WAVE PHENOMENA IN INHOMOGENEOUS MEDIA*

BY

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Summary. The difference between the fundamental solutions of differential equations governing the same physical phenomenon in two different physical media is investigated. A stationary expression of this difference is established, leading to a Ritz–Galerkin procedure. The Ritz–Galerkin system is solved analytically, providing a representing series for the difference of the two fundamental solutions as a functional of either one or the other of these functions. The convergence of the series is a consequence of its construction itself. Physical examples are considered which show that the convergence rate can be partially controlled.

Introduction. Many physical phenomena are well described by partial differential equations with variable coefficients—for instance, the propagation of acoustical waves in the sea, of light in the high atmosphere, or the vibrations of complex mechanical structures. The literature on this topic is full of different methods leading to various approximations [1]. Nevertheless, no general method seems to have been developed up to now.

It is well known that the solution of any boundary-value problem can be reached when the fundamental solution of the governing equation, defined in the whole space and satisfying a suitable Sommerfeld condition, is known. For this reason, this paper will deal with the construction of such a fundamental solution; that is, the field of a point source in the indefinite space is investigated.

Let two physical media be described by two elliptic operators $\mathcal{L}$ and Λ governing the same physical phenomenon, but corresponding to two different physical data. Let $\mathcal{G}$ and Γ be the fundamental solutions of $\mathcal{L}$ and Λ respectively which satisfy a Sommerfeld condition expressing that no energy is sent back by points at infinity. Finally, assume that $\mathcal{L}$ and Λ differ on a bounded space domain, say ω.

Generally, the different methods consider that one of the two operators, say $\mathcal{L}$, is the perturbation of the other, the fundamental solution of which is known. A perturbation parameter is pointed out which is assumed to be small according to some norm. The unknown solution is expressed, as a functional of Γ, by a series of the successive powers of the perturbation parameter. The principal disadvantage of such a method is that the convergence of the series is obtained for the perturbation parameter less than a certain bound.

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Central to the present analysis is the fact that it is not possible to decide which of the operators $\mathcal{L}$ and Λ is the perturbation of the other. This remark implies that the quantity to be investigated is the difference $\mathcal{G} - \Gamma$ between the two fundamental solutions; furthermore, $\mathcal{G} - \Gamma$ must have an expression symmetric with respect to the two media.

It is well known that the difference $\mathcal{G} - \Gamma$ can be interpreted as the field of fictitious sources lying in ω. But the sources appearing in the first medium have to be associated with the sources in the second medium in such a way that their respective radiations are identical, i.e. described by a unique function.

The main result of this paper is the construction of a convergent series representing the difference $\mathcal{G} - \Gamma$ between the two fundamental solutions. To reach such a series, two bases $\{\psi_n\}$ and $\{\varphi_n\}$ of $L^2(\omega)$ are associated according to the fact that the radiation in the first medium of a source ψ_n must be identical to that of φ_n in the second medium. Next an approximation of $\mathcal{G} - \Gamma$ is sought by a Ritz–Galerkin method applied to a suitable variational formulation of the problem. The Ritz–Galerkin system is solved analytically which leads to a formal series representing $\mathcal{G} - \Gamma$, the convergence of which is proved. If one of the two fundamental solutions is known, this series provides the other.

In the first section, a variational formulation of the problem is proposed. The second section deals with the construction of the basis $\{\psi_n\}$ as a functional of a given basis $\{\varphi_n\}$. The solution of the Ritz–Galerkin system giving an approximation of $\mathcal{G} - \Gamma$ is investigated in the third section; the series representing $\mathcal{G} - \Gamma$ is derived. The fourth section proposes several examples in acoustics and mechanics; furthermore, it is shown how a suitable choice of the first basis function φ_0 leads to a good approximation of $\mathcal{G} - \Gamma$ by the first term of the series.

1. A variational formulation of the problem. Let Λ be the governing operator of some physical phenomenon in the space $\mathbf{R}^n (n = 1, 2, \text{ or } 3)$. The field $\Gamma_s(X)$ due to a point source at S and satisfying a suitable Sommerfeld condition at infinity is the fundamental solution given by:

$$\Lambda_X \Gamma_s(X) = \delta_s(X) + \text{Sommerfeld condition} \tag{1}$$

(the subscript X in Λ_X stands for derivations with respect to the variable X).

Let $\mathcal{L}$ be the governing operator of the same physical phenomenon but corresponding to different physical data. The corresponding fundamental solution $\mathcal{G}_s(X)$ is given by:

$$\mathcal{L}_X \mathcal{G}_s(X) = \delta_s(X) + \text{Sommerfeld condition.} \tag{2}$$

1. *Hypotheses and notation.* The operator Λ and $\mathcal{L}$ are elliptic partial differential ones of order $2m$ with C^∞-coefficients (they can include integral operators). The difference of these operators, say $l = \mathcal{L} - \Lambda$, is assumed of order less than $2m$ and, for simplicity, with bounded support ω.

Let ${}^t\mathcal{L}$, ${}^t\Lambda$, and ${}^tl$ be the transposed operators of $\mathcal{L}$, Λ, and l, respectively; they are defined by:

$$(f, {}^t\mathcal{L}\varphi) = (\mathcal{L}f, \varphi),$$
$$(f, {}^t\Lambda\varphi) = (\Lambda f, \varphi),$$
$$(f, {}^tl\varphi) = (lf, \varphi),$$

where f is any distribution on $\mathbf{R}^n$ and φ any infinitely differentiable function with bounded support ($\varphi \in \mathcal{D}(\mathbf{R}^n)$).

2. *A variational formula.* From Eqs. (1) and (2) the following relationships can be derived:

$$\mathcal{G}_s(X) - \Gamma_s(X) = -\Gamma_y(X)[l_y\mathcal{G}_s(y)] = -\mathcal{G}_y(X)[l_y\Gamma_s(y)] \tag{3}$$

where

$$\Gamma_y(X)[l_y\mathcal{G}_s(y)] = \int_\omega \Gamma_y(X)l_y\mathcal{G}_s(y)\,dy,$$

$$\mathcal{G}_y(X)[l_y\Gamma_s(y)] = \int_\omega \mathcal{G}_y(X)l_y\Gamma_s(y)\,dy.$$

When X belongs to ω, the first equality (3) is an integrodifferential equation in $\mathcal{G}_s(X)$, $\Gamma_y(X)$ being known; the second equality is an equation in $\Gamma_s(X)$, $\mathcal{G}_y(X)$ being known.

Eqs. (3) make possible the construction of several stationary expressions of $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$; in particular, one gets:

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) = -\mathcal{G}_s(X)[{}^tl_X\Gamma_X(\Sigma)] - \mathcal{G}_X(\Sigma)[l_X\Gamma_s(X)]$$
$$+\begin{cases}\Gamma_X(\Sigma)[l_X\Gamma_s(X)] \\ \Gamma_s(X)[{}^tl_X\Gamma_X(\Sigma)]\end{cases} - [\mathcal{G}_s(X)[{}^tl_X\Gamma_X(\Sigma)]][l_z\Gamma_s(z)] \tag{4'}$$

(the third term may have the two expressions mentioned above). Eq. (4′) is obtained with the help of the trivial equality:

$$[\mathcal{G}_s(X) - \Gamma_s(X) + \mathcal{G}_Z(X)[l_Z\Gamma_s(Z)]][{}^tl_X\Gamma_X(\Sigma)] = 0.$$

Expressed as in (4′), the difference $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ is stationary with respect to $l_X\Gamma_s(X)$ and to ${}^tl_X\Gamma_X(\Sigma)$ because of the equality

$$\Gamma_X(\Sigma)[l_X\Gamma_s(X)] = \Gamma_s(X)[{}^tl_X\Gamma_X(\Sigma)].$$

Similarly, an expression of $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ can be derived which is stationary with respect to $l_X\mathcal{G}_s(X)$ and ${}^tl_X\mathcal{G}_X(\Sigma)$:

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) = -\Gamma_s(X)[{}^tl_X\mathcal{G}_X(\Sigma)] - \Gamma_X(\Sigma)[l_X\mathcal{G}_s(X)]$$
$$+\begin{cases}\mathcal{G}_X(\Sigma)[l_X\mathcal{G}_s(X)] \\ \mathcal{G}_s(X)[{}^tl_X\mathcal{G}_X(\Sigma)]\end{cases} - [\Gamma_X(z)[{}^tl_z\mathcal{G}_s(\Sigma)]][l_X\mathcal{G}_s(X)]. \tag{4''}$$

When $\Gamma_s(\Sigma)$ is assumed to be known, the use of the stationarity of (4′) will provide a series representing $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$; when $\mathcal{G}_s(\Sigma)$ is known, expression (4″) must be used.

2. The basis $\{\psi_n\}$ associated with any basis $\{\varphi_n\}$. Let $\{\varphi_n\}$ be any orthonormal basis of $L^2(\omega)$, the space of all square-integrable functions in ω. The relationship:

$$\psi_n(X) = \varphi_n(X) + \Gamma_y(X)[l_y\varphi_n(y)]$$
$$= \varphi_n(X) + \int_\omega \Gamma_y(X)l_y\varphi_n(y)\,dy \tag{5}$$

defines a set of functions $\psi_n(X)$. Let us prove that the set $\{\psi_n(X)\}$ is another basis of $L^2(\omega)$. This is true if any function $f(X) \in L^2(\omega)$ orthogonal to all the $\psi_n(X)$ is identically zero. The orthogonality relationship

$$\int_\omega f^*(X)\psi_n(X)\,dX = \int_\omega f^*(X)\left[\varphi_n(X) + \int_\omega \Gamma_y(X) l_y \varphi_n(y)\,dy\right] dX = 0, \qquad \forall\, n \tag{6}$$

($f^*(X)$ = imaginary conjugate of $f(X)$) can be written in the form:

$$\int_\omega \varphi_n(X)\left[f^*(X) + {}^t l_X \int_\omega f^*(y)\Gamma_X(y)\,dy\right] dX = 0, \qquad \forall\, n. \tag{7}$$

Because of the completeness of the $\{\varphi_n(X)\}$ basis, this implies:

$$f^*(X) + {}^t l_X \int_\omega f^*(y)\Gamma_X(y)\,dy = 0, \qquad \forall\, X \in \omega. \tag{8}$$

It is easily seen that a solution of any equation of the form:

$$u(X) + {}^t l_X \int_\omega u(y)\Gamma_X(y)\,dy = v(X), \qquad X \in \omega \tag{9}$$

is given by

$$u(X) = {}^t\Lambda_X \int_\omega \mathcal{G}_X(y)v(y)\,dy$$

because of the facts that ${}^t l = {}^t\mathcal{L} - {}^t\Lambda$ and ${}^t\mathcal{L}_X \mathcal{G}_X(y) = \delta_y(X)$, ${}^t\Lambda_X\Gamma_X(y) = \delta_y(X)$. The unicity of the solution of (9) can be proved as follows. Let the existence of two different solutions u_1 and u_2 be assumed; the function

$$U(X) = \int_\omega [u_1(y) - u_2(y)]\Gamma_X(y)\,dy$$

must be a solution of the equation

$${}^t\mathcal{L}_X U(X) = 0$$

satisfying the same Sommerfeld condition that $\Gamma_X(y)$ does; $U(X)$ is consequently zero, and so is the function ${}^t\Lambda_X U(X)$, which is $u_1(X) - u_2(X)$. This last conclusion is inconsistent with the starting hypothesis. The only solution of Eq. (8) is $f^*(X) \equiv 0$, which proves that the set $\{\psi_n(X)\}$ is a basis of $L^2(\omega)$.

In a similar way we can define a third basis of $L^2(\omega)$ by:

$$\begin{aligned} {}^t\psi_n(X) &= \varphi_n(X) + \Gamma_X(y)[{}^t l_y \varphi_n(y)] \\ &= \varphi_n(X) + \int_\omega \Gamma_X(y)\,{}^t l_y \varphi_n(y)\,dy. \end{aligned} \tag{10}$$

Finally, let $\hat{\psi}_n(X)$ and ${}^t\hat{\psi}_n(X)$ be the functions

$$\hat{\psi}_n(X) = \Gamma_y(X)[l_y\varphi_n(y)] = \int_\omega \Gamma_y(X) l_y \varphi_n(y)\,dy, \tag{11}$$

$${}^t\hat{\psi}_n(X) = \Gamma_X(y)[{}^t l_y \varphi_n(y)] = \int_\omega \Gamma_X(y)\,{}^t l_y \varphi_n(y)\,dy. \tag{12}$$

It is easy to see that:

$$\hat{\psi}_n(X) = \mathcal{G}_y(X)[l_y \psi_n(y)], \tag{11'}$$

$${}^t\hat{\psi}_n(X) = \mathcal{G}_X(y)[{}^t l_y\, {}^t\psi_n(y)]. \tag{12'}$$

3. Approximation and expression of $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$. As a first step of the present investigation, a Ritz–Galerkin method is applied to provide a linear algebraic system, the solution of which enables the construction of an approximate representation of $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$. The second step is devoted to the expression of the analytic solution of this system, leading to a convergent representing series for $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$.

1. *A Ritz–Galerkin method.* Let us look for approximations in the distribution sense of $\Gamma_s(X)$ and $\Gamma_X(\Sigma)$ in the form:

$$\Gamma_s(X) \simeq \sum_{j=0}^{N} b_s{}^j \psi_j(X)$$
$$\Gamma_X(\Sigma) \simeq \sum_{j=0}^{N} b_\Sigma{}^j\, {}^t\psi_j(X) \tag{13}$$

Expressions (13) lead to:

$$\mathcal{G}_s(X)[{}^t l_X \Gamma_X(\Sigma)] \simeq \sum_{j=0}^{N} b_\Sigma{}^j\, {}^t\hat{\psi}_j(s)$$
$$\mathcal{G}_X(\Sigma)[l_X \Gamma_s(X)] \simeq \sum_{j=0}^{N} b_s{}^j \hat{\psi}_j(\Sigma)$$
$$\Gamma_X(\Sigma)[l_X \Gamma_s(X)] \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} b_s{}^i b_\Sigma{}^j \int_\omega {}^t\psi_j(X) l_X \psi_i(X)\, dX \tag{14}$$
$$\simeq \sum_{i=0}^{N}\sum_{j=0}^{N} b_s{}^i b_\Sigma{}^j \int_\omega {}^t l_X {}^t\psi_j(X)\psi_i(X)\, dX.$$

Eq. (4′) is approximated by:

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) \simeq -\sum_{j=0}^{N} b_s{}^j\, {}^t\hat{\psi}_j(s) - \sum_{i=0}^{N} b_s{}^i \hat{\psi}_i(\Sigma)$$
$$+ \sum_{i=0}^{N}\sum_{j=0}^{N} b_s{}^i b_\Sigma{}^j \int_\omega {}^t\psi_j(X) l_X \psi_i(X)\, dX - \sum_{i=0}^{N}\sum_{j=0}^{N} b_s{}^i b_\Sigma{}^j \int_\omega \left\{ \begin{matrix} {}^t\psi_j(X) l_X \hat{\psi}_i(X) \\ {}^t\hat{\psi}_j(X) l_X \psi_i(X) \end{matrix} \right. dx. \tag{15}$$

Now let relation (15) be stationary with respect to $b_s{}^i$ ($i = 0, 1, \cdots, N$), one gets:

$$\sum_{j=0}^{N} b_\Sigma{}^j \int_\omega \left. \begin{matrix} {}^t\psi_j(X) l_X \varphi_i(X) \\ \varphi_j(X) l_X \psi_i(X) \end{matrix} \right\} dX = \hat{\psi}_i(\Sigma), \qquad i = 0, 1, \cdots, N. \tag{16}$$

(In (15) and (16) the integrals over ω have the two mentioned expressions). The linear algebraic system (16) determines the $b_\Sigma{}^j$ insuring the stationarity of (15). The corresponding approximation of $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ is given by:

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) \simeq -\sum_{j=0}^{N} b_\Sigma{}^j\, {}^t\hat{\psi}_j(s). \tag{17}$$

In a similar way, it is possible to determine the $b_s{}^i$ by the system

$$\sum_{i=0}^{N} b_s{}^i \int_\omega {}^t l_X\, {}^t\psi_j(X)\varphi_i(X)\, dX = {}^t\hat{\psi}_j(s), \qquad j = 0, 1, \cdots, N, \tag{16′}$$

leading to the approximation:

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) \simeq -\sum_{i=0}^{N} b_s{}^i \hat{\psi}_i(\Sigma). \tag{17′}$$

Remark. It is important to notice that the accuracy of the approximations (17) or (17′) can be characterized by the accuracy of the approximations (13).

2. *Explicit solution of system* (16) *and representation of* $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ *by a series.* Let $\gamma_s^N(\Sigma)$ be the approximation of $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ given by formula (17). Noting the coefficients of system (16) by D_{ij}, we define the following determinants:
D_N = determinant of the D_{ij} matrix,

$$\mathfrak{D}_N(\hat{\psi}_i(\Sigma)) = \text{deter} \begin{vmatrix} D_{00} & D_{01} \cdots D_{0N-1} & \hat{\psi}_0(\Sigma) \\ D_{10} & D_{11} \cdots D_{1N-1} & \hat{\psi}_1(\Sigma) \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ D_{N0} & D_{N1} \cdots D_{NN-1} & \hat{\psi}_N(\Sigma) \end{vmatrix}, \tag{18}$$

$${}^t\mathfrak{D}_N({}^t\hat{\psi}_i(s)) = \text{deter} \begin{vmatrix} D_{00} & D_{01} \cdots D_{0N-1} & {}^t\hat{\psi}_0(s) \\ D_{10} & D_{11} \cdots D_{1N-1} & {}^t\hat{\psi}_1(s) \\ & & \\ D_{N0} & D_{N1} \cdots D_{NN-1} & {}^t\hat{\psi}_N(s) \end{vmatrix}. \tag{18'}$$

Let $b_\Sigma'^i$ be the solution of system (16) corresponding to $(i = 0, 1, \cdots, N-1;\ j = 0, 1, \cdots, N-1)$, and let β_Σ^i and B^i be defined by:

$$\beta_\Sigma^i = b_\Sigma^i - b_\Sigma'^i, \qquad j = 0, 1, \cdots, N-1,$$
$$\beta_\Sigma^N = b_\Sigma^N,$$
$$B_\Sigma^i = 0, \qquad i = 0, 1, \cdots N-1,$$
$$B_\Sigma^N = \hat{\psi}_N(\Sigma) - \sum_{j=0}^{N-1} b_\Sigma'^i D_{Nj}\,.$$

Similarly, the β_s^i and B_s^i are defined by:

$$\beta_s^i = b_s^i - b_s'^i, \qquad i = 0, 1, \cdots, N-1,$$
$$\beta_s^N = \tilde{b}_s^N,$$
$$B_s^i = 0, \qquad j = 0, 1, \cdots, N-1,$$
$$B_s^N = {}^t\hat{\psi}_N(s) - \sum_{i=0}^{N-1} b_s'^i D_{iN}\,.$$

It is easy to show that the β_Σ^i and the β_s^i are the respective solutions of:

$$\sum_{j=0}^{N} D_{ij}\beta_\Sigma^i = B_\Sigma^i, \qquad i = 0, 1, \cdots, N, \tag{19}$$

and

$$\sum_{i=0}^{N} D_{ij}\beta_s^i = B_s^i, \qquad j = 0, 1, \cdots, N, \tag{19'}$$

these systems being an immediate consequence of the systems that the b_Σ^i, $b_\Sigma'^i$, b_s^i, and $b_s'^i$ satisfy. The equality

$$\gamma_s^N(\Sigma) - \gamma_s^{N-1}(\Sigma) = -B_\Sigma^N \beta_s^N \tag{20}$$

is true because of

$$\sum_{i=0}^{N} \sum_{j=0}^{N} \beta_\Sigma{}^i D_{ij} b_s{}^i = \sum_{j=0}^{N} \beta_\Sigma{}^i \, {}^t\hat{\psi}_j(s) = \sum_{i=0}^{N} B_\Sigma{}^i b_s{}^i = B_\Sigma{}^N \beta_s{}^N.$$

The expression of β_s^N as a function of B_s^N is easily found because all the $B_s^j (j = 0, 1, \cdots, N - 1)$ are zero:

$$\beta_s^N = B_s^N \frac{\overline{D_{N-1}}}{\overline{D_N}}. \tag{21}$$

Now the B_Σ^N and B_s^N are to be determined. For this purpose, let $f_\Sigma{}^i$ and $g_s{}^i$ be defined by:

$$b_\Sigma{}'^i = -f_\Sigma{}^i/f, \qquad b_s{}'^i = -g_s{}^i/g,$$

f and g being arbitrary constants. The constants $f_\Sigma{}^i$, f, $g_s{}^i$, and g satisfy:

$$\sum_{j=0}^{N-1} D_{ij} f_\Sigma{}^i + \hat{\psi}_i(\Sigma) f = 0, \qquad i = 0, 1, \cdots, N - 1,$$

$$\sum_{j=0}^{N-1} D_{Nj} f_\Sigma{}^i + (\hat{\psi}_N(\Sigma) - B_\Sigma{}^N) f = 0,$$

$$\sum_{i=0}^{N-1} D_{ij} g_s{}^i + {}^t\hat{\psi}_i(s) g = 0, \qquad j = 0, 1, \cdots, N - 1,$$

$$\sum_{i=0}^{N-1} D_{iN} g_s{}^i + ({}^t\hat{\psi}_N(s) - B_s{}^N) g = 0$$

because of (19) and (19′). These two systems will have a nonzero solution if the determinants of their respective matrices are zero. This implies that B_Σ^N and B_s^N are given by:

$$B_\Sigma{}^N = \frac{\mathfrak{D}_N(\hat{\psi}_i(\Sigma))}{\overline{D_{N-1}}}, \qquad B_s{}^N = \frac{{}^t\mathfrak{D}_N({}^t\hat{\psi}_i(s))}{\overline{D_{N-1}}}.$$

By using this last result, expression (20) becomes:

$$\gamma_s{}^N(\Sigma) - \gamma_s{}^{N-1}(\Sigma) = -\frac{\mathfrak{D}_N(\hat{\psi}_i(\Sigma)) \, {}^t\mathfrak{D}_N({}^t\hat{\psi}_i(s))}{\overline{D_{N-1}} \, \overline{D_N}}. \tag{22}$$

Finally, making use of the trivial equality

$$\gamma^N = (\gamma^N - \gamma^{N-1}) + (\gamma^{N-1} - \gamma^{N-2}) + \cdots + (\gamma^1 - \gamma^0) + \gamma^0$$

and letting N grow to infinity, we get the representing series desired:

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) = -\frac{\hat{\psi}_0(\Sigma) \, {}^t\hat{\psi}_0(s)}{\int_\omega \varphi_0(X) l_X \psi_0(X) \, dX} - \sum_{i=1}^{\infty} \frac{\mathfrak{D}_i(\hat{\psi}_i(\Sigma)) \, {}^t\mathfrak{D}_i({}^t\hat{\psi}_i(s))}{\overline{D_{i-1}} \, \overline{D_i}}, \tag{23}$$

the convergence of which will be proved later.

Remarks. 1) This result is very close to that given by S. Bergmann [2] for the Green functions of the inner boundary-value problems concerning the Helmholtz equation with purely imaginary wave parameter. The present investigation avoids Bergmann's assumption that the operator $\mathfrak{L}$ defines a positive definite L^2-norm.

2) In formula (23), the two media described respectively by the operators Λ and $\mathfrak{L}$ play exactly the same role because the $\hat{\psi}_i$ and the ${}^t\psi_i$ can be expressed either with Γ or with $\mathcal{G}$. This is in agreement with the fact that each medium can be considered as the perturbation of the other.

3. *Proof of the convergence of the representing series.* The convergence of (23) is nothing other than the convergence of the Ritz–Galerkin procedure here proposed. The proof is obtained by showing that the sequence of approximations of ${}^t l_X \Gamma_X(\Sigma)$ by

$$ {}^t l_X \Gamma_X(\Sigma) \simeq \sum_{j=0}^{N} b_\Sigma{}^i \; {}^t l_X \; {}^t\psi_i(X) $$

is convergent in the distribution sense; that is:

$$ \lim_{N\to\infty} \int_\omega dX \int_{\mathbf{R}^n} \left[{}^t l_X \Gamma_X(\Sigma) - \sum_{j=0}^{N} b_\Sigma{}^i \; {}^t l_X \; {}^t\psi_i(X) \right] w(X, \Sigma)\, d\Sigma = 0, $$

$$ \forall\ w(X, \Sigma) \in \mathfrak{D}(\omega \times \mathbf{R}^n). $$

If ${}^t l_X \Gamma_X(\Sigma) \in L^2(\omega)$, the convergence is taken in the L^2 sense because $\{{}^t\psi_i(X)\}$ is a basis of $L^2(\omega)$.

Using system (16), it is obvious that:

$$ \sum_{i=0}^{N} \varphi_i{}^*(y) \int_\omega dX \sum_{j=0}^{N} {}^t l_X \; {}^t\psi_i(X)\varphi_i(X) \int_{\mathbf{R}^n} b_\Sigma{}^i w(y, \Sigma)\, d\Sigma $$

$$ = \sum_{i=0}^{N} \varphi_i{}^*(y) \int_\omega dX\, \varphi_i(X) \int_{\mathbf{R}^n} {}^t l_X \Gamma_X(\Sigma) w(y, \Sigma)\, d\Sigma \qquad (24) $$

($\varphi_i{}^*(y)$ is the imaginary conjugate of $\varphi_i(y)$). But, because $\{\varphi_n\}$ is an orthonormal basis of $L^2(\omega)$, for large enough N the functions defined by the first and second member of (24) are as closed as desired (in the $L^2(\omega)$ sense) to respectively:

$$ \sum_{j=0}^{N} {}^t l_y \; {}^t\psi_i(y) \int_{\mathbf{R}^n} b_\Sigma{}^i w(y, \Sigma)\, d\Sigma, \qquad \int_{\mathbf{R}^n} {}^t l_y \Gamma_y(\Sigma) w(y, \Sigma)\, d\Sigma \qquad (25) $$

Eq. (24) thus implies that the functions given by (25) can be made arbitrarily closed to each other by choosing N large enough. As a consequence, for any given small number $\epsilon > 0$, a M exists such that:

$$ \left| \int_\omega \int_{\mathbf{R}^n} \left[\sum_{j=0}^{N} b_\Sigma{}^i \; {}^t l_y \; {}^t\psi_i(y) - {}^t l_y \Gamma_y(\Sigma) \right] w(y, \Sigma)\, d\Sigma\, dy \right| < \epsilon, \qquad \forall\ N > M. $$

This proves the required convergence, and, consequently, that

$$ \lim_{N\to\infty} \sum_{j=0}^{N} b_\Sigma{}^i \hat{\psi}_i(s) = \mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) $$

in the distribution sense.

This result can be improved, leading to a uniform convergence. As a well-known result about the behavior at $S = \Sigma$ of fundamental solutions of elliptic partial differential equations, the difference is a continuous $(m - 1)$ times continuously differentiable function; the functions defining series (23) are continuous functions of their arguments; this implies the uniform convergence of this representing series.

4. *Remark on the case of an unbounded domain ω.* If ω is an unbounded domain,

the convergence of the integrals in the variational formula (4′) is not necessary insured. However, it is possible to consider (4′) as a formal variational principle and to make use of it in solving the problem in the way proposed formerly.

Let $\{\varphi_n(X)\}$ be a basis of $L^2(\omega)$ orthogonal with a weight $\bar{\omega}(X)$. As in Sec. 2, a new basis $\{\psi_n(X)\}$ can be defined

$$\psi_n(X) = \bar{\omega}(X)\varphi_n(X) + \int_\omega \Gamma_y(X)l_y(\bar{\omega}(y)\varphi_n(y))\,dy \tag{5*}$$

and associated with the set

$$\begin{aligned}\hat{\psi}_n(X) &= \int_\omega \Gamma_y(X)l_y(\bar{\omega}(y)\varphi_n(y))\,dy\\ &= \int_\omega \mathcal{G}_y(X)l_y\psi_n(y)\,dy.\end{aligned} \tag{11*}$$

In the same way the sets ${}'\psi_n(X)$ and ${}'\hat{\psi}_n(X)$ are defined by

$${}'\psi_n(X) = \bar{\omega}(X)\varphi_n(X) + \int_\omega \Gamma_X(y)\ {}'l_y(\bar{\omega}(y)\varphi_n(y))\,dy \tag{10*}$$

$$\begin{aligned}{}'\hat{\psi}_n(X) &= \int_\omega \Gamma_X(y)\ {}'l_y(\bar{\omega}(y)\varphi_n(y))\,dy\\ &= \int_\omega \mathcal{G}_X(y)\ {}'l_y\ {}'\psi_n(y)\,dy\end{aligned} \tag{12*}$$

The convergence of all the above integrals is insured by the presence of the weight function. Now let D_{ij} be the integrals

$$D_{ij} = \int_\omega \bar{\omega}(X)\varphi_i(X)\ {}'l_X{}'\ \psi_j(X)\,dX.$$

$\overline{D_N}$, $\mathfrak{D}_N(\hat{\psi}_i(\Sigma))$ and ${}'\mathfrak{D}_N({}'\hat{\psi}_i(S))$ are defined as in Sec. 3.2. We get for $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ the representing series given in (23). The convergence of this series is proved again.

4. Physical examples. This section is mainly devoted to Helmholtz equation; one-dimensional examples are considered and it is shown that φ_0 can be chosen in such a way that $\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma)$ is approximated well by the only first term of the representing series. The first physical example is devoted to the plate equation: in the case of a point mass density perturbation of a constant thickness and mass density plate the first term of series (23) corresponds to the exact solution, whatever the φ_n are.

1. *The plate equation for transverse vibrations* [3]. The plate operator corresponding to a constant thickness and mass density plate driven by a harmonic force is given by:

$$\Lambda. = \Delta.^2 - \frac{m_0\tilde{\omega}^2}{D_0}, \tag{26}$$

with $\tilde{\omega}$ = circular frequency, m_0 = mass density per unit area, and D_0 = rigidity parameter.

For a plate with variable thickness and mass density, the corresponding operator takes the following form in rectangular coordinates (x_1, x_2):

$$\mathcal{L}. = \Delta.^2 + 2\,\frac{\operatorname{grad}(D - D_0)}{D}\operatorname{grad}\Delta. + \frac{\Delta(D\nu - D\nu_0)}{D}\,\Delta.$$

$$+ \sum_{i,j=1}^{2} \frac{1}{D}\,\frac{\partial^2[D(1-\nu) - D_0(1-\nu_0)]}{\partial x_i\,\partial x_j}\,\frac{\partial^2 .}{\partial x_i\,\partial x_j} - \frac{m\omega^2 \cdot}{D} \tag{27}$$

In this expression the rigidity D, the Poisson's ratio ν and the mass density m are assumed to be constant and equal to D_0, ν_0, m_0 respectively outside some bounded domain ω.

The fundamental solution of (27) satisfying the Sommerfeld condition deduced from the energy conservation principle can be expressed in term of the fundamental solution of (26) by the series (23). Assume that the rigidity of the plate and its Poisson ratio are constant; the mass density is assumed constant, different from m_0, and equal to m in a small circular domain ω. Let ω decrease to zero and m increase to infinity in such a way that the total excess mass $M = (m \cdot \text{area of } \omega)$ is constant; the limit is obtained for a perturbating point mass M at a point X_0. It is important to notice that the approximation obtained by taking into account only the first term of the representing series is the exact solution of the problem

$$\mathcal{G}_s(\Sigma) - \Gamma_s(\Sigma) = \frac{M^2\tilde{\omega}^4}{D^2}\,\frac{\Gamma_s(X.)}{1 - \dfrac{M^2\tilde{\omega}^4}{D^2}\,\Gamma_{X_0}(X_0)}\,\Gamma_{X_0}(\Sigma),$$

whatever the function φ_0. When a perturbation method is considered, it is generally impossible to obtain approximate formulas providing an exact solution in this case.

Before ending this section, let us mention that the case of a fluid-loaded plate can be solved in the same way by using the solution given in [4] for constant coefficients. This problem has a great importance in mechanical and acoustical engineering.

2. *The Helmholtz equation with variable index.* In this section one-dimensional examples are considered and the function φ_0 is chosen in each case to provide a one-term approximation in good agreement with the exact solution. The operator $\mathcal{L}$ is defined by

$$\mathcal{L}. = \frac{d.^2}{dX^2} + k^2[1 + p(X)].\,,$$

the corresponding operator Λ being

$$\Lambda. = \frac{d.^2}{dX^2} + k.^2$$

$\mathcal{L}$ and Λ being self-transposed operators, the function ${}^t\psi_n$, ${}^t\hat{\psi}_n$, and ${}^t\mathcal{D}_i({}^t\hat{\psi}_n)$ are respectively equal to ψ_n, $\hat{\psi}_n$ and $\mathcal{D}_i(\hat{\psi}_i)$.

The important question is how to choose the function φ_0. This choice is derived from the fact that the approximation of $\mathcal{G}(s, \Sigma) - \Gamma(s, \Sigma)$ by the N first terms of series (23) corresponds to the approximation of $\Gamma(s, \Sigma)$ by:

$$\Gamma(s, \Sigma) \simeq \sum_{i=0}^{N} \frac{\mathcal{D}_i(\psi_i(X))\mathcal{D}_i(\hat{\psi}_i(s))}{\overline{D_{N-1}}\;\overline{D_N}}. \tag{28}$$

So, the first step of the analysis is to check that (28) is a good approximation formula.

First case: the perturbed region is $X > 0$, and $p(X) = aX$; the source S is in the negative region. Let $\varphi_0(X)$ be defined by

$$\varphi_0(X) = \exp(lkX - \beta X), \qquad \mathrm{Re}(\beta) > 0$$

The function $\hat{\psi}_0(X)$ is thus:

$$\hat{\psi}_0(X) = -k^2 \int_0^\infty ay \exp(iky - \beta y) \frac{\exp(ik|y - X|)}{2ik} dy$$

$$= \frac{\exp(-ikX)}{ik} \frac{k^2 a}{\beta^2} \text{ for } X < 0,$$

$$= \frac{\exp(ikX)}{2ik} k^2 a \left\{ \frac{1}{(\beta - 2ik)^2} + \frac{4ik(\beta - ik)}{\beta^2(\beta - 2ik)^2} \exp(-\beta X) + X \frac{2ik}{\beta(\beta - 2ik)} \exp(-\beta X) \right\} \text{ for } X > 0.$$

The corresponding approximation of $\Gamma(s, X)$ for $s < 0$ and $X > 0$ is given by:

$$\Gamma(s, X) \simeq \frac{\exp(ik(X - s))}{2ik} \frac{\beta - ik}{(\beta - 2ik)^2} \left\{ \exp(-\beta X) + \frac{ka}{2i} \left[\frac{1}{\beta^2} + \frac{4ik(\beta - ik)}{\beta^2(\beta - 2ik)^2} \exp(-\beta X) + \frac{2ikX}{\beta(\beta - 2ik)} \exp(-\beta X) \right] \right\} \left\{ \frac{1}{4(\beta - ik)} + \frac{ka}{2i\beta(\beta - 2ik)} \left[\frac{1}{\beta - 2ik} + \frac{ik}{(\beta - ik)^2} \right] \right\}^{-1}. \quad (29)$$

The corresponding approximation for $\mathcal{G}(s, X < 0)$ is:

$$\mathcal{G}(s, X < 0) = \frac{\exp(ik|X - s|)}{2ik} - \frac{\exp(-ik(X + s))}{2ik} \cdot \frac{ka}{2i(\beta - 2ik)^4} \cdot \left\{ \frac{1}{4(\beta - ik)^2} + \frac{ka}{2i(\beta - ik)(\beta - 2ik)\beta} \left[\frac{1}{\beta - 2ik} + \frac{ik}{\beta - ik)^2} \right] \right\}^{-1}$$

If a/k is assumed to be small compared to 1 (slowly varying index), one can choose β real satisfying the double inequality

$$a/k \ll \beta^2/k^2 \ll 1,$$

and (29) becomes:

$$\Gamma(s, X) \simeq \frac{\exp(ik(X - s))}{2ik} \left\{ \exp(-\beta X) + \frac{a}{2ik} \left[\frac{k^2}{\beta^2} - \frac{k^2}{\beta^2} \exp(-\beta X) - \frac{k^2 X}{\beta} \exp(-\beta X) \right] \right\} \quad (30)$$

which is a good approximation for small kX. This yields

$$\mathcal{G}(s_1\Sigma) - \Gamma(s_1\Sigma) \simeq -\frac{i}{8} \frac{a}{k} \frac{\exp(-ik(s + \Sigma))}{2ik} \quad \text{for} \quad \Sigma < 0, \quad (31)$$

which is in agreement with the asymptotic value of the exact solution given in [5] by

$$\mathcal{G}(s_1\Sigma) - \Gamma(s_1\Sigma) = \frac{\exp(-ik(s + \Sigma))}{2ik} \frac{iH_{1/3}^{(1)}(w) - H_{-2/3}^{(1)}(w)}{iH_{1/3}^{(1)}(w) + H_{-2/3}^{(1)}(w)},$$

$$w = \frac{2}{3} \frac{k}{a}.$$

Conversely a/k can be assumed to be large compared to unity (rapidly varying index); with the assumption

$$1 \ll \beta^2/k^2 \ll a/k,$$

expression (29) gives:

$$\Gamma(s_1 X) \simeq \frac{\exp(ik(X - s))}{2ik} (1 + 2ikX \exp(-\beta X)) \tag{32}$$

This is a good approximation for small and large kX. The corresponding approximate expression of $\mathcal{G}(s, \Sigma) - \Gamma(s, \Sigma)$ is:

$$\mathcal{G}(s, \Sigma) - \Gamma(s, \Sigma) \simeq -\frac{\exp(-ik(s + \Sigma))}{2ik}. \tag{33}$$

Here again we get the asymptotic value of the exact expression. As a conclusion of this example, the following choice of $\varphi_0(X)$:

$$\varphi_0(X) = \exp\left[i - \left(\frac{a}{k}\right)^\alpha\right]kX, \qquad 0 < \alpha < 1/2,$$

provided an approximation of $\mathcal{G}(s, X) - \Gamma(s, X)$ which corresponds to the exact solution for the two asymptotic cases $a/k \ll 1$ and $a/k \gg 1$.

Second case: Here, again, the perturbed region is $X > 0$; but

$$p(X) = [\exp(\sqrt{2}\, akX) - 1]/2$$

which is a index variation closer to physical data than that considered above; the source S is in the negative region. Using well-known results (for example see [5]), it can be proved that the solution is given with the help of Bessel functions of imaginary index by:

$$\mathcal{G}(s, X) - \Gamma(s, X) = \frac{\sqrt{2}\, J - iJ'}{\sqrt{2}\, J + iJ'} \frac{\exp(-ik(s + X))}{2ik}, \qquad X < 0 \tag{32}$$

with:

$$J = J_{-i/a}(1/a), \qquad J' = J'_{-i/a}(1/a).$$

For the two asymptotic cases $a \ll 1$ and $a \gg 1$, formula (32) becomes respectively

$$\mathcal{G}(s, X) - \Gamma(s, X) = \frac{ia}{8\sqrt{2}} \frac{\exp(-ik(s + X))}{2ik}, \tag{32'}$$

$$\mathcal{G}(s, X) - \Gamma(s, X) = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \frac{\exp(-ik(s + X))}{2ik}. \tag{32''}$$

Let $\varphi_0(X)$ have, here again, the form:

$$\varphi_0(X) = \exp(+ikX - \beta kX), \qquad \mathrm{Re}(\beta) > 0.$$

The expression for $\hat{\psi}_0(X)$ takes the following form:

$$\hat{\psi}_0(X) = k^2 \int_0^\infty \frac{\exp(ik|y - X|)}{2ik} \exp(iky) \exp(-\beta ky) \frac{\exp(-a\sqrt{2}\, ky) - 1}{2}\, dy$$

$$= -\frac{\exp(-ikX)}{4i} \frac{a\sqrt{2}}{(\beta + a\sqrt{2} - 2i)(\beta - 2i)}, \qquad X < 0,$$

$$= \frac{\exp(ikX)}{4i}\left\{\frac{-a\sqrt{2}}{\beta(\beta + a\sqrt{2})} - \frac{2i\exp(-\beta kX)}{\beta(\beta - 2i)} + \frac{2i\exp(-ak\sqrt{2}\,X)\exp(-\beta kX)}{(\beta + a\sqrt{2})(\beta + a\sqrt{2} - 2i)}\right\}, \qquad X > 0.$$

This yields the following approximation of $\Gamma(s, X)$ for $s < 0$ and $X > 0$:

$$\Gamma(s, X) \simeq \frac{\exp(ik(X - s))}{2ik}\cdot\frac{-a\sqrt{2}}{2(\beta - 2i)(\beta + a\sqrt{2} - 2i)}\left\{\exp(-\beta kX) + \frac{1}{4i}\left[\frac{-a\sqrt{2}}{\beta(\beta + a\sqrt{2})} - \frac{2i\exp(-\beta kX)}{\beta(\beta - 2i)} + \frac{2i\exp(-a\sqrt{2}\,kX)\exp(-\beta kX)}{(\beta + a\sqrt{2})(\beta + a\sqrt{2} - 2i)}\right]\right\}$$
$$\cdot\left\{\frac{-a\sqrt{2}}{4(\beta - i)(2\beta + a\sqrt{2} - 2i)} - \frac{1}{8i}\left[\frac{2a^2}{\beta(\beta + a\sqrt{2})(\beta - 2i)(\beta + a\sqrt{2} - 2i)} + \frac{ia\sqrt{2}}{\beta(\beta + 2i)(\beta - i)(2\beta + a\sqrt{2} - 2i)} - \frac{ia\sqrt{2}}{(\beta + a\sqrt{2})(\beta + a\sqrt{2} - 2i)(\beta + a\sqrt{2} - i)(2\beta + a\sqrt{2} - 2i)}\right]\right\}^{-1} \quad (33)$$

and the approximation for $\mathcal{G}(s, X < 0)$:

$$\mathcal{G}(s, X < 0) = \frac{\exp(ik\,|X - s|)}{2ik} - \frac{\exp(-ik(s + X))}{2ik}\cdot\frac{2a^2}{8i(\beta + a\sqrt{2} - 2i)^2(\beta - 2i)^2}$$
$$\cdot\left\{-\frac{a\sqrt{2}}{(4\beta - i)(2\beta + a\sqrt{2} - 2i)} + \frac{a}{8i}\left[\frac{2a}{\beta(\beta + a\sqrt{2})(\beta - 2i)(\beta + a\sqrt{2} - 2i)} + \frac{i\sqrt{2}}{\beta(\beta - 2i)(\beta - i)(2\beta + a\sqrt{2} - 2i)} - \frac{i\sqrt{2}}{(\beta + a\sqrt{2})(\beta + a\sqrt{2} - 2i)(\beta + a\sqrt{2} - i)(2\beta + a\sqrt{2} - 2i)}\right]\right\}^{-1}.$$

Let $a \ll 1$; (33) gives:

$\Gamma(s, X) \simeq \exp(ik(X - s)/2ik \exp(-\beta kX)$, $a \ll \beta$ real $\ll 1$, which is a good approximation. The corresponding approximation of $\mathcal{G}(s, X) - \Gamma(s, X)$ is

$$\mathcal{G}(s, X) - \Gamma(s, X) \simeq \frac{ia}{8\sqrt{2}}\,\frac{\exp(-ik(X + s))}{2ik}, \qquad s < 0, \qquad X < 0.$$

This is the asymptotic value of the exact solution.

Conversely, assume $a \gg 1$; formula (33) will become

$$\Gamma(s, X) \simeq \frac{\exp(ik(X - s)}{2ik}\,\frac{4}{3}\left[\frac{-1}{4i\beta} + \frac{2\beta(\beta - 2i) - 1}{2\beta(\beta - 2i)}\exp(-\beta kX)\right]$$

Under the hypothesis β real $\ll 1$ and for small enough kX, this last expression takes the form:

$$\Gamma(s, X) \simeq \frac{\exp(ik(X - S))}{2ik}\,\frac{7}{6}.$$

The corresponding approximate solution is

$$\mathcal{G}(s, \Sigma) - \Gamma(s, \Sigma) \simeq \frac{1}{6} \frac{\exp(-ik(s + X))}{2ik}.$$

The reflexion coefficient (1/6) here obtained is not equal to the exact one $(\sqrt{2} - 1)/(\sqrt{2} + 1)$, but is pretty close to it: the error is about 3%.

In this second case it is again possible to derive an approximate expression of $\mathcal{G}(s, \Sigma) - \Gamma(s, \Sigma)$ which is in good agreement with the exact formula for the two asymptotic values of a: in fact the function

$$\varphi_0(X) = \exp k[i - e^{-a} a^{\alpha}]X, \qquad \alpha \ll 1$$

is consistent with the conditions imposed above.

5. Conclusion. In this paper the author has looked for a series representing the fundamental solution of a partial differential equation with variable coefficients as a functional of the known fundamental solution of another partial differential equation of the same kind. Central to the investigation, a Ritz–Galerkin procedure is developed.

The first important result is that the functions used to establish the Ritz–Galerkin system are derived in a suitable way from any basis of the L^2-space constructed on the domain where the two differential operators differ.

Because of its construction, the Ritz–Galerkin system can be solved analytically, leading to a series representation of the unknown fundamental solution. The convergence of this series is proved, whatever the starting basis. This is the second important result.

The third result is a consequence of the arbitrariness of the starting basis. Because of it, it is possible to increase the convergence of the representing series by a suitable choice of the basis. As has been shown in two examples, the first term of the series can provide a good (even perfect) approximation of the solution.

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