^3). \qquad (2.23)$$

Using (2.23) in the Taylor-series expansion of (2.22) about $z = 0$,

$$\boldsymbol{\Phi}_{tt} + g\boldsymbol{\Phi}_z = \frac{1}{g}\boldsymbol{\Phi}_t(\boldsymbol{\Phi}_{ttz} + g\boldsymbol{\Phi}_{zz}) - 2\boldsymbol{V}\cdot\boldsymbol{V}_t$$

$$-\frac{1}{g}\left(\frac{1}{g}\boldsymbol{\Phi}_t\boldsymbol{\Phi}_{tz} - \tfrac{1}{2}V^2\right)(\boldsymbol{\Phi}_{ttz} + g\boldsymbol{\Phi}_{zz}) - \frac{1}{2g^2}\boldsymbol{\Phi}_t^2(\boldsymbol{\Phi}_{ttzz} + \boldsymbol{\Phi}_{zzz})$$

$$+\frac{2}{g}\boldsymbol{\Phi}_t(\boldsymbol{V}_z\cdot\boldsymbol{V}_t + \boldsymbol{V}\cdot\boldsymbol{V}_{tz}) - \tfrac{1}{2}\boldsymbol{V}\cdot\nabla(V^2) + O(\boldsymbol{\Phi}^4) \quad \text{on } z = 0. \qquad (2.24)$$

For the same potentials associated with (2.19)–(2.21) the corresponding free-surface boundary conditions can be written in the forms

$$g\phi_{2z}^{(0)} = f_2^{(0)}, \qquad (2.25)$$

$$g\phi_{1jz}^{(\pm)} - (\omega \pm \sigma)^2 \phi_{1j}^{(\pm)} = f_{1j}^{(\pm)}, \qquad (2.26)$$

$$g\phi_{2jz}^{(0)} - \sigma^2 \phi_{2j}^{(0)} = f_{2j}^{(0)} \quad \text{on } z = 0. \qquad (2.27)$$

The inhomogeneous function on the right-hand side of (2.25) is

$$f_2^{(0)} = \mathrm{Re}\left\{\frac{\mathrm{i}\omega}{2g}\phi_1 \frac{\partial}{\partial z}(-\omega^2\phi_1^* + g\phi_{1z}^*)\right\} = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathrm{i}\omega\phi_1\,\phi_{1zz}^*\}. \qquad (2.28)$$

Here (2.4) has been used to eliminate the imaginary contribution from the first derivative. The right-hand side of (2.26) can be evaluated in the form

$$f_{1j}^{(\pm)} = \frac{\mathrm{i}\omega}{2g}\phi_1(-\sigma^2\phi_{0jz} + g\phi_{0jzz})^{\pm} \pm \frac{\mathrm{i}\sigma}{2g}(\phi_{0j})^{\pm}(-\omega^2\phi_{1z} + g\phi_{1zz}) - \mathrm{i}(\omega \pm \sigma)\,\nabla\phi_1 \cdot \nabla(\phi_{0j})^{\pm}. \qquad (2.29)$$

Here the superscript ($\pm$) following a function in parentheses denotes the function or its complex conjugate, respectively. The corresponding result for the right-hand side of (2.27) will be derived in §3, under the approximation of small σ.

The boundary condition (2.25) implies that $\phi_2^{(0)}$ is non-wavelike. In deep water, the right-hand side of (2.28) is $o(1/R)$, for large horizontal radius R, since the terms in parentheses vanish for plane waves. A more careful analysis in the Appendix shows that $\phi_2^{(0)} = O(R^{-2})$.

3. Low-frequency analysis

It is appropriate to consider the asymptotic forms of the potentials for $\sigma \ll \omega$ and $\sigma^2 l \ll g$, where l is the characteristic lengthscale of the body and $\omega^2 l/g = O(1)$ is implied. In the limit $\sigma = 0$, (2.7) reduces to the rigid-free-surface condition. In view of the boundary condition (2.5) the potential is re-scaled in the form

$$\phi_{0j} \sim \sigma\varphi_j, \qquad (3.1)$$

where the canonical potentials φ_j are real and satisfy the boundary conditions

$$\varphi_{jn} = n_j \quad \text{on } S_b, \qquad (3.2)$$

$$\varphi_{jz} = 0 \quad \text{on } z = 0. \qquad (3.3)$$

These are the velocity potentials for translation or rotation of the body, with unit velocity, in the presence of the 'rigid' free surface. For small values of $\sigma^2 l/g$, (3.1) applies throughout an inner domain which is large compared to l and the wavelength $2\pi g/\omega^2$ of the diffraction problem, but small compared to the wavelength $2\pi g/\sigma^2$ of the low-frequency oscillations. Hereafter our attention will be restricted to this inner region. From the free-surface condition (2.7) it follows that the imaginary part of ϕ_{0j} is of order σ^3.

Next we consider (2.26) and the associated functions (2.29), which define the free-surface boundary conditions for the potentials $\phi_{1j}^{(\pm)}$. In the low-frequency analysis it is convenient to define the auxiliary potentials

$$\phi_{1j}^{(+)} - \phi_{1j}^{(-)} = P_j, \qquad (3.4)$$

$$\phi_{1j}^{(+)} + \phi_{1j}^{(-)} = \sigma Q_j. \qquad (3.5)$$

From (2.20), the corresponding boundary conditions on the mean position S_b of the body surface are

$$P_{jn} = \mathrm{i}\mathscr{D}_{jn}(\phi_1), \qquad (3.6)$$

$$Q_{jn} = 0. \qquad (3.7)$$

Similarly, using (2.26) and (2.29) on $z = 0$, and neglecting terms of order σ^2,

$$gP_{jz} - \omega^2 P_j = 0, \qquad (3.8)$$

$$gQ_{jz} - \omega^2 Q_j = 2\omega P_j + \mathrm{i}\omega\phi_1\varphi_{jzz} - 2\mathrm{i}\omega\nabla\phi_1 \cdot \nabla\varphi_j. \qquad (3.9)$$

Since the boundary conditions (2.20) and (3.6) do not involve σ, it follows that the functions Q_j and P_j defined by (3.4) and (3.5) also are independent of σ. The error in neglecting σ is a factor $1 + O(\sigma^2)$.

A potential which satisfies the boundary conditions (3.6) and (3.8) is easily constructed in the form $P_j = \mathrm{i}\mathscr{D}_j(\phi_1)$, but this violates the radiation condition since the incident wave is part of ϕ_1. For $j = 1, 2$ this problem can be overcome simply by adding an extra term proportional to ϕ_1, with the results

$$P_1 = \mathrm{i}\phi_{1x} - K\cos\beta\phi_1, \quad P_2 = \mathrm{i}\phi_{1y} - K\sin\beta\phi_1. \qquad (3.10)$$

For $j = 6$ the appropriate extra term involves the derivative $\phi_{1\beta}$ with respect to the wave heading angle β:

$$P_6 = \mathrm{i}\mathscr{D}_6(\phi_1) + \mathrm{i}\phi_{1\beta}. \qquad (3.11)$$

To confirm these solutions, note that each sum vanishes for the incident-wave potential (2.2), hence (3.10) and (3.11) satisfy the far-field radiation condition. In view of the boundary condition (2.3) there is no contribution from the second terms to the boundary condition (3.6), and thus the validity of (3.10)–(3.11) is established.

In the analysis of Q_j attention is first given to the term $2\omega P_j$ on the right-hand side of the free-surface boundary condition (3.9). The general solution can be expressed in the form

$$Q_j = (2\omega/g)P_{jK} + q_j, \qquad (3.12)$$

where the subscript K denotes differentiation with respect to the wavenumber. The potential q_j is subject to the boundary conditions

$$q_{jn} = -(2\omega/g)P_{jKn} \quad \text{on } S_b, \qquad (3.13)$$

and

$$gq_{jz} - \omega^2 q_j = \mathrm{i}\omega\phi_1\varphi_{jzz} - 2\mathrm{i}\omega\,\nabla\phi_1 \cdot \nabla\varphi_j \quad \text{on } z = 0. \qquad (3.14)$$

Since P_j is a solution of the homogeneous free-surface condition, its effect on the right-hand side of (3.9) is secular. For large values of the horizontal radius R the radiation condition implies that $P_j \approx R^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}KR}$, and thus the solution (3.12) is non-uniform in the far field, with the asymptotic behaviour $Q_j \sim R^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}KR}$. This does not result in practical difficulties in the analysis to follow, provided the domain considered is suitably restricted.

Since φ_j is asymptotic to a Rankine dipole, the right-hand side of (3.14) is of order $1/R^3$ in the far field. The solution of this boundary condition is non-trivial, but uniform at infinity with the same far-field form as a first-order radiating wave. Thus the conventional far-field radiation condition is applicable for the regular component q_j.

Next we consider the function $f_{2j}^{(0)}$ defined by (2.27). The possible contributions are indicated from the complete third-order free-surface condition (2.24), and involve the following combinations of lower-order potentials and their drivatives:

$$(\phi_1, \phi_{1j}), \quad (\phi_2, \phi_{0j}), \quad (\phi_1^2, \phi_{0j}).$$

Only the first combination is independent of ϕ_{0j}, and hence of order one as $\sigma \to 0$. After substituting the potentials ϕ_1 and $\phi_{1j}^{(\pm)}$ with appropriate time-dependent factors in the terms on the first line of the right side of (2.24), and collecting all components with the time dependence $e^{i\sigma t}$, the limit for $f_{2j}^{(0)}$ as $\sigma \to 0$ is obtained in the form

$$f_{2j}^{(0)} \sim \tfrac{1}{2}i\omega[\phi_1^*(K\phi_{1jz}^{(+)} - \phi_{1jzz}^{(+)}) - \phi_{1j}^{(+)}(K\phi_{1z}^* - \phi_{1zz}^*) - \phi_1(K\phi_{1jz}^{(-)*} - \phi_{1jzz}^{(-)*}) + \phi_{1j}^{(-)*}(K\phi_{1z} - \phi_{1zz})]$$
$$\sim \tfrac{1}{4}i\omega[-\phi_1^* P_{jzz} + P_j\phi_{1zz}^* - \phi_1 P_{jzz}^* + P_j^*\phi_{1zz}]. \qquad (3.15)$$

In the far field the second derivatives in (3.15) can be replaced by K^2, and it follows that $f_{2j}^{(0)} = o(1/R)$. Thus in the limit as $\sigma \to 0$ the boundary condition (2.27) implies that $\phi_{2j}^{(0)}$ is non-wavelike, vanishing algebraically in the far field in the same manner as $\phi_2^{(0)}$.

The solutions (3.10) can be substituted in (3.15) for $j = 1, 2$ with the result

$$f_{2j}^{(0)} \sim \tfrac{1}{4}\omega\mathscr{D}_j(\phi_1^*\phi_{1zz} - \phi_1\phi_{1zz}^*) = i\mathscr{D}_j(f_2^{(0)}). \qquad (3.16)$$

Thus, in the limit $\sigma \to 0$, the potentials

$$\phi_{2j}^{(0)} = i\mathscr{D}_j(\phi_2^{(0)}) \qquad (3.17)$$

are solutions of the boundary conditions (2.21) and (2.27) for ($j = 1, 2$). Similarly, for $j = 6$,

$$f_{26}^{(0)} \sim i\mathscr{D}_6(f_2^{(0)}) + if_{2\beta}^{(0)}, \qquad (3.18)$$

and

$$\phi_{26}^{(0)} = i\mathscr{D}_6(\phi_2^{(0)}) + i\phi_{2\beta}. \qquad (3.19)$$

4. The hydrodynamic pressure force and damping coefficients

Perturbation expansions similar to (2.1) can be assumed for the pressure and the resulting force (and moment) acting on the body. The appropriate terms to consider for the component of the force or moment in the same direction as the mode ξ_i, due to the diffraction field, the oscillatory motion in the mode ξ_i, and their interaction, are

$$F_i(t) = \mathrm{Re}\,\{F_{1i}e^{i\omega t} + F_{2i}^{(0)} + F_{2i}^{(2)}e^{2i\omega t} + \dots$$
$$+ \xi_j[F_{0ij}e^{i\sigma t} + F_{1ij}^{(+)}e^{i(\omega+\sigma)t} + F_{1ij}^{(-)}e^{i(\omega-\sigma)t}$$
$$+ F_{2ij}^{(0)}e^{i\sigma t} + F_{2ij}^{(2+)}e^{i(2\omega+\sigma)t} + F_{2ij}^{(2-)}e^{i(2\omega-\sigma)t} + \dots]\}. \qquad (4.1)$$

Here F_{1i} is the first-order exciting force and $F_{2i}^{(0)}$ is the second-order mean drift force for the fixed body. F_{0ij} is the first-order force due to a unit motion ξ_j, which can be expressed in the usual form

$$F_{0ij} = -(i\sigma^2 A_{0ij} + \sigma B_{0ij}), \qquad (4.2)$$

where the real coefficients A_{0ij} and B_{0ij} are the added-mass and damping coefficients. The higher-order force component $F_{2ij}^{(0)}$ can be expressed in the analogous form

$$F_{2ij}^{(0)} = -(i\sigma^2 A_{2ij} + \sigma B_{2ij}). \qquad (4.3)$$

This force is of second order in the wave amplitude, and first order in the motion amplitude.

For asymptotically small values of the frequency σ, $A_{0ij} = O(1)$, whereas $B_{0ij} = O(\sigma^7)$ as noted in the Introduction. By comparison, the highest-order added-mass coefficient A_{2ij} is of secondary importance, but the corresponding damping coefficient B_{2ij} is significant since B_{0ij} is asymptotically small.

The most direct approach to evaluate $F_{2ij}^{(0)}$ is from pressure integration on the body. The analysis is carried out for a floating body with a time-varying wetted surface $\tilde{S}_b$, mean surface S_b, waterline contour $\tilde{C}_b$, and mean waterline contour C_b, with the restriction that the body surface is smooth and vertical at the waterline. Integration around the closed contour C_b is defined in the positive sense with respect to the enclosed boundary surface S_b, i.e. in the counter-clockwise direction when viewed from above the origin. For a submerged body the integrals over the waterline can be neglected. Ultimately a fixed control surface S_c also will be used, which surrounds the body in the far field. The portion of the mean free surface between S_b and S_c will be denoted S_f. The intersection of S_c and S_f is the contour C_c.

The horizontal components of the pressure force and the vertical component of the moment are evaluated using Bernoulli's equation, in the form

$$F_i(t) = \iint_{\tilde{S}_b} pn_i\,dS = -\rho\iint_{\tilde{S}_b}(\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + gz)\,n_i\,dS \quad (i = 1, 2, 6). \qquad (4.4)$$

As in the case of the body boundary condition (2.13), the pressure is transferred from $\tilde{S}_b$ to its mean position S_b. In addition, the contribution from the time-varying intersection of $\tilde{S}_b$ with the free surface $z = \zeta$ is expanded as a line integral on C_b.

The contribution from the transfer of the pressure involves the expansion

$$p|_{\tilde{S}_b} = p|_{S_b} + \xi_j(t)\,\mathscr{D}_j(p) + O(\xi_j^2).$$

The linear correction does not affect the damping coefficients in (4.2) or (4.3) since it is out of phase with the body velocity, and the hydrostatic pressure does not contribute to (4.4). Thus the only contribution from integration over the mean surface is

$$-\rho\iint_{S_b}(\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi)\,n_i\,dS. \qquad (4.5)$$

The contribution from the contour C_b includes the vertically integrated hydrostatic pressure contribution $-\frac{1}{2}\rho g\zeta^2$, and the Taylor expansion of the dynamic pressure. The resulting line integral is

$$-\rho\oint_{C_b}[\zeta(\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi) + \tfrac{1}{2}\zeta^2\phi_{zt} + \dots + \tfrac{1}{2}g\zeta^2]\,n_i\,dl$$
$$= \frac{1}{2}\frac{\rho}{g}\oint_{C_b}\left(\phi_t^2 + \phi_t\nabla\phi\cdot\nabla\phi - \frac{1}{g}\phi_t^2\phi_{zt}\right)n_i\,dl. \qquad (4.6)$$

The last term in the first integral accounts for the integrated hydrostatic pressure; in the second integral (2.23) is used. Terms of higher order than (4.3) are neglected. The last term in (4.6), which contributes terms of order $\sigma\phi_{0j} = O(\sigma^2)$ to (4.3), is neglected hereafter.

Our objective is to evaluate B_{2ij}, the wave-drift damping coefficient in the direction i due to a velocity in mode j. It is convenient to simplify the notation by defining the new coefficients

$$\mathscr{B}_{ij} = B_{2ij}/\rho. \qquad (4.7)$$

Considering only the real part of the force coefficient $F_{2j}^{(0)}$, and substituting the appropriate components of the velocity potential (2.1) in (4.5)–(4.6),

$$\mathscr{B}_{ij} = \mathrm{Re}\iint_{S_b} (\mathrm{i}\phi_{2j}^{(0)} + \nabla\phi_2^{(0)}\cdot\nabla\varphi_j + \tfrac{1}{2}\nabla\phi_1^*\cdot\nabla Q_j)\, n_i\, \mathrm{d}S - \frac{1}{2}\frac{\omega}{g}\mathrm{Re}\oint_{C_b}\phi_1^*(\omega Q_j + P_j - \mathrm{i}\nabla\phi_1\cdot\nabla\varphi_j)\, n_i\, \mathrm{d}l. \quad (4.8)$$

Hereafter the symbol Re is deleted, with the understanding that the real part is implied in all of the following equations.

Equation (4.8) provides an explicit relation for the wave-drift damping coefficients. The principal difficulty is in evaluating the various higher-order potentials including P_j, Q_j, $\phi_2^{(0)}$, and $\phi_{2j}^{(0)}$. The dependence on the last two can be removed by further analysis, using Green's and Stokes' theorems together with appropriate boundary conditions on the body and free surface. After substituting the boundary condition (3.2) in the first term of the surface integral, and Stokes' theorem in the form (A 2) for the second and third terms,

$$\mathscr{B}_{ij} = \iint_{S_b} (\mathrm{i}\phi_{2j}^{(0)}\varphi_{in} + \varphi_j\mathscr{D}_{in}(\phi_2^{(0)}) + \tfrac{1}{2}Q_j\mathscr{D}_{in}(\phi_1^*))\,\mathrm{d}S + \frac{1}{g}\oint_{C_b}[\varphi_j f_2^{(0)} - \tfrac{1}{2}\omega\phi_1^*(P_j - \mathrm{i}\nabla\phi_1\cdot\nabla\varphi_j)]\, n_i\,\mathrm{d}l. \quad (4.9)$$

To evaluate the contribution from the first term in the surface integral of (4.9), Green's theorem is applied using the boundary conditions (2.21), (2.27), and (3.3), with the result

$$\iint_{S_b}(\mathrm{i}\phi_{2j}^{(0)}\varphi_{in} + \varphi_j\mathscr{D}_{in}(\phi_2^{(0)}))\,\mathrm{d}S = \iint_{S_b}(\varphi_j\mathscr{D}_{in}(\phi_2^{(0)}) - \varphi_i\mathscr{D}_{jn}(\phi_2^{(0)}))\,\mathrm{d}S + \frac{1}{g}\mathrm{i}\iint_{S_f}\varphi_i f_{2j}^{(0)}\,\mathrm{d}S$$
$$= I_{ij} + \frac{1}{g}\mathrm{i}\iint_{S_f}\varphi_i f_{2j}^{(0)}\,\mathrm{d}S + \iint_{S_f}(\varphi_i\mathscr{D}_{jn}(\phi_2^{(0)}) - \varphi_j\mathscr{D}_{in}(\phi_2^{(0)}))\,\mathrm{d}S, \quad (4.10)$$

where the integral which remains over S_b is defined as

$$I_{ij} = \iint_{S_b}(\varphi_{jn}\mathscr{D}_i(\phi_2^{(0)}) - \varphi_{in}\mathscr{D}_j(\phi_2^{(0)}))\,\mathrm{d}S. \quad (4.11)$$

Various alternative formulae for evaluating (4.11) are given in the Appendix. The only non-zero elements are I_{16}, I_{26}, I_{61}, and I_{62}.

The contribution from the last integral of (4.10) is

$$\frac{1}{g}\iint_{S_f}(\varphi_i\mathscr{D}_j(f_2^{(0)}) - \varphi_j\mathscr{D}_i(f_2^{(0)}))\,\mathrm{d}S = \frac{1}{g}\iint_{S_f}(\varphi_i\mathscr{D}_j(f_2^{(0)}) + \mathscr{D}_i(\varphi_j)f_2^{(0)})\,\mathrm{d}S - \frac{1}{g}\oint_{C_b}\varphi_j f_2^{(0)} n_i\,\mathrm{d}l$$
$$= \frac{1}{g}\iint_{S_f}[\varphi_i\mathscr{D}_j(f_2^{(0)})\,\mathrm{d}S + \tfrac{1}{2}\mathrm{i}\omega\mathscr{D}_i(\varphi_j)\,\phi_1\,\phi_{1zz}^*]\,\mathrm{d}S - \frac{1}{g}\oint_{C_b}\varphi_j f_2^{(0)} n_i\,\mathrm{d}l, \quad (4.12)$$

where (A 4) and (2.28) have been used.

Substituting these results in (4.9),

$$\mathscr{B}_{ij} = \frac{1}{2}\iint_{S_b}Q_j\mathscr{D}_{in}(\phi_1^*)\,\mathrm{d}S + I_{ij} + \tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\iint_{S_f}\mathscr{D}_i(\varphi_j)\,\phi_1\,\phi_{1zz}^*\,\mathrm{d}S - \frac{1}{2}\frac{\omega}{g}\oint_{C_b}\phi_1^*(P_j - \mathrm{i}\nabla\phi_1\cdot\nabla\varphi_j)\,n_i\,\mathrm{d}l + \frac{1}{g}\iint_{S_f}\varphi_i(\mathrm{i}f_{2j}^{(0)} + \mathscr{D}_j(f_2^{(0)}))\,\mathrm{d}S. \quad (4.13)$$

The integrand in the last integral vanishes for $j = 1, 2$, as a result of (3.16). For $j = 6$, it follows from (3.18) that

$$\frac{1}{g}\iint_{S_f}\varphi_i(\mathrm{i}f_{26}^{(0)} + \mathscr{D}_6(f_2^{(0)}))\,\mathrm{d}S = -\frac{1}{g}\frac{\partial}{\partial\beta}\iint_{S_f}\varphi_i f_2^{(0)}\,\mathrm{d}S$$
$$= -\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\frac{\partial}{\partial\beta}\iint_{S_f}\varphi_i\,\phi_1\,\phi_{1zz}^*\,\mathrm{d}S = -\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\frac{\partial}{\partial\beta}\iint_{S_f}\phi_1\,\nabla\phi_1^*\cdot\nabla\varphi_i\,\mathrm{d}S, \quad (4.14)$$

where (2.28) and (A 3) have been used.

Comparing (4.13) and (4.14) with the original expression (4.8), integrals over the free surface are introduced but the higher-order potentials $\phi_{2j}^{(0)}$ have been removed and the second-order diffraction potential $\phi_2^{(0)}$ only contributes via the dipole moments μ_i in the evaluation of I_{ij} from (A 16). The latter contribution is present only for the coefficients $\mathscr{B}_{16}$, $\mathscr{B}_{26}$, $\mathscr{B}_{61}$, and $\mathscr{B}_{62}$. The special role of $\phi_2^{(0)}$ has been emphasized by Grue & Palm (1992, 1993) with respect to the coefficients $\mathscr{B}_{16}$, $\mathscr{B}_{26}$.

5. Far-field analysis

Momentum relations have been used in the quasi-steady analyses by Grue & Palm (1992, 1993) and Emmerhoff & Sclavounos (1992), to relate the wave-drift damping coefficients to integrals over a control surface S_c in the far field. Similarly, in the present analysis of unsteady body motions, energy conservation could be used to relate the work done by the damping coefficients to the rate of energy flux in the far field. However, higher-order potentials and higher-order terms in the low-frequency approximation must be considered in both the momentum and energy approaches. Alternatively, integral theorems can be applied to the results of §4 with the objective of replacing integrals in the near field, over the body and free surface, by integrals over S_c. This analysis is carried out below.

From the boundary condition (3.7) and Green's theorem, the integral over S_b in (4.13) can be expressed as

$$\frac{1}{2}\iint_{S_b}(\mathscr{D}_{in}(\phi_1^*)\,Q_j - \mathscr{D}_i(\phi_1^*)\,Q_{jn})\,\mathrm{d}S = -\frac{1}{2}\iint_{S_f+S_c}(\mathscr{D}_{in}(\phi_1^*)\,Q_j - \mathscr{D}_i(\phi_1^*)\,Q_{jn})\,\mathrm{d}S. \quad (5.1)$$

Invoking the inhomogeneous condition (3.9) for Q_j, the contribution from S_f is

$$-\frac{1}{2}\iint_{S_f}(\mathscr{D}_i(\phi_{1z}^*)\,Q_j - \mathscr{D}_i(\phi_1^*)\,Q_{jz})\,\mathrm{d}S = -\mathrm{i}\frac{\omega}{g}\iint_{S_f}\mathscr{D}_i(\phi_1^*)(\mathrm{i}P_j + \nabla\phi_1\cdot\nabla\varphi_j - \tfrac{1}{2}\phi_1\varphi_{jzz})\,\mathrm{d}S. \quad (5.2)$$

The last term in (5.2) can be transformed using (A 3). With the substitutions $\phi = \phi_1 \mathscr{D}_i(\phi_1^*)$ and $\psi = \varphi_j$, and the boundary condition $\varphi_{jz} = 0$ imposed, (5.2) is equal to

$$-\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\iint_{S_f}(\mathscr{D}_i(\phi_1^*)\nabla\phi_1\cdot\nabla\varphi_j-\phi_1\nabla\mathscr{D}_i(\phi_1^*)\cdot\nabla\varphi_j)\,\mathrm{d}S$$

$$+\frac{\omega}{g}\iint_{S_f}\mathscr{D}_i(\phi_1^*)P_j\,\mathrm{d}S-\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\oint_{C_b}\mathscr{D}_i(\phi_1^*)\phi_1 n_j\,\mathrm{d}l. \quad (5.3)$$

In the contour integral (3.2) has been used, and there is no contribution from C_c since φ_j is $o(1/R)$ in the far field. After substituting (5.1) and (5.3) in (4.13),

$$\mathscr{B}_{ij} = -\frac{1}{2}\iint_{S_c}(\mathscr{D}_{in}(\phi_1^*)Q_j-\mathscr{D}_i(\phi_1^*)Q_{jn})\,\mathrm{d}S+I_{ij}$$

$$+\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\iint_{S_f}\phi_1\nabla\phi_1^*\cdot\nabla\mathscr{D}_i(\varphi_j)\,\mathrm{d}S+\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\oint_{C_b}\phi_1^*\nabla\phi_1\cdot\nabla\varphi_j n_i\,\mathrm{d}l$$

$$+\frac{1}{g}\iint_{S_f}\varphi_i(\mathrm{i}f_{2j}^{(0)}+\mathscr{D}_j(f_2^{(0)}))\,\mathrm{d}S$$

$$-\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\iint_{S_f}(\mathscr{D}_i(\phi_1^*)\nabla\phi_1\cdot\nabla\varphi_j-\phi_1\nabla\mathscr{D}_i(\phi_1^*)\cdot\nabla\varphi_j)\,\mathrm{d}S$$

$$+\frac{\omega}{g}\iint_{S_f}\mathscr{D}_i(\phi_1^*)P_j\,\mathrm{d}S+\frac{1}{2}\frac{\omega}{g}\oint_{C_b}\phi_1^*(\mathrm{i}\mathscr{D}_i(\phi_1)n_j-P_j n_i)\,\mathrm{d}l. \quad (5.4)$$

Here in the first integral over S_f (A 3) has been used, together with the boundary conditions (2.3) and (3.3).

After using Stokes' theorem in the form (A 4) to transform the two contour integrals over C_b,

$$\mathscr{B}_{ij} = -\frac{1}{2}\iint_{S_c}(\mathscr{D}_{in}(\phi_1^*)Q_j-\mathscr{D}_i(\phi_1^*)Q_{jn})\,\mathrm{d}S+I_{ij}+\frac{1}{g}\iint_{S_f}\varphi_i(\mathrm{i}f_{2j}^{(0)}+\mathscr{D}_j(f_2^{(0)}))\,\mathrm{d}S$$

$$+\frac{1}{2}\frac{\omega}{g}\iint_{S_f}[\mathscr{D}_i(\phi_1^*)P_j-\phi_1^*\mathscr{D}_i(P_j)+\mathrm{i}\mathscr{D}_i(\phi_1)\mathscr{D}_j(\phi_1^*)+\mathrm{i}\phi_1^*\mathscr{D}_j(\mathscr{D}_i(\phi_1))]\,\mathrm{d}S$$

$$-\frac{1}{2}\frac{\omega}{g}\oint_{C_c}\phi_1^*(\mathrm{i}\mathscr{D}_i(\phi_1)n_j-P_j n_i)\,\mathrm{d}l. \quad (5.5)$$

The first integral over S_f vanishes for $j = 1, 2$, as noted following (4.13). Similarly, after using (3.10) to evaluate P_j, the second integral over S_f vanishes for $i = 1, 2$. For $i = 6$ the contribution from the second integral cancels I_{ij}, except for the last two terms in (A 16). For $j = 6$ the first integral over S_f must be evaluated using (4.14), and (3.11) may be used to evaluate P_j. The final result for all cases except $i = j = 6$ can be expressed in the form

$$\mathscr{B}_{ij} = -\frac{1}{2}\iint_{S_c}(\mathscr{D}_{in}(\phi_1^*)Q_j-\mathscr{D}_i(\phi_1^*)Q_{jn})\,\mathrm{d}S-\frac{1}{2}\frac{\omega}{g}\oint_{C_c}\phi_1^*(\mathrm{i}\mathscr{D}_i(\phi_1)n_j-P_j n_i)\,\mathrm{d}l$$

$$+(\delta_{j6}-\delta_{i6})\left(\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\oint_{C_c}\phi_1\phi_{1n}^*\begin{pmatrix}y\\-x\end{pmatrix}\mathrm{d}l+2\pi\begin{pmatrix}\mu_2\\-\mu_1\end{pmatrix}\right)+\tfrac{1}{2}\delta_{j6}\mathrm{i}\frac{\omega}{g}\frac{\partial}{\partial\beta}\oint_{C_c}\phi_1\phi_{1n}^*\begin{pmatrix}x\\y\end{pmatrix}\mathrm{d}l. \quad (5.6)$$

Here δ_{ij} is the Kroenecker delta function, equal to one if $i = j$ and zero otherwise, and μ_1, μ_2 are the horizontal components of the effective dipole moment associated with the potential $\phi_2^{(0)}$ as defined by (A 13). Except for this dipole moment, (5.6) depends only on the first-order diffraction potential ϕ_1, and on the interactions (3.4)–(3.5) between ϕ_1 and φ_j. The same dependence was noted at the end of §4, with respect to the near-field analysis.

Finally, in the case $i = j = 6$,

$$\mathscr{B}_{66} = -\frac{1}{2}\iint_{S_c}(\mathscr{D}_{6n}(\phi_1^*)Q_6-\mathscr{D}_6(\phi_1^*)Q_{6n})\,\mathrm{d}S-\frac{1}{2}\frac{\omega}{g}\oint_{C_c}\phi_1^*(\mathrm{i}\mathscr{D}_6(\phi_1)n_6-P_6 n_6)\,\mathrm{d}l$$

$$+\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\frac{\partial}{\partial\beta}\iint_{S_f}\phi_1(\mathscr{D}_6(\phi_1^*)-\nabla\phi_1^*\cdot\nabla\varphi_6)\,\mathrm{d}S. \quad (5.7)$$

Note that in (5.6) the only integrals which remain are in the far field, but in (5.7) an integral remains over the free surface.

If $i = j \neq 6$, (5.6) reduces to the form

$$\mathscr{B}_{ii} = -\frac{1}{2}\iint_{S_c}(\mathscr{D}_{in}(\phi_1^*)Q_i-\mathscr{D}_i(\phi_1^*)Q_{in})\,\mathrm{d}S-\frac{1}{2}\frac{\omega}{g}K\begin{pmatrix}\cos\beta\\\sin\beta\end{pmatrix}\oint_{C_c}\phi_1^*\phi_1 n_i\,\mathrm{d}l. \quad (5.8)$$

After applying Stokes' theorem over S_c, in a form analogous to (A 2),

$$\mathscr{B}_{ii} = \frac{1}{2}\iint_{S_c}(\mathscr{D}_i(\phi_1^*)Q_{in}+\phi_{1n}^*\mathscr{D}_i(Q_i)-\nabla\phi_1^*\cdot\nabla Q_i n_i)\,\mathrm{d}S$$

$$+\frac{1}{2}\frac{\omega}{g}K\oint_{C_c}\phi_1^*\left[Q_i-\begin{pmatrix}\cos\beta\\\sin\beta\end{pmatrix}\phi_1 n_i\right]\mathrm{d}l, \quad (5.9)$$

where (2.4) is used. Except for differences of notation this formula is identical to equation (72) of Emmerhoff & Sclavounos (1992).

Another variant of the far-field representation is derived by using the function P_i in place of $\mathscr{D}_i(\phi_1^*)$ in (5.1). For $i = j$ this leads to the relatively simple result

$$\mathscr{B}_{ii} = -\tfrac{1}{2}\mathrm{i}\iint_{S_c}(P_{in}^*Q_i-P_i^*Q_{in})\,\mathrm{d}S, \quad (5.10)$$

which is valid for all three values of i. A feature of (5.10) is that the contribution from the secular component of (3.12) can be evaluated by differentiation of a non-singular integral:

$$\mathscr{B}_{ii} = -\tfrac{1}{2}\mathrm{i}\iint_{S_c}(P_{in}^*q_i-P_i^*q_{in})\,\mathrm{d}S-\mathrm{i}\frac{\omega}{g}\iint_{S_c}(P_{in}^*P_{iK}-P_i^*P_{iKn})\,\mathrm{d}S$$

$$= -\tfrac{1}{2}\mathrm{i}\iint_{S_c}(P_{in}^*q_i-P_i^*q_{in})\,\mathrm{d}S-\tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\frac{\partial}{\partial K}\iint_{S_c}(P_{in}^*P_i-P_i^*P_{in})\,\mathrm{d}S. \quad (5.11)$$

In this form explicit dependence on the derivatives P_{iK} and P_{iKn} is removed, but the latter derivative still must be evaluated on the body surface in the boundary condition (3.13). Since the functions which remain in the integrand of (5.11) satisfy the radiation condition, far-field asymptotic approximations can be substituted for each potential in the same manner as in the evaluation of the mean drift force (cf. Newman 1967).

Collectively, (5.6) and (5.10) can be used to evaluate all of the damping coefficients $\mathscr{B}_{ij}$ from far-field integrations. However the apparent computational advantage of these relations over (4.13) is offset by the fact that, regardless of which approach is followed, the higher-order potentials q_j must be evaluated on the body surface.

6. Discussion

The approach followed in the present work is to consider the low-frequency motion of a floating body as a perturbation of the incident-wave diffraction problem where the body is fixed in position. Two timescales are involved, one associated with the body motions at the frequency σ and the other corresponding to the incident wave frequency ω. The analysis is based on the assumptions that the incident-wave amplitude A and body motions ξ are both small, and that $\sigma \ll \omega$. Unlike other works which use a quasi-steady analysis, the relevant damping force acting on the body is derived without introducing a vanishingly small forward velocity of the body and considering the derivative of the force with respect to this velocity.

The results confirm that, whereas low-frequency oscillations in calm water result in wave radiation and damping asymptotically small with respect to the frequency σ, the same quantities are of order one in σ if the oscillations are superposed upon an incident-wave field. More specifically, for horizontal low-frequency oscillations of the body, the matrix of calm-water damping coefficients $B_{0ij} = O(\sigma^7)$, whereas the analogous wave-drift coefficient $B_{2ij} = O(A^2)$. Thus the relative importance of these two damping coefficients is in proportion to the ratio σ^7/A^2. The small value of the calm-water damping coefficient can be associated with the fact that the far-field energy flux is associated with long waves, the amplitude of which is negligible due to their asymptotically large lengthscale relative to the body. The situation is different in the presence of the diffraction field, where the basic wavelength is comparable to the body scale and the interaction of these waves with the body motions leads to a modification of the energy flux associated with the scattered field.

It is interesting to compare the present analysis with that of the quasi-steady approach, e.g. the work of Emmerhoff & Sclavounos (1992). The linear potentials ϕ_1 and ϕ_{0j} are the same, but the interaction potential ϕ_{1j} is somewhat different. In the quasi-steady analysis the difference component (3.4) does not appear, and the interaction potential is formally equivalent to (3.5). Although the potential P_j is absent, an equivalent term is included in the inhomogeneous free-surface condition (3.9) to account for the effect of steady forward velocity on the frequency of encounter. As shown in §5 the final results for the wave-drift damping coefficients $\mathscr{B}_{11}$ and $\mathscr{B}_{22}$ are equivalent in these different approaches.

The quasi-steady analysis of Nossen *et al.* (1991) is somewhat different. Instead of the first inhomogeneous term on the right-hand side of (3.9), the secular component of the interaction potential appears as a consequence of differentiating the zero-speed Green function to obtain a linearized correction for forward velocity. This construction was first suggested by Huijsmans & Hermans (1985). It has been extended by Grue & Palm (1992, 1993) to include the coefficients $\mathscr{B}_{21}$ and $\mathscr{B}_{61}$, with similarities to the present results for these coefficients.

A significant feature of the present method is the ability to include angular (yaw) oscillations of the body, about the vertical axis. This mode of motion cannot be accounted for in the quasi-steady approach, except possibly by considering a slow steady rotation of the body. Unlike the conventional added-mass and damping coefficients in (4.2), the matrix of wave-drift damping coefficients $\mathscr{B}_{ij}$ derived here appears to be asymmetric, and there is no obvious way of relating the forces due to yaw motions to the corresponding moments due to surge or sway.

Our results for yaw ((4.14) and (5.6)–(5.7)) involve differentiation of the first-order diffraction solution with respect to the angle of incidence β; this can be interpreted in the quasi-steady sense as the correction of the incidence angle due to the body's yaw oscillations. Equation (5.7), for the yaw damping moment, includes an integral over the free surface, but the same coefficient is expressed completely in terms of a far-field integration in the forms (5.10)–(5.11). The latter formulae do not involve explicit differentiation with respect to the heading angle, but this is implicit via the functional P_6 defined by (3.11).

In the present work it is assumed that the unsteady motions $\xi(t)$ are sufficiently small to justify perturbation expansions about the stationary mean position of the body. This assumption is not made explicitly in the quasi-steady analyses, which assume only that the corresponding velocities $\dot{\xi}(t)$ are small. At first glance this distinction appears to be significant, since low-frequency horizontal excursions of offshore platforms generally occur with substantial amplitudes. However, in the context of deriving only the wave-drift damping coefficients, i.e. the component of the total hydrodynamic force which is linear in $\dot{\xi}(t)$, the magnitude of $\xi(t)$ is irrelevant. Thus, despite the different initial assumptions concerning the order of $\xi(t)$, our results (5.8)–(5.9) are identical to those of Emmerhoff & Sclavounos (1992).

Far-field integration is generally considered to be simpler or more accurate than direct pressure integration on the body or intermediate results such as (4.13). Asymptotic relations can be used to evaluate the components of the velocity potential, and integrals over the control surface S_c can be reduced to azimuthal integrals in terms of the far-field scattering amplitude. A more specific advantage of the far-field evaluations here is that the secular component of the higher order potential Q_j can be evaluated as the derivative of an integral with respect to the wavenumber, as in (5.11). On the other hand, the most difficult task envisaged in numerical implementation is the evaluation of the potential q_j, as the solution of the boundary conditions (3.13)–(3.14). Even in the far-field analysis this solution is required locally on the body, in order to evaluate the far-field scattering amplitude. Thus there is no obvious advantage in evaluating the wave-drift damping coefficients in the far field, and direct use of (4.13) may in fact be simpler. Numerical implementation of the present analysis is required to confirm this conjecture, and to demonstrate the practical value of our results.

Several restrictions should be recognized in the present analysis. These include the assumptions of infinite fluid depth, no first-order body motions, and the consideration of low-frequency motions only in the horizontal plane (modes $j = 1, 2, 6$). The effects of finite depth are relatively simple to account for, including the second-order component of the incident-wave potential required in the diffraction solution. First-order body motions can be accommodated by including the corresponding linear radiation potentials in ϕ_1, but the boundary condition (2.3) must be modified and this will affect much of the subsequent analysis. Low-frequency vertical motions of the body may be important in certain applications; the principal difficulty anticipated in this extension of the analysis is that the corresponding components of the operator (2.17), including vertical derivatives, will complicate the reduction of the integrals over the free surface. Each of these possible extensions will be useful in practical applications.

This work was conducted under the Joint Industry Project 'Wave effects on offshore structures'. The author is grateful to the sponsors, including the Chevron Oil Field Research Company, Conoco Norway, the National Research Council of Canada, Norsk Hydro, Petrobrás, Saga Petroleum, the Shell Development Company, Statoil, and Veritas Research. Several errors in the original manuscript were noted by Dr P. J. Clark, and one of the referees corrected a fundamental error in the Appendix.

Appendix

A variant of Stokes' theorem (Milne Thomson 1955, §2.51) can be applied in the form

$$\iint_S (\boldsymbol{n}(\nabla\cdot\boldsymbol{q})-\nabla(\boldsymbol{n}\cdot\boldsymbol{q}))\,\mathrm{d}S=\oint_C \boldsymbol{q}\times \mathrm{d}\boldsymbol{l}. \qquad \text{(A 1)}$$

Here S is an open surface, C the boundary contour, and the integration around C is in the positive (counterclockwise) direction with respect to the normal vector $\boldsymbol{n}$.

For application to the second and third terms in the surface integral of (4.8) the substitution $\boldsymbol{q}=\phi\nabla\psi$ is made, with (A 1) applied on the body surface S_b and the contour C_b in the plane $z=0$. The x,y-components of (A 1) are then given respectively by setting $i=1,2$ in the equation

$$\iint_{S_b}(\nabla\phi\cdot\nabla\psi)\,n_i\,\mathrm{d}S=\iint_{S_b}[\mathscr{D}_i(\phi)\,\psi_n+\phi\mathscr{D}_{in}(\psi)]\,\mathrm{d}S+\oint_{C_b}\phi\psi_z n_i\,\mathrm{d}l. \qquad \text{(A 2)}$$

Here it is assumed that $\nabla^2\psi=0$. To verify that (A 2) also holds in the case $i=6$, the products $y\phi$ and $x\phi$ are substituted for ϕ with $i=1,2$, respectively; the difference between these two results is equivalent to setting $i=6$ in (A 2).

Alternatively, consider the vertical component of (A 1) on the free surface,

$$\iint_{S_f}(\nabla\phi\cdot\nabla\psi)\,\mathrm{d}S=\iint_{S_f}(\phi_z\psi_z+\phi\psi_{zz})\,\mathrm{d}S+\oint_{C_b+C_c}\phi\psi_n\,\mathrm{d}l. \qquad \text{(A 3)}$$

In the contour integral the normal derivative is directed out of S_f in the same plane. This integral is to be evaluated over both contours C_b and C_c in the positive direction (counterclockwise when viewed from above).

In the special cases where $\psi=x_i$, (A 3) is reduced to the more familiar form of Stokes' theorem,

$$\iint_{S_f}\mathscr{D}_i(\phi)\,\mathrm{d}S=\oint_{C_b+C_c}\phi n_i\,\mathrm{d}l. \qquad \text{(A 4)}$$

This formula also can be extended to include the case $i=6$, using the same procedure described following (A 2).

Next we consider the integrals I_{ij} as defined by (4.11). From the boundary conditions (3.2),

$$I_{ij}=\iint_{S_b}(n_j\mathscr{D}_i(\phi_2^{(0)})-n_i\mathscr{D}_j(\phi_2^{(0)}))\,\mathrm{d}S. \qquad \text{(A 5)}$$

The integrals I_{11}, I_{22}, and I_{66} are obviously equal to zero. For I_{21} the integrand in (A 5) is equal to the vertical component of $\boldsymbol{n}\times\nabla\phi_2^{(0)}$, and the integral vanishes by another variant of Stokes' theorem (Milne Thomson 1955, §2.51, equation 2). Thus $I_{21}=-I_{12}=0$. The only non-zero cases are

$$\begin{pmatrix} I_{61}=-I_{16}\\ I_{62}=-I_{26}\end{pmatrix}=\iint_{S_b}(n_j\mathscr{D}_6(\phi_2^{(0)})-n_6\mathscr{D}_j(\phi_2^{(0)}))\,\mathrm{d}S$$

$$=\iint_{S_b}\begin{pmatrix} n_y\\ -n_x\end{pmatrix}\phi_2^{(0)}\,\mathrm{d}S=\iint_{S_f}\begin{pmatrix}\varphi_2\\ -\varphi_1\end{pmatrix}\phi_{2z}^{(0)}\,\mathrm{d}S. \qquad \text{(A 6)}$$

Here Stokes' and Green's theorems have been used, together with the boundary conditions (3.2) and (3.3). After reversing the signs for the lower elements of (A 6) and using (A 3) with the boundary conditions (2.3), (2.25) and (2.28),

$$\begin{pmatrix} I_{61}=-I_{16}\\ -I_{62}=I_{26}\end{pmatrix}=\frac{1}{2}\frac{\omega}{g}\mathrm{Re}\,\mathrm{i}\iint_{S_f}\begin{pmatrix}\varphi_2\\ \varphi_1\end{pmatrix}\phi_1\phi_{1zz}^{*}\,\mathrm{d}S=\frac{1}{2}\frac{\omega}{g}\mathrm{Re}\,\mathrm{i}\iint_{S_f}\phi_1\nabla\phi_1^{*}\cdot\nabla\begin{pmatrix}\varphi_2\\ \varphi_1\end{pmatrix}\mathrm{d}S. \qquad \text{(A 7)}$$

An alternative representation can be derived by applying Green's theorem to $\phi_2^{(0)}$ and (φ_i-x_i), for $i=1,2$. Since both potentials satisfy homogeneous Neumann conditions on the body,

$$\iint_{S_f}\phi_{2z}^{(0)}(\varphi_i-x_i)\,\mathrm{d}S+\iint_{S_c}(\phi_{2n}^{(0)}(\varphi_i-x_i)-\phi_2^{(0)}(\varphi_{in}-n_i))\,\mathrm{d}S=0. \qquad \text{(A 8)}$$

The asymptotic form of $\phi_2^{(0)}$ is required to ascertain the contribution from the last integral in (A 8). For this purpose it is convenient to assume that S_c is a circular cylinder of large radius R about the vertical axis. If the divergence theorem is used with the boundary conditions (2.19) and (2.25),

$$\iint_{S_c}\phi_{2n}^{(0)}\,\mathrm{d}S=-\iint_{S_f}\phi_{2z}^{(0)}\,\mathrm{d}S$$

$$=\frac{1}{2}\frac{\omega}{g}\mathrm{Re}\,\mathrm{i}\iint_{S_f}\phi_1\phi_{1zz}^{*}\,\mathrm{d}S=-\frac{1}{2}\frac{\omega}{g}\mathrm{Re}\,\mathrm{i}\oint_{C_c}\phi_1\phi_{1n}^{*}\,\mathrm{d}l, \qquad \text{(A 9)}$$

where (A 3) is used to derive the last contour integral. To estimate this contour integral Green's theorem is applied to ϕ_1 and its conjugate. Since the boundary conditions on the body and free surface are homogeneous,

$$\mathrm{Re}\,\mathrm{i}\iint_{S_c}\phi_1\phi_{1n}^{*}\,\mathrm{d}S=0. \qquad \text{(A 10)}$$

In the far field the asymptotic form of ϕ_1 is such that, for large R,

$$\phi_1(R,\theta,z)=\phi_1(R,\theta,0)\,\mathrm{e}^{Kz}+O(R^{-3}). \qquad \text{(A 11)}$$

Here the estimate of the error follows from the spherical-harmonic 'wave-free' potentials (Havelock 1955, equation 8), and from the far-field asymptotic expansion of the corresponding Green function (Newman 1985, equation 6). With (A 11) substituted in (A 10), the vertical integration over S_c can be performed, with the resulting estimate

$$\mathrm{Re}\,\mathrm{i}\oint_{C_c}\phi_1\phi_{1n}^{*}\,\mathrm{d}l=O(R^{-3}). \qquad \text{(A 12)}$$

Thus (A 9) is of order (R^{-2}), implying that $\phi_2^{(0)}$ behaves like a Rankine dipole in the far field, with the asymptotic approximation

$$\phi_2^{(0)} \sim \boldsymbol{\mu}\cdot\nabla(1/r). \tag{A 13}$$

Here $\boldsymbol{\mu}$ is the dipole and $r = (R^2+z^2)^{\frac{1}{2}}$. The only non-vanishing contribution to the integral in (A 8) over S_c, as the radius of the control surface is increased to infinity, is

$$\iint_{S_c} (\phi_2^{(0)} n_l - \phi_{2n}^{(0)} x_l)\,\mathrm{d}S = -2\pi\mu_l. \tag{A 14}$$

Evaluating the last integral in (A 8) and using (A 3),

$$\iint_{S_f} \phi_{2z}^{(0)}(\varphi_l - x_l)\,\mathrm{d}S = \frac{1}{2}\frac{\omega}{g}\,\mathrm{Re}\,\mathrm{i}\iint_{S_f} (\varphi_l - x_l)\,\phi_1\,\phi_{1zz}^{*}\,\mathrm{d}S$$

$$= \frac{1}{2}\frac{\omega}{g}\,\mathrm{Re}\,\mathrm{i}\iint_{S_f} \phi_1(\nabla\varphi_l\cdot\nabla\phi_1^{*} - \mathscr{D}_l(\phi_1^{*}))\,\mathrm{d}S - \frac{1}{2}\mathrm{i}\frac{\omega}{g}\oint_{C_0} \phi_1\,\phi_{1n}^{*}\begin{pmatrix} y \\ x \end{pmatrix}\mathrm{d}l = 2\pi\mu_l. \tag{A 15}$$

Substituting (A 15) in (A 7) gives the alternative expressions

$$\begin{pmatrix} I_{61} = -I_{16} \\ -I_{62} = I_{26} \end{pmatrix} = \tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\iint_{S_f} \begin{pmatrix} y \\ x \end{pmatrix}\phi_1\,\phi_{1zz}^{*}\,\mathrm{d}S + 2\pi\begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}$$

$$= \tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\iint_{S_f} \phi_1\begin{pmatrix} \phi_{1y}^{*} \\ \phi_{1x}^{*} \end{pmatrix}\mathrm{d}S - \tfrac{1}{2}\mathrm{i}\frac{\omega}{g}\oint_{C_c} \phi_1\,\phi_{1n}^{*}\begin{pmatrix} y \\ x \end{pmatrix}\mathrm{d}l + 2\pi\begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}. \tag{A 16}$$

REFERENCES

Aonon, Y. & Mei, C. C. 1985 Slow-drift motion of a two-dimensional block in beam seas. *J. Fluid Mech.* **151**, 279–294.

Chakrabarti, S. K. & Cotter, D. C. 1992 Hydrodynamic coefficients of a moored semi-submersible in waves. *J. Offshore Mech. Arctic Engng* **114**, 9–15.

Emmerhoff, O. J. & Sclavounos, P. D. 1992 The slow-drift motion of arrays of vertical cylinders. *J. Fluid Mech.* **242**, 31–50.

Faltinsen, O. M. 1991 *Sea Loads on Ships and Offshore Structures*. Cambridge University Press.

Grue, J. & Palm, E. 1992 The effect of second order velocities on drift forces and drift moments. In *7th Intl Workshop on Water Waves and Floating Bodies, Val de Reuil, France* (ed. R. Cointe).

Grue, J. & Palm, E. 1993 The mean drift force and yaw moment on marine structures in waves and current. *J. Fluid Mech.* **250**, 121–142.

Havelock, T. H. 1955 Waves due to a floating sphere making periodic heaving oscillations. *Proc. R. Soc. Lond.* A **231**, 1–7.

Huijsmans, R. H. M. & Hermans, A. J. 1989 A fast algorithm for computation of 3-D ship motions at moderate forward speed. In *Proc. 4th Intl Conf. on Numerical Ship Hydrodynamics, Washington* (ed. J. H. McCarthy), pp. 24–31. National Academy of Science Press.

Lee, C.-H., Newman, J. N., Kim, M.-H. & Yue, D. K. P. 1991 The computation of second-order wave loads. In *Proc. 10th Intl Conf. on Offshore Mechanics and Arctic Engineering, Stavanger, Norway*, Vol. 1 (A) (ed. S. K. Chakrabarti, H. Maeda, A. N. Williams, D. G. Morrison & M. Hendriksen), pp. 113–123. ASME.

Milne-Thomson, L. M. 1955 *Theoretical Hydrodynamics*. Macmillan.

Newman, J. N. 1967 The drift force and moment on ships in waves. *J. Ship Res.* **11**, 1, 51–60.

Newman, J. N. 1977 *Marine Hydrodynamics*. MIT Press.

Newman, J. N. 1978 The theory of ship motions. *Adv. Appl. Mech.* **18**, 221–283.

Newman, J. N. 1985 Algorithms for the free-surface Green function. *J. Engng Maths* **19**, 57–67.

Nossen, J., Grue, J. & Palm, E. 1991 Wave forces on three-dimensional floating bodies with small forward speed. *J. Fluid Mech.* **227**, 135–160.

Wichers, J. E. W. & Sluijs, M. F. van 1979 The influence of waves on the low-frequency hydrodynamic coefficients of moored vessels. *Proc. Offshore Technology Conf., Houston*, OTC 3625.

Zhao, R. & Faltinsen, O. M. 1989 Interaction between current, waves and marine structures. In *Proc. 5th Intl Conf. on Numerical Ship Hydrodynamics, Hiroshima, Japan* (ed. K. Mori), pp. 513–525. National Academy of Science Press.