DEPARTMENT OF THE ARMY CORPS OF ENGINEERS

BEACH EROSION BOARD
OFFICE OF THE CHIEF OF ENGINEERS

WAVE FORCES ON PILES: A DIFFRACTION THEORY

TECHNICAL MEMORANDUM NO. 69

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FOREWORD

Although circular piling is a much-used structural element in shore protection, harbor, and other maritime structures, only recently have significant advances been made toward gaining a quantitative understanding of the forces developed by wave action against piling. The present report deals with this subject.

The report was prepared at the University of California, Berkeley, California. The work on which the report is based was sponsored by the Office of Naval Research, U. S. Department of the Navy. The authors of the report are R. C. MacCamy and R. A. Fuchs of the Institute of Engineering Research, University of California. Because of its applicability to the research and investigation program of the Beach Erosion Board, and through the courtesy of the authors, the report is being published at this time in the technical memorandum series of the Beach Erosion Board. Views and conclusions stated in the report are not necessarily those of the Beach Erosion Board.

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WAVE FORCES ON PILES: A DIFFRACTION THEORY

by
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<u>Introduction</u>. This report contains two main results. In the first section an exact mathematical solution is presented for the linearized problem of water waves of small steepness incident on a circular cylinder. The fluid is assumed to be frictionless and the motion irrotational. This section includes, in addition to the formal mathematical treatment, some simple deductions based on the assumption of very small ratio of cylinder diameter to incident wave-length. The principal results of the theory are summarized, for convenience in calculations, in the second section. Also presented are some suggestions as to possible extensions of the theory to take care of more extreme wave conditions and other obstacle shapes.

The second result is an attempt to apply the theory to the computation of actual wave forces on cylindrical piles. The basis of comparison is a series of tests performed in the wave channel. The agreement is found to be quite good in the region in which the assumptions of the theory are fairly closely realized.

<u>Theory</u>. The problem of diffraction of plane waves from a circular cylinder of infinite extent has been solved both for electromagnetic and sound waves. Only slight modifications are necessary to obtain a corresponding solution for water waves incident on a circular pile. Reference is made to Morse[(1)*] especially for the expansions in equations 2, 3, 5, and for a survey of the asymptotic developments of the Bessel's Functions.

The following assumptions are made. The fluid is frictionless and moving irrotationally. The ratio of the height of the waves to their length is sufficiently small so that all quantities involving the parameter (H/L) in the second or higher powers may be neglected without sensible error, thus giving rise to the so-called linear theory. The waves are indident on a vertical circular cylinder which extends to the bottom. The depth of the water is d, finite.

A set of axes x , y, z is chosen with z directed positively upward from the still-water level. The cylinder of radius, a, is assumed to lie along the z-axis and cylindrical waves are incident from the negative x-direction. The velocity potential of the incident wave then may be written,

$$\phi^{(i)} = - \frac{g\,H}{2\sigma} \, \frac{\cosh k\,(d+z)}{\cosh kd} \, e^{i\,(kx - \sigma t)} \qquad (1)$$

*Numbers in parentheses refer to list of references on page 11 .

It is understood here that the actual potential is the real part of this complex expression, and that in order to find the physical solution in what follows, it is necessary to take real parts.

Introducing polar co-ordinates r and θ, equation 1 admits of an expansion in cylindrical harmonics, having the form:

$$\phi^{(i)} = -\frac{gH}{2}\frac{\cosh k(d+z)}{\cosh k d}\left[J_o(kr) - \sum_{m=1}^{\infty} 2i^m \cos m\theta\, J_m(kr)\quad e^{-i\sigma t}\right] \quad (2)$$

The assumption is now made that the reflected wave admits of a similar expansion. The particular combination appropriate to a wave moving outward, symmetrically with respect to θ, that is such that $\phi(-\theta) = \phi(\theta)$, is,

$$\phi_{m'} = A_m \cos m\theta \left[J_m(kr) + i\, Y_m(kr)\right] e^{-i\sigma t} \quad (3)$$

This combination of Bessel Functions is known as the Hankel function of the first kind, $H_m^{(1)}(kr)$, and, for large values of r, has the asymptotic form:

$$H_m^{(1)}(kr) \sim \sqrt{\frac{2}{kr}}\; e^{i\left(kr - \frac{2m-1}{4}\pi\right)} \quad (4)$$

Hence equation 3 has, for large values of r, the form of a periodic disturbance moving outward in the r direction, with frequency σ and wave number k, and vanishing at $r = \infty$.

For the total velocity potential, ϕ, there is taken a superposition of $\phi(i)$ and an infinite series of terms like the quantities A_m are then determined by setting the particle velocity normal to the cylinder, that is $\frac{\partial \phi}{\partial r}$, equal to zero at the surface, $r = a$.

The result of this calculation is,

$$\phi = \frac{gH}{2\sigma} e^{-i\sigma t}\, \frac{\cosh k(d+z)}{\cosh kd}\left[J_o(kr) - \frac{J_o'(ka)}{H_o^{(2)'}(ka)} H_o^{(2)}(kr) + 2\sum_{m=1}^{\infty} i^m\left(J_m(kr) - \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}(kr)\right)\cos m\theta\right] \quad (5)$$

where $H_m^{(2)}(kr)$ is the Hankel Function of the second kind and equals $J_m - i\,Y_m$. This result is given by Havelock[2] for the special case of infinite depth.

The pressure exerted on the cylinder is computed from Bernoulli's equation,

$$p = \rho \frac{\partial \phi}{\partial t} - \frac{\rho}{2} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] - g\rho z \qquad (6)$$

where, in the linear theory, the squared terms are neglected.

The x-component of the force, per unit length in the z-direction, is,

$$F_z = \mathrm{Re}\ 2 \int_0^{\pi} p\,(\theta)\quad a \cos\,(\pi - \theta)\ d\theta$$

Only the term in cos θ will contribute to this integral and the result after taking the real part may be written as,

$$F_z = \frac{2\rho\ g\ H}{k} \frac{\cosh\ k\ (d+z)}{\cosh\ kd} A\ (ka)\ \cos\ (\sigma\ t - \alpha) \qquad (7)$$

where

$$\tan \alpha = \frac{J_1'\ (ka)}{Y_1'\ (ka)} \quad ; \quad A\ (ka) = \frac{1}{\sqrt{J_1'^2\ (ka) + Y_1'^2\ (ka)}}$$

These functions are plotted in Figures 1 and 2, ka being equal to π D/L.

The moment about a point z = u, on a cylinder extending to depth v below the still-water level may be easily computed from equation 7, assuming that the motion of the fluid is the same as if the cylinder extended to the bottom. The expression for the moment is,

$$m_{u,v} = \int_{-v}^{\eta} (z - u)\ F_z\ dz \qquad (8)$$

To be consistent with the linear theory the integration need only be carried up to the still-water level z = 0, the result being

$$m_{u,v} = -\frac{2g\rho H}{k^3} A(ka) \left[\frac{uk\ \sinh\ kd - \sinh\ k(d-v) - vk\ \sinh\ k(d-v) + \cosh\ kd}{\cosh\ kd} - \frac{\cosh\ k(d-v)}{\cosh\ kd} \right] \cos\ (\sigma\ t - \alpha) \qquad (9)$$

The special case of a pile hinged about the bottom is evaluated by setting u = -d, v = d.

$$m_o = \frac{2\, g\rho H}{k^3}\, A(ka)\, \frac{(k\, d \sinh kd - \cosh kd - 1)}{\cosh kd} \cos(\sigma t - \alpha) \qquad (10)$$

The function $D\,(kd) = \dfrac{1 - \cosh kd + kd \sinh kd}{\cosh kd}$ giving the dependence on depth is plotted in Figure 3.

An estimate of the effect of second order terms on the moment $m_{u,v}$ may be immediately obtained from equation 8 by evaluating that portion of the integral from zero to η. To the second order, for,

$$\eta = \frac{H}{2} \sin \sigma t \qquad (11)$$

$$\Delta m_u = \int_0^{\eta} (z - u)\, F_z\, dz = \frac{\rho g\, H^2 u}{k} A(ka) \sin\sigma t \cos(\sigma t - \alpha) \qquad (12)$$

This calculation omits that portion of the second-order terms arising from the second term in the velocity, but this latter term may be expected to be small. It is noted that the result (12) may be obtained by assuming that the force and lever arm are constant over the range $o \leq z \leq \eta$, having the value at $z = 0$ and multiplying these constant values by the length, η. For the special case of a cylinder hinged at the bottom the total moment becomes

$$m_o + \Delta m_o = \frac{2\rho\, gH}{k^3} A(ka)\, D(kd) \cos\sigma t \left[1 + \frac{k^2 Hd}{2D(kd)} \sin\sigma t\right] \qquad (13)$$

From equation 13 it is seen that the maximum moment occurs for,

$$\sin(\sigma t)_{max} = \frac{1 - \sqrt{1 + 2\left[\dfrac{k^2\, H\, d}{D\,(kd)}\right]^2}}{2\, \dfrac{k^2 Hd}{D\,(kd)}} \qquad (14)$$

and has the value obtained by substituting $(\sigma t)_{max}$ into equation 13.

For cylinders, the diameters of which are small compared to the length of the waves, the foregoing theory admits of several simplifications. Asymptotic values of the Bessel's Functions and their derivatives are presented for reference in Table IV. These lead immediately to the approximate formulas,

$$\left.\begin{aligned} A(ka) &\cong \frac{\pi}{2} (ka)^2 \\ \alpha(ka) &\cong \frac{\pi}{4} (ka)^2 \end{aligned}\right\} \qquad (16)$$

In particular, equation 7 may then be replaced by

$$F_z = \pi \rho g H k a^2 \frac{\cosh k\,(d+z)}{\cosh kd} \cos \sigma t \qquad (7')$$

In this form the force F_z admits of a much simpler derivation. For a wave incident on a vertical wall at an arbitrary angle there is complete reflection without loss of energy, resulting in a total pressure equal to twice that of the incident wave. Assuming that this result holds for the cylinder also, an incident wave with velocity potential given by equation 1 will give rise to a real pressure,

$$p = -\rho g H \frac{\cosh k\,(d+z)}{\cosh k\,d} \sin (k\,x - \sigma t). \qquad (17)$$

The resulting force, F, is then obtained by integration as for equation 7, giving the relationship

$$F_z = -2\, a \rho g H \frac{\cosh k\,(d+z)}{\cosh kd} \int_0^{\pi} \sin (ka \cos \theta - \sigma t) \cos \theta \; d\theta \qquad (18)$$

But now for small values of ka, expanding the integrand in equation 18 gives

$$F_z \simeq -4\, a \rho g H \frac{\cosh k\,(d+z)}{\cosh kd} \int_0^{\frac{\pi}{2}} \cosh \sigma t \quad ka \cos^2 \theta \;\; d\theta, \qquad (19)$$

which leads again to equation 7'. It is to be noted in connection with this equation, that the force F_z is equal to the so-called "virtual mass force" in Morison's result[3] provided the experimentally determined constant C_M is taken as two. The result is to be expected since an essential assumption of Morison's theory is that the form of the incident wave is little affected by the presence of the cylinder. From equation 23 it is seen that this assumption is equivalent to the smallness of the ratio of pile diameter to wave length. It is to be noted in this connection that the exact theory of the present report represents an extension since its accuracy does not depend on the relative size of the cylinder. The value of C_M quoted by Morison for a series of model studies is nearly 1.5.

This type of analysis admits of certain extensions. For example the same technique might be used to obtain forces on more complicated shapes, the dimensions of which are small compared to the wave length, since a knowledge of the form of the reflected wave is not necessary. It is also shown in the next section how an estimate of the effect of steeper waves may be obtained in a similar manner.

A more exact analysis of the relative effects of the incident and reflected waves is possible from the small cylinder theory and will offer justification for the developments of the preceding paragraphs. The surface profile may be obtained from the velocity potential, ϕ, given by equation 5 from the formula,

$$\eta = \frac{1}{g}\frac{\partial \phi}{\partial t} \quad z = 0 \tag{20}$$

this gives

$$(\eta)_r = a = \frac{-He^{-i\sigma t}}{\pi\, ka}\left[\frac{1}{H_o^{(2)'}(ka)} + 2\sum_{n=1}^{\infty} i^n \frac{1}{H_n^{(2)'}(ka)}\cos n\theta\right] \tag{21}$$

where use has been made of the identity,

$$J_m(x)\, H_m^{(2)'}(x) - J_m'(x)\, H_m^{(2)}(x) = -\frac{2i}{\pi x} \tag{22}$$

Using the asymptotic formulas for the Bessel Functions for small values of ka, equation 21 becomes, on taking the real part,

$$(\eta)_r = a \simeq \frac{H}{2}\sqrt{1 + 4\,(ka)^2\cos^2\theta}\;\sin\;(\sigma t - \psi) \tag{23}$$

where

$$\tan\psi = 2ka\cos\theta$$

In the same notation the pressure, at the surface of the pile, is,

$$p = \frac{g\rho H}{\pi\, ka}\left[\frac{1}{H_o^{(2)'}(ka)} + 2\sum_{n=1}^{\infty} i^n \frac{1}{H_n^{(2)'}(ka)}\cos n\,\theta\right]\frac{\cosh k\,(d+z)}{\cosh kd}\, e^{-i\sigma t} \tag{24}$$

or for small piles, the real part of equation 24 gives

$$p \simeq g\rho H\,(\sin\sigma t + 2ka\cos\theta\cos\sigma t)\,\frac{\cosh k\,(d+z)}{\cosh kd} \tag{25}$$

It can easily be shown that the pressure due to the incident wave only is to the same degree of approximation,

$$p^{(i)} = g\rho H\,(\sin\sigma t + k\,a\cos\theta\cos\sigma\, t)\,\frac{\cosh k\,(d+z)}{\cosh kd} \tag{26}$$

so that the pressure due to the reflected wave is,

$$p^{(i)} = g\rho H\, ka\cos\theta\cos\sigma t\;\frac{\cosh k\,(d+z)}{\cosh kd} \tag{27}$$

It is observed that the first, and largest, term of equation 26 is independent of θ and hence will contribute nothing to the force, F_z. Hence the "effective" pressures due to the incident wave and the reflected waves are identical. This is in contrast to the effect on the surface elevation, since equation 23 shows that the deviation from that of the incident wave alone is small.

Summary. The diffraction of long-crested waves incident on vertical circular cylinders extending from above the water surface to the bottom is treated exactly within the framework of the linearized irrotational theory. The essential results are summarized below.

Letting z be the distance along the cylinder, in the direction of its axis, with positive direction upward from the still-water level, the x-component of the force on the cylinder per unit length in the z-direction and at depth z, is,

$$F_z = \frac{2 \rho g H}{k} \frac{\cosh k (d+z)}{\cosh kd} A(\tfrac{D}{L}) \cos (\sigma t - \alpha) \tag{28}$$

where

$$\tan \alpha = \frac{J_1' (\pi \frac{D}{L})}{Y_1' (\pi \frac{D}{L})}$$

$$A (\tfrac{D}{L}) = \frac{1}{\sqrt{J_1'^2 (\pi \frac{D}{L}) + Y_1'^2 (\pi \frac{D}{L})}}$$

when the surface elevation is given by,

$$\eta = \frac{H}{2} \sin (k x - \sigma t) \tag{29}$$

J_1 and Y_1 are the Bessel's Functions of the first and second kinds, respectively, and primes indicate differentation. The functions A and α are plotted in Figures 1 and 2. Additional values can be obtained from a set of tables published by the Mathematical Tables Project(4).

The corresponding movement on a cylinder extending to depth v below the still-water level and hinged at depth u relative to the still-water level is given by

$$m_{u,v} = \frac{-2g \rho H}{k^3} A(\tfrac{D}{L}) \left[\frac{uk \sinh kd - \sinh k(d-v) - vk \sinh k(d-v)}{\cosh kd} + \frac{\cosh kd - \cosh k(d-v)}{\cosh kd} \right] \cos (\sigma t - \alpha) \tag{30}$$

In the special case of a cylinder extending to the bottom and hinged at the bottom, $u = -d$ and $v = d$ and equation 30 becomes

$$m_o = \frac{2\, g \rho H}{k^3} \quad D\,(kd) \quad A\ \left(\frac{D}{A}\right) \cos\,(\sigma t - \alpha\,) \tag{31}$$

where

$$D\,(kd) = \frac{1 - \cosh kd + kd \sinh kd}{\cosh kd}$$

The function D (kd) is plotted in Figure 3. The moments in this case may be easily computed through the use of Figures 4 and 5. Assuming H, T, d, D to be known, the ratio $\frac{D}{L}$ is found from Figure 4 and then m_o computed from Figure 5.

For the case of small cylinders, that is, such that the ratio of the diameter to the wave length is small, these formulas may be greatly simplified. This appears to be the most important case as is seen by considering Figure 2. For a 150 foot ocean wave, the cylinder diameter could exceed fifteen feet without appreciable deviation from the approximate formulas. For this condition the functions

$A\ \left(\frac{D}{L}\right)$ and $\alpha\left(\frac{D}{L}\right)$ may be replaced by

$$A\ \left(\frac{D}{L}\right) \cong \frac{\pi^3}{2} \left(\frac{D}{L}\right)^2 \quad ; \qquad \alpha\ \left(\frac{D}{L}\right) \cong \frac{\pi^3}{4} \quad \left(\frac{D}{L}\right)^2 \tag{32}$$

The force, F_z, then becomes

$$F_z \cong \frac{\pi^2\ \rho\ \ g\ \ D^2}{2} \left(\frac{H}{L}\right) \quad \frac{\cosh k\ (d + z)}{\cosh kd} \quad \cos \sigma t \tag{28'}$$

and the surface elevation at the circumference of the pile may be written,

$$\eta = \frac{H}{2} \sqrt{1 + \frac{\pi^2 D^2}{L^2} \cos^2 \theta} \sin\,(\sigma t - \psi\,) \tag{33}$$

where

$$\tan \psi = \frac{\pi D}{L} \cos \theta\ ,$$

while the pressure, at depth z, as a function of θ, is

$$p = \frac{\rho g\ H}{2} \quad \frac{\cosh k\ (d{+}z)}{\cosh kd} \sqrt{1 + \frac{4\pi^2\ D^2}{L^2} \quad \cos^2\theta} \sin\,(\sigma t + \delta) \tag{34}$$

where

$$\tan \delta = \frac{2\pi D}{L} \quad \cos \theta$$

A comparison of equations 29 and 30 indicates the maximum force and moment occur almost ninety degrees out of phase with the crest of the wave, that is approximately at the time the wave is passing through the still-water level.

In the formulas thus far presented the linear theory has been strictly followed. Approximations to the effects of steeper waves may be obtained by making some additional assumptions. It has been shown previously that in the case of small piles the force, F_z, given by equation 7' is exactly twice that of the incident wave alone. Assuming that this is a general result, the second and higher order terms in the parameter $(\frac{H}{L})$ may be introduced into the force calculations. To the second order, the force obtained in this manner is,

$$\frac{2\, F_z}{\rho \pi g\, D\, H} = \frac{\cosh k\,(d+z)}{\cosh kd} \left(\frac{\pi D}{L}\right) \sin \sigma t + \left(\frac{\pi H}{L}\right) \left(\frac{3 \cosh 2\, k\,(d+z)}{4 \sinh^3 kd \cosh kd} - \frac{1}{2 \sinh 2\, kd}\right) \left(\frac{2\pi D}{L}\right) \sin 2\,\sigma\, t \quad (34)$$

for the surface elevation of,

$$\eta/H = \frac{1}{2} \cos \sigma\, t + \frac{1}{4}\, \pi \, \left(\frac{H}{L}\right) \operatorname{ctnh} kd \left(1 + \frac{3}{2 \sinh^2 kd}\right) \cos 2\sigma t \quad (35)$$

For purpose of calculation a set of force distribution curves has been presented in Figures 6 and 7. The corresponding moments may be computed graphically according to the following procedure. For the moment about a hinge at depth z_1 compute z_1/d on the vertical scale. A new curve then may be plotted with abscissa $(\frac{z}{d} - \frac{z_1}{d})d$ times the old abscissa, and the corresponding moment will be equal to the area under this curve, after multiplication by the respective numerical factors. The coefficients of the $\sin \sigma t$, and $\sin 2\,\sigma\, t$ terms in the force equation have been designated $F_z^{(1)}$ and $F_z^{(2)}$, respectively.

The finite height of the waves introduces a second correction to the calculated moments, namely the contribution to the total moment of that portion of the wave above or below the still-water level. For a pile hinged at position u this correction term is, approximately,

$$\Delta m_u = \frac{\rho g H^2 u}{k} A \left(\frac{D}{L} \right) \cos (\sigma t - \alpha) \sin \sigma t \qquad (36)$$

Comparison with Experiment. A series of experiments has been carried out by Morison[5] in the wave channel to measure moments on cylindrical piles under varying sets of wave conditions. The cylinders were hinged at varying depths and subjected to regular wave trains which were of essentially three types; moderately steep waves in shallow water, steep waves in deep water and low waves in deep water. In Table I are presented the results of these experiments for piles hinged on the bottom in low waves in deep water. The theoretical moments, m_o, computed from the graphs of Figures 4 and 5, are corrected for the finite height of the waves by adding Δm_{-d} as given by equation 36.

In Table II the results for the largest cylinder in the same wave conditions are presented with the pile hinged at varying depths, z. It is seen that in both of these tables, in which the actual conditions approximate the assumptions made in solution, the agreement is good.

In Table III the results for the first two types of waves are presented. The first three entries correspond to moderately steep waves in shallow water, and the last three to steep waves in deep water. The deviations here are seen to be quite large, reflecting the fact that the waves cannot be closely approximated by sine waves in this range of $\frac{H}{L}$ and $\frac{d}{L}$.

Conclusions and Recommendations for Further Work. The rather large deviations of the experimental results from calculated values, which are indicated in Table III, give rise to the need for a consideration of possible sources of error together with possible modifications. In order to obtain agreement with experiment Morison[3] has introduced a second component of force on the pile which he designates as a "drag" force. It has been previously pointed out that his accelerative force, in the special case of small piles, may be identified with the diffraction theory of this report, provided C_m is taken equal to two. The introduction of the drag force is then equivalent to the assumption that drag and diffraction forces may be separated, each being considered to act independently of the other, an assumption which may not be well justified.

The force attributed to "drag" is essentially of two parts. One arises from the viscosity of the fluid and the corresponding frictional drag exerted by the fluid moving past the cylinder. This problem has been considered approximately using Schlicting's theory of periodic boundary layers and the results indicate that frictional effects are unimportant. The second part of the drag force is due to the separation of the lines of flow, with the resultant decrease in pressure behind the cylinder. The wake behind the cylinder is then essentially a region of no-motion except for the possible formation of vortices. The exact nature of this wake is not well understood even for the case of steady flow and very little is known about periodic motion, for then there is continual change with increasing and decreasing velocity.

For the diffraction theory presented in this report the motion is symmetrical around the cylinder. Hence no separation occurs and the wake drag must be zero. In light of this result it seems doubtful that the correction due to drag could be made simply by addition of a term corresponding to a wake while still maintaining the same diffraction force.

Drag forces are determined experimentally for the case of steady flow past a cylinder in the following manner. The assumption is made that the drag force is proportional to the square of the velocity, the diameter of the cylinder and the density of the fluid. The constant of proportionality, called C_D, is then determined empirically for various values of the Reynolds number. Morison has assumed that this result will also hold for periodic motion, an assumption which needs considerable investigation, since the flow behind the cylinder may not be able to adjust rapidly enough to maintain steady state conditions. Some calculations have been made, however, using Morison's assumptions and it is found that the introduction of drag does not appreciably improve the results.

The results of this report indicate that a great deal of additional work might profitably be carried out. In particular a detailed experimental study, with photographs, of the actual state of motion behind the cylinder would be of considerable value in estimating the effect of drag. It might be expected that for moderately small velocities the motion up to a certain point on the cylinder is well approximated by the diffraction theory, while beyond that point the flow separates leaving a dead water region. If this should prove to be the case, additional theoretical results are possible.

REFERENCES

(1) Morse, P. M. Vibrations and Sound. First Edition 1936

(2) Havelock, T.H. The Pressure of Water Waves on a Fixed Obstacle. Proc. Roy. Soc. of London, A, volume 175, pages 409-421,1940

(3) Morison, J. R.;O'Brien, M.P.; Johnson, J.W.; Schaaf, S.A. The Force Exerted by Surface Waves on Piles. Petroleum Transactions, A.I.M.E., vol. 189, 1950

(4) Mathematical Tables Project and M.I.T. Underwater Sound Lab. Scattering and Radiation from Circular Cylinders and Spheres. July, 1946.

(5) Morison, J. R. The Forces Exerted by Waves on Marine Structures. Wave Project Report. Series 35, Issue 3, Oct. 1950.

TABLE I

D (in.)	$\frac{d}{L}$	$\frac{H}{L}$	(m_o) exp. (ft. lbs.)	(m_o) theo. (ft. lbs.)	$m_o + \Delta m_o$ (ft. lbs.)
$\frac{1}{2}$	0.40	0.037	0.0207	0.0202	0.0205
$\frac{1}{2}$	0.41	0.038	0.0203	0.0207	0.0211
1	0.40	0.036	0.0903	0.0804	0.0816
1	0.40	0.037	0.0998	0.0813	0.0825
2	0.40	0.037	0.2910	0.314	0.320
2	0.40	0.037	0.2905	0.310	0.315

TABLE II

D = 2 inches

$\frac{z}{d}$	$\frac{d}{L}$	$\frac{H}{L}$	(m_o) exp. (ft. lbs.)	(m_o) theo. (ft. lbs.)	$(m_o + \Delta m_o)$ (ft. lbs.)
0.25	0.39	0.037	0.0335	0.0311	0.0373
0.42	0.40	0.039	0.0836	0.0808	0.0894
0.52	0.40	0.037	0.112	0.110	0.120
0.68	0.40	0.038	0.154	0.176	0.187
0.78	0.39	0.037	0.205	0.221	0.232
0.98	0.40	0.037	0.291	0.315	0.320
0.98	0.40	0.037	0.291	0.310	0.315

TABLE III

D (in.)	$\frac{d}{L}$	$\frac{H}{L}$	(m_o) exp. (ft. lbs.)	(m_o) theo. (ft. lbs.)	$(m + \Delta m)$ (ft. lbs.)
$\frac{1}{2}$	0.15	0.044	0.1158	0.0650	0.0703
1	0.16	0.044	0.0785	0.0408	0.0428
1	0.39	0.093	0.3900	0.2065	0.2240
2	0.16	0.045	0.2205	0.1530	0.1610
2	0.40	0.090	0.962	0.822	0.886

TABLE IV

Asymptotic Expansions for Bessel's Functions and Their Derivatives for Small x.

$$J_o(x) \sim 1 \qquad J_o'(x) \sim -\frac{x}{2}$$

$$J_1(x) \sim \frac{x}{2} \qquad J_1'(x) \sim \frac{1}{2}$$

$$J_m(x) \sim \frac{1}{m!}\left(\frac{x}{2}\right)^m \qquad J_m'(x) \sim \frac{1}{(m-1)!}\left(\frac{x}{2}\right)^{m-1}$$

$$Yo(x) \sim \frac{2}{\pi}(\ln x - \gamma) \qquad Y_o'(x) \sim \frac{2}{\pi x} \qquad \gamma = 0.1159$$

$$Y_1(x) \sim -\frac{2}{\pi x} \qquad Y_1'(x) \sim \frac{2}{\pi x^2}$$

$$Y_m(x) \sim -\frac{(m-1)!}{\pi}\left(\frac{2}{x}\right)^m \qquad Y_m'(x) \sim \frac{m!}{2\pi}\left(\frac{2}{x}\right)^{m-1}$$

$$m > 0$$

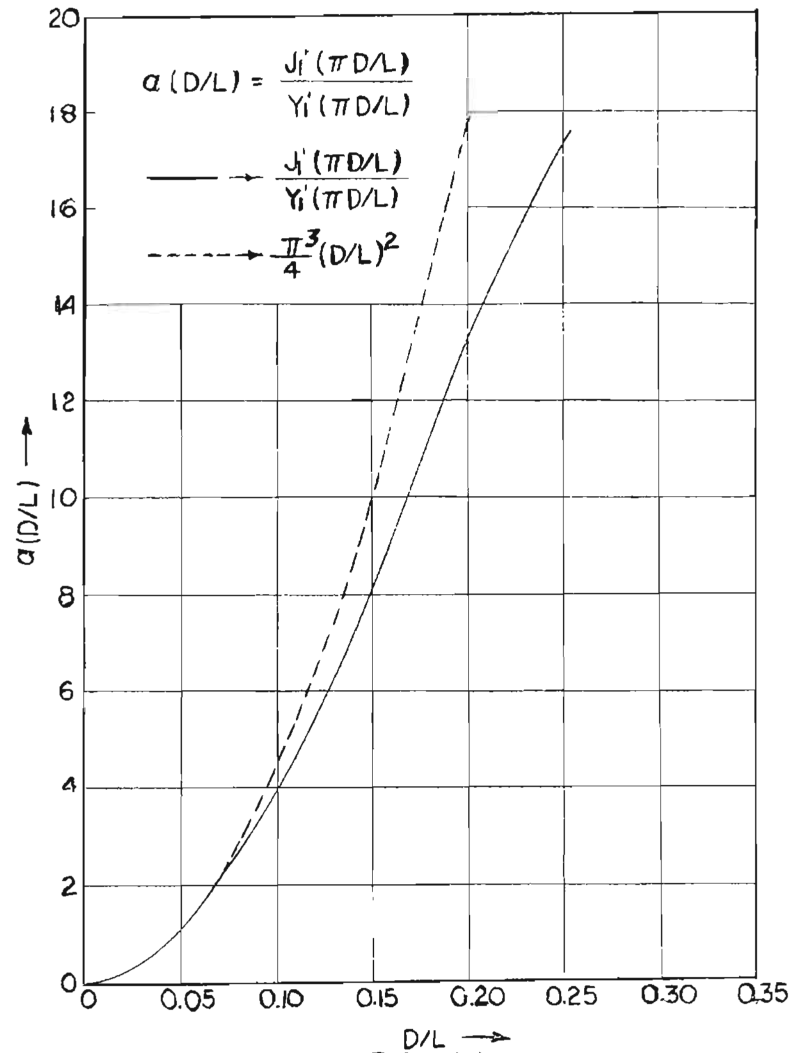

FIGURE 1

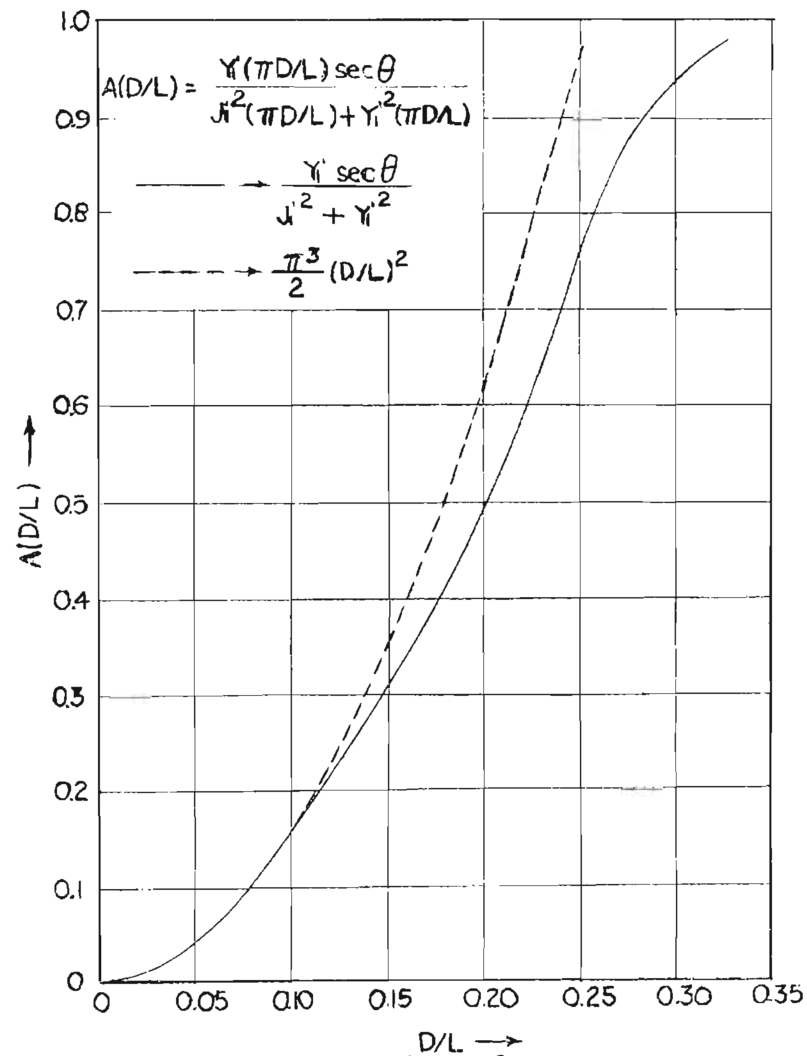

FIGURE 2

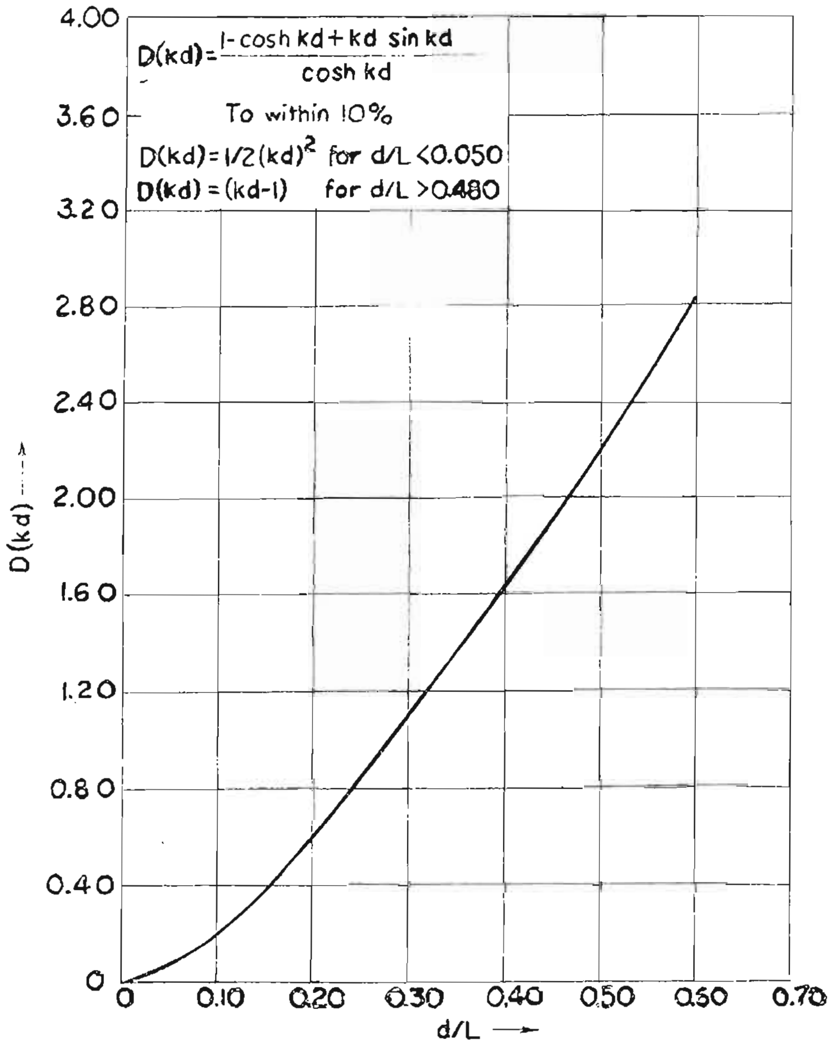

FIGURE 3

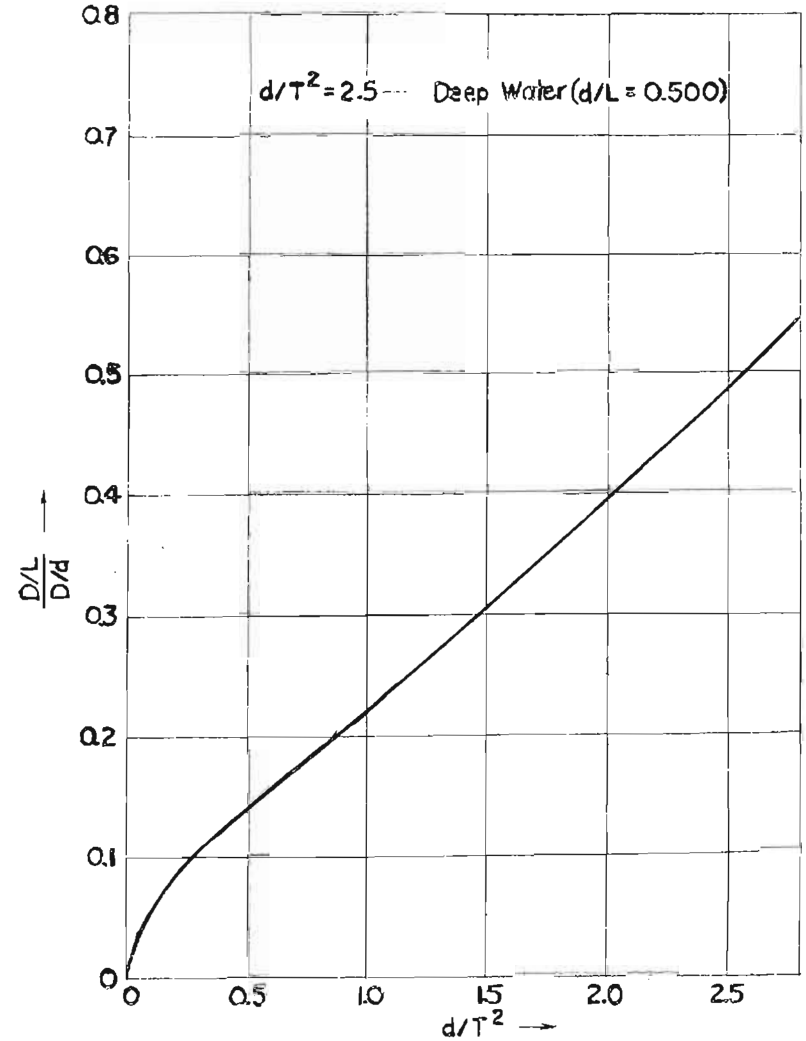

FIGURE 4

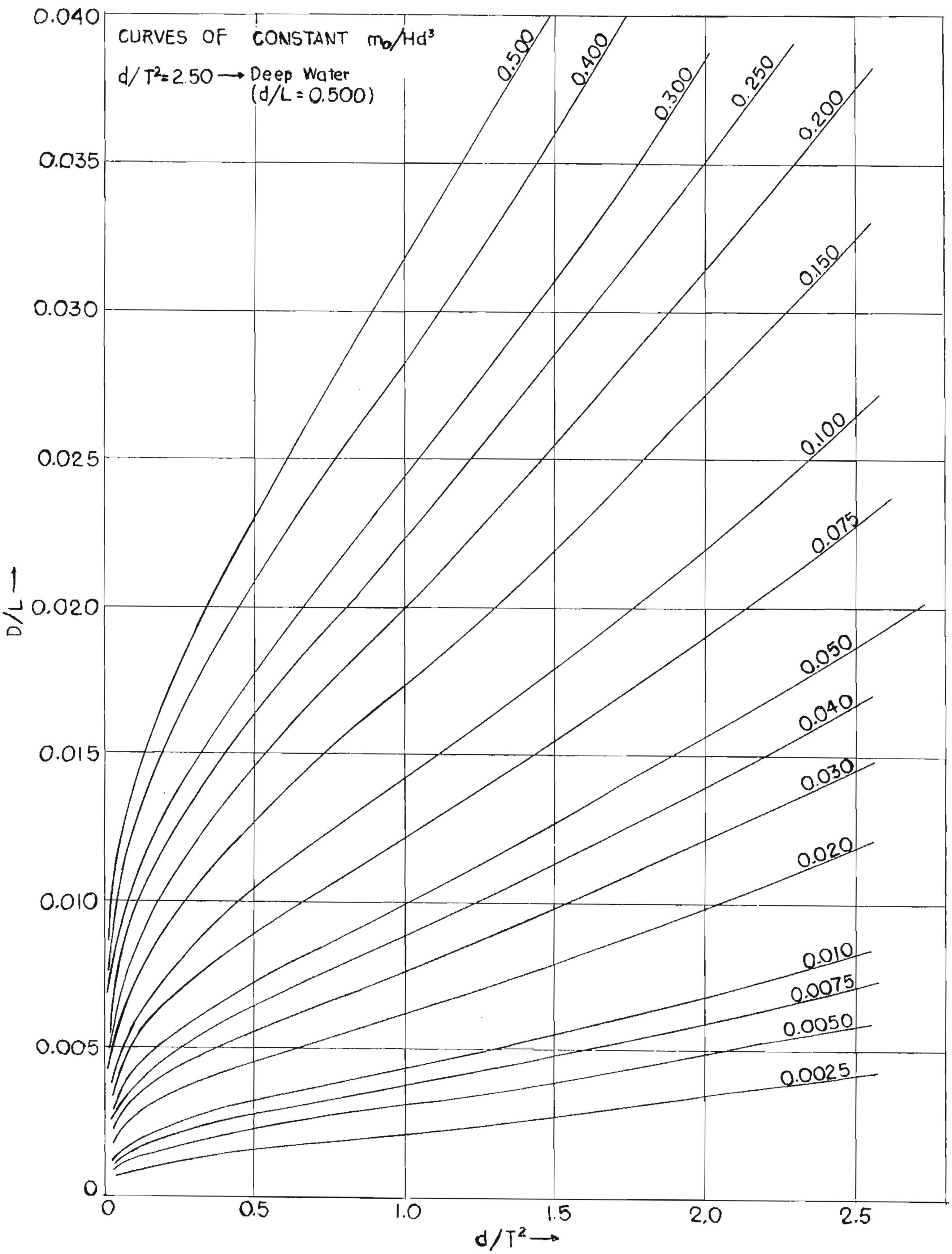
CURVES OF CONSTANT m_o/Hd^3
$d/T^2 = 2.50 \rightarrow$ Deep Water
$(d/L = 0.500)$
0.500
0.400
0.300
0.250
0.200
0.150
0.100
0.075
0.050
0.040
0.030
0.020
0.010
0.0075
0.0050
0.0025
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0
D/L
0
0.5
1.0
1.5
2.0
2.5
d/T^2

FIGURE 5

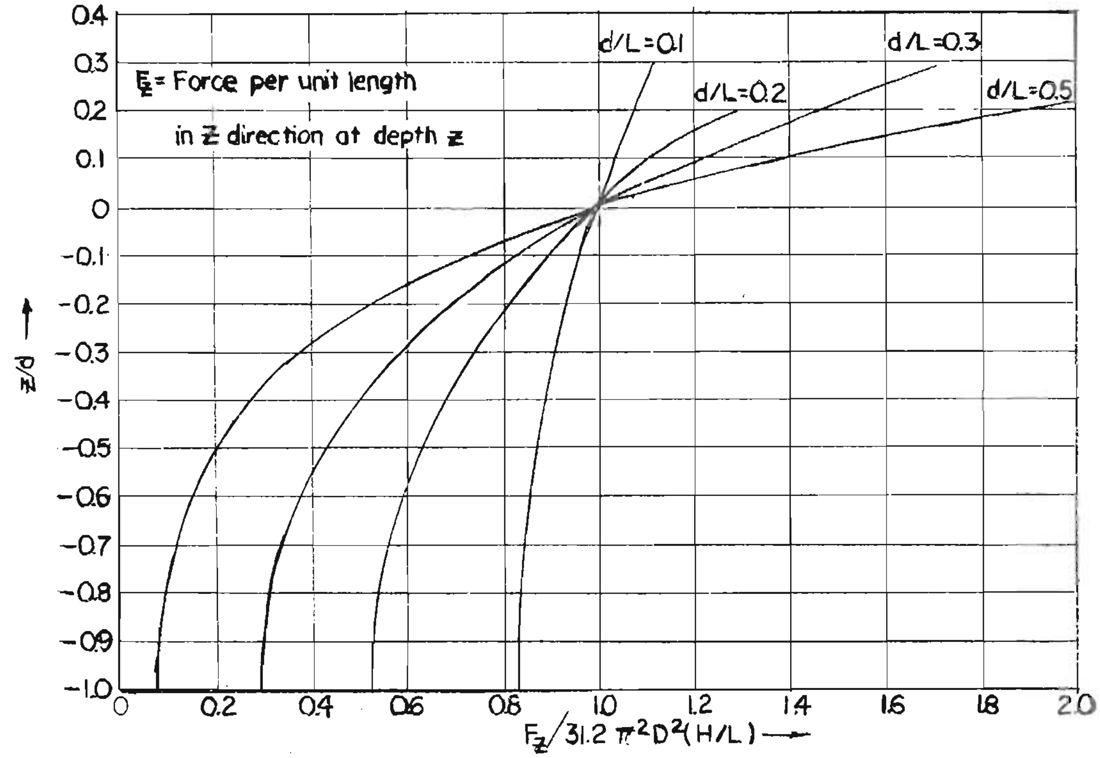

FIGURE 6

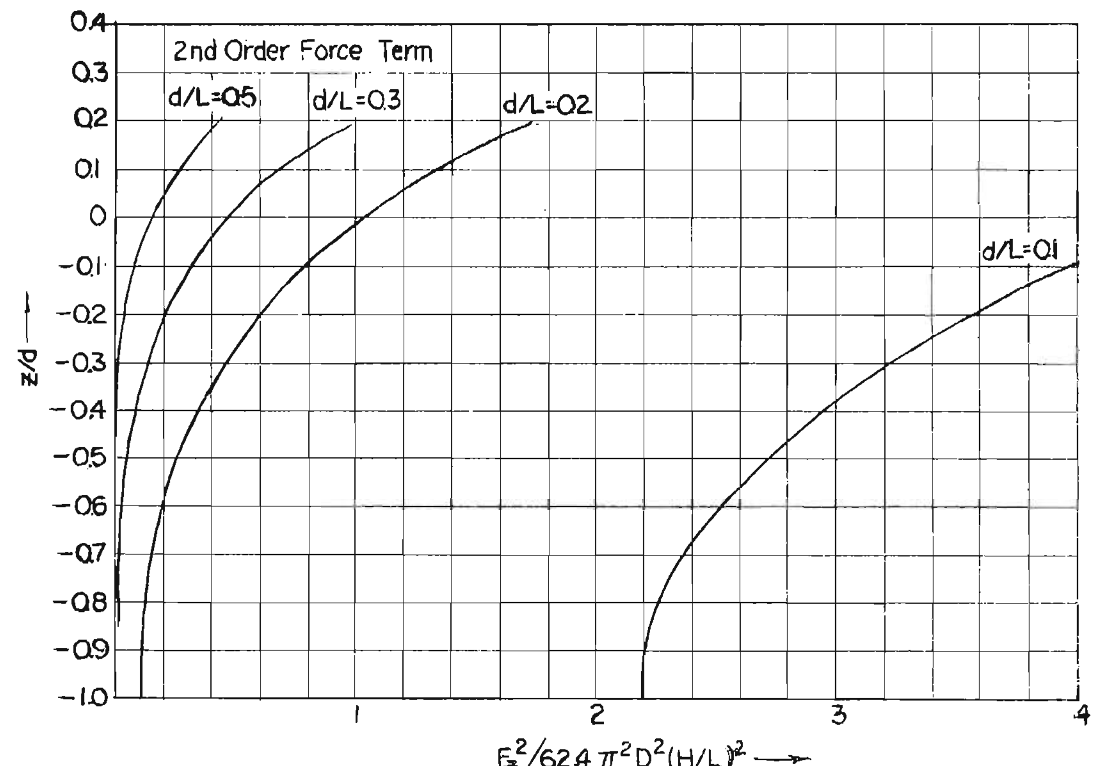

FIGURE 7