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WAVE PROPAGATION IN A ONE-DIMENSIONAL RANDOM MEDIUM*

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Abstract. Wave propagation in a slab of random medium is considered. The index of refraction is assumed to fluctuate randomly about a mean value, the fluctuations being small. Using a recent result of Hashminskii we give a description of the statistical characteristics of the reflection and transmission coefficients.

1. Introduction. We consider propagation of scalar waves through a slab of random medium. The slab is assumed to be plane stratified and the index of refraction is taken to be a random function with small fluctuations about a mean value. Our purpose is to compute the statistical characteristics of the reflection and transmission coefficients and therefore to characterize the scattering properties of the slab.

The analysis followed here is based on a result of Hashminskii [1]. In § 2 the problem under consideration is stated along with an abridged version of Hashminskii's theorem. In § 3 the theorem is applied and the main results are obtained. A somewhat similar approach to the problem of scattering by a random medium was followed by Gertsenshtein and Vasiliev [2], who employed a result of Karpelevich et al. [3]. Here, instead of discretizing the medium we treat it as a continuum directly and the random refractive index can have, within certain limitations, an arbitrary correlation function. By assuming that the fluctuations in the refractive index are Markovian it is possible to obtain special cases of the results stated here by using perturbation theory on the relevant Kolmogorov equation [4]. A discussion of the relation of this problem to wave propagation in a wave guide with inhomogeneities can be found in [2].

Wave propagation in a random medium has been studied by several investigators [5], [7], and an extensive survey of the literature is included in the latter reference. In these works and the works to which they refer there is no adequate information concerning the mean of the square of the magnitude of the reflection and transmission coefficients. However, these quantities are important in this problem, for they measure the mean power reflected and transmitted by the slab. Here we obtain explicit formulas, (3.12) and (3.14), for the mean power reflection and transmission coefficients.

2. Formulation of the problem. Let $v(x)$ be the time harmonic wave field at location x with the time factor $e^{-i\omega t}$ omitted and assume that the slab occupies the interval $[0, L]$. Then $v(x)$ satisfies the equations

$$v_{xx} + k^2 v = 0, \qquad -\infty < x < 0, \quad L < x < \infty, \tag{2.1}$$

$$v_{xx} + k^2[1 + \varepsilon m(x)]v = 0, \qquad 0 < x < L, \tag{2.2}$$

and at $x = 0$, $x = L$ the field v and its derivative v_x are continuous. In (2.2), $m(x) = m(x, \alpha)$ is a real random function with zero mean and with $\alpha \in \Omega$, (Ω, U, P)

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a probability space, ε a small dimensionless parameter characterizing the size of the fluctuations, k the free space wave number, and L the width of the slab. Clearly $v(x) = v(x, \alpha)$ is also a random function. In what follows we suppress the dependence on α as is customary.

Let Ae^{-ikx}, $L < x < \infty$, represent a wave incident on the slab from the right and Be^{ikx} the reflected wave in the same space interval. A and B are, for each L, complex-valued random variables. We may assume that the wave transmitted into the region $-\infty < x < 0$ is of the form e^{-ikx}. By definition, the reflection coefficient $R(L)$ is given by

$$R(L) = B(L)/A(L). \tag{2.3}$$

Let us consider L variable and therefore $R(L) = R(L, \alpha)$ a random function. We shall now derive a stochastic differential equation satisfied by $R(L)$.

From the continuity of v and v_x at $x = L$ and the assumed form of the field in $L < x < \infty$ it follows that

$$\frac{v_x(L)}{v(L)} = ik\frac{[R(L)e^{ikL} - e^{-ikL}]}{[R(L)e^{ikL} + e^{-ikL}]}. \tag{2.4}$$

Here $v(x)$ is the solution of (2.2) with initial conditions at $x = 0$:

$$v(0) = 1, \qquad v_x(0) = -ik. \tag{2.5}$$

These conditions follow from the continuity of v and v_x at zero and the assumed form of v in $-\infty < x < 0$. Solving (2.4) for $R(L)$ we obtain

$$R(L) = e^{-2ikL}\frac{ikv(L) + v_x(L)}{ikv(L) - v_x(L)}, \qquad L \geqq 0. \tag{2.6}$$

Let us now differentiate (2.6) and eliminate $v(L)$ and its derivatives from the resulting expression by using (2.2) and (2.4). Then we obtain

$$\frac{dR(L)}{dL} = \varepsilon\frac{ikm(L)}{2}(e^{ikL}R(L) + e^{-ikL})^2, \qquad L \geqq 0, \quad R(0) = 0. \tag{2.7}$$

The initial condition in (2.7) follows from (2.5) and (2.6).

Equation (2.7) is the desired stochastic differential equation for the stochastic process $R(L)$, to which Hashminskii's theorem will be applied. The statement of this theorem [1] is rather lengthy, but it will be presented here for completeness in a form adequate for treating (2.7). It is convenient to separate real and imaginary parts in (2.7) so that it can be written in system form:

$$\begin{aligned} dy_i/dL &= \varepsilon m(L)F_i(y(L), L), \qquad y_i(0) = 0, \quad i = 1, 2, \\ R(L) &= y_1(L) + iy_2(L). \end{aligned} \tag{2.8}$$

Let us now make the following assumptions about $m(L)$ and F_i:

(i) $m(L)$, $L \geqq 0$, is a real-valued stochastic process on a probability space (Ω, U, P), almost surely bounded and such that

$$E\{m(L)\} = 0, \qquad E\{m(L)m(L')\} = \rho(|L - L'|). \tag{2.9}$$

Here $E\{\cdot\}$ denotes taking expected values, that is, integration over Ω with respect to the measure P.

(ii) $m(L)$ satisfies the strong mixing condition; that is, there exists a family $U_s^t, 0 \leqq s \leqq t \leqq \infty$, of σ-algebras of subsets of Ω such that $U_s^t \in U, U_{s_1}^{t_1} \subset U_s^t$, $s \leqq s_1, t_1 \leqq t$, and for any $t > 0$, $B \in U_{t+T}^{\infty}$ and some $\beta(T) \downarrow 0$ as $T \to \infty$,

$$|P\{B|U_0^t\} - P\{B\}| < \beta(T). \tag{2.10}$$

(iii) The vector function $F = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ satisfies for all y and L, $y \in D \subset R^2$, the condition

$$|F| < C, \quad \left|\frac{\partial F}{\partial y_i}\right| < C, \quad \left|\frac{\partial^2 F}{\partial y_i \partial y_j}\right| < C, \qquad i, j = 1, 2. \tag{2.11}$$

Here D is the range of the y variables, C is a positive constant and $|\cdot|$ is the vector norm.

(iv) The following limits exist uniformly in y and t_0:

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{s} \rho(s - \sigma) F_i(y, s) F_j(y, \sigma)\, d\sigma\, ds = a_{ij}(y), \tag{2.12}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{s} \rho(s - \sigma) \sum_{j=1}^{2} \frac{\partial F_i}{\partial y_j}(y, s) F_j(y, \sigma)\, d\sigma\, ds = b_i(y), \qquad i, j = 1, 2. \tag{2.13}$$

We can now state Hashminskii's theorem:

Let $m(L)$ and $F(y, L)$ satisfy (i)–(iv) *and suppose that the function $\beta(T)$ in* (2.10) *is such that*

$$T^6 \beta(T) \downarrow 0 \quad as\ T \to \infty. \tag{2.14}$$

Let $y^{(\varepsilon)}(L)$ be the solution of (2.8) *and τ_0 be an arbitrary positive number. Then on the interval $0 \leqq \varepsilon^2 L < \tau_0$ the process $y^{(\varepsilon)}(\varepsilon^2 L)$ converges weakly as $\varepsilon \to 0, L \to \infty$, $\varepsilon^2 L =$* const., *to a Markov process $y^{(0)}(\varepsilon^2 L)$ which is continuous with probability* 1 *and whose infinitesimal generator A is given by*

$$A = \sum_{i,j=1}^{2} a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{j=1}^{2} b_j(y) \frac{\partial}{\partial y_j}. \tag{2.15}$$

3. The mean of the power reflection coefficient. In this section we shall verify that Hashminskii's theorem applies to (2.7) and then proceed to use it. As usual, we shall interpret this theorem to mean that, for ε sufficiently small, the statistics of $y^{(\varepsilon)}$ and those of $y^{(0)}$ are approximately the same.

Let us assume that the random fluctuations $m(L)$ of the refractive index satisfy conditions (i), (ii) and (2.14) of the previous section. These conditions are quite reasonable for a model of a random medium. Moreover, it is well known that

$$|R(L)| \leqq 1, \qquad L \geqq 0. \tag{3.1}$$

From (3.1) it follows that the right-hand side in (2.7) is bounded and hence (2.11) holds with D defined by

$$D = \{(y_1, y_2) | y_1^2 + y_2^2 \leqq 1\}. \tag{3.2}$$

Condition (iv) also holds in view of (2.10) and (2.14). Thus all conditions in Hashminskii's theorem are satisfied and hence it may be applied to (2.7).

Before proceeding further, let us define new dependent variables u and ϕ by

$$R = \sqrt{\frac{u-1}{u+1}}e^{i\phi}, \qquad u \geqq 1, \quad 0 \leqq \phi \leqq 2\pi. \tag{3.3}$$

Variables quite similar to u and ϕ are also used in [2], where, in addition, an important geometrical interpretation for $R(L)$ is given. Let us substitute (3.3) into (2.7) and separate real and imaginary parts. This yields the system of equations

$$\frac{du(L)}{dL} = \varepsilon k m(L)\sqrt{u^2(L) - 1}\sin(\phi(L) + 2kL), \qquad u(0) = 1, \tag{3.4}$$

$$\frac{d\phi(L)}{dL} = \varepsilon k m(L)\left[\frac{u(L)}{\sqrt{u^2(L) - 1}}\cos(\phi(L) + 2kL) + 1\right]. \tag{3.5}$$

Now it appears that Hashminskii's theorem does not apply to (3.4), (3.5) because (2.11) does not hold. However, it is sufficient that (2.11) hold in some coordinate representation. The generator (2.15) can be shown to be invariant so that applying the theorem to (3.4)–(3.5) is equivalent to applying it to (2.7) and subsequently using (3.3) in (2.15). Moreover, information about the statistics of the phase ϕ of the reflection coefficient is not of primary importance in the physical problem. Therefore we shall concentrate on the statistics of $|R|$, hence of u, which lead to considerable simplification.

Let us now compute the coefficients $a_{ij}(u, \phi)$, $b_i(u, \phi)$, $i, j = 1,2$, of the generator A in (2.15) corresponding to (3.4), (3.5). From (2.12) and (2.13) it follows that these coefficients do not depend on ϕ. Hence we can integrate out the ϕ-dependence in the generator A and restrict attention to the marginal generator $\tilde{A}$ that involves u only:

$$\tilde{A} = a_{11}(u)\frac{\partial^2}{\partial u^2} + b_1(u)\frac{\partial}{\partial u}. \tag{3.6}$$

Here $a_{11}(u)$ and $b_1(u)$ are easily shown to reduce to

$$\begin{aligned} a_{11}(u) &= \frac{k^2}{2}(u^2 - 1)\int_0^\infty \rho(\tau)\cos 2k\tau\, d\tau, \\ b_1(u) &= k^2 u \int_0^\infty \rho(\tau)\cos 2k\tau\, d\tau. \end{aligned} \tag{3.7}$$

Therefore, $\tilde{A}^*$, the adjoint of $\tilde{A}$, is given by

$$\tilde{A}^* = s\frac{\partial}{\partial u}\left[(u^2 - 1)\frac{\partial}{\partial u}\right], \qquad s = \frac{k^2}{2}\int_0^\infty \rho(\tau)\cos 2k\tau\, d\tau. \tag{3.8}$$

According to Hashminskii's theorem and the interpretation given in the beginning of this section, the transition probability density $P(L, u)$ of $u(L)$ given $u(0) = 1$ is, for ε small, approximately equal to $P_\varepsilon(L, u)$, which is the solution of

$$\frac{\partial P_\varepsilon}{\partial L} = \varepsilon^2 s \frac{\partial}{\partial u}\left[(u^2 - 1)\frac{\partial P_\varepsilon}{\partial u}\right], \qquad P_\varepsilon(0, u) = \delta(u - 1). \tag{3.9}$$

This equation, except for the constant $\varepsilon^2 s$, was derived by entirely different considerations in [2]. Its solution, also given in [2], can be obtained by using the Mehler transform [6] and subsequently transforming the result in a manner analogous to that used in the Poisson summation formula [4]. It is given by

$$P_\varepsilon(L, u) = \frac{e^{-\varepsilon^2 sL/4}}{2\sqrt{2\pi}(\varepsilon^2 Ls)^{3/2}} \int_u^\infty \frac{x e^{-x^2/(4\varepsilon^2 sL)}}{\sqrt{\cosh x - \cosh u}}\, dx. \tag{3.10}$$

Let us use (3.10) to compute approximately the mean of the square of the magnitude of the reflection coefficient $E\{|R(L)|^2\}$. From (3.3) it follows that the approximate value of $E\{|R(L)|^2\}$, which we will denote by $E_\varepsilon\{|R(L)|^2\}$, is given by

$$E_\varepsilon\{|R(L)|^2\} = \int_1^\infty \left(\frac{u - 1}{u + 1}\right) P_\varepsilon(L, u)\, du. \tag{3.11}$$

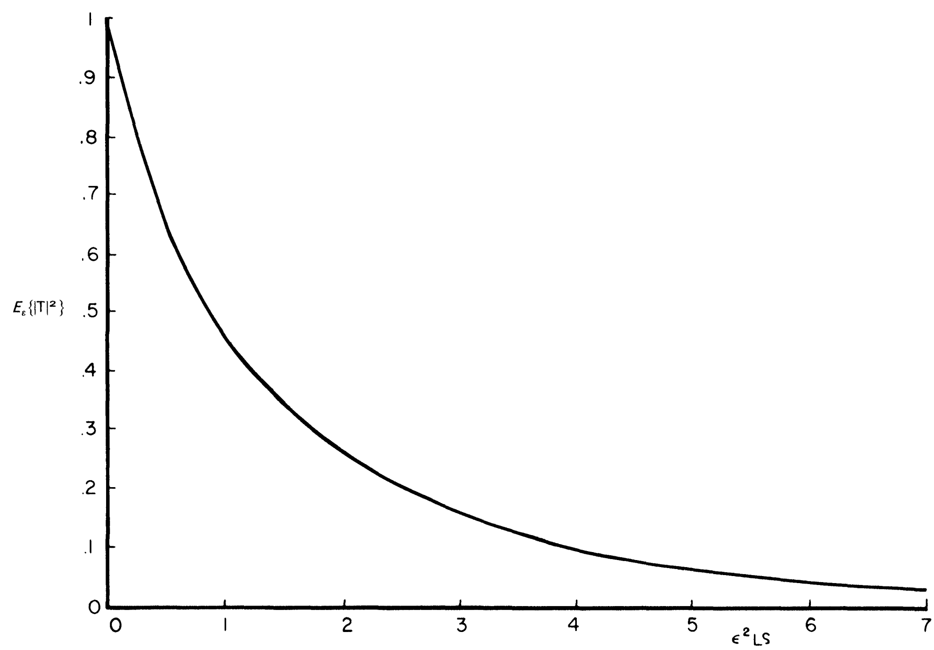

FIG. 1. *The approximate value of the mean power transmission coefficient plotted as a function of* $\varepsilon^2 Ls$, *where s is defined by* (3.8)

Substituting (3.10) in (3.11) and performing one integration we obtain

$$E_\varepsilon\{|R(L)|^2\} = 1 - \frac{4}{\sqrt{\pi}} e^{-\varepsilon^2 sL} \int_0^\infty \frac{x^2 e^{-x^2}\,dx}{\cosh(\varepsilon\sqrt{sL}x)}. \tag{3.12}$$

Concerning the transmission coefficient $T(L)$ we can use the well-known relation

$$|R|^2 = 1 - |T|^2 \tag{3.13}$$

to deduce the approximate statistical properties of $|T|$ from those of $|R|$. In particular, in view of (3.13) and (3.12), we have

$$E_\varepsilon\{|T(L)|^2\} = \frac{4}{\sqrt{\pi}} e^{-\varepsilon^2 sL} \int_0^\infty \frac{x^2 e^{-x^2}\,dx}{\cosh(\varepsilon\sqrt{sL}x)}. \tag{3.14}$$

In Fig. 1 a graph of (3.14) is shown which was obtained by numerical integration. The result (3.14) was also obtained in [4].

To obtain the approximate transition density of the modulus of the reflection coefficient $r = |R|$ we use the transformation (3.3) in (3.10). Thus

$$P_\varepsilon(L, r) = \frac{2re^{-\varepsilon^2 sL/4}}{\sqrt{2\pi}(\varepsilon^2 sL)^{3/2}(1 - r^2)^2} \int_{(1+r^2)/(1-r^2)}^\infty \frac{xe^{-x^2/(4\varepsilon^2 sL)}}{\sqrt{\cosh x - \cosh((1 + r^2)/(1 - r^2))}}\,dx. \tag{3.15}$$

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REFERENCES

[1] R. Z. Hashminskii, *A limit theorem for the solutions of differential equations with random right-hand sides*, Theor. Probability Appl., 11 (1966), pp. 390–406.

[2] M. E. Gertsenshtein and V. B. Vasiliev, *Wave guides with random inhomogeneities and Brownian motion in the Lobachevsky plane*, Ibid., 4 (1959), pp. 392–398.

[3] F. I. Karpelevich, V. N. Tutubalin and M. G. Shur, *Limit theorems for the composition of distributions in the Lobachevsky plane and space*, Ibid., 4 (1959), pp. 399–401.

[4] J. A. Morrison, G. C. Papanicolaou and J. B. Keller, *Mean power transmission through a slab of random medium*, Comm. Pure Appl. Math., 24 (1971), to appear.

[5] I. Kupiec, L. B. Felsen, S. Rosenbaum, J. B. Keller and P. Chow, *Reflection and transmission by a random medium*, Radio Sci., 4 (1969), pp. 1067–1077.

[6] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.

[7] U. Frisch, *Wave propagation in a random medium*, Probabilistic Methods in Applied Mathematics, vol. 1, A. T. Bharucha-Reid, ed., Academic Press, New York, 1968.