Wave Equations on Lorentzian Manifolds and Quantization

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joint work with

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Outline

Wave Operators

Throughout let M denote a timeoriented Lorentzian manifold.
Let $E \to M$ be a vector bundle.
Denote the smooth sections in E by $C^\infty(M,E)$.

Definition

A wave operator or normally hyperbolic operator is a linear differential operator $P : C^\infty(M,E) \to C^\infty(M,E)$ of second order which looks locally like

$$P = -\sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x)\frac{\partial}{\partial x^j} + B(x)$$

Wave Operators; Examples

- d'Alembert operator (functions)

$$P = \Box$$

- Klein-Gordon operator (functions)

$$P = \Box + m^2 \text{or } P = \Box + m^2 + \kappa \cdot \mathrm{scal}$$

- Wave operator in electro-dynamics (1-forms)

$$P = d\delta + \delta d$$

- Square of Dirac operator (spinors)

$$P = D^2$$

Cauchy Problem

Let M be globally hyperbolic and let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S.

Theorem

For each $u_0, u_1 \in C_c^\infty(S,E)$ and for each $f \in C_c^\infty(M,E)$ there exists a unique $u \in C^\infty(M,E)$ satisfying

$$\begin{cases} Pu = f, & \text{on } M \\ u|_S = u_0, & \text{along } S \\ \nabla_\nu u = u_1, & \text{along } S \end{cases}$$

Cauchy Problem

Well-posedness

The solution u depends continuously on the data f, u_0, and u_1.

Finite propagation speed

Moreover,
$\mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$
where $K =$
$\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

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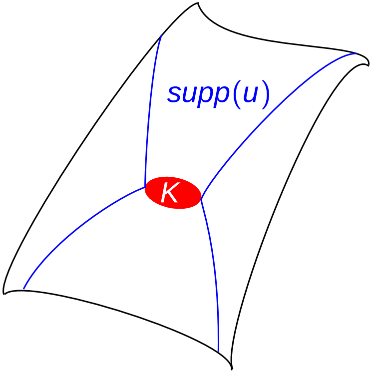

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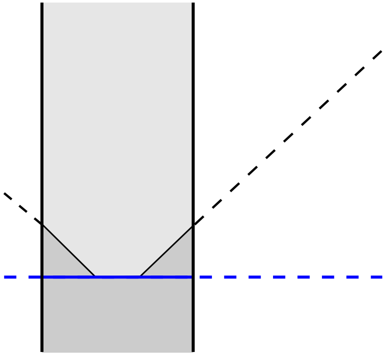
Cauchy Problem; What Can Go Wrong



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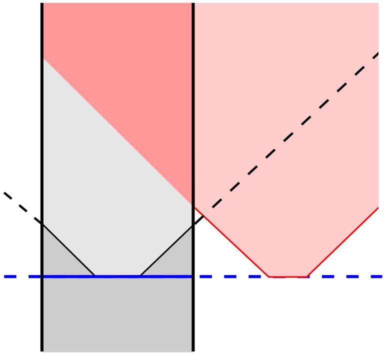

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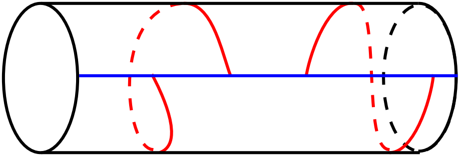

Green's Operators

Definition

A linear operator $G : C_c^\infty(M,E) \to C^\infty(M,E)$ is called a Green's operator for P if

$$P \circ G = G \circ P = \mathrm{id}_{C_c^\infty(M,E)}$$

Definition

A Green's operator G is called advanced or retarded resp. if

$$\mathrm{supp}(G(u)) \subset J_+(\mathrm{supp}(u)) \text{ or } J_-(\mathrm{supp}(u))$$

resp. for any $u \in C_c^\infty(M,E)$.

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Green's Operators

Theorem

Let P be a wave operator over a globally hyperbolic manifold M.

Then there exist unique advanced and retarded Green's operators for P.
These Green's operators are continuous.
The sequence of linear maps

$$0 \to C_c^\infty(M,E) \xrightarrow{P} C_c^\infty(M,E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M,E) \xrightarrow{P} C_{sc}^\infty(M,E)$$

is exact.

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Outline

1. Wave Equations
2. Quantization

Fock Space

H complex Hilbert space, $\bigodot^n H$ completion of $\bigodot^n_{\mathrm{alg}} H$

(Bosonic or symmetric) Fock space $\mathfrak{F}(H)$ is the completion of

$$\mathfrak{F}_{\mathrm{alg}}(H) := \bigoplus_{n=0}^{\infty} \bigodot^n H.$$

Fix $v \in H$. Define the creation operator

$$a^*(v) v_1 \odot \ldots \odot v_n := v \odot v_1 \odot \ldots \odot v_n$$

and the annihilation operator

$$a(v)(w_0 \odot \cdots \odot w_n) := \sum_{k=0}^{n} (v, w_k) w_0 \odot \cdots \odot \hat{w}_k \odot \cdots \odot w_n$$

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Canonical Commutator Relations

Canonical commutator relations:

$$[a(v), a(w)] = [a^*(v), a^*(w)] = 0,$$

$$[a(v), a^*(w)] = (v, w) \cdot \mathrm{id}.$$

Definition

Segal operator:

$$\theta(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v))$$

The Segal operator on $\mathfrak{F}_{\mathrm{alg}}(H)$ is essentially self-adjoint in $\mathfrak{F}(H)$.

$$[\theta(v), \theta(w)] = i \cdot \mathfrak{Im}(v, w)$$

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Geometric Setup

- Globally hyperbolic Lorentzian manifold M
- Real vector bundle $E \to M$ with non-degenerate metric
- Formally selfadjoint wave operator P on E

Definition

A twist structure of spin $k/2$ on E is a smooth section $Q \in C^\infty(M, \mathrm{Hom}(\bigodot^k TM, \mathrm{End}(E)))$ such that:

- $\langle Q(X_1 \odot \cdots \odot X_k)e, f\rangle = \langle e, Q(X_1 \odot \cdots \odot X_k)f\rangle$
- If X is future directed timelike, then the bilinear form $\langle \cdot, \cdot \rangle_X$ defined by

$$\langle f, g\rangle_X := \langle Q_X f, g\rangle$$

is positive definite where $Q_X = Q(X \odot \cdots \odot X)$.

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Example

If the metric on E is positive definite, one can choose $k = 0$ and $Q = \mathrm{id}$

Example

For spinor bundle E let $k = 1$ and $Q(X)$ be Clifford multiplication by X

Example

For $E = \Lambda^q T^*M$ let $k = 2$ and

$$Q(X \odot Y)\alpha := X^\flat \wedge \iota_Y \alpha + Y^\flat \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$

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- Twist structure Q
- Cauchy hypersurface $S \subset M$

We get real Hilbert space $L^2(S, E^*)$ where

$$(u, v)_S := \int_S \langle u, v \rangle_\nu \, dA = \int_S \langle Q_\nu^* u, v \rangle \, dA$$

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Quantum Field

- Apply Fock space construction to $H_S := L^2(S, E^*) \otimes_{\mathbb{R}} \mathbb{C}$
- Get Segal field θ

Definition

Quantum field: For $f \in C_c^\infty(S, E^*)$ put

$$\Phi_S(f) := \theta(\underbrace{i(G_+^* - G_-^*)f|_S - (Q_\nu^*)^{-1}\nabla_\nu((G_+^* - G_-^*)f)}_{\in H_S}).$$

Haag-Kastler Axioms

Theorem

- $C_c^\infty(M, E^*) \to \mathfrak{F}(H_S), \quad f \mapsto \Phi_S(f)\omega$, *is continuous for any* $\omega \in \mathfrak{F}_{\mathrm{alg}}(H_S)$
- $P\Phi_S = 0$ *in the distributional sense*
- $[\Phi_S(f), \Phi_S(g)] = 0$ *if the supports of* f *and* g *are causally independent.*
- *The linear span of the vectors* $\Phi_S(f_1)\cdots\Phi_S(f_n)\Omega$ *is dense in* $\mathfrak{F}(H_S)$ *where* $\Omega = 1 \in \bigodot^0 H_S = \mathbb{C}$ *is the vacuum vector.*

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- Construction depends on choice of Cauchy hypersurface
- Microlocal spectrum condition is violated

Algebraic quantum field theory:

- Forget Fock space (and particles)
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- To each (reasonable) spacetime region associate an algebra of observables

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CCR-algebras

Let (V, ω) be a symplectic vector space.

Definition

A CCR-algebra of (V, ω) consists of a C^*-algebra A with unit and a map $W : V \to A$ such that for all $\phi, \psi \in V$ we have

- $W(0) = 1$
- $W(-\phi) = W(\phi)^*$
- $W(\phi) \cdot W(\psi) = e^{-i\omega(\phi,\psi)/2} W(\phi + \psi)$
- A is generated by the $W(\phi)$

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Construction of the Symplectic Vector Space

Let M be globally hyperbolic, let P a formally self-adjoint wave operator acting on sections in E.
Let G_+ and G_- be the Green's operators of P.

$$\tilde{\omega}(\phi,\psi) := \int_M \langle (G_+ - G_-)\phi, \psi\rangle \, dVol$$

defines a degenerate symplectic form on $C_c^\infty(M,E)$.
It induces a (nondegenerate) symplectic form ω on

$$\begin{aligned} V(M,E,P) &:= C_c^\infty(M,E)/P(C_c^\infty(M,E)) \\ &= C_c^\infty(M,E)/ker(G_+ - G_-) \end{aligned}$$

Quantization Functor

$\mathfrak{A}_M := CCR(M, E, P) := CCR(V(M, E, P), \omega)$ defines a functor

globally hyperbolic manifolds equipped with a formally self-adjoint wave operator	$\Longrightarrow$	C^*-algebras with unit

Haag-Kastler Axioms, II

Theorem

- *If* $O_1 \subset O_2$*, then* $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$ *for all* $O_1, O_2 \in I$.
- $\mathfrak{A}_M = \overline{\bigcup_{\substack{O \in I \\ O \neq \emptyset, O \neq M}} \mathfrak{A}_O}$.
- $\mathfrak{A}_M$ *is simple.*
- *The* $\mathfrak{A}_O$*'s have a common unit* 1.
- *For all* $O_1, O_2 \in I$ *with* $J(\overline{O_1}) \cap \overline{O_2} = \emptyset$ *the subalgebras* $\mathfrak{A}_{O_1}$ *and* $\mathfrak{A}_{O_2}$ *of* $\mathfrak{A}_M$ *commute:* $[\mathfrak{A}_{O_1}, \mathfrak{A}_{O_2}] = \{0\}$.
- *Time-slice axiom. Let* $O_1 \subset O_2$ *be nonempty elements of* I *admitting a common Cauchy hypersurface. Then* $\mathfrak{A}_{O_1} = \mathfrak{A}_{O_2}$.
- *Let* $O_1, O_2 \in I$ *and let the Cauchy development* $D(O_2)$ *be relatively compact in* M*. If* $O_1 \subset D(O_2)$*, then* $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$.

Comparison of the Two Approaches

Given a Cauchy hypersurface $S \subset M$, a twist structure, and the corresponding quantum field Φ_S

$$W_S(f) := \exp(i\Phi_S(f))$$

defines a CCR-representation for $V(M, E, P)$.

Problems

- Construct physically satisfactory representations (Hadamard states)
- Construct n-point functions (Singularities, renormalization)
- Construct nonlinear fields (Energy-momentum tensor)

Applications in Physics

- Hawking radiation of black holes
- Unruh effect

Brunetti, Dimock, Fewster, Fredenhagen, Hollands, Radzikowski, Verch, Wald, ...