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WAVE PACKET TRANSFORMS OVER FINITE FIELDS*

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Abstract. This article introduces the abstract notion of finite wave packet groups over finite fields as the finite group of dilations, translations, and modulations. Then it presents a unified theoretical linear algebra approach to the theory of wave packet transforms (WPT) over finite fields. It is shown that each vector defined over a finite field can be represented as a finite coherent sum of wave packet coefficients as well.

Key words. Finite field, Wave packet group, Wave packet representation, Wave packet transform, Dilation operator, Periodic (finite size) data, Prime integer.

AMS subject classifications. 42C40, 12E20, 13B05, 12F10, 81R05, 20G40.

1. Introduction. The mathematical theory of finite fields has significant roles and applications in computer science, information theory, communication engineering, coding theory, cryptography, finite quantum systems and number theory [17, 28]. Discrete exponentiation can be computed quickly using techniques of fast exponentiation such as binary exponentiation within a finite field operations and also in coding theory, many codes are constructed as subspaces of vector spaces over finite fields, see [18, 20, 27] and references therein.

The finite dimensional data analysis and signal processing is the basis of digital signal processing, information theory, and large scale data analysis. In data processing, time-frequency (resp., time-scale) analysis comprises those techniques that analyze a vector in both the time and frequency (resp., time and scale) domains simultaneously, called time-frequency (resp., time-scale) methods or representations, see [4, 5, 16] and references therein. Commonly used coherent (structured) methods and techniques in such analysis are time-frequency analysis which is sometimes called as Gabor analysis [6], time-scale analysis which is called as wavelet analysis [25], and scale-time-frequency analysis which is mostly called as wave packet methods, see [11] and references therein. The theory of Gabor analysis is based on the modulations and translations of a given window vector and the phase space has a unified group structure, see [2, 10, 12, 19] and references therein. The wavelet theory is based on affine group as the group of dilations and translation, see [25] and references therein.

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Wavelet analysis of periodic data rely on embedding the vector space of finite size data in the Hilbert space of all complex valued sequences with finite $\|\cdot\|_2$-norm which is not on finite dimensional analogous to the continuous setting as is the case in Gabor analysis [3, 24, 29]. Some different approaches to the wavelet analysis over finite fields studied in [7, 8, 13, 15].

Wave packet analysis is a shrewd coherent state analysis which is an extension of the two most important and prominent coherent state methods. The mathematical theory of wave packet analysis over the local field $\mathbb{R}$ is originated from dyadic dilations, integer translations, and integer modulations of a given window vector. The structure of wave packet groups over prime fields (finite Abelian groups of prime orders) and the notion of wave packet representation on these wave packet groups are recently presented in [11].

In this article, we introduce the notion of wave packet group $\mathrm{WP}_{\mathbb{F}}$ associated to the finite field $\mathbb{F}$ as the group of dilation, translation and modulation and we present the abstract theory of wave packet transform over $\mathbb{F}$. If $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ is a window vector, we define the wave packet transform (WPT) $\mathcal{V}_{\mathbf{y}}$ as the voice transform defined on $\mathbb{C}^{\mathbb{F}}$ with complex values which are indexed in the finite wave packet group $\mathrm{WP}_{\mathbb{F}}$. These techniques imply a unified group theoretical based scale-time-frequency (dilation, translation and modulation) representations for vectors in $\mathbb{C}^{\mathbb{F}}$. It is shown that the wave packet transform $\mathcal{V}_{\mathbf{y}}$ as a windowed transform satisfies the isometric property and inversion formula as well.

2. Preliminaries and notation. Let $\mathbb{H}$ be a finite dimensional complex Hilbert space and $\dim \mathbb{H} = N$. A finite system (sequence) $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\} \subset \mathbb{H}$ is called a frame (or finite frame) for $\mathbb{H}$, if there exist positive constants $0 < A \leq B < \infty$ such that [4]

$$A\|\mathbf{x}\|^2 \leq \sum_{j=0}^{M-1} |\langle \mathbf{x}, \mathbf{y}_j \rangle|^2 \leq B\|\mathbf{x}\|^2 \quad \text{for all } \mathbf{x} \in \mathbb{H}. \tag{2.1}$$

If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a frame for $\mathbb{H}$, the synthesis operator $F : \mathbb{C}^M \to \mathbb{H}$ is $F\{c_j\}_{j=0}^{M-1} = \sum_{j=0}^{M-1} c_j \mathbf{y}_j$ for all $\{c_j\}_{j=0}^{M-1} \in \mathbb{C}^M$. The adjoint (analysis) operator $F^* : \mathbb{H} \to \mathbb{C}^M$ is $F^*\mathbf{x} = \{\langle \mathbf{x}, \mathbf{y}_j \rangle\}_{j=0}^{M-1}$ for all $\mathbf{x} \in \mathbb{H}$. By composing F and F^*, we get the positive and invertible frame operator $S : \mathbb{H} \to \mathbb{H}$ given by

$$\mathbf{x} \mapsto S\mathbf{x} = FF^*\mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle \mathbf{y}_j \quad \text{for all } \mathbf{x} \in \mathbb{H}. \tag{2.2}$$

In terms of the analysis operator we have $A\|\mathbf{x}\|_2^2 \leq \|F^*\mathbf{x}\|_2^2 \leq B\|\mathbf{x}\|_2^2$ for $\mathbf{x} \in \mathbb{H}$. If $\mathfrak{A}$ is a finite frame for $\mathbb{H}$, the set $\mathfrak{A}$ spans the complex Hilbert space $\mathbb{H}$ which implies

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$M \geq N$, where $M = |\mathfrak{A}|$. It should be mentioned that each finite spanning set in $\mathbb{H}$ is a finite frame for $\mathbb{H}$. The ratio between M and N is called as redundancy of the finite frame $\mathfrak{A}$ (i.e., $\mathrm{red}_{\mathfrak{A}} = M/N$), where $M = |\mathfrak{A}|$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a finite frame for $\mathbb{H}$, each $\mathbf{x} \in \mathbb{H}$ satisfies the following reconstruction formulas

$$\mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, S^{-1}\mathbf{y}_j \rangle \mathbf{y}_j = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle S^{-1}\mathbf{y}_j. \tag{2.3}$$

In this case, the complex numbers $\langle \mathbf{x}, S^{-1}\mathbf{y}_j \rangle$ are called frame coefficients and the finite sequence $\mathfrak{A}^\bullet := \{S^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ which is a frame for $\mathbb{H}$ as well, is called the canonical dual frame of $\mathfrak{A}$. A finite frame $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ for $\mathbb{H}$ is called tight if we have $A = B$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a tight frame for $\mathbb{H}$ with frame bound A, then the canonical dual frame $\mathfrak{A}^\bullet$ is exactly $\{A^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ and for $\mathbf{x} \in \mathbb{H}$ we have [4]

$$\mathbf{x} = \frac{1}{A} \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle \mathbf{y}_j. \tag{2.4}$$

For a finite group G, the finite dimensional complex vector space $\mathbb{C}^G = \{\mathbf{x} : G \to \mathbb{C}\}$ is a $|G|$-dimensional Hilbert space with complex vector entries indexed by elements in the finite group G[1]. The inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{g \in G} \mathbf{x}(g)\overline{\mathbf{y}(g)}$, and the induced norm is the $\|\cdot\|_2$-norm of $\mathbf{x}$, that is $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. For $\mathbb{C}^{\mathbb{Z}_N}$, where $\mathbb{Z}_N$ denotes the cyclic group of N elements $\{0, \dots, N-1\}$, we simply write $\mathbb{C}^N$ at times.

Time-scale analysis and time-frequency analysis on finite Abelian group G as modern computational harmonic analysis tools are based on three basic operations on $\mathbb{C}^G$. The translation operator $T_k : \mathbb{C}^G \to \mathbb{C}^G$ given by $T_k\mathbf{x}(g) = \mathbf{x}(g-k)$ with $g, k \in G$. The modulation operator $M_\ell : \mathbb{C}^G \to \mathbb{C}^G$ given by $M_\ell \mathbf{x}(g) = \overline{\ell(g)}\mathbf{x}(g)$ with $g \in G$ and $\ell \in \widehat{G}$, where $\widehat{G}$ is the character/dual group of G. As the fundamental theorem of finite Abelian groups provides a factorization of G into cyclic groups, that is, $G \cong \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_d}$ as groups, which implies $\widehat{G} \cong G$, we can assume that the action of $\ell = (\ell_1, \dots, \ell_d) \in \widehat{G}$ on $g = (g_1, \dots, g_d) \in G$ is given by

$$\ell(g) = ((\ell_1, \ell_2, \dots, \ell_d), (g_1, \dots, g_d)) = \prod_{j=1}^{d} \mathbf{e}_{\ell_j}(g_j),$$

where $\mathbf{e}_{\ell_j}(g_j) = e^{2\pi i \ell_j g_j / N_j}$ for all $1 \leq j \leq d$. Thus,

$$\ell(g) = ((\ell_1, \ell_2, \dots, \ell_d), (g_1, \dots, g_d)) = e^{2\pi i (\ell_1 g_1/N_1 + \ell_2 g_2/N_2 + \cdots + \ell_d g_d/N_d)}.$$

[1] $|G|$ denotes the order of the group G, or, more generally, the cardinality of a set G.

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The character/dual group $\widehat{G}$ of any finite Abelian group G is isomorphic with G via the canonical group isomorphism $\ell \mapsto \mathbf{e}_\ell$, where the character $\mathbf{e}_\ell : G \to \mathbb{T}$ is given by $\mathbf{e}_\ell(g) = \ell(g)$ for all $g \in G$. The third fundamental operator is the discrete Fourier Transform (DFT) $\mathcal{F}_G : \mathbb{C}^G \to \mathbb{C}^{\widehat{G}} = \mathbb{C}^G$ which allows us to pass from time representations to frequency representations. It is defined as a function on $\widehat{G}$ by

$$\mathcal{F}_G(\mathbf{x})(\ell) = \widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \mathbf{x}(g)\overline{\ell(g)} \tag{2.5}$$

for all $\ell \in \widehat{G}$ and $\mathbf{x} \in \mathbb{C}^G$. That is equivalently

$$\mathcal{F}_G(\mathbf{x})(\ell) = \widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{|G|}} \sum_{g_1=0}^{N_1-1} \cdots \sum_{g_d=0}^{N_d-1} \mathbf{x}(g_1,\ldots,g_d)\overline{((\ell_1,\ldots,\ell_d),(g_1,\ldots,g_d))}$$

for all $\ell = (\ell_1,\ldots,\ell_d) \in \widehat{G}$ and $\mathbf{x} \in \mathbb{C}^G$. Translation, modulation, and the Fourier transform on the Hilbert space $\mathbb{C}^G = \mathbb{C}^{\widehat{G}}$ are unitary operators with respect to the $\|\cdot\|_2$-norm. For $\ell, k \in G \cong \widehat{G}$ we have $(T_k)^* = (T_k)^{-1} = T_{-k}$ and $(M_\ell)^* = (M_\ell)^{-1} = M_{-\ell}$. The circular convolution of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ is defined by

$$\mathbf{x} * \mathbf{y}(k) = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \mathbf{x}(g)\mathbf{y}(k-g) \quad \text{for } k \in G.$$

In terms of the translation operators, we have $\mathbf{x} * \mathbf{y}(k) = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \mathbf{x}(g) T_g \mathbf{y}(k)$ for $k \in G$. The circular involution or circular adjoint of $\mathbf{x} \in \mathbb{C}^G$ is given by $\mathbf{x}^*(k) = \overline{\mathbf{x}(-k)}$. The complex linear space $\mathbb{C}^G$ equipped with the $\|\cdot\|_1$-norm, that is $\|\mathbf{x}\|_1 = \sum_{g\in G} |\mathbf{x}(g)|$, the circular convolution, and involution is a Banach $*$-algebra, which means that for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ we have

$$\|\mathbf{x} * \mathbf{y}\|_1 \le \frac{1}{\sqrt{|G|}} \|\mathbf{x}\|_1 \|\mathbf{y}\|_1 \ \text{ and } \ \|\mathbf{x}^*\|_1 = \|\mathbf{x}\|_1.$$

The unitary DFT (2.5) satisfies

$$\widehat{T_k\mathbf{x}} = M_k\widehat{\mathbf{x}}, \quad \widehat{M_\ell\mathbf{x}} = T_{-\ell}\widehat{\mathbf{x}}, \quad \widehat{\mathbf{x}^*} = \overline{\widehat{\mathbf{x}}}, \quad \widehat{\mathbf{x} * \mathbf{y}} = \widehat{\mathbf{x}}.\widehat{\mathbf{y}}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$, $k \in G$ and $\ell \in \widehat{G}$. See standard references of harmonic analysis such as [9, 22, 29] and references therein.

Let $\mathbb{H}$ be a complex finite dimensional inner product space with $\dim \mathbb{H} = N$. Let $\mathcal{U}(\mathbb{H})$ be the group of all unitary operators on $\mathbb{H}$, which is precisely the matrix group of all unitary $N \times N$-matrices with complex entries. A projective group representation

$$\pi : G \to \mathcal{U}(\mathbb{H}) \cong \mathbf{U}_{N\times N}(\mathbb{C})$$

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of G is a family of unitary operators $\{\pi(g) : g \in G\}$ such that

$$\pi(gg') = c_G(g, g')\pi(g)\pi(g') \quad \text{for } g, g' \in G$$

for unimodular numbers $c_G(g, g')$. The projective group representation π is called irreducible on $\mathbb{H}$, if $\{0\}$ and $\mathbb{H}$ are the only π-invariant subspaces of $\mathbb{H}$.

3. Harmonic analysis over finite fields. Throughout this section, we present a summary of basic and classical results concerning harmonic analysis over finite fields. For proofs we refer readers to see [14, 17, 21, 23, 28] and references therein.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of order q. Then there is a prime number p and an integer number $d \geq 1$ in which $q = p^d$. Every finite field of order $q = p^d$ is isomorphic as a field to every other field of order q. From now on, when it is necessary we denote any finite field of order $q = p^d$ by $\mathbb{F}_q$ otherwise we just denote it by $\mathbb{F}$. The prime number p is called the characteristic of $\mathbb{F}$, which means that

$$p.\tau = \sum_{k=1}^{p} \tau = 0 \quad \text{for all } \tau \in \mathbb{F}.$$

The absolute trace map $\mathbf{t} : \mathbb{F} \to \mathbb{Z}_p$ is given by $\tau \mapsto \mathbf{t}(\tau)$ where

$$\mathbf{t}(\tau) = \sum_{k=0}^{d-1} \tau^{p^k} \quad \text{for all } \tau \in \mathbb{F}.$$

The absolute trace map $\mathbf{t}$ is a $\mathbb{Z}_p$-linear transform from $\mathbb{F}$ onto $\mathbb{Z}_p$. It should be mentioned that in the case of prime fields, the trace map is readily the identity map.

There exists an irreducible polynomial $P \in \mathbb{Z}_p[t]$ of degree d and a root $\theta \in \mathbb{F}$ of P such that the set

$$\mathcal{B}_\theta := \{\theta^j : j = 0, \dots, d-1\} = \{1, \theta, \theta^2, \dots, \theta^{d-2}, \theta^{d-1}\}$$

is a linear basis of $\mathbb{F}$ over $\mathbb{Z}_p$. Then $\mathcal{B}_\theta$ is called as a polynomial basis of $\mathbb{F}$ over $\mathbb{Z}_p$ and θ is called as a defining element of $\mathbb{F}$ over $\mathbb{Z}_p$. Let $\mathbf{H} = \mathbf{H}_\theta \in \mathbb{Z}_p^{d\times d}$ be the $d \times d$ matrix with entries in the field $\mathbb{Z}_p$ given by $\mathbf{H}_{jk} := \mathbf{t}(\theta^{j+k})$ for all $0 \leq j, k \leq d-1$, which is invertible with the inverse $\mathbf{S} \in \mathbb{Z}_p^{d\times d}$. Then the dual polynomial basis

$$\widetilde{\mathcal{B}_\theta} := \{\Theta_k : k = 0, \dots, d-1\}, \tag{3.1}$$

given by

$$\Theta_k = \sum_{j=0}^{d-1} \mathbf{S}_{kj} \theta^j, \tag{3.2}$$

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satisfies the following orthogonality relation

$$\mathbf{t}(\theta^k\Theta_j) = \delta_{k,j}, \tag{3.3}$$

for all $j, k = 0, \ldots, d-1$.

Proposition 3.1. *Let $\mathbb{F}$ be a finite field of order $q = p^d$ with trace map $\mathbf{t} : \mathbb{F} \to \mathbb{Z}_p$. Then:*

1. *For $\tau \in \mathbb{F}$, we have the following decompositions*

$$\tau = \sum_{k=0}^{d-1} \tau_{(k)}\theta^k = \sum_{k=0}^{d-1} \tau_{[k]}\Theta_k,$$

where for all $k = 0, \ldots, d-1$, we have

$$\tau_{(k)} := \mathbf{t}(\tau\Theta_k), \quad \tau_{[k]} := \mathbf{t}(\tau\theta^k).$$

2. *For $\tau \in \mathbb{F}$, the coefficients (components) $\{\tau_{(k)} : k = 0, \ldots, d-1\}$ and $\{\tau_{[k]} : k = 0, \ldots, d-1\}$ satisfy*

$$\tau_{(k)} = \sum_{j=0}^{d-1} \mathbf{S}_{kj}\tau_{[j]}, \quad \tau_{[k]} = \sum_{j=0}^{d-1} \mathbf{H}_{kj}\tau_{(j)},$$

for all $k = 0, \ldots, d-1$.

Let $\theta \in \mathbb{F}$ be a defining element of $\mathbb{F}$ over $\mathbb{Z}_p$. Then θ defines a $\mathbb{Z}_p$-linear isomorphism $J_\theta : \mathbb{F} \to \mathbb{Z}_p^d$ by

$$\gamma \mapsto J_\theta(\tau) := \tau_\theta = (\tau_{(k)})_{k=1}^d \quad \text{for all } \tau \in \mathbb{F}. \tag{3.4}$$

Then the additive group of the finite field $\mathbb{F}$, $\mathbb{F}^+$, is isomorphic with the finite elementary group $\mathbb{Z}_p^d$ via J_θ. Thus, using classical dual theory on the ring $\mathbb{Z}_p^d$ we get

$$\mathbf{e}_{\tau_\theta}(\tau'_\theta) = \mathbf{e}_{1,p}\left(\tau_\theta.\tau'_\theta\right) = \mathbf{e}_{1,p}\left(\sum_{k=1}^d \tau_{(k)}\tau'_{(k)}\right) \quad \text{for all } \tau, \tau' \in \mathbb{F}.$$

Remark 3.2. The dual (character) group of the finite elementary group $\mathbb{Z}_p^d$, that is $\widehat{\mathbb{Z}_p^d}$, is precisely

$$\left\{\mathbf{e}_\ell : \ell = (\ell_1, \ldots, \ell_d) \in \mathbb{Z}_p^d\right\},$$

where the additive character $\mathbf{e}_\ell : \mathbb{Z}_p^d \to \mathbb{T}$ is given by

$$\mathbf{e}_\ell(g) = \mathbf{e}_{1,p}(\ell \cdot g) = \exp\left(\frac{2\pi i\ \ell \cdot g}{p}\right) = \prod_{k=1}^d \mathbf{e}_{\ell_k,p}(g_k) \quad \text{for all } g = (g_1, \ldots, g_d) \in \mathbb{Z}_p^d,$$

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with $\ell \cdot g = \sum_{k=1}^{d} \ell_k g_k$.

Let $\chi : \mathbb{F} \to \mathbb{T}$ be given by

$$\chi(\tau) := \exp\left(\frac{2\pi i \mathbf{t}(\tau)}{p}\right) = \mathbf{e}_{1,p}(\mathbf{t}(\tau)) \quad \text{for all } \tau \in \mathbb{F}.$$

Since the trace map is $\mathbb{Z}_p$-linear, we deduce that χ is a character on the additive group of $\mathbb{F}$ (i.e $\chi \in \widehat{\mathbb{F}^+}$).

Proposition 3.3. *Let $\mathbb{F}$ be a finite field of order $q = p^d$ with trace map $\mathbf{t} : \mathbb{F} \to \mathbb{Z}_p$. Then:*

1. *For $\tau, \tau' \in \mathbb{F}$, we have*

$$\mathbf{t}(\tau\tau') = \sum_{j=0}^{d-1}\sum_{k=0}^{d-1} \mathbf{H}_{jk}\tau_{(j)}\tau'_{(k)} = \sum_{j=0}^{d-1}\sum_{k=0}^{d-1} \mathbf{S}_{jk}\tau_{[j]}\tau'_{[k]} = \sum_{k=0}^{d-1} \tau_{(k)}\tau'_{[k]} = \sum_{k=0}^{d-1} \tau_{[k]}\tau'_{(k)}.$$

2. *For $\tau, \tau' \in \mathbb{F}$, we have*

$$\chi(\tau\tau') = \mathbf{e}_{1,p}\left(\sum_{k=1}^{d} \tau_{(k)}\tau'_{[k]}\right) = \mathbf{e}_{1,p}\left(\sum_{k=1}^{d} \tau_{[k]}\tau'_{(k)}\right).$$

For $\gamma \in \mathbb{F}$, let $\chi_\gamma : \mathbb{F} \to \mathbb{T}$ be given by

$$\chi_\gamma(\tau) := \chi(\gamma\tau) = \exp\left(\frac{2\pi i \mathbf{t}(\gamma\tau)}{p}\right) = \mathbf{e}_{1,p}(\mathbf{t}(\gamma\tau)) \quad \text{for all } \tau \in \mathbb{F}.$$

Then χ_γ is a character on the additive group of $\mathbb{F}$ (i.e $\chi_\gamma \in \widehat{\mathbb{F}^+}$). For $\gamma = 1$, we get $\chi = \chi_1$.

If $\alpha \in \mathbb{F}^*$, the character χ_α is called as a non-principal character. The interesting property of non-principal characters is that any non-principal character can parametrize the full character group of the additive group of $\mathbb{F}$. In details, if $\alpha \in \mathbb{F}^*$, then we have

$$\widehat{\mathbb{F}^+} = \{\chi_{\alpha\gamma} \ : \ \gamma \in \mathbb{F}\}.$$

Thus, the mapping $\gamma \mapsto \chi_{\alpha\gamma}$ is group isomorphism of $\mathbb{F}$ onto $\widehat{\mathbb{F}^+}$. Then for $\alpha = 1$, we get

$$\widehat{\mathbb{F}^+} = \{\chi_\gamma \ : \ \gamma \in \mathbb{F}\}. \tag{3.5}$$

Remark 3.4. The characterization (3.5) for the character group of finite fields is a consequence of applying the trace map in duality theory over finite fields. This

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characterization plays significant role in structure of dual action, and hence, wave packet groups over finite fields; see Section 4.

Then the Fourier transform of a vector $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ at $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ is

$$\widehat{\mathbf{x}}(\chi_\gamma) = \frac{1}{\sqrt{p^d}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau)\overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{p^d}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau)\overline{\mathbf{F}(\gamma, \tau)},$$

where the matrix $\mathbf{F} : \mathbb{F} \times \mathbb{F} \to \mathbb{C}$ is given by

$$\mathbf{F}(\gamma, \tau) := \chi(\gamma\tau) = \exp\left(\frac{2\pi i \mathbf{t}(\gamma\tau)}{p}\right) \quad \text{for all } \gamma, \tau \in \mathbb{F}.$$

Remark 3.5. (i) For $\beta \in \mathbb{F}$, the translation operator $T_\beta : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is

$$T_\beta \mathbf{x}(\tau) := \mathbf{x}(\tau - \beta) \quad \text{for all } \tau \in \mathbb{F} \text{ and } \mathbf{x} \in \mathbb{C}^{\mathbb{F}}.$$

(ii) For $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$, the modulation operator $M_\gamma : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is

$$M_\gamma \mathbf{x}(\tau) := \overline{\chi_\gamma(\tau)}\mathbf{x}(\tau) \quad \text{for all } \tau \in \mathbb{F} \text{ and } \mathbf{x} \in \mathbb{C}^{\mathbb{F}}.$$

4. Wave packet groups over finite fields. The abstract notion of wave packet groups over prime fields (finite Abelian groups of prime order) introduced in [11]. The algebraic structure of wave packet groups over prime fields based on the canonical action of the multiplicative group of nonzero elements on the associated time-frequency groups, that is the group consists of all time-frequency shifts over prime fields. This action is originated from the canonical affine action of the multiplicative group of nonzero elements on the prime field (as time domain) and the induced dual action on the character group (as frequency domain) of the underlying additive group of prime fields. Thus, to extend the notion of wave packet groups over finite fields we need to present generalized version of dilation operators on both the time and the frequency domain. To this end, first we present properties of affine action of the multiplicative group of nonzero elements and then we will discuss various aspects of the induced dual action. Finally we introduce algebraic structure of wave packet groups over finite fields.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of order $q = p^d$. The finite multiplicative group

$$\mathbb{F}^* := \mathbb{F} - \{0\} = \{\alpha \in \mathbb{F} : \alpha \neq 0\} \tag{4.1}$$

of nonzero elements of $\mathbb{F}$ is a finite cyclic group of order $q - 1 = p^d - 1$. Any generator of the finite cyclic group $\mathbb{F}^*$ is called a primitive element or primitive root of $\mathbb{F}$ over $\mathbb{Z}_p$.

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For $\alpha \in \mathbb{F}^*$, define the *dilation operator* $D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ by

$$D_\alpha \mathbf{x}(\tau) := \mathbf{x}(\alpha^{-1}\tau)$$

for all $\tau \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$.

Hence, we state basic algebraic properties of dilation operators.

PROPOSITION 4.1. *Let $\mathbb{F}$ be a finite field. Then:*

1. *For $(\alpha,\beta) \in \mathbb{F}^* \times \mathbb{F}$, we have $D_\alpha T_\beta = T_{\alpha\beta} D_\alpha$.*
2. *For $\alpha, \alpha' \in \mathbb{F}^*$, we have $D_{\alpha\alpha'} = D_\alpha D_{\alpha'}$.*
3. *For $(\alpha,\beta),(\alpha',\beta') \in \mathbb{F}^* \times \mathbb{F}$, we have $T_{\beta+\alpha\beta'} D_{\alpha\alpha'} = T_\beta D_\alpha T_{\beta'} D_{\alpha'}$.*

Proof. Let $\mathbb{F}$ be a finite field and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then:

(1) For $(\alpha,\beta) \in \mathbb{F}^* \times \mathbb{F}$ and $\tau \in \mathbb{F}$, we can write

$$\begin{aligned}
D_\alpha T_\beta \mathbf{x}(\tau) &= T_\beta \mathbf{x}(\alpha^{-1}\tau) \\
&= \mathbf{x}(\alpha^{-1}\tau - \beta) \\
&= \mathbf{x}(\alpha^{-1}\tau - \alpha^{-1}\alpha\beta) \\
&= \mathbf{x}(\alpha^{-1}(\tau - \alpha\beta)) \\
&= D_\alpha \mathbf{x}(\tau - \alpha\beta) = T_{\alpha\beta} D_\alpha \mathbf{x}(\tau).
\end{aligned}$$

(2) For $\alpha, \alpha' \in \mathbb{F}^*$ and $\tau \in \mathbb{F}$, we can write

$$\begin{aligned}
D_{\alpha\alpha'} \mathbf{x}(\tau) &= \mathbf{x}((\alpha\alpha')^{-1}\tau) \\
&= \mathbf{x}(\alpha'^{-1}\alpha^{-1}\tau) \\
&= D_{\alpha'} \mathbf{x}(\alpha^{-1}\tau) = D_\alpha D_{\alpha'} \mathbf{x}(\tau).
\end{aligned}$$

(3) It is straightforward from (1) and (2). ∎

Next proposition summarizes analytic properties of dilation operators.

PROPOSITION 4.2. *Let $\mathbb{F}$ be a finite field and $\alpha \in \mathbb{F}^*$. Then:*

1. *$D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is a $*$-isometric isomorphism of the Banach $*$-algebra $\mathbb{C}^{\mathbb{F}}$.*
2. *$D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is unitary in $\|\cdot\|_2$-norm and satisfies $(D_\alpha)^* = (D_\alpha)^{-1} = D_{\alpha^{-1}}$.*

Proof. (1) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ and $\tau \in \mathbb{F}$. Then we have

$$D_\alpha(\mathbf{x} * \mathbf{y})(\tau) = \mathbf{x} * \mathbf{y}(\alpha^{-1}\tau) = \frac{1}{\sqrt{q}} \sum_{\tau' \in \mathbb{F}} \mathbf{x}(\tau')\mathbf{y}(\alpha^{-1}\tau - \tau').$$

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Replacing τ' with $\alpha^{-1}\tau'$, we get

$$\begin{aligned}\frac{1}{\sqrt{q}}\sum_{\tau'\in\mathbb{F}}\mathbf{x}(\tau')\mathbf{y}(\alpha^{-1}\tau-\tau') &= \frac{1}{\sqrt{q}}\sum_{\tau'\in\mathbb{F}}\mathbf{x}(\alpha^{-1}\tau')\mathbf{y}(\alpha^{-1}\tau-\alpha^{-1}\tau')\\ &= \frac{1}{\sqrt{q}}\sum_{\tau'\in\mathbb{F}}\mathbf{x}(\alpha^{-1}\tau')\mathbf{y}(\alpha^{-1}(\tau-\tau'))\\ &= \frac{1}{\sqrt{q}}\sum_{\tau'\in\mathbb{F}}D_\alpha\mathbf{x}(\tau')D_\alpha\mathbf{y}(\tau-\tau') = (D_\alpha\mathbf{x})*(D_\alpha\mathbf{y})(\tau),\end{aligned}$$

which implies that $D_\alpha(\mathbf{x}*\mathbf{y}) = (D_\alpha\mathbf{x})*(D_\alpha\mathbf{y})$.

We can also write

$$\begin{aligned}(D_\alpha\mathbf{x})^*(\tau) &= \overline{D_\alpha\mathbf{x}(-\tau)}\\ &= \overline{\mathbf{x}(-\alpha^{-1}\tau))}\\ &= \mathbf{x}^*(\alpha^{-1}\tau) = D_\alpha\mathbf{x}^*(\tau),\end{aligned}$$

which guarantees $(D_\alpha\mathbf{x})^* = D_\alpha\mathbf{x}^*$.

(2) Let $\mathbf{x}\in\mathbb{C}^{\mathbb{F}}$. Then we can write

$$\begin{aligned}\|D_\alpha\mathbf{x}\|_2^2 &= \sum_{\tau\in\mathbb{F}}|D_\alpha\mathbf{x}(\tau)|^2\\ &= \sum_{\tau\in\mathbb{F}}|\mathbf{x}(\alpha^{-1}\tau)|^2\\ &= \sum_{\tau\in\mathbb{F}}|\mathbf{x}(\tau)|^2 = \|\mathbf{x}\|_2^2,\end{aligned}$$

which implies that $D_\alpha:\mathbb{C}^{\mathbb{F}}\to\mathbb{C}^{\mathbb{F}}$ is unitary in $\|\cdot\|_2$-norm and satisfies

$$(D_\alpha)^* = (D_\alpha)^{-1} = D_{\alpha^{-1}}. \quad \square$$

Remark 4.3. Let $\mathbb{F}=\mathbb{F}_q$ be a finite field of order $q=p^d$, where p is a positive prime integer and $d\geq 1$ is an integer.

(i) Let $d=1$. Then $\mathbb{F}=\mathbb{Z}_p$, and hence, the affine action of $\mathbb{F}^*=\mathbb{Z}_p=\{0\}$ canonically induces the dual action on $\widehat{\mathbb{F}^+}=\mathbb{Z}_p$, see [11].

(ii) Let $d>1$ and also let $\theta\in\mathbb{F}$ be a defining element of $\mathbb{F}$ over $\mathbb{Z}_p$. Then $\mathbb{F}^+$, the additive group of $\mathbb{F}$, is isomorphic with the elementary group $\mathbb{Z}_p^d$ via the $\mathbb{Z}_p$-linear isomorphism $J_\theta:\mathbb{F}\to\mathbb{Z}_p^d$ given in (3.4). Then $\widehat{J_\theta}:\widehat{\mathbb{Z}_p^d}\to\widehat{\mathbb{F}^+}$ given by $\widehat{J_\theta}(\mathbf{e}_\ell):=\mathbf{e}_\ell\circ J_\theta$ for all $\mathbf{e}_\ell\in\widehat{\mathbb{Z}_p^d}$, is a group isomorphism. Thus, $\ell\mapsto\mathbf{e}_\ell\circ J_\theta$, defines a group isomorphism from $\mathbb{Z}_p^d$ onto $\widehat{\mathbb{F}^+}$. Since $\mathbb{Z}_p^d$ is not a (finite) field, if multiplication

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and addition are defined coordinatewise, the affine action of the multiplicative group $\mathbb{F}^*$, on the dual group $\widehat{\mathbb{F}^+}$ does not make sense via the group isomorphisms J_θ. Thus, we deduce that replacing the prime field $\mathbb{Z}_p$ with the ring $\mathbb{Z}_p^d$ does not characterize a unified version of the dual action of the multiplicative group $\mathbb{F}^*$ on the character group $\widehat{\mathbb{F}^+}$.

In the remainder of this article, we use the explicit characterization of the character group given by (3.5). Using (3.5), which can be considered as a consequence of analytic and algebraic properties of the trace map, the finite field $\mathbb{F}$ parametrizes the full character group $\widehat{\mathbb{F}^+}$. This parametrization implies a unified labelling on the character group $\widehat{\mathbb{F}^+}$ with $\mathbb{F}$.

Then we can present the following proposition.

Proposition 4.4. *Let $\mathbb{F}$ be a finite field and $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$. Then:*

1. *$M_\gamma : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is a unitary operator in $\|\cdot\|_2$-norm and satisfies $(M_\gamma)^* = (M_\gamma)^{-1} = M_{-\gamma}$.*
2. *For $\alpha \in \mathbb{F}^*$, we have $D_\alpha M_\gamma = M_{\alpha^{-1}\gamma} D_\alpha$.*
3. *For $\beta \in \mathbb{F}$, we have $T_\beta M_\gamma = \chi_\gamma(\beta) M_\gamma T_\beta$.*

Proof. (1) This statement is evident invoking definition of modulation operators.

(2) Let $\alpha \in \mathbb{F}^*$. Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and $\tau \in \mathbb{F}$. Then we can write

$$\begin{aligned} D_\alpha M_\gamma \mathbf{x}(\tau) &= M_\gamma \mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi_\gamma(\alpha^{-1}\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi(\gamma\alpha^{-1}\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi(\alpha^{-1}\gamma\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi_{\alpha^{-1}\gamma}(\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi_{\alpha^{-1}\gamma}(\tau)} D_\alpha \mathbf{x}(\tau) = M_{\alpha^{-1}\gamma} D_\alpha \mathbf{x}(\tau), \end{aligned}$$

which implies $D_\alpha M_\gamma = M_{\alpha^{-1}\gamma} D_\alpha$.

(3) Let $\beta \in \mathbb{F}$. Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and $\tau \in \mathbb{F}$. Then we have

$$\begin{aligned} T_\beta M_\gamma \mathbf{x}(\tau) &= M_\gamma \mathbf{x}(\tau - \beta) \\ &= \overline{\chi_\gamma(\tau - \beta)}\mathbf{x}(\tau - \beta) \\ &= \overline{\chi_\gamma(-\beta)\chi_\gamma(\tau)}\mathbf{x}(\tau - \beta) \\ &= \overline{\chi_\gamma(-\beta)\chi_\gamma(\tau)} T_\beta \mathbf{x}(\tau) = \chi_\gamma(\beta) M_\gamma T_\beta \mathbf{x}(\tau), \end{aligned}$$

which implies $T_\beta M_\gamma = \chi_\gamma(\beta) M_\gamma T_\beta$. $\square$

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For $\alpha \in \mathbb{F}^*$, let $\widehat{D}_\alpha : \mathbb{C}^{\widehat{\mathbb{F}^+}} \to \mathbb{C}^{\widehat{\mathbb{F}^+}}$ be given by

$$\widehat{D}_\alpha \mathbf{x}(\chi_\gamma) := \mathbf{x}(\chi_{\alpha^{-1}\gamma}),$$

for all $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ and $\mathbf{x} \in \mathbb{C}^{\widehat{\mathbb{F}}}$. Since $\mathbb{F}$ and $\widehat{\mathbb{F}^+}$ are isomorphic as finite Abelian groups, we may use D_α instead of $\widehat{D}_\alpha$ at times.

The following proposition presents some analytic properties of dilation operators on the frequency domain.

Proposition 4.5. *Let $\mathbb{F}$ be a finite field and $\alpha \in \mathbb{F}^*$. Then:*

1. *$D_\alpha : \mathbb{C}^{\widehat{\mathbb{F}^+}} \to \mathbb{C}^{\widehat{\mathbb{F}^+}}$ is a $*$-isometric isomorphism of the Banach $*$-algebra $\mathbb{C}^{\widehat{\mathbb{F}^+}}$*
2. *$D_\alpha : \mathbb{C}^{\widehat{\mathbb{F}^+}} \to \mathbb{C}^{\widehat{\mathbb{F}^+}}$ is unitary in $\|\cdot\|_2$-norm and satisfies $(D_\alpha)^* = (D_\alpha)^{-1} = D_{\alpha^{-1}}$.*

Next result states analytic properties of dilation operators and also connections with the Fourier transform.

Proposition 4.6. *Let $\mathbb{F}$ be a finite field of order q. Then:*

1. *For $\beta \in \mathbb{F}$, we have $\mathcal{F}_\mathbb{F} T_\beta = M_\beta \mathcal{F}_\mathbb{F}$.*
2. *For $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$, we have $\mathcal{F}_\mathbb{F} M_\gamma = T_{-\gamma}\mathcal{F}_\mathbb{F}$.*
3. *For $\alpha \in \mathbb{F}^*$, we have $\mathcal{F}_\mathbb{F} D_\alpha = \widehat{D}_{\alpha^{-1}}\mathcal{F}_\mathbb{F}$.*

Proof. (1) Let $\beta \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{C}^\mathbb{F}$. Then for $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$, we have

$$\mathcal{F}_\mathbb{F}(T_\beta \mathbf{x})(\chi_\gamma) = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} T_\beta \mathbf{x}(\tau)\overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau - \beta)\overline{\chi_\gamma(\tau)}.$$

Replacing τ with $\tau + \beta$, we get

$$\frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau - \beta)\overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau)\overline{\chi_\gamma(\tau + \beta)} = \frac{\overline{\chi_\gamma(\beta)}}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau)\overline{\chi_\gamma(\tau)}.$$

Then we can write

$$\begin{aligned} \mathcal{F}_\mathbb{F}(T_\beta \mathbf{x})(\chi_\gamma) &= \overline{\chi_\gamma(\beta)}\mathcal{F}_\mathbb{F}(\mathbf{x})(\chi_\gamma) \\ &= \overline{\chi_\gamma(\beta)}\mathcal{F}_\mathbb{F}(\mathbf{x})(\chi_\gamma) = \overline{\chi_\beta(\gamma)}\mathcal{F}_\mathbb{F}(\mathbf{x})(\chi_\gamma), \end{aligned}$$

implying $\mathcal{F}_\mathbb{F} T_\beta = M_\beta \mathcal{F}_\mathbb{F}$.

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(2) Let $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then for all $\gamma' \asymp \chi_{\gamma'} \in \widehat{\mathbb{F}^+}$, we have

$$\begin{aligned}\mathcal{F}_{\mathbb{F}}(M_\gamma \mathbf{x})(\gamma') &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} M_\gamma \mathbf{x}(\tau) \overline{\chi_\gamma(\tau)} \\ &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \overline{\chi_\gamma(\tau)} \mathbf{x}(\tau) \overline{\chi_{\gamma'}(\tau)} \\ &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau) \overline{\chi_{\gamma+\gamma'}(\tau)} \\ &= \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\gamma + \gamma') = T_{-\gamma} \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\gamma').\end{aligned}$$

(3) Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$. Then we have

$$\mathcal{F}_{\mathbb{F}}(D_\alpha \mathbf{x})(\gamma) = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} D_\alpha \mathbf{x}(\tau) \overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\alpha^{-1}\tau) \overline{\chi_\gamma(\tau)}.$$

Replacing τ with $\alpha\tau$, we achieve

$$\begin{aligned}\frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\alpha^{-1}\tau) \overline{\chi_\gamma(\tau)} &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau) \overline{\chi_\gamma(\alpha\tau)} \\ &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau) \overline{\chi_{\alpha\gamma}(\tau)} = \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\alpha\gamma),\end{aligned}$$

which implies $\mathcal{F}_{\mathbb{F}}(D_\alpha \mathbf{x}) = \widehat{D}_{\alpha^{-1}}(\mathcal{F}_{\mathbb{F}} \mathbf{x})$. $\square$

The underlying set $\mathbb{F}^* \times \mathbb{F} \times \mathbb{F} = \mathbb{F}^* \times \mathbb{F} \times \widehat{\mathbb{F}^+}$ equipped with group operations given by

$$(\alpha, \beta, \gamma) \ltimes (\alpha', \beta', \gamma') := (\alpha\alpha', \alpha'^{-1}\beta + \beta', \alpha'\gamma + \gamma'), \tag{4.2}$$

$$(\alpha, \beta, \gamma)^{-1} := (\alpha^{-1}, \alpha^{-1}.(-\beta), \alpha.(-\gamma)) \tag{4.3}$$

for all $(\alpha, \beta, \gamma), (\alpha', \beta', \gamma') \in \mathbb{F}^* \times \mathbb{F} \times \widehat{\mathbb{F}}$, is a finite non-Abelian group of order $q^2(q-1)$ which is denoted by $\mathrm{WP}_{\mathbb{F}}$. The group $\mathrm{WP}_{\mathbb{F}}$ is called as *finite wave packet group* over the finite field $\mathbb{F}$. Since any two field of order $q = p^d$ are isomorphic as finite field, we deduce that the notion of $\mathrm{WP}_{\mathbb{F}}$ just depends on q. In details, if $\mathbb{F}$ and $\mathbb{F}'$ are two finite field of order q, then the groups $\mathrm{WP}_{\mathbb{F}}$ and $\mathrm{WP}_{\mathbb{F}'}$ are isomorphic as finite groups of order $q^2(q-1)$. Thus, we may use the notation WP_q instead of $\mathrm{WP}_{\mathbb{F}}$ at times.

Next theorem guarantees that the group structure of the wave packet group $\mathrm{WP}_{\mathbb{F}}$ is canonically connected with a projective group representation.

THEOREM 4.7. *Let $\mathbb{F}$ be a finite field of order $q > 2$. Then:*

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1. $\mathrm{WP}_{\mathbb{F}}$ *is a non-Abelian group of order* $q^2(q-1)$ *which contains* $\mathbb{F}\times\widehat{\mathbb{F}^+}\cong\mathbb{F}\times\mathbb{F}$ *as a normal Abelian subgroup and* $\mathbb{F}^*$ *as a non-normal Abelian subgroup.*
2. *The map* $\Gamma:\mathrm{WP}_{\mathbb{F}}\to\mathcal{U}(\mathbb{C}^{\mathbb{F}})\cong\mathbf{U}_{q\times q}(\mathbb{C})$ *defined by*

$$(\alpha,\beta,\gamma)\mapsto\Gamma(\alpha,\beta,\gamma):=D_\alpha T_\beta M_\gamma\quad\text{for }(\alpha,\beta,\gamma)\in\mathrm{WP}_{\mathbb{F}},\tag{4.4}$$

is a projective group representation of the finite wave packet group $\mathrm{WP}_{\mathbb{F}}$ *on the finite dimensional Hilbert space* $\mathbb{C}^{\mathbb{F}}$.

Proof. Let $\mathbb{F}$ be a finite field of order $q>2$. Then:

(1) It is straightforward from the group structure given in (4.2) that $\mathbb{F}\times\widehat{\mathbb{F}^+}\cong\mathbb{F}\times\mathbb{F}$ is a normal Abelian subgroup and $\mathbb{F}^*$ is a non-normal Abelian subgroup of $\mathrm{WP}_{\mathbb{F}}$.

(2) It is evident to check that $\Gamma(1,0,0)=I$ and $\Gamma(\alpha,\beta,\gamma):\mathbb{C}^{\mathbb{F}}\to\mathbb{C}^{\mathbb{F}}$ is a unitary operator for all $(\alpha,\beta,\gamma)\in\mathrm{WP}_{\mathbb{F}}$. Now let $(\alpha,\beta,\gamma),(\alpha',\beta',\gamma')\in\mathrm{WP}_{\mathbb{F}}$. Then using Proposition 4.1, we can write

$$\begin{aligned}
D_{\alpha\alpha'}T_{\alpha'^{-1}\beta+\beta'}M_{\alpha'\gamma+\gamma'}&=D_\alpha D_{\alpha'}T_{\alpha'^{-1}\beta}T_{\beta'}M_{\alpha'\gamma}M_{\gamma'}\\
&=D_\alpha(D_{\alpha'}T_{\alpha'^{-1}\beta})T_{\beta'}M_{\alpha'\gamma}M_{\gamma'}\\
&=D_\alpha(T_\beta D_{\alpha'})T_{\beta'}M_{\alpha'\gamma}M_{\gamma'}\\
&=D_\alpha T_\beta D_{\alpha'}T_{\beta'}M_{\alpha'\gamma}M'_\gamma\\
&=D_\alpha T_\beta D_{\alpha'}(T_{\beta'}M_{\alpha'\gamma})M_{\gamma'}\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}D_\alpha T_\beta D_{\alpha'}(M_{\alpha'\gamma}T_{\beta'})M_{\gamma'}\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}D_\alpha T_\beta D_{\alpha'}M_{\alpha'\gamma}T_{\beta'}M_{\gamma'}\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}D_\alpha T_\beta(D_{\alpha'}M_{\alpha'\gamma})T_{\beta'}M_{\gamma'}\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}D_\alpha T_\beta(M_\gamma D_{\alpha'})T_{\beta'}M_{\gamma'}\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}D_\alpha T_\beta M_\gamma D_{\alpha'}T_{\beta'}M_{\gamma'}\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}(D_\alpha T_\beta M_\gamma)(D_{\alpha'}T_{\beta'}M_{\gamma'}),
\end{aligned}$$

where $\chi_{\alpha'\gamma}(\beta')=\chi(\alpha'\gamma\beta)$. Thus, we get

$$\begin{aligned}
\Gamma\left((\alpha,\beta,\gamma)\ltimes(\alpha',\beta',\gamma')\right)&=\Gamma(\alpha\alpha',{\alpha'}^{-1}\beta+\beta',\alpha'\gamma+\gamma')\\
&=\overline{\chi_{\alpha'\gamma}(\beta')}\Gamma(\alpha,\beta,\gamma)\Gamma(\alpha',\beta',\gamma'),
\end{aligned}$$

which implies that Γ is a projective group representation of the finite wave packet group $\mathrm{WP}_{\mathbb{F}}$ on the finite dimensional Hilbert space $\mathbb{C}^{\mathbb{F}}$. ∎

REMARK 4.8. The restriction of the wave packet representation $\Gamma:\mathrm{WP}_{\mathbb{F}}\to\mathcal{U}(\mathbb{C}^{\mathbb{F}})$ to the subgroup $\mathbb{F}\times\widehat{\mathbb{F}^+}$ is unitarily equivalent with the projective Schrödinger representation of the group $\mathbb{F}\times\widehat{\mathbb{F}^+}$ on $\mathbb{C}^{\mathbb{F}}$ (see [6] and references therein) and similarly

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restriction of the wave packet representation $\Gamma : \mathrm{WP}_{\mathbb{F}} \to \mathcal{U}(\mathbb{C}^{\mathbb{F}})$ to the subgroup $\mathbb{F}^* \times \mathbb{F}$ is unitarily equivalent with the unitary quasi-regular representation of the wavelet group $\mathbb{F}^* \times \mathbb{F}$ on $\mathbb{C}^{\mathbb{F}}$, see [1, 8, 15] and references therein. Thus, we deduce that the wave packet representation $\Gamma : \mathrm{WP}_{\mathbb{F}} \to \mathcal{U}(\mathbb{C}^{\mathbb{F}})$ contains both projective Schrödinger representation and quasi-regular representation.

5. Wave packet transform (WPT) over finite fields. In this section, we present abstract theory of wave packet transforms on finite fields and we study analytic properties of this transform. Throughout this section, it is still assumed that $\mathbb{F}$ is a finite field of order $q = p^d$.

Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window vector. The *wave packet transform* (*WPT*) of a given vector $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ with respect to the window vector $\mathbf{y}$ ($\mathbf{y}$-wave packet transform) is defined on the finite wave packet group $\mathrm{WP}_{\mathbb{F}}$ by

$$\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) := \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau) e^{2\pi i \mathbf{t}(\gamma(\alpha^{-1}\tau-\beta))/p}\overline{\mathbf{y}(\alpha^{-1}\tau-\beta)} \quad \text{for all } (\alpha,\beta,\gamma) \in \mathrm{WP}_{\mathbb{F}}. \tag{5.1}$$

Then $\mathcal{V}_{\mathbf{y}} : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathrm{WP}_{\mathbb{F}}}$ given by $\mathbf{x} \mapsto \mathcal{V}_{\mathbf{y}}\mathbf{x}$ is linear.

By (5.1), we can write

$$\begin{aligned}
\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) &= \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau) e^{2\pi i \mathbf{t}(\gamma(\alpha^{-1}\tau-\beta))/p}\overline{\mathbf{y}(\alpha^{-1}\tau-\beta)} \\
&= \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau)\chi_\gamma(\alpha^{-1}\tau-\beta)\overline{\mathbf{y}(\alpha^{-1}\tau-\beta)} \\
&= \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau)\overline{M_\gamma\mathbf{y}(\alpha^{-1}\tau-\beta)} \\
&= \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau)\overline{T_\beta M_\gamma\mathbf{y}(\alpha^{-1}\tau)} = \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau)\overline{D_\alpha T_\beta M_\gamma\mathbf{y}(\tau)}.
\end{aligned}$$

Thus, in terms of the inner product of the Hilbert space $\mathbb{C}^{\mathbb{F}}$, for $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ we can write

$$\mathcal{W}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) = \langle \mathbf{x}, \Gamma(\alpha,\beta,\gamma)\mathbf{y}\rangle = \langle \mathbf{x}, D_\alpha T_\beta M_\gamma \mathbf{y}\rangle \quad \text{for } (\alpha,\beta,\gamma) \in \mathrm{WP}_{\mathbb{F}}. \tag{5.2}$$

Then using basic properties of dilation, translation and modulation operators, we have

$$\langle D_{\alpha^{-1}}\mathbf{x}, T_\beta M_\gamma \mathbf{y}\rangle = \langle T_{-\beta} D_{\alpha^{-1}}\mathbf{x}, M_\gamma \mathbf{y}\rangle = \langle M_{-\gamma} T_{-\beta} D_{\alpha^{-1}}\mathbf{x}, \mathbf{y}\rangle. \tag{5.3}$$

Using the Plancherel formula and also (5.3), we get

$$\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) = \langle D_{\alpha^{-1}}\mathbf{x}, T_\beta M_\gamma \mathbf{y}\rangle = \langle \widehat{D_{\alpha^{-1}}\mathbf{x}}, \widehat{T_\beta M_\gamma \mathbf{y}}\rangle. \tag{5.4}$$

Then invoking (5.4) and Propositions 4.6, we achieve

$$\langle \widehat{D_{\alpha^{-1}}\mathbf{x}}, \widehat{T_\beta M_\gamma \mathbf{y}}\rangle = \langle D_\alpha \widehat{\mathbf{x}}, M_\beta \widehat{M_\gamma \mathbf{y}}\rangle = \langle D_\alpha \widehat{\mathbf{x}}, M_\beta T_{-\gamma} \widehat{\mathbf{y}}\rangle. \tag{5.5}$$

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REMARK 5.1. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window vector and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then Remark 4.8 implies that the restriction of the wave packet transform $\mathcal{V}_{\mathbf{y}}\mathbf{x}$ to the subgroup $\mathbb{F} \times \widehat{\mathbb{F}^+}$ coincides with the Gabor transform of $\mathbf{x}$ with respect to $\mathbf{y}$ and also similarly, the restriction of the wave packet transform $\mathcal{V}_{\mathbf{y}}\mathbf{x}$ to the subgroup $\mathbb{F}^* \times \mathbb{F}$ is the wavelet transform of $\mathbf{x}$ with respect to $\mathbf{y}$. Thus, we deduce that the wave packet transform unifies both wavelet and Gabor (short time Fourier) transform over finite fields.

In the following, we present some representations for the wave packet transform defined in (5.1).

PROPOSITION 5.2. *Let $\mathbb{F}$ be a finite field of order q. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ and $(\alpha, \beta, \gamma) \in \mathrm{WP}_{\mathbb{F}}$. Then:*

1. $\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) = \sqrt{q}\mathcal{F}_{\widehat{\mathbb{F}^+}}\left(D_\alpha\widehat{\mathbf{x}}.T_{-\gamma}\overline{\widehat{\mathbf{y}}}\right)(-\beta)$.
2. $\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) = D_{\alpha^{-1}}\mathbf{x} * (M_\gamma \mathbf{y})^*(\beta)$.

The representation (1) is called a *Fourier representation* of the WPT and the representation (2) is called a *circular convolution representation* of the WPT.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ and $(\alpha, \beta, \gamma) \in \mathrm{WP}_{\mathbb{F}}$. Then:

(1) Using (5.5), we can write

$$\begin{aligned}
\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) &= \langle D_\alpha\widehat{\mathbf{x}}, M_\beta T_{-\gamma}\widehat{\mathbf{y}}\rangle \\
&= \sum_{\gamma'\in\widehat{\mathbb{F}^+}} D_\alpha\widehat{\mathbf{x}}(\gamma')\overline{M_\beta T_{-\gamma}\widehat{\mathbf{y}}(\gamma')} \\
&= \sum_{\gamma'\in\widehat{\mathbb{F}^+}} D_\alpha\widehat{\mathbf{x}}(\gamma') M_{-\beta} T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma') \\
&= \sum_{\gamma'\in\widehat{\mathbb{F}^+}} \chi_\beta(\gamma') D_\alpha\widehat{\mathbf{x}}(\gamma') T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma') \\
&= \sum_{\gamma'\in\widehat{\mathbb{F}^+}} \overline{\chi_{-\beta}(\gamma')}(D_\alpha\widehat{\mathbf{x}}.T_{-\gamma}\overline{\widehat{\mathbf{y}}})(\gamma') = \sqrt{q}\mathcal{F}_{\widehat{\mathbb{F}^+}}\left(D_\alpha\widehat{\mathbf{x}}.T_{-\gamma}\overline{\widehat{\mathbf{y}}}\right)(-\beta).
\end{aligned}$$

(2) Similarly, using the Plancherel formula and (5.5), we have

$$\begin{aligned}
\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) &= \langle \widehat{D_{\alpha^{-1}}\mathbf{x}}, M_\beta \widehat{M_\gamma \mathbf{y}}\rangle \\
&= \sum_{\gamma'\in\widehat{\mathbb{F}^+}} \widehat{D_{\alpha^{-1}}\mathbf{x}}(\gamma')\overline{\widehat{M_\gamma\mathbf{y}}(\gamma')}\chi_\beta(\gamma') \\
&= \sum_{\gamma'\in\widehat{\mathbb{F}^+}} \widehat{D_{\alpha^{-1}}\mathbf{x}}(\gamma')\widehat{(M_\gamma\mathbf{y})^*}(\gamma')\chi_\beta(\gamma') = D_{\alpha^{-1}}\mathbf{x} * (M_\gamma\mathbf{y})^*(\beta). \quad \square
\end{aligned}$$

The following theorem presents a concrete formulation for the $\|\cdot\|_2$-norm of the

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wave packet transform $\mathcal{V}_{\mathbf{y}}\mathbf{x}$.

THEOREM 5.3. *Let $\mathbb{F}$ be a finite field of order q. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window vector and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then*

$$\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}\sum_{\gamma\in\widehat{\mathbb{F}^+}} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 = q(q-1)\|\mathbf{y}\|_2^2\|\mathbf{x}\|_2^2. \tag{5.6}$$

Proof. Let $\mathbb{F}$ be a finite field of order q. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window vector and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Let $\alpha \in \mathbb{F}^*$ and $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}}$ be given. Using Proposition 5.2 and Plancherel formula, we have

$$\begin{aligned}
\sum_{\beta\in\mathbb{F}} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 &= q\sum_{\beta\in\mathbb{F}} \left|\mathcal{F}_{\widehat{\mathbb{F}}}\left(D_\alpha\widehat{\mathbf{x}}.T_{-\gamma}\overline{\widehat{\mathbf{y}}}\right)(-\beta)\right|^2 \\
&= q\sum_{\beta\in\mathbb{F}} \left|\mathcal{F}_{\widehat{\mathbb{F}}}\left(D_\alpha\widehat{\mathbf{x}}.T_{-\gamma}\overline{\widehat{\mathbf{y}}}\right)(\beta)\right|^2 \\
&= q\sum_{\gamma'\in\widehat{\mathbb{F}^+}} \left|\left(D_\alpha\widehat{\mathbf{x}}.T_{-\gamma}\overline{\widehat{\mathbf{y}}}\right)(\gamma')\right|^2 \\
&= q\sum_{\gamma'\in\widehat{\mathbb{F}^+}} \left|D_\alpha\widehat{\mathbf{x}}(\gamma').T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma')\right|^2 = q\sum_{\gamma'\in\widehat{\mathbb{F}^+}} |D_\alpha\widehat{\mathbf{x}}(\gamma')|^2|T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma')|^2.
\end{aligned}$$

Then we get

$$\begin{aligned}
\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}\sum_{\gamma\in\widehat{\mathbb{F}^+}} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 &= \sum_{\alpha\in\mathbb{F}^*}\sum_{\gamma\in\widehat{\mathbb{F}^+}}\sum_{\beta\in\mathbb{F}} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 \\
&= q\sum_{\alpha\in\mathbb{F}^*}\sum_{\gamma\in\widehat{\mathbb{F}^+}}\left(\sum_{\gamma'\in\widehat{\mathbb{F}^+}} |D_\alpha\widehat{\mathbf{x}}(\gamma')|^2|T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma')|^2\right) \\
&= q\sum_{\gamma'\in\widehat{\mathbb{F}^+}}\left(\sum_{\alpha\in\mathbb{F}^*} |D_\alpha\widehat{\mathbf{x}}(\gamma')|^2\right)\left(\sum_{\gamma\in\widehat{\mathbb{F}^+}} |T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma')|^2\right).
\end{aligned}$$

Replacing γ by $\gamma-\gamma'$, we have

$$\begin{aligned}
\sum_{\gamma\in\widehat{\mathbb{F}^+}} |T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma')|^2 &= \sum_{\gamma\in\widehat{\mathbb{F}^+}} |\overline{\widehat{\mathbf{y}}}(\gamma'+\gamma)|^2 \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} |\widehat{\mathbf{y}}(\gamma)|^2 = \|\widehat{\mathbf{y}}\|_2^2 = \|\mathbf{y}\|_2^2.
\end{aligned}$$

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Thus, we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}\sum_{\gamma\in\widehat{\mathbb{F}^+}}|\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 &= q\sum_{\gamma'\in\widehat{\mathbb{F}^+}}\left(\sum_{\alpha\in\mathbb{F}^*}|D_\alpha\widehat{\mathbf{x}}(\gamma')|^2\right)\left(\sum_{\gamma\in\widehat{\mathbb{F}^+}}|T_{-\gamma}\overline{\widehat{\mathbf{y}}}(\gamma')|^2\right)\\ &= q\sum_{\gamma'\in\widehat{\mathbb{F}^+}}\left(\sum_{\alpha\in\mathbb{F}^*}|D_\alpha\widehat{\mathbf{x}}(\gamma')|^2\right)\|\mathbf{y}\|_2^2\\ &= q\|\mathbf{y}\|_2^2\left(\sum_{\gamma'\in\widehat{\mathbb{F}^+}}\sum_{\alpha\in\mathbb{F}^*}|D_\alpha\widehat{\mathbf{x}}(\gamma')|^2\right).\end{aligned}$$

Replacing the summation, we get

$$\begin{aligned}\sum_{\gamma'\in\widehat{\mathbb{F}^+}}\sum_{\alpha\in\mathbb{F}}|D_\alpha\widehat{\mathbf{x}}(\gamma')|^2 &= \sum_{\alpha\in\mathbb{F}^*}\sum_{\gamma'\in\widehat{\mathbb{F}^+}}|D_\alpha\widehat{\mathbf{x}}(\gamma')|^2\\ &= \sum_{\alpha\in\mathbb{F}^*}\|D_\alpha\widehat{\mathbf{x}}\|_2^2\\ &= \sum_{\alpha\in\mathbb{F}^*}\|\widehat{\mathbf{x}}\|_2^2\\ &= (q-1)\|\widehat{\mathbf{x}}\|_2^2 = (q-1)\|\mathbf{x}\|_2^2.\end{aligned}$$

Therefore, we achieve

$$\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}\sum_{\gamma\in\widehat{\mathbb{F}^+}}|\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 = q\|\mathbf{y}\|_2^2\left(\sum_{\gamma'\in\widehat{\mathbb{F}^+}}\sum_{\alpha\in\mathbb{F}}|D_\alpha\widehat{\mathbf{x}}(\gamma')|^2\right) = q(q-1)\|\mathbf{y}\|_2^2\|\mathbf{x}\|_2^2.$$

which implies (5.6). ∎

As a consequence of (5.6), we can deduce the following orthogonality relation.

Corollary 5.4. *Let $\mathbb{F}$ be a finite field of order q and $\mathbf{v},\mathbf{y}\in\mathbb{C}^{\mathbb{F}}$ be window vectors. Then, for every $\mathbf{x},\mathbf{z}\in\mathbb{C}^{\mathbb{F}}$, we have*

(5.7) $$\langle\mathcal{V}_{\mathbf{v}}\mathbf{x},\mathcal{V}_{\mathbf{y}}\mathbf{z}\rangle_{\mathbb{C}^{\mathrm{WP}_{\mathbb{F}}}} = q(q-1)\langle\mathbf{y},\mathbf{v}\rangle_{\mathbb{C}^{\mathbb{F}}}\langle\mathbf{x},\mathbf{z}\rangle_{\mathbb{C}^{\mathbb{F}}}.$$

In particular, we have

(5.8) $$\|\mathcal{V}_{\mathbf{y}}\mathbf{x}\|_2 = \sqrt{q(q-1)}\|\mathbf{y}\|_2\|\mathbf{x}\|_2.$$

The following result states an inversion formula for the windowed transform given in (5.1).

Proposition 5.5. *Let $\mathbb{F}$ be a finite field of order q. Let $\mathbf{y}\in\mathbb{C}^{\mathbb{F}}$ be non-zero*

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window vector and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. *Then*

$$\mathbf{x}(\tau) = \frac{\|\mathbf{y}\|_2^{-2}}{q(q-1)} \sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \quad \text{for } \tau \in \mathbb{F}. \tag{5.9}$$

Proof. For $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and a non-zero window vector $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$, define

$$\widetilde{\mathbf{x}}(\tau) := \sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \quad \text{for } \tau \in \mathbb{F}.$$

Let $\mathbf{z} \in \mathbb{C}^{\mathbb{F}}$ be given. Using (5.7), we have

$$\begin{aligned}
\langle \widetilde{\mathbf{x}}, \mathbf{z}\rangle_{\mathbb{C}^{\mathbb{F}}} &= \sum_{\tau\in\mathbb{F}} \widetilde{\mathbf{x}}(\tau)\overline{\mathbf{z}(\tau)} \\
&= \sum_{\tau\in\mathbb{F}} \left(\sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \right) \overline{\mathbf{z}(\tau)} \\
&= \sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) \left(\sum_{\tau\in\mathbb{F}} \overline{\mathbf{z}(\tau)} D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \right) \\
&= \sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) \left(\sum_{\tau\in\mathbb{F}} \mathbf{z}(\tau) \overline{D_\alpha T_\beta M_\gamma \mathbf{y}(\tau)} \right)^{-} \\
&= \sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) \overline{\mathcal{V}_{\mathbf{y}}\mathbf{z}(\alpha,\beta,\gamma)} \\
&= \langle \mathcal{V}_{\mathbf{y}}\mathbf{x}, \mathcal{V}_{\mathbf{y}}\mathbf{z}\rangle_{\mathbb{C}^{\mathrm{WP}_{\mathbb{F}}}} = q(q-1)\|\mathbf{y}\|_2^2 \langle \mathbf{x}, \mathbf{z}\rangle_{\mathbb{C}^{\mathbb{F}}},
\end{aligned}$$

implying

$$\mathbf{x}(\tau) = \frac{\|\mathbf{y}\|_2^{-2}}{q(q-1)} \widetilde{\mathbf{x}}(\tau) \quad \text{for } \tau \in \mathbb{F},$$

which yields (5.9). ∎

Corollary 5.6. *Let* $\mathbb{F}$ *be a finite field of order* q. *Let* $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ *be a non-zero window vector with* $\|\mathbf{y}\|_2 = 1$ *and* $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. *Then*

$$\mathbf{x}(\tau) = \frac{1}{q(q-1)} \sum_{\alpha\in\mathbb{F}^*} \sum_{\beta\in\mathbb{F}} \sum_{\gamma\in\widehat{\mathbb{F}^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \quad \text{for } \tau \in \mathbb{F}. \tag{5.10}$$

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In terms of the abstract frame theory, we can summarize Theorem 5.3 and Proposition 5.5 as follows.

Corollary 5.7. *Let $\mathbb{F}$ be a finite field of order q and $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a non-zero window vector. The finite system*

$$\mathfrak{A}_{\mathbf{y}} := \{\Gamma(\alpha,\beta,\gamma)\mathbf{y} : (\alpha,\beta,\gamma) \in \mathrm{WP}_{\mathbb{F}}\}$$

constitutes a tight frame for the Hilbert space $\mathbb{C}^{\mathbb{F}}$ with the redundancy $q(q-1)$ and the frame bound $q(q-1)\|\mathbf{y}\|_2^2$.

Next theorem states an analytic property of the projective representation Γ.

Theorem 5.8. *Let $\mathbb{F}$ be a finite field of order q. The unitary projective group representation $\Gamma : \mathrm{WP}_{\mathbb{F}} \to \mathcal{U}(\mathbb{C}^{\mathbb{F}})$ is irreducible.*

Proof. Let $\mathcal{H}$ be a non-zero Γ-invariant subspace of $\mathbb{C}^{\mathbb{F}}$. We claim that $\mathcal{H} = \mathbb{C}^{\mathbb{F}}$. It is enough to show that $\mathcal{H}^{\perp} = \{0\}$. Let $\mathbf{x} \in \mathcal{H}^{\perp}$ be arbitrary. Let $\mathbf{y} \in \mathcal{H}$ be a non-zero vector. Then for all $(\alpha,\beta,\gamma) \in \mathrm{WP}_{\mathbb{F}}$ we get $\langle \mathbf{x}, \Gamma(\alpha,\beta,\gamma)\mathbf{y}\rangle = 0$. Thus, using (5.6) we can write

$$\begin{aligned} q(q-1)\|\mathbf{y}\|_2^2\|\mathbf{x}\|_2^2 &= \sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}\sum_{\gamma\in\widehat{\mathbb{F}^+}} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 \\ &= \sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}\sum_{\gamma\in\widehat{\mathbb{F}^+}} |\langle \mathbf{x}, \Gamma(\alpha,\beta,\gamma)\mathbf{y}\rangle|^2 = 0, \end{aligned}$$

which implies that $\mathbf{x} = 0$. ◻

6. Examples. In this section, we present examples of finite fields and we study the theory of wave packet transform over them.

6.1. The finite field $\mathbb{Z}_p$. Let p be a positive prime integer and $\mathbb{F} = \mathbb{Z}_p$ be the prime field of order p. Thus, readily the trace map is the identity map. Then $\mathbb{F}^* = \mathbb{Z}_p - \{0\}$ and for $1 \le \alpha \le p-1$ the dilation operator $D_\alpha : \mathbb{C}^p \to \mathbb{C}^p$ is $D_\alpha\mathbf{x}(\tau) = \mathbf{x}(\alpha^{-1}\tau)$ for all $0 \le \tau \le p-1$, where α^{-1} is the multiplicative inverse of $\alpha \in \mathbb{F}^*$ (i.e., an element $\alpha^{-1} \in \mathbb{F}^*$ with $\alpha\alpha^{-1} \overset{p}{\equiv} \alpha^{-1}\alpha \overset{p}{\equiv} 1$) which satisfies $\alpha^{-1}\alpha + np = 1$ for some $n \in \mathbb{Z}$, which can be done by Bezout lemma [14, 23]. The finite wave packet group WP_p over the field $\mathbb{Z}_p$ has the underlying set

$$\{1,\ldots,p-1\} \times \{0,1,\ldots,p-1\} \times \{0,1,\ldots,p-1\}.$$

Let $\mathbf{y} \in \mathbb{C}^p$ be a window vector. Then the wave packet transform of a given vector $\mathbf{x} \in \mathbb{C}^p$ with respect to the window vector $\mathbf{y}$ ($\mathbf{y}$-wave packet transform) is

$$\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) = \sum_{\tau=0}^{p-1} \mathbf{x}(\tau)e^{2\pi i\gamma(\alpha^{-1}\tau-\beta)/p}\overline{\mathbf{y}(\alpha^{-1}\tau-\beta)} \quad \text{for } (\alpha,\beta,\gamma) \in \mathrm{WP}_p. \tag{6.1}$$

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Let $\mathbf{y} \in \mathbb{C}^p$ be a window vector and $\mathbf{x} \in \mathbb{C}^p$. Then

$$\sum_{\alpha=1}^{p-1}\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 = p(p-1)\|\mathbf{y}\|_2^2\|\mathbf{x}\|_2^2, \tag{6.2}$$

$$\mathbf{x}(\tau) = \frac{\|\mathbf{y}\|_2^{-2}}{p(p-1)}\sum_{\alpha=1}^{p-1}\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \quad \text{for } 0 \le \tau \le p-1. \tag{6.3}$$

Remark 6.1. Dyadic dilations of signals on the real line preserve the geometry of signals but dilations over $\mathbb{C}^p$ destroy geometric properties and the localization of signals. Dilations operators over $\mathbb{C}^p$ imply sculptured and permuted rearrangement of signal or data entries. Invoking Proposition 4.5, dilation operators lead to permutation of spectra as well. This property of dilations over $\mathbb{C}^p$ have recently been used in implementation of algorithms for sparse fast Fourier transform, see [26] and references therein.

6.2. The finite field $\mathbb{F}_4$. The finite field $\mathbb{F}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the smallest finite field which does not have prime order. It can be considered as the polynomial ring $\mathbb{Z}_2[t]$ over an indeterminate variable t with addition and multiplication defined module the irreducible polynomial $t^2 + t + 1$. That is the classic polynomial addition and multiplication with this note that field operations (addition and multiplication) are done module 2 and the relation $t + 1 \equiv t^2$ holds as well.

The finite wave packet group WP_4 over the field $\mathbb{F}_4$ has the underlying set $\mathbb{F}^* \times \mathbb{F} \times \widehat{\mathbb{F}^+}$. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window vector. Then the wave packet transform of a given vector $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ with respect to the window vector $\mathbf{y}$ is

$$\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma) = \sum_{\tau\in\mathbb{F}_4} \mathbf{x}(\tau')e^{2\pi i\mathbf{t}(\gamma(\alpha^{-1}\tau-\beta))/p}\overline{\mathbf{y}(\alpha^{-1}\tau-\beta)} \quad \text{for } (\alpha,\beta,\gamma) \in \mathrm{WP}_4. \tag{6.4}$$

Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a nonzero window vector and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then

$$\sum_{\alpha\in\mathbb{F}_4^*}\sum_{\beta\in\mathbb{F}_4}\sum_{\gamma\in\widehat{\mathbb{F}_4^+}} |\mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)|^2 = 12\|\mathbf{y}\|_2^2\|\mathbf{x}\|_2^2, \tag{6.5}$$

which implies the following reconstruction formula

$$\mathbf{x}(\tau) = \frac{\|\mathbf{y}\|_2^{-2}}{12}\sum_{\alpha\in\mathbb{F}_4^*}\sum_{\beta\in\mathbb{F}_4}\sum_{\gamma\in\widehat{\mathbb{F}_4^+}} \mathcal{V}_{\mathbf{y}}\mathbf{x}(\alpha,\beta,\gamma)D_\alpha T_\beta M_\gamma \mathbf{y}(\tau) \quad \text{for } \tau \in \mathbb{F}_4. \tag{6.6}$$

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REFERENCES

[1] A. Arefijamaal and R.A. Kamyabi-Gol. On the square integrability of quasi regular representation on semidirect product groups. *J. Geom. Anal.*, 19:541–552, 2009.

[2] A. Arefijamaal and E. Zekaee. Signal processing by alternate dual Gabor frames. *Appl. Comput. Harmon. Anal.*, 35:535–540, 2013.

[3] G. Caire, R.L. Grossman, and H. Vincent Poor. Wavelet transforms associated with finite cyclic groups. *IEEE Trans. Inform. Theory*, 39:113–119, 1993.

[4] P. Casazza and G. Kutyniok. *Finite Frames Theory and Applications.* Applied and Numerical Harmonic Analysis, Springer-Birkhauser, Boston, 2013.

[5] L. Cohen. *Time-Frequency Analysis.* Prentice-Hall, New York, 1995.

[6] H.G. Feichtinger, W. Kozek, and F. Luef. Gabor analysis over finite Abelian groups. *Appl. Comput. Harmon. Anal.* 26:230–248, 2009.

[7] F. Fekri, R.M. Mersereau, and R.W. Schafer. Theory of wavelet transform over finite fields. *Proceedings of International Conference on Acoustics, Speech, and Signal Processing*, 3:1213–1216, 1999.

[8] K. Flornes, A. Grossmann, M. Holschneider, and B. Torrésani. Wavelets on discrete fields. *Appl. Comput. Harmon. Anal.*, 1:137–146, 1994.

[9] G.B. Folland. *A Course in Abstract Harmonic Analysis.* CRC Press, 1995.

[10] A. Ghaani Farashahi. Continuous partial Gabor transform for semi-direct product of locally compact groups. *Bull. Malays. Math. Sci. Soc.*, 38:779–803, 2015.

[11] A. Ghaani Farashahi. Cyclic wave packet transform on finite Abelian groups of prime order. *Int. J. Wavelets Multiresolut. Inf. Process.*, 12:1450041, 2014.

[12] A. Ghaani Farashahi and R. Kamyabi-Gol. Gabor transform for a class of non-abelian groups. *Bull. Belg. Math. Soc. Simon Stevin.*, 19:683–701, 2012.

[13] A. Ghaani Farashahi and M. Mohammad-Pour. A unified theoretical harmonic analysis approach to the cyclic wavelet transform (CWT) for periodic signals of prime dimensions. *Sahand Commun. Math. Anal.*, 1:1–17, 2014.

[14] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers.* Oxford University Press, 1979.

[15] C.P. Johnston. On the pseudodilation representations of flornes, grossmann, holschneider, and torrésani. *J. Fourier Anal. Appl.*, 3:377–385, 1997.

[16] J.B. Lima and R.M. Campello de Souza. Fractional cosine and sine transforms over finite fields. *Linear Algebra Appl.*, 438:3217–3230, 2013.

[17] G.L. Mullen and D. Panario. *Handbook of Finite Fields.* Discrete Mathematics and Its Applications, Chapman and Hall/CRC, 2013.

[18] R.J. McEliece. *Finite Fields for Computer Scientists and Engineers.* The Kluwer International Series in Engineering and Computer Science, Kluwer Academic Publishers, Boston, 1987.

[19] G. Pfander. Gabor Frames in Finite Dimensions. Finite Frames Theory and Applications. In: G. Pfander, P.G. Casazza, and G. Kutyniok (editors), *Finite Frames, Applied and Numerical Harmonic Analysis*, Birkhauser, Boston, 193–239, 2013.

[20] O. Pretzel. *Error-Correcting Codes and Finite Fields.* Applied Mathematics and Computing Science Series, Oxford University Press, New York, 1996.

[21] D. Ramakrishnan and R.J. Valenza. *Fourier Analysis on Number Fields.* Springer-Verlag, New York, 1999.

[22] R. Reiter and J.D. Stegeman. *Classical Harmonic Analysis*, second edition. Oxford University Press, New York, 2000.

[23] H. Riesel. *Prime Numbers and Computer Methods for Factorization*, second edition. Birkhauser, Boston, 1994.

[24] S. Sarkar and H. Vincent Poor. Cyclic wavelet transforms for arbitrary finite data lengths. *Signal Processing*, 80:2541–2552, 2000.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 507-529, September 2015

[25] G. Strang and T. Nguyen. *Wavelets and Filter Banks*. Wellesley-Cambridge Press, Wellesley, 1996.

[26] B. Sun, Q. Chen, X. Xu, Y. He, and J. Jiang. Permuted&Filtered Spectrum Compressive Sensing. *IEEE Signal Processing Letters*, 20:685–688, 2013.

[27] S.A. Vanstone and P.C. Van Oorschot. *An Introduction to Error Correcting Codes with Applications*. The Springer International Series in Engineering and Computer Science, New York, 1989.

[28] A. Vourdas. Harmonic analysis on a Galois field and its subfields. *J. Fourier Anal. Appl.*, 14:102–123, 2008.

[29] M.W. Wong. *Discrete Fourier Analysis*. Pseudo-differential Operators Theory and Applications, Vol. 5, Springer, Birkhauser, 2010.