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WAVE GENERATION BY TURBULENT CONVECTION

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ABSTRACT

We consider wave generation by turbulent convection in a plane parallel, stratified atmosphere that sits in a gravitational field, g. The atmosphere consists of two semi-infinite layers, the lower adiabatic and polytropic and the upper isothermal. The adiabatic layer supports a convective energy flux given by mixing length theory; $F_c \sim \rho v_H^3$, where ρ is mass density and v_H is the velocity of the energy bearing turbulent eddies.

Acoustic waves with $\omega > \omega_{ac}$ and gravity waves with $\omega < 2k_h H_i \omega_b$ propagate in the isothermal layer whose acoustic cutoff frequency, ω_{ac}, and Brunt-Väisälä frequency, ω_b, satisfy $\omega_{ac}^2 = \gamma g/4H_i$ and $\omega_b^2 = (\gamma - 1)g/\gamma H_i$, where γ and H_i denote the adiabatic index and scale height. The atmosphere traps acoustic waves in upper part of the adiabatic layer (p-modes) and gravity waves on the interface between the adiabatic and isothermal layers (f-modes). These modes obey the dispersion relation

$$\omega^2 \approx \frac{2}{m} g k_h \left(n + \frac{m}{2}\right),$$

for $\omega < \omega_{ac}$. Here, m is the polytropic index, k_h is the magnitude of the horizontal wave vector, and n is the number of nodes in the radial displacement eigenfunction; $n = 0$ for f-modes.

Wave generation is concentrated at the top of the convection zone since the turbulent Mach number, $M = v_H/c$, peaks there; we assume $M_t \ll 1$. The dimensionless efficiency, η, for the conversion of the energy carried by convection into wave energy is calculated to be $\eta \sim M_t^{15/2}$ for p-modes, f-modes, and propagating acoustic waves, and $\eta \sim M_t$ for propagating gravity waves. Most of the energy going into p-modes, f-modes, and propagating acoustic waves is emitted by inertial range eddies of size $h \sim M_t^{3/2} H_t$ at $\omega \sim \omega_{ac}$ and $k_h \sim 1/H_t$. The energy emission into propagating gravity waves is dominated by energy bearing eddies of size $\sim H_t$ and is concentrated at $\omega \sim v_t/H_t \sim M_t \omega_{ac}$ and $k_h \sim 1/H_t$.

We find the power input to individual p-modes, $\dot{E}_p$, to vary as $\omega^{(2m^2+7m-3)/(m+3)}$ at frequencies $\omega \ll v_t/H_t$. Libbrecht has shown that the amplitudes and linewidths of the solar p-modes imply $\dot{E}_p \propto \omega^8$ for $\omega \ll 2 \times 10^{-2}\ \mathrm{s}^{-1}$. The theoretical exponent matches the observational one for $m \approx 4$, a value obtained from the density profile in the upper part of the solar convection zone. This agreement supports the hypothesis that the solar p-modes are stochastically excited by turbulent convection.

Subject headings: convection — Sun: atmosphere — Sun: oscillations — turbulence — wave motions

I. INTRODUCTION

Lighthill (1952) wrote the seminal paper on the generation of acoustic waves by turbulence in homogeneous fluids. Stein (1967) extended Lighthill's techniques to stratified fluids and also treated the emission of gravity waves. We reconsider Stein's problem for a more realistic model atmosphere and relate the turbulent spectrum to the convective energy flux via the Kolmogorov scaling and the mixing length hypothesis. Our goal is to estimate efficiencies for the conversion of the convective energy flux into both trapped and propagating waves. We treat mode excitation but not mode damping. Thus, we cannot estimate the energies of trapped modes which depend upon the balance between these two effects.

The plan of our paper is as follows. In § II we describe the model atmosphere and its eigenmodes. Next, in § III, we derive expressions for the rates at which individual modes gain energy from turbulent convection. In § IV, we estimate the total emissivities for the different wave types, p-modes, f-modes, propagating acoustic waves, and propagating gravity waves. A comparison of our results with those obtained in earlier studies, and a discussion of their implications, is given in § V.

II. ATMOSPHERE AND EIGENMODES

a) Static Atmosphere

Our model atmosphere is plane parallel, sits in a constant gravitational field, **g**, and consists of two semi-infinite layers, the lower adiabatic and polytropic and the upper isothermal. The pressure, p, density, ρ, and temperature, T, are continuous across the interface between the two layers. In the lower layer the adiabatic and polytropic indices are related by $\Gamma = 1 + 1/m$. The adiabatic

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index in the upper layer, γ, may differ from Γ. The z coordinate measures *depth* below the level at which the adiabatic layer would terminate in the absence of the isothermal layer. We denote quantities evaluated at the top of the adiabatic layer by a subscript t. Parameters in the isothermal layer are distinguished by a subscript i. Note, the ratio of the sound speeds $c_i/c_t = (\gamma/\Gamma)^{1/2}$.

In the adiabatic layer the thermodynamic variables exhibit a power-law behavior with depth:

$$p = p_t\left(\frac{z}{z_t}\right)^{m+1}, \quad \rho = \rho_t\left(\frac{z}{z_t}\right)^{m}, \quad T = T_t\left(\frac{z}{z_t}\right). \tag{1}$$

The sound speed, c, and the pressure scale height, H, satisfy $c^2 = gz/m$ and $H = z/(m+1)$.

The isothermal atmosphere is still simpler: $T = T_i$, $c = c_i$, and $H = H_i$ are all constant, whereas p and ρ are proportional to $\exp(z/H_i)$.

b) *Normal Modes*

We choose the Eulerian enthalpy perturbation, $Q \equiv p_1/\rho$, as the dependent variable in the linear wave equations. These read

$$\frac{d^2Q}{dz^2} + \frac{m}{z}\frac{dQ}{dz} + \left(\frac{\omega^2}{c^2} - k_h^2\right)Q = 0, \tag{2}$$

in the adiabatic layer, and

$$\frac{d^2Q}{dz^2} + \frac{1}{H_i}\frac{dQ}{dz} + \left[\frac{\omega^2}{c_i^2} - k_h^2\left(1 - \frac{\omega_b^2}{\omega^2}\right)\right]Q = 0, \tag{3}$$

in the isothermal layer (Kumar and Goldreich 1989). Here, ω is the wave frequency and $\boldsymbol{k}_h$ is the horizontal wavevector ($l \equiv k_h R_\odot$).

The displacement vector, $\boldsymbol{\xi}$, is related to Q by

$$\boldsymbol{\xi}_h = i\frac{\boldsymbol{k}_h}{\omega^2}Q, \quad \xi_z = \frac{1}{\omega^2}\frac{\partial Q}{\partial z}, \tag{4}$$

in the adiabatic layer, and by

$$\boldsymbol{\xi}_h = i\frac{\boldsymbol{k}_h}{\omega^2}Q, \quad \xi_z = \frac{1}{(\omega^2 - \omega_b^2)}\left[\frac{\partial Q}{\partial z} + \frac{(\gamma - 1)}{\gamma H_i}Q\right], \tag{5}$$

in the isothermal layer.

The normal modes are obtained by solving equations (2) and (3) subject to $Q \to 0$ as $z \to \infty$, Q and ξ_z continuous across the interface at z_t, and the appropriate boundary conditions as $z \to -\infty$. The continuity of $\boldsymbol{\xi}_h$ follows from that of Q.

The modes are classified as trapped or propagating, and as composed of acoustic or gravity waves. The adiabatic layer supports acoustic waves, but not gravity waves. Moreover, it refracts acoustic waves upward. Thus, propagating modes must be traveling waves in the isothermal atmosphere.

Solutions of the wave equation in the isothermal atmosphere are proportional to $\exp(-\kappa_\pm z)$, with

$$\kappa_\pm = \left\{\frac{1}{2H_i} \pm i\sqrt{\left[\left(\frac{\omega}{\omega_{ac}}\right)^2 - 1\right]\frac{1}{(2H_i)^2} + \left[\left(\frac{\omega_b}{\omega}\right)^2 - 1\right]k_h^2}\right\}, \tag{6}$$

where ω_{ac} and ω_b are the acoustic cutoff and Brunt-Väisälä frequencies:

$$\omega_{ac}^2 = \frac{\gamma g}{4H_i}, \tag{7}$$

and

$$\omega_b^2 = \frac{(\gamma - 1)g}{\gamma H_i}. \tag{8}$$

Thus $\omega_b^2 = 4(\gamma - 1)\omega_{ac}^2/\gamma^2$. There are two branches to the dispersion curve for traveling waves. For $2k_h H_i \ll 1$, these are a high frequency, acoustic wave, branch with $\omega > \omega_{ac}$, and a low-frequency, gravity-wave, branch with $\omega < 2k_h H_i \omega_b$.

Wave excitation by turbulent convection is concentrated in the upper adiabatic layer where the convective velocity peaks. We seek analytic expressions for the normalized eigenfunctions in this region. Since the dominant interactions are proportional to $\partial^2 Q/\partial z^2$ (see § III*b*), we explicitly evaluate this quantity for each mode. In doing so, we drop factors of order unity including, in places, γ, Γ, and m.

i) *Trapped Modes*

Trapped modes correspond to evanescent solutions in the isothermal layer and are restricted to a discrete set of eigenfrequencies for fixed k_h. In the limit that the adiabatic layer extends to vanishing surface pressure, the eigenfunctions may be expressed in terms of associated Laguerre polynomials and the dispersion relation reads

$$\omega^2 = \frac{2}{m}gk_h\left(n + \frac{m}{2}\right), \tag{9}$$

where the integer n denotes the number of nodes in the radial displacement eigenfunction (Christensen-Dalsgaard 1980; Christensen-Dalsgaard and Gough 1980). Trapped acoustic modes, or p-modes, correspond to $n \neq 0$. Modes with $n = 0$ are surface gravity waves, or f-modes. Trapped g-modes with $n \neq 0$ do not exist since the adiabatic layer is neutrally stratified, that is, its Brunt-Väisälä frequency vanishes. Equation (9) remains a good approximation for $\omega < \omega_{ac}$ even with finite surface pressure.

Only the physical solution, the one that grows less rapidly with height in the isothermal layer, is normalizable. The normalization condition reads

$$I \equiv \omega^2 \int_{-\infty}^{\infty} dz\, \rho \boldsymbol{\xi}_\omega \cdot \boldsymbol{\xi}^*_{\omega'} = \delta_{\omega,\omega'}\,, \tag{10}$$

at fixed $\boldsymbol{k}_h$. For modes with $2k_h H_i \ll 1$, most of the contribution to the energy integral comes from the adiabatic layer. This enables us to reexpress the normalization condition, using equation (2), in terms of the enthalpy perturbation as

$$I \approx \int_{z_t}^{\infty} dz\, \frac{\rho}{c^2}\, Q_\omega Q^*_{\omega'} = \delta_{\omega,\omega'}\,. \tag{11}$$

For $\omega = \omega'$, this integral evaluates the potential energy of a trapped mode in the adiabatic layer. The potential energy is equal to the kinetic energy for all modes. This accounts for the relation between equations (10) and (11).

1. *P*-Modes

A p-mode is a standing acoustic wave trapped between an upper reflecting layer at z_1, where $\omega/c(z_1) = 1/2H(z_1)$, and a lower turning point at z_2, where $\omega/c(z_2) = k_h$. The requirement that there be an upper reflecting layer restricts p-modes to frequencies below ω_{ac}.

It is easily shown that

$$\frac{z_1}{z_t} \sim \left(\frac{\omega_{ac}}{\omega}\right)^2, \tag{12}$$

and

$$\frac{z_2}{z_1} \sim \left(n + \frac{m}{2}\right)^2. \tag{13}$$

Outside the interval $z_1 \lesssim z \lesssim z_2$, the mode is evanescent. Both Q and $\boldsymbol{\xi}$ increase slowly with height above z_1. Below z_2 the k_h term in equation (2) dominates and $Q \propto \exp(-k_h z)$.

We study the p-mode eigenfunctions in the dual limit $\omega \ll \omega_{ac}$ and $2k_h H_t \ll 1$. In a polytropic layer with vanishing surface pressure, the eigenfunctions are solutions of equation (2) that are analytic at $z = 0$. These solutions may be expressed in terms of associated Laguerre polynomials. When the polytropic layer is overlane by an isothermal layer, the eigenfunctions include a contribution from the solution that is singular at $z = 0$. However, the boundary conditions at the interface between the two layers ensure that the contribution from the singular solution is small for $\omega \ll \omega_{ac}$.

We can approximate the eigenfunction in the region of propagation, $z_1 \ll z \ll z_2$, by the WKB solution

$$Q \sim \left(\frac{z_t}{z}\right)^{(m-1)/2} B_p \sin\left[2\omega\left(\frac{mz}{g}\right)^{1/2} + \phi_p\right]. \tag{14}$$

Below the lower turning point at z_2, the eigenfunction is exponentially small. In the evanescent zone above z_1 the atmosphere responds stiffly. Thus B_p is approximately equal to the surface amplitude, $Q(z_t)$, for $\omega \ll \omega_{ac}$.

The z derivaties of Q in the evanescent region enter into the expressions we derive for wave generation. For $\omega \ll \omega_{ac}$, $\partial/\partial z$ has magnitude $\omega^2/g \sim (\omega/\omega_{ac})^2 H^{-1}$, as follows directly from equation (2). This equation has a singular point at $z = 0$, and its regular solution is given by a power series in $\omega^2 z/g$. This verifies our assertion about the magnitude of $\partial/\partial z$. Of course, the polytropic atmosphere does not extend to $z = 0$. However, this is of little consequence for the eigenfunctions that become evanescent well below $z = z_t$.

Given the properties of the eigenfunction described above, it follows from the normalization equation (11) that

$$B_p^2 \sim \frac{z_t^m \omega^{2(m-1)} k_h}{g^{(m-2)} \rho_t}\,. \tag{15}$$

Evaluating $\partial^2 Q/\partial z^2$ we obtain

$$\frac{\partial^2 Q}{\partial z^2} \sim \left(\frac{\omega^2}{g}\right)^2 B_p\,, \tag{16}$$

for $z_t \lesssim z \ll z_1$.

2. *F*-Modes

Direct substitution into equations (2) and (3) verifies that $Q = B_f \exp(-k_h z)$, with $\omega^2 = gk_h$, is an exact solution of the wave equations in both the adiabatic and the isothermal layers. Moreover, ξ_z formed from equations (4) and (5) is continuous across z_t. This family of normal modes consists of gravity waves confined near the surface of the convection zone; they are known as f-modes.

The f-modes are incompressible, $\nabla \cdot \boldsymbol{\xi} = 0$, which accounts for their simple dispersion relation. The amplitude, B_f, is determined from the normalization equation (11) to be

$$B_f^2 \sim \frac{z_t^m \omega^{2(m-1)} k_h}{g^{(m-2)} \rho_t} . \tag{17}$$

For all z,

$$\frac{\partial^2 Q}{\partial z^2} = k_h^2 B_f \exp(-k_h z) . \tag{18}$$

ii) *Propagating Waves*

Modes that propagate in the isothermal layer have continuous spectra. They are chosen to have no net flux in the isothermal layer; that is, they are composed of pairs of inward- and outward-propagating waves of equal amplitude. This choice ensures that propagating modes have real frequencies and are orthogonal to trapped modes. These modes are normalized such that

$$\omega^2 \int_{-\infty}^{\infty} dz \, \rho \boldsymbol{\xi}_\omega \cdot \boldsymbol{\xi}_{\omega'}^* = \delta(\omega - \omega') , \tag{19}$$

at fixed $\boldsymbol{k}_h$. The upper limit on the integral in equation (19) may be taken to be z_t, since the contribution from the adiabatic layer is finite, and therefore negligible.

1. Acoustic Waves

These modes have $\omega > \omega_{ac}$ and propagate in the isothermal atmosphere and in the upper part of the adiabatic layer. They are evanescent below the lower turning point at $z_2 \sim \omega^2/gk_h^2$. We deduce the properties of the eigenfunctions in the joint limit $\omega \gg \omega_{ac}$ and $k_h \ll \omega/c_i$.

In the isothermal layer

$$Q = C_a \sin [K_z(z_t - z) + \zeta_a] \exp\left[\frac{(z_t - z)}{2H_i}\right] , \tag{20}$$

where $K_z \approx \omega/c_i$. Application of the normalization condition given by equation (19) to equation (20) yields

$$C_a^2 \sim \frac{g^{1/2} z_t^{1/2}}{\rho_t} . \tag{21}$$

We approximate the eigenfunctions in the adiabatic layer by the WKB solutions

$$Q \sim \left(\frac{z_t}{z}\right)^{(m-1)/2} B_a \sin\left[2\omega\left(\frac{m}{g}\right)^{1/2}(z^{1/2} - z_t^{1/2}) + \phi_a\right] , \tag{22}$$

for $z_t \leq z \ll z_2$. The continuity of Q and ξ_z across z_t is used to relate B_a and ϕ_a to C_a and ζ_a. The phase, ϕ_a, is determined by the condition that $Q \propto \exp(-k_h z)$ for $z \to \infty$. For ω just above ω_{ac}, $B_a(\omega, k_h)$ displays sharp ridges along extensions of the p-mode dispersion curves. These correspond to resonances for the scattering of incoming waves by the atmosphere. These ridges flatten for $\omega \gg \omega_{ac}$ and

$$B_a^2 \sim \frac{\Gamma C_a^2}{[1 + (\Gamma - \gamma)\cos^2 \phi_a]} . \tag{23}$$

For later use we record

$$\frac{\partial^2 Q}{\partial z^2} \approx -\frac{m\omega^2}{gz} Q , \tag{24}$$

for $z_t \leq z \ll z_2$.

2. Gravity Waves

Gravity modes with $\omega < 2k_h H_i \omega_b$ propagate in the isothermal atmosphere but are evanescent in the neutrally stable adiabatic layer. We detail their properties in the double limit $\omega \ll 2k_h H_i \omega_b$ and $2k_h H_i \ll 1$.

In the isothermal atmosphere

$$Q = C_g \sin [K_z(z_t - z) + \zeta_g] \exp\left[\frac{(z_t - z)}{2H_i}\right] , \tag{25}$$

where $K_z \approx (\omega_b/\omega)k_h$.

The amplitude C_g is determined by the normalization equation (19) to be

$$C_g^2 \sim \frac{g^{1/2}}{\rho_t z_t^{1/2} k_h} . \tag{26}$$

In the adiabatic layer, for $z_t \leq z \ll k_h^{-1}$, the last term in the wave equation (2) is much smaller than the first and second terms and may be ignored. The reduced wave equation yields

$$Q \approx B_g \left(\frac{z_t}{z}\right)^{(m-1)} + D_g . \tag{27}$$

The ratio D_g/B_g is determined by fitting Q in the upper part of the adiabatic layer to $Q \propto \exp(-k_h z)$ at $z \gg k_h^{-1}$. For $2k_h H_t \ll 1$, $D_g/B_g \ll 1$.

The continuity of Q and ξ_z across z_t is used to relate B_g and D_g to C_g and ζ_g. We find

$$\tan \zeta_g \sim -\frac{\omega k_h z_t^{3/2}}{g^{1/2}} . \tag{28}$$

The small value of $\tan \zeta_g$ is due to the change in orientation of the velocity field from almost vertical in the top of the adiabatic layer to almost horizontal in the isothermal layer. From equation (28) it follows that

$$B_g^2 \sim \frac{z_t^{5/2} \omega^2 k_h}{g^{1/2} \rho_t} . \tag{29}$$

For later use we note that

$$\frac{\partial^2 Q}{\partial z^2} \sim \frac{m(m-1)Q}{z^2} , \tag{30}$$

holds for $z \ll k_h^{-1}$.

c) *Turbulent Convection*

In the absence of a reliable theory for turbulent convection, we are guided by the mixing length hypothesis. According to this hypothesis, the convective energy flux, F_c, is carried by turbulent eddies whose dimensions are of order the local pressure scale height, $H(z) = z/(m+1)$. The velocity and entropy fluctuations associated with these energy bearing eddies, $v_H(z)$ and $s_H(z)$, are related to the mean entropy gradient, ds/dz, by

$$v_H^2 \sim \frac{gH^2}{c_p} \frac{ds}{dz} , \tag{31}$$

where c_p is the specific heat at constant pressure per unit mass, and

$$s_H \sim H \frac{ds}{dz} , \tag{32}$$

These relations lead to

$$F_c \sim \rho T v_H s_H \sim \rho v_H^3 . \tag{33}$$

Since F_c is independent of z,

$$v_H(z) = v_t \left(\frac{z_t}{z}\right)^{m/3} , \tag{34}$$

where $v_t \equiv v_H(z_t)$.

In treating the convection zone as adiabatic we have been neglecting the superadiabaticity of the temperature gradient, $c_p^{-1} T\, ds/dz$, with respect to the adiabatic temperature gradient, g/c_p. From equation (32) it follows that the ratio of these gradients may be expressed as

$$\frac{T}{g} \frac{ds}{dz} \sim M^2 , \tag{35}$$

where the Mach number of the turbulence, $M \equiv v_H/c$. Appeal to equation (33) establishes that

$$M \sim \left(\frac{F_c}{\rho c^3}\right)^{1/3} . \tag{36}$$

We assume that the turbulent velocities are substantially subsonic even near the top of the convection zone, that is, $M_t \ll 1$. Under these conditions we are justified in approximating the convection zone as adiabatic when calculating eigenfunctions for the normal modes.

The characteristic time scale of the energy bearing eddies is

$$\tau_H \sim \frac{H}{v_H} . \tag{37}$$

It is smallest at the top of the convection zone where

$$\tau_t \sim \frac{1}{M_t \omega_{ac}} . \tag{38}$$

The velocities of smaller, $h < H$, inertial range eddies are related to those of the energy bearing eddies by the Kolmogorov scaling (Tennekes and Lumley 1972),

$$\frac{v_h}{v_H} = \left(\frac{h}{H}\right)^{1/3} , \tag{39}$$

at fixed z. The Kolmogorov spectrum applies to turbulent convection because, below the scale of the energy bearing eddies, the Reynolds stress provides greater accelerations than the buoyancy forces (Goldreich and Keeley 1977*a*). This implies that entropy mixes like a passive scalar contaminant in the inertial range. Thus,

$$\frac{s_h}{s_H} \sim \left(\frac{h}{H}\right)^{1/3} . \tag{40}$$

The depth dependence of the properties of eddies of fixed size h follows from equations (32), (34), (37), and (40). We find

$$\begin{aligned} v_h(z) &\sim v_t\left[\frac{h z_t^m}{z^{(m+1)}}\right]^{1/3} \\ \tau_h(z) &\sim \tau_t\left[\frac{z^{(m+1)} h^2}{z_t^{(m+3)}}\right]^{1/3} . \\ s_h(z) &\sim s_t\left[\frac{h z_t^{(2m+3)}}{z^{2(m+2)}}\right]^{1/3} \end{aligned} \tag{41}$$

III. MODE EXCITATION

a) Source Terms

We begin this section by adding source terms due to turbulent convection to the linear wave equation (2) for the adiabatic layer. Next, we classify the individual terms as sources of monopole, dipole, and quadrupole radiation. Then we evaluate the excitation of wave modes by these sources.

We distinguish three principal sources of wave excitation by turbulent convection. They are, the expansion and contraction of fluid due to the gain and loss of specific entropy, buoyancy force variations associated with these entropy changes, and momentum transport by the fluctuating Reynold's stress.

We derive the inhomogeneous wave equation from the linearized versions of the equations for mass and momentum conservation supplemented by the equation of state for a perfect adiabatic gas. We augment the momentum equation by the divergence of the turbulent Reynolds stress, and the adiabatic equation of state by the entropy fluctuations associated with turbulent convection. These equations now read:

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 , \tag{42}$$

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla p_1 - \rho_1 \boldsymbol{g} = -\nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v}) \equiv \boldsymbol{F} , \tag{43}$$

and

$$\frac{p_1}{p} - \frac{\Gamma \rho_1}{\rho} = \frac{s}{c_v} , \tag{44}$$

where ρ_1, p_1, $\boldsymbol{v}$, and s are the Eulerian density, pressure, velocity, and entropy perturbations associated with the turbulent convection and the waves it generates. The subscript 1 attached to the density and pressure perturbations denotes that only the lowest order variations of these quantities need be retained. Equation (44), the Eulerian form of the perturbed equation of state, holds because the background state is isentropic.

Eliminating ρ_1 and $\boldsymbol{v}$ from the left-hand sides of equations (42)–(44), we obtain the inhomogeneous wave equation

$$\nabla^2 Q + \frac{g}{c^2}\frac{\partial Q}{\partial z} - \frac{1}{c^2}\frac{\partial^2 Q}{\partial t^2} = \frac{S^{(1)} + S^{(2)}}{\rho} , \tag{45}$$

where

$$S^{(1)} = -\rho \frac{\partial^2}{\partial t^2}\left(\frac{s}{c_p}\right) - g\frac{\partial}{\partial z}\left(\frac{\rho s}{c_p}\right) , \qquad S^{(2)} = \nabla \cdot \boldsymbol{F} . \tag{46}$$

The interpretation of equation (45) is somewhat subtle. Provided we drop the final $c^{-2}\partial^2 Q/\partial t^2$ term on the left-hand side as a first approximation in the limit of subsonic turbulence, it determines the near field turbulent pressure perturbations from the turbulent velocity and entropy perturbations. The $c^{-2}\partial^2 Q/\partial t^2$ term connects the near field perturbations to the wave field perturbations. The latter may be expanded in terms of the normal modes.

The identification of sources by multipole order is a useful device in estimating wave emission by turbulent convection. It helps to separate the sources that must be retained from those that may be safely discarded. For homogeneous and isotropic turbulence the multipole expansion may be carried out in several equivalent ways. In our application the turbulence is z-dependent, and therefore inhomogeneous, and the atmosphere is stratified, and therefore anisotropic. Under these circumstances the method of choice is to identify sources according to whether they involve a change in fluid volume (monopole terms), a source of external momentum (dipole terms), or merely internal stresses (quadrupole terms).[2] Classification based on the angular dependence of the wave amplitude in the radiation zone is not useful, because the angular dependence results, in part, from the anisotropy of the medium.[3] Identification of sources by the number of their spatial derivatives also leads to ambiguity, since it differs according to the choice of dependent variable.

The first term in $S^{(1)}$ arises directly from the volume change due to an entropy change at fixed pressure. It is a monopole source. The second term in $S^{(1)}$ reflects the buoyancy force variation associated with this volume change. It involves a variation of the density of momentum supplied by the external gravitational force and is a dipole source. The double divergence of the Reynolds stress in $S^{(2)}$ reflects the redistribution of momentum by internal stresses. It is a quadrupole source.

One might suspect that the monopole and dipole terms in $S^{(1)}$ produce more acoustic radiation than the quadrupole term in $S^{(2)}$. Treating these three terms independently appears to confirm this suspicion; the monopole and dipole terms are found to excite comparably greater amounts of acoustic radiation than the quadrupole term. However, the correct solution is more subtle. As we demonstrate shortly, destructive interference causes the total monopole plus dipole acoustic emission to be of the same order as the quadrupole emission.

b) Amplitude Equation

The total enthalpy perturbation, $Q(x, t)$, is expanded in terms of the normal modes, $Q_\alpha(z)$, as

$$Q = \frac{1}{\sqrt{2\mathscr{A}}} \sum_\alpha [A_\alpha Q_\alpha \exp(-i\omega t + i\boldsymbol{k}_h \cdot \boldsymbol{x}) + A_\alpha^* Q_\alpha^* \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x})] , \tag{47}$$

where $\mathscr{A}$ is the horizontal cross section of the atmosphere.[4] The mode amplitudes, $A_\alpha(t)$, are slowly varying functions of time, $|dA_\alpha/dt| \ll \omega |A_\alpha|$. Substituting this expansion into equation (45), multiplying both sides by $Q_\alpha^* \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x})$, and integrating over space and time, we obtain

$$A_\alpha(t) = \frac{1}{2i\omega \mathscr{A}^{1/2}} \int_{-\infty}^{t} dt \int d^3x Q_\alpha^*(S^{(1)} + S^{(2)}) \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x}) . \tag{48}$$

Taking $-\infty$ for the lower limit on the integral over t involves the implicit assumption that damping erases the memory of excitations from the distant past.

Next, we integrate by parts to transfer all time and space derivatives to the eigenfunctions. The contributions due to the individual source terms are discussed separately below.

The monopole plus dipole terms contribute

$$A_\alpha^{(1)}(t) = \frac{1}{2i\omega \mathscr{A}^{1/2}} \int_{-\infty}^{t} dt \int d^3x \frac{\rho s}{c_p} \left(\omega^2 Q_\alpha^* + g \frac{\partial Q_\alpha^*}{\partial z}\right) \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x}) . \tag{49}$$

With the aid of the homogeneous wave equation (2), we transform equation (49) to

$$A_\alpha^{(1)}(t) \approx -\frac{1}{2i\omega \mathscr{A}^{1/2}} \int_{-\infty}^{t} dt \int d^3x \frac{\rho c^2 s}{c_p} \left(\frac{\partial^2 Q_\alpha^*}{\partial z^2} - k_h^2 Q_\alpha^*\right) \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x}) . \tag{50}$$

The contribution due to the quadrupole term is

$$A_\alpha^{(2)}(t) \approx \frac{1}{2i\omega \mathscr{A}^{1/2}} \int_{-\infty}^{t} dt \int d^3x \rho \boldsymbol{v}\boldsymbol{v} : \nabla\nabla Q_\alpha^* \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x}) . \tag{51}$$

The normal mode eigenfunctions share the property that $k_h |Q_\alpha| \lesssim |\partial Q_\alpha/\partial z|$ near the top of the adiabatic layer. More precisely, other than the f-modes for which $\partial Q_\alpha/\partial z = -k_h Q_\alpha$, the approximate mode eigenfunctions calculated in § IIb satisfy the strict inequality. This implies that

$$A_\alpha^{(1)}(t) \approx -\frac{1}{2i\omega \mathscr{A}^{1/2}} \int_{-\infty}^{t} dt \int d^3x \frac{\rho c^2 s}{c_p} \frac{\partial^2 Q_\alpha^*}{\partial z^2} \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x}) , \tag{52}$$

[2] This method preserves the ordering of source terms by the efficiency with which they generate radiation.

[3] For example, a spherically symmetric point source radiates anisotropically in a stratified atmosphere.

[4] For the moment we are treating the atmosphere as being of finite horizontal extent.

and

$$A_\alpha^{(2)}(t) \approx \frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\infty}^{t} dt \int d^3x \rho v_z^2 \frac{\partial^2 Q_\alpha^*}{\partial z^2} \exp(i\omega t - i\boldsymbol{k}_h \cdot \boldsymbol{x}) , \tag{53}$$

provide order of magnitude estimates for $A_\alpha^{(1)}(t)$ and $A_\alpha^{(2)}(t)$. However, $A_\alpha^{(1)}(t) = 0$ for f-modes as a consequence of their incompressibility.

Now we compare the relative sizes of $A_\alpha^{(1)}(t)$ and $A_\alpha^{(2)}(t)$. We start with the contributions made by energy bearing eddies and go on to investigate those due to smaller, inertial range eddies.

According to equations (31)–(32), $c^2 s_H/c_p \sim v_H^2$. Thus, except for the f-modes, the entropy and the Reynolds stress sources associated with energy bearing eddies make comparable contributions to $A_\alpha(t)$. This illustrates the destructive interference between the monopole and dipole amplitudes to which we referred earlier; for energy bearing eddies and acoustic modes with $\omega \sim v_H/H$, the monopole and dipole terms in equation (49) are each larger by a factor $\sim(c/v_H)^2$ than the combined term in equation (50). The destructive interference between monopole and dipole amplitudes is a consequence of the anisotropy of the adiabatic layer. This is expressed by the anisotropic form of equation (2) which transforms equation (49) into equation (50).

For inertial range eddies, $c^2 s_h/c_p \sim v_h^2(H/h)^{1/3}$. This suggests that, unlike energy bearing eddies, inertial range eddies might excite waves more by their entropy sources than by their Reynolds stress sources. In fact, this is not the case. From equation (50) we see that wave excitation by the entropy source depends upon the time variability of the Eulerian entropy field. Inertial range eddies contribute to this time variation in different ways. The kinetic energy in an eddy of size $h \lesssim H$ may dissipate raising the local value of s_h. Neighboring eddies of similar size having opposite signs of s_h may collide and mix their fluid thereby smoothing the spatial variation of the entropy field on scale h. An eddy of size h carrying an entropy fluctuation s_h may be advected at speeds up to v_H. Of these possibilities, the dissipation of kinetic energy into heat produces the largest entropy source. However, this source is just equal to that provided by the Reynolds stress. Thus, from here on we use equation (53) to estimate the total excitation rate of normal modes.

Destructive interference between monopole and dipole radiation fields holds the acoustic emissivity of turbulent convection at the level characteristic of free turbulence[5] for which the emissivity is dominated by acoustic quadruples. We did not appreciate this point in our earlier treatment of acoustic emission by turbulent fluids (Goldreich and Kumar 1988). There we discussed the emissivity of turbulent pseudo-convection, a surrogate for turbulent convection. Since this model has acoustic dipoles but not acoustic monopoles, its emissivity is greater than that of free turbulence.

c) *Excitation Rate*

Turbulent convection consists of a hierarchy of critically damped eddies. Different eddies of similar size are assumed to be uncorrelated. This assumption enables us to divide into several steps the calculation of the rate at which turbulent convection pumps energy into a wave mode.

To begin, we estimate the magnitude of the incremental amplitude, ΔA_α^h, produced by a single eddy of size h located at depth z over its lifetime $\tau_h \sim h/v_h$.

$$\Delta A_\alpha^h \sim \frac{\rho v_h h^4}{2i\omega\mathscr{A}^{1/2}} \frac{\partial^2 Q_\alpha^*}{\partial z^2} , \qquad \omega \lesssim \tau_h^{-1} . \tag{54}$$

In arriving at the above equation we have assumed that the eigenfunction does not vary dramatically over $\Delta z = h \leq H$. This is a good approximation for all the modes we are concerned with. At frequencies much greater than τ_h^{-1}, ΔA_α^h declines exponentially with increasing ω.

Next, we note that

$$\dot{E}_\alpha^h \sim \frac{|\Delta A_\alpha^h|^2}{\tau_h} , \tag{55}$$

is the mean rate at which one eddy supplies energy to mode α.

Then, summing over eddies of all sizes and depths, we obtain

$$\dot{E}_\alpha \sim \frac{1}{\omega^2} \int_{z_t}^{\infty} dz \rho^2 \left| \frac{\partial^2 Q_\alpha}{\partial z^2} \right|^2 \int_0^{h_{\max}} \frac{dh}{h} v_h^3 h^4 , \tag{56}$$

where

$$h_{\max}(z) \sim \frac{H(z)}{1 + [\omega \tau_H(z)]^{3/2}} . \tag{57}$$

In deriving equation (56) from (55), we include a factor $\mathscr{A} dz/h^3$, the number of eddies in the horizontal slice of cross-sectional area $\mathscr{A}$ between vertical depths z and $z + dz$. The appearance of dh/h in equation (56) denotes that each inertial range eddy accounts for a finite range of scale size $dh/h \sim 1$. Carrying out the integration over h yields

$$\dot{E}_\alpha \sim \frac{\rho_t^2 H_t^8}{\omega^2 \tau_t^3} \int_{z_t}^{\infty} \frac{dz}{z_t} W\left(\frac{z}{z_t}\right) \left| \frac{\partial^2 Q_\alpha}{\partial z^2} \right|^2 , \tag{58}$$

[5] Free turbulence is turbulence that is not subject to external forces.

where the weight factor, W, is given by

$$W(u) = \frac{u^{m+4}}{[1 + (\omega\tau_t)^{3/2} u^{(m+3)/2}]^5} . \tag{59}$$

The weight factor is sharply peaked about

$$u_* \sim 1 + \frac{1}{(\omega\tau_t)^{3/(m+3)}} , \tag{60}$$

where

$$W(u_*) \approx \frac{1}{(\omega\tau_t)^{3(m+4)/(m+3)}[1 + (\omega\tau_t)^{3(3m+7)/2(m+3)}]} ; \tag{61}$$

it decays as $u^{(m+4)}$ for $u \ll u_*$ and as $u^{-(3m+7)/2}$ for $u \gg u_*$.

The peak in W is so sharp that $\dot{E}_\alpha$ is dominated by contributions from $z \sim z_*$ for all wave modes. Physically, this means that the excitation is concentrated in the layer where the turnover time of the energy bearing eddies is most nearly equal to the mode period. This enables us to further simplify the expression for $\dot{E}_\alpha$ to

$$\dot{E}_\alpha \sim \frac{\rho_t^2 H_t^8}{\tau_t} \frac{1}{(\omega\tau_t)^{(5m+21)/(m+3)}[1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}]} \left| \frac{\partial^2 Q_\alpha(z_*)}{\partial z^2} \right|^2 . \tag{62}$$

IV. FLUXES OF ENERGY

To evaluate the total excitation rate for each type of mode, we substitute the relevant expression for $\partial^2 Q_\alpha(z_*)/\partial z^2$ given in § II*b*) into equation (62). Following that, we integrate $\dot{E}_\alpha$ over all modes of the family to determine the fraction of the convective energy flux that family receives.

The frequencies of trapped modes satisfy equation (9). The flux of energy going into modes of a given family is

$$F_\alpha = \frac{1}{\mathscr{A}} \sum_\alpha \dot{E}_\alpha = \frac{1}{2\pi} \sum_n \int dk_h \, k_h \, \dot{E}_\alpha , \tag{63}$$

where the sum over α includes all modes in the family, the sum over n includes all dispersion ridges in the family, and $\int dk_h$ is over all modes along a ridge. The last equality follows because the spacing between adjacent k_h modes in a box of horizontal area, $\mathscr{A}$, is equal to $2\pi/\sqrt{\mathscr{A}}$. Therefore, the number of modes in d^2k_h is $\mathscr{A} d^2k_h/(2\pi)^2 = (\mathscr{A}/2\pi) dk_h \, k_h$.

For propagating modes, ω and k_h are independently specified. The flux of energy into a family of modes is computed from

$$F_\alpha = \frac{1}{\mathscr{A}} \sum_\alpha \dot{E} = \frac{1}{2\pi} \int d\omega \int dk_h \, k_h \, \dot{E}_\alpha , \tag{64}$$

where the double integral is over all modes in the family.

a) *P-Modes*

From equations (16) and (62), we obtain

$$\dot{E}_p \sim \rho_t H_t^3 v_t^3 M_t^{2(m+2)} k_h \frac{(\omega\tau_t)^{(2m^2+7m-3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} . \tag{65}$$

At fixed k_h, $\dot{E}_p$ varies as $\omega^{(2m^2+7m-3)/(m+3)}$ for $\omega\tau_t < 1$ and as $\omega^{(4m-7)/2}$ for $\omega\tau_t > 1$. To obtain the energy input rate per mode along the n'th p-mode ridge, we eliminate k_h from equation (65) by using equation (9). This procedure yields

$$\dot{E}_p \sim \frac{\rho_t H_t^2 v_t^3 M_t^{2(m+3)}}{(n + m/2)} \frac{(\omega\tau_t)^{(2m^2+9m+3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} . \tag{66}$$

The total flux of energy going into p-modes follows from substituting equation (66) into equation (63):

$$F_p \sim \rho_t v_t^3 M_t^{15/2} = M_t^{15/2} F_c . \tag{67}$$

From equation (66), we note that for $\omega\tau_t \gg 1$ the energy input rate is proportional to $(\omega\tau_t)^{(4m-3)/2}$, which increases with increasing ω for $m > \frac{3}{4}$. Since the maximum frequency for trapped p-modes is ω_{ac}, most of the energy flux goes into modes whose frequencies lie just below the acoustic cutoff, $\omega \lesssim \omega_{ac}$, and is emitted by inertial range eddies with $h \sim M_t^{3/2} H_t$ located in the top scale height of the convection zone.

b) *F-Modes*

The calculations for the f-modes are similar to those for the p-modes. We substitute equation (18) into equation (62) and find

$$\dot{E}_f \sim \rho_t H_t^3 v_t^3 M_t^{2(m+2)} k_h \frac{(\omega\tau_t)^{(2m^2+7m-3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} , \tag{68}$$

where we have set $\exp(-k_h z_*) \sim 1$ since $k_h z_* \sim M_t^2(\omega\tau_t)^{(2m+3)/(m+3)}[1 + (\omega\tau_t)^{3/(m+3)}] \lesssim 1$ for $\omega \le \omega_{ac}$. The rate of energy input per mode along the f-mode ridge reads

$$\dot{E}_f \sim \rho_t H_t^2 v_t^3 M_t^{2(m+3)} \frac{(\omega\tau_t)^{(2m^2+9m+3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} . \tag{69}$$

The total flux of energy going into f-modes is

$$F_f \sim \rho_t v_t^3 M_t^{15/2} = M_t^{15/2} F_c . \tag{70}$$

c) *Acoustic Waves*

From equations (24) and (62), we obtain

$$\dot{E}_a \sim \rho_t H_t^3 v_t^2 \frac{M_t^3}{(\omega\tau_t)^{11/2}} , \tag{71}$$

after averaging over the phase ϕ_a. Substituting equation (71) into equation (64), we derive the total flux of energy carried by the acoustic waves:

$$F_a \sim \rho_t v_t^3 M_t^{15/2} = M_t^{15/2} F_c . \tag{72}$$

Most of this energy is emitted by inertial eddies of size $h \lesssim M_t^{3/2} H_t$ located in the top scale height of the convection zone. It is carried by waves with $\omega \gtrsim \omega_{ac}$ and $k_h \lesssim 1/H_t$.

d) *Gravity Waves*

Equations (30) and (62) yield

$$\dot{E}_g \sim \rho_t H_t^4 v_t^2 M_t k_h \frac{(\omega\tau_t)^{3(m-3)/(m+3)}}{[1 + (\omega\tau_t)^{3(7m+9)/2(m+3)}]} , \tag{73}$$

so the power input into gravity waves peaks for $\omega\tau_t \sim 1$. Equation (73) holds for k_h in the range $\omega/\omega_b < 2k_h H_t < (\omega\tau_t)^{3/(m+3)}/[1 + (\omega\tau_t)^{3/(m+3)}]$. Substituting equation (73) into equation (64), we find the total flux of energy carried by the gravity waves:

$$F_g \sim \rho_t v_t^3 M_t = M_t F_c . \tag{74}$$

Most of this energy is emitted by energy bearing eddies located in the top scale height of the convection zone. It is carried by waves with $\omega\tau_t \sim 1$ and $k_h \lesssim 1/H_t$. The vertical wave vector of these waves in the isothermal layer is $k_z \sim 1/(M_t H_t)$.

V. DISCUSSION

a) *Previous Results*

Our principal results are dimensional efficiencies, η, for the conversion of the convective energy flux into the energy flux in different types of wave modes; $\eta_p \sim \eta_f \sim \eta_a \sim M_t^{15/2}$, and $\eta_g \sim M_t$. It is illuminating to compare these efficiencies to those obtained in previous investigations.

The classic result for the efficiency of emission of acoustic waves by homogeneous, isotropic turbulence is that of Lighthill (1952). Translated into our notation it is $\eta_a \sim M_t^5$. Here we are thinking of the acoustic emission from a layer of turbulent fluid of thickness, H_t, embedded in an otherwise uniform atmosphere. The energy bearing eddies are characterized by size, H_t, and velocity, v_t. In this system, the acoustic emission is dominated by the energy bearing eddies, and is concentrated at $\omega \sim v_t/H_t$, $k \sim M_t H_t$. We find $\eta_p \sim \eta_a \sim M_t^{15/2}$, with the emission dominated by inertial range eddies of size $h \sim M_t^{3/2} H_t$ and concentrated at $\omega \sim c_i/H_t$, $k_h \sim 1/H_t$. There are two relevant comparisons between our results and those of Lighthill.

First, we can redo the estimate for η_a from Lighthill's treatment restricting attention to emission from inertial range eddies having $h \lesssim M_t^{3/2} H_t$. These eddies, whose lifetimes $\tau_h \lesssim \omega_{ac}^{-1}$, dominate the emission of energy into p-modes and acoustic waves in the stratified atmosphere. A simple calculation yields $\eta_a \sim M_t^{15/2}$. This result agrees with ours showing that the acoustic emission from eddies with $h \lesssim M_t^{3/2} H_t$ is not affected by stratification.

Second, we can modify our calculation of η_p so that only the emission by energy bearing eddies is included. This is accomplished by repeating the procedure described in § IVa) but now limiting the integration over frequency along the p-mode ridges to $\omega \lesssim v_t/H_t$. This exercise yields $\eta_p \sim M_t^{10}$. The factor M_t^5 by which this result differs from Lighthill's may be accounted for as follows. Both in a homogeneous atmosphere and in our stratified atmosphere, the acoustic emissivity is proportional to $|\nabla\nabla Q|^2$. However, for $\omega \sim v_t/H_t$, $|\nabla\nabla Q|^2 \sim (M_t/H_t)^4 |Q|^2$ in the homogeneous atmosphere, whereas $|\nabla\nabla Q|^2 \sim (M_t^2/H_t)^4 |Q|^2$ in the stratified atmosphere. This difference, which accounts for four factors of M_t, arises because p-modes with $\omega \sim v_t/H_t \sim M_t \omega_{ac}$ are evanescent near the top of the convection zone in the stratified atmosphere.[6] The fifth factor of M_t arises from differences in phase space mode densities. In a uniform atmosphere, the number density of modes having $\omega \sim v_t/H_t$ is approximately $(M_t/H_t)^3$. This becomes M_t^3/H_t^2 per unit area for a layer H_t thick. The corresponding area density of p-modes in the stratified atmosphere is M_t^4/H_t^2, just one power of M_t smaller.

[6] For acoustic waves with $\omega \gtrsim \omega_{ac}$, $|\nabla\nabla Q|^2$ is of the same order in the stratified atmosphere as in a homogeneous atmosphere.

Stein (1967) investigated the emission of acoustic and gravity waves by turbulent convection in a stratified atmosphere. He paid proper attention to the roles of ω_{ac} and ω_b and to the shapes of the mode eigenfunctions. However, Stein considered an isothermal atmosphere whereas we treat a two level atmosphere with the turbulent convection confined to the lower, adiabatic layer. Finally, we relate the properties of the turbulence to the convective energy flux using the mixing length hypothesis and the Kolmogorov scaling. The differences between out model assumptions and those of Stein preclude a meaningful comparison between his results and ours.

Milkey (1970) commented on the relation between Stein's calculation of acoustic spectral emissivity, $\epsilon_a(\omega)$, and that for free turbulence.[7] He showed that the Kolmogorov spectrum implies $\epsilon_a \propto \omega^{-7/2}$ in the dual limit $\omega \gg \omega_{ac}$ and $\omega \gg 1/\tau_t$. Equation (13) in Goldreich and Kumar (1988) confirms this simple result and, written in our notation, reads

$$\epsilon_a(\omega) \sim \rho_t v_t^2 \frac{M_t^5}{(\omega\tau_t)^{7/2}} . \tag{75}$$

Our equation (71) giving $\dot{E}_a$ also leads to equation (75) since $\epsilon_a(\omega) \sim (\omega/c_t)^2 \dot{E}_a/H_t \sim (\omega\tau_t)^2 M_t^2 \dot{E}_a/H_t^3$.

b) *Solar p-Modes*

Libbrecht (1988) has determined $\dot{E}_p(\omega)$ from his solar *p*-mode observations. He finds $\dot{E}_p \propto \omega^8$ for $\omega \ll 2 \times 10^{-2}\ \mathrm{s}^{-1}$. Equation (65) gives $\dot{E}_p \propto \omega^{(2m^2+7m-3)/(m+3)}$ for $\omega\tau_t \ll 1$, in agreement with the observational result for $m \approx 4$, the polytropic index that fits the average density profile in the hydrogen ionization zone. Our formula fails for $\omega\tau_t \gg 1$; it gives $\dot{E}_p \propto \omega^{(4m-7)/4}$, or $\dot{E}_p \propto \omega^{4.5}$ for $m = 4$, while Libbrecht finds $\dot{E}_p \propto \omega^{-5}$ for $\omega \gg 2 \times 10^{-2}\ \mathrm{s}^{-1}$. The resolution of this discrepancy is in hand. It involves modification of the eigenfunctions in the polytropic layer for ω close to ω_{ac} by the boundary conditions imposed at the interface with the isothermal layer. These modifications, which are ignored here, will be described in a subsequent paper devoted to a detailed examination of the excitation of the solar *p*-modes.

Even the limited success of our theoretical calculations in matching the frequency dependence of $\dot{E}_p$ lends support to the hypothesis that the solar *p*-modes are stochastically excited by turbulent convection (Goldreich and Keeley 1977*b*).

c) *General Applications*

Wave emission by turbulent convection is a common process in stellar and planetary atmospheres. It is clearly implicated in the heating of stellar chromospheres and coronas. Our results provide a foundation for the theory of wave emission in stratified atmospheres. However, several additional factors need to be examined before serious applications to real systems are contemplated. Several of these are mentioned below.

Real atmospheres differ from our model atmosphere in ways that may have important practical implications. The upper part of the convective zone, where much of the wave generation occurs, may not be well approximated by an isentropic layer of constant adiabatic index. Instead, as in the Sun, it may be significantly superadiabatic and possess ionization zones through which Γ undergoes substantial variations. The model atmosphere makes an abrupt transition from an adiabatic layer to an isothermal layer. The emission of gravity waves is likely reduced by the gradual rise of ω_b with height in a real atmosphere. Moreover, radiative smoothing of temperature perturbations may damp waves and also modify their propagation by lowering the effective adiabatic index. Both effects are most likely to be relevant for the dominant gravity waves because of their low frequencies and short vertical wavelengths.

The scope of our investigation is restricted to linear waves in unmagnetized media. Wave heating depends upon the behavior of nonlinear waves. It may also involve the coupling of acoustic and gravity waves to magnetosonic and Alfv́en waves. These issues remain to be addressed by future studies.

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[7] Spectral emissivity is the energy emission rate, per unit volume, per unit frequency.

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