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Wave Propagation in an Anisotropic Column with Ring Source Excitation

by
SURENDRA NATH SAMADDAR

December 1961

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WAVE PROPAGATION IN AN ANISOTROPIC COLUMN WITH RING SOURCE EXCITATION

by

Surendra Nath Samaddar

December 1961

This work was submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy at
The University of Michigan

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ABSTRACT

Propagation of electromagnetic waves through a homogeneous anisotropic column of a medium of infinite length is considered. The anisotropy of the medium is characterized by the dyadic form of the permittivity ϵ and permeability μ . This anisotropic column is surrounded by a coaxial homogeneous isotropic medium characterized by scalars ϵ_2 and μ_2 , this complete structure being enclosed by a perfectly conducting metallic circular cylindrical waveguide. A magnetic current ring source is inserted symmetrically in the isotropic medium. Throughout the analysis the strength of the source is considered to be an arbitrary function of the polar angle θ. For this general problem the complete expressions for fields (due to the source), power flow, and the dispersion relation have been studied.

To solve a source problem, dyadic Green's functions for both point electric current source and point magnetic current source have been constructed in a formal way from the source free solutions of the appropriate Maxwell's equations. These dyadic Green's functions can be used for any arbitrary source.

From the general expressions for the transverse fields in terms of the longitudinal fields, in any arbitrary cylindrical region (unbounded) the propagation wave number of a TEM mode travelling in the longitudinal direction z has

been obtained. The wave numbers for a TEM wave propagating in a direction perpendicular to z, can be obtained from the general expressions for the transverse wave numbers, using

$$\frac{\partial}{\partial z} = -j\chi = 0 \ .$$

The results of the above general problem have been used to study the wave propagation in an anisotropic plasma column and an anisotropic ferrite column separately. The various possible passbands for the propagation of electromagnetic waves in an anisotropic plasma column have been obtained and a special case is considered for numerical computation of the longitudinal electric field in a plasma. The analysis for the plasma problem emphasizes the slow wave propagation.

I
INTRODUCTION

Electromagnetic wave propagation through an anisotropic medium has been studied by many authors in different situations - from ionosphere to conventional waveguides. Ionospheric anisotropy is due to the presence of the earth's static magnetic field. Ionized, but macroscopically neutral gases of any kind are known as plasmas. An ionized and neutral stationary plasma in a weak electromagnetic field can be represented as an equivalent dielectric medium. Moreover, when this plasma is situated in a uniform static magnetic field, the strength of which is not necessarily small, its equivalent dielectric "constant" behaves as a dyadic (tensor). It is well known that radio wave propagation through the ionosphere depends on the frequency of the electromagnetic wave, electron density, the collision frequency, the ion gyrofrequency and the electron gyrofrequency. The same is true for a wave propagating through a plasma waveguide. This knowledge of the wave propagation is essential for satisfactory long-distance radio communication through the ionosphere.

The study of a plasma is also of vital importance in the field of thermonuclear reactions. In a thermonuclear reactor a static magnetic field is used to confine and to heat a plasma. Many times it is desirable to obtain information on the temperature, density, etc. of the plasma. Such investigations which determine the characteristics and behavior of the plasma are known as "plasma diagnostics."

Besides the above, the propagation of electromagnetic waves in plasma-filled or partially plasma-filled waveguides has aroused considerable interest in recent years [6] [7] [8], primarily because of possible applications to the generation or amplification of microwaves. A medium whose dielectric "constant" is a tensor is called gyroelectric.

On the other hand, there is another class of materials known as ferrites which exhibit ferro-magnetic properties. The chemical composition of the ferrites may be expressed [3] [18] by the formula $MO\,Fe_2O_3$, where M represents a metal, such as Mn, Fe, Ni, Cu, Mg, Al, Co, etc. Although ordinary iron (Fe) and nickel (Ni) possess ferromagnetic properties, they are of little use as microwave components due to their high losses. But the ferrite materials mentioned above, whose specific resistances are above 10^6 times higher than those of the metals, with relative permeabilities ranging up to several thousands and rel. ve dielectric constants varying from 5 to 25, have extensive use in microwave devices. In the presence of a static magnetic field the permeability μ of a ferrite becomes anisotropic, i.e., μ becomes a dyadic, which is the characteristic of a gyromagnetic medium. The medium whose dielectric constant and permeability both are tensors, is known as gyrotropic.

In connection with the Faraday rotation of guided electromagnetic waves in a gyromagnetic medium with a uniform static magnetic field in a circular cylindrical waveguide (the axis of the waveguide coincides with the gyro-axis which is also the direction of the static magnetic field and has been taken as the z-axis), Suhl and Walker [2] have shown that only circularly polarized modes exist, if $E_z \neq 0$, and $H_z \neq 0$, and pure TE and TM modes do not exist. However, pure TE and TM modes can exist if $\frac{\partial}{\partial z} = 0$. It should be noted that in this case as well as throughout the present work, the anisotropic medium under consideration is homogeneous. For an anisotropic plasma medium TE and TM modes can exist [7] independently in another special case when the axial-static magnetic field is infinite. Besides the above mentioned work, a number of investigators including Van Trier [1], Gamo [19], Fainberg and Gorbatenko [4], Agdur [6], Epstein [3], Trivelpiece [8] etc. have carried out research in connection with wave propagation in gyroelectric, gyromagnetic, or gyrotropic media with various configurations. All of the research work cited above except that in [7] considered only the source free resonance behavior of electromagnetic waves. In the present problem, however, a source of electromagnetic waves which interact with the anisotropic medium is included. In Appendix A, a general formulation of the source-free problem is presented for any cylindrical geometry with arbitrary cross section. This formulation is suitable even for an unbounded anisotropic medium, provided cylindrical symmetry is assumed. In Appendix B, dyadic Green's functions for a point source (electric current or magnetic current or both) are constructed from the general source free solutions of the Maxwell's equations for both dissipative and non-dissipative media. For such a construction of Dyadic Green's functions references [9], [11], and [12] have been found very useful. An alternative method using a transmission line formulation can be devised for the construction of Dyadic Green's functions.

In chapter I the problem considered is to find the dispersion relations and the complete fields due to an excitation by a magnetic current ring source situated in a cylindrical isotropic homogeneous medium characterized by a relative dielectric constant ϵ_2 and a relative permeability μ_2, which encloses a central cylindrical column of a homogeneous anisotropic medium characterized by dyadics $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$, this whole structure being enclosed by a perfectly conducting cylindrical waveguide. This general analysis has avoided specifying any particular medium, say a plasma or a ferrite, and also it does not necessarily consider a ring source of constant strength. A ring source (magnetic current) represents an idealization of a possible excitation, for example, a circumferential slot in the waveguide wall, or an annular slot on a thin metallic disc* fitting

* In this example one must also consider the boundary condition for the conducting metallic disc.

tightly across the waveguide. Although a ring source is taken for analysis, any other type of source can also be handled adequately since the formal expressions for dyadic Green's functions for an electric current source and a point magnetic current source are given in Appendix B. A magnetic current ring source is more appropriate for a plasma problem, whereas for a ferrite problem an electric dipole at the center of a cross section of a circular waveguide is more appropriate.

In Appendix C, the general dispersion relation of Chapter I has been evaluated in a number of interesting special situations with appropriate limiting processes. Although in these special cases the procedures are also applicable to obtain expressions for the total fields due to the source from the general expressions given in Chapter I, no attempt has been made to obtain these expressions owing to the laborious task they involve.

Chapter II deals with a problem in which the anisotropic column is taken to be a plasma in an axial static magnetic field, using the results of Chapter I. In this case necessary conditions for slow wave propagation (which give maximum passbands) have been obtained. The sufficient condition and hence the actual passbands can be obtained from the solution of the dispersion relation. Since it is not possible to study a dispersion relation in general, a few special cases have been discussed. In addition to these a more general dispersion relation for slow wave propagation has been considered for numerical computation in Appendix D. In this case the lowest eigenvalues are found and the corresponding longitudinal electric field is calculated.

In Chapter III, the results of Chapter I have been applied to study wave propagation through a ferrite column with a uniform axial static magnetic field. It may be noted here that results for a ferrite problem can be obtained by using duality on the corresponding results for a plasma problem when the boundary conditions on H in the ferrite are the same as those on E in the plasma, for example an unbounded plasma and a ferrite.

From the formal expressions of the transverse electric and magnetic fields as functions of the longitudinal fields E_z and H_z , and using the expressions for the transverse propagation wave numbers obtained in Appendix A, conditions for TEM wave propagation in the direction parallel to or perpendicular to the d.c. magnetic field have been obtained. These conditions provide expressions for the propagation wave numbers in the respective cases. The conditions which give the possibility of a TEM wave propagation in an unbounded medium cannot be valid in a bounded medium or in waveguides (except those bounded by two non-connecting metallic boundaries). Therefore these discussions suggest that the study of a dispersion relation for a bounded anisotropic medium, under the condition of TEM wave propagation, is meaningless and inconsistent. It can be shown that the condition of TEM wave propagation in the direction of the static magnetic field, is equivalent to zero-value of the product of the two transverse wave numbers. However, results for such a situation have been presented in the literature mistakenly.* A reason for such inconsistent results may be that the authors overlooked the direct equivalent relations between the conditions of TEM waves travelling in the direction of the static magnetic field and the vanishing conditions of the product of the transverse wave numbers.

*Agdur, [6] pages 183 to 185, also [4], page 497.

I

GENERAL PROBLEM

Statement of the Problem

An infinitely long column of an anisotropic medium characterized by tensors ϵ and μ, of radius a, is situated coaxially inside a perfectly conducting circular cylindrical waveguide of radius b. The annular space between the column of the anisotropic medium and the cylindrical waveguide is filled with an isotropic medium characterized by scalars ϵ_2 and μ_2. The electromagnetic fields are introduced into this system by a magnetic current ring source of radius c, the center of which lies on the axis of the waveguide, such that $a \leqslant c \leqslant b$. The total fields and their behaviors are studied. Figure 1 shows the geometry of the problem.

General Formulation of the Problem

For convenience the plane of the ring source will be chosen as $z = 0$, where the axis of the cylinder lies along the z-axis. Due to the cylindrical symmetry of the structure, cylindrical coordinates r, θ, and z will be used here. If the ring source is very thin both in the radial and in the axial direction, it can be represented in the following way

$$\underline{I_m} = \underline{\theta_o} I_m e^{j\omega t} = \underline{\theta_o}\, m(\theta)\, \delta(r - c)\,\delta(z)\, e^{j\omega t} \tag{1}$$

where ω = exciting angular frequency

$\underset{\sim}{\theta}_o$ = unit vector in the θ-direction

$m(\theta)$ = strength of the source in volts

$\delta(r - c)$ and $\delta(z)$ are well known Dirac-delta functions.

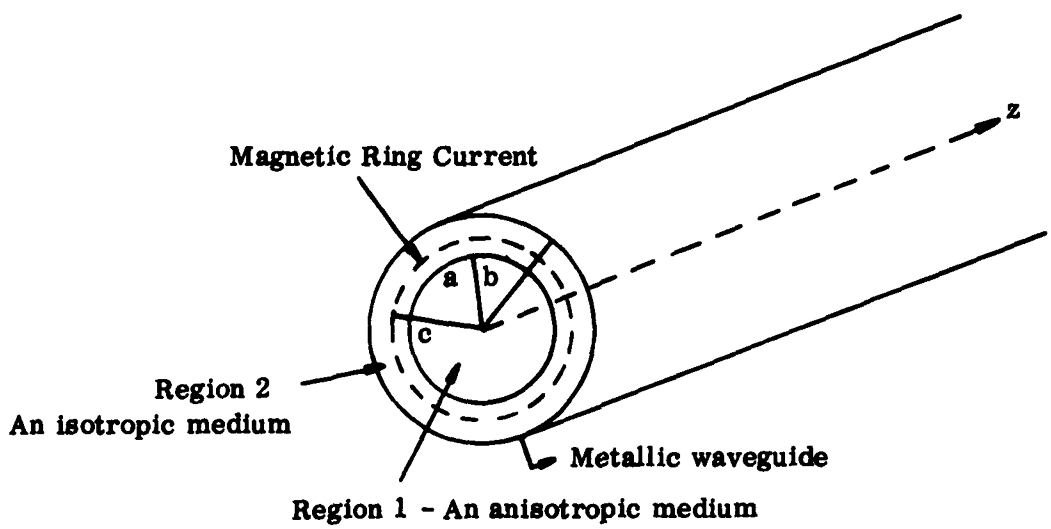


FIGURE 1

Although ultimately $m(\theta)$ will be chosen as a constant for numerical computational facility, the present formulation of the problem is valid for $m(\theta)$, any arbitrary function of its argument θ.

The Maxwell's equations for this problem can be expressed (the time dependence is assumed to be $e^{j\omega t}$) as

$$\left.\begin{aligned} \nabla \times \underline{E}(\underline{r}) &= -j\omega\mu_0 \underset{\sim}{\mu}(r) \cdot \underline{H}(\underline{r}) - \underline{\theta}_0 I_m \\ \nabla \times \underline{H}(\underline{r}) &= j\omega\epsilon_0 \underset{\sim}{\epsilon}(r) \cdot \underline{E}(\underline{r}) \end{aligned}\right\} \tag{2}$$

where the relative dielectric constant $\underset{\sim}{\epsilon}(r)$ and the relative permeability $\underset{\sim}{\mu}(r)$ are defined in the following way.

$$\underset{\sim}{\epsilon}(r) = \begin{Vmatrix} \epsilon_{rr} & j\epsilon_{r\theta} & 0 \\ -j\epsilon_{\theta r} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{Vmatrix}, \text{ for } 0 \leqslant r \leqslant a \tag{3a}$$

with constant elements

$$= \quad \epsilon_2 \text{ (constant)}, \quad \text{for } a \leqslant r \leqslant b \tag{3b}$$

$$\left.\begin{aligned} \epsilon_{rr} &= \epsilon_{\theta\theta} \\ \epsilon_{r\theta} &= \epsilon_{\theta r} \end{aligned}\right\} \tag{3c}$$

$$\underset{\sim}{\mu}(r) = \begin{Vmatrix} \mu_{rr} & j\mu_{r\theta} & 0 \\ -j\mu_{\theta r} & \mu_{\theta\theta} & 0 \\ 0 & 0 & \mu_{zz} \end{Vmatrix}, \text{ for } 0 \leqslant r \leqslant a \tag{4a}$$

with constant elements

$$= \mu_2 \text{ (constant)}, \qquad \text{for } a \leqslant r \leqslant b \tag{4b}$$

$$\left.\begin{aligned} \mu_{rr} &= \mu_{\theta\theta} \\ \mu_{r\theta} &= \mu_{\theta r} \end{aligned}\right\} \tag{4c}$$

$\underline{r}$ in eqs. (2) represents a three dimensional position vector, and r is the radial coordinate. The results developed in Appendices A and B will be frequently used in the following.

A method of solving any source problem in terms of Green's function will be presented here. To construct a Green's function for a problem with some given boundary conditions, it is sufficient to find corresponding eigenfunctions which form a complete orthogonal set. These eigenfunctions are solutions of the source-free problem subject to the same boundary conditions.

In the present problem where the waveguide is uniform (independent of z) and the medium is also homogeneous in the axial direction z (i.e., components of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are not functions of the coordinate z, with $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ having forms shown in (3) and (4) respectively), one can assume that there will be waves propagating in the z-direction, having z-dependence as $e^{-j\varkappa z}$, where $\varkappa$ is a propagation wave number for a particular mode. This assumption leads Maxwell's equations with appropriate boundary conditions, to an eigenvalue problem, with $\varkappa$ as an eigenvalue (see Appendix B and [9] to [13]).

Thus the source-free solutions $\underline{\mathcal{E}}_{\ell}(\underline{r})$ and $\underline{\mathcal{H}}_{\ell}(\underline{r})$ satisfying the following Maxwell's equations (5), form a complete orthogonal set of eigenfunctions.

$$\left.\begin{aligned}\nabla \times \underline{\mathcal{E}}_{\ell}(\underline{r}) &= -j\omega\mu_0 \underset{\sim}{\mu}(r) \cdot \underline{\mathcal{H}}_{\ell}(\underline{r}) \\ \nabla \times \underline{\mathcal{H}}_{\ell}(\underline{r}) &= j\omega\epsilon_0 \underset{\sim}{\epsilon}(r) \cdot \underline{\mathcal{E}}_{\ell}(\underline{r})\end{aligned}\right. \qquad (5)^{+}$$

The orthogonality relation can be obtained by choosing another set $\underline{\mathcal{E}}''_{\ell'}(\underline{r})$ and $\underline{\mathcal{H}}''_{\ell'}(\underline{r})$, which satisfy the same boundary conditions and the following Maxwell's equations,

$$\left.\begin{aligned}\nabla \times \underline{\mathcal{E}}''_{\ell'}(\underline{r}) &= j\omega\mu_0 \underset{\sim}{\mu}^{+*}(r) \cdot \underline{\mathcal{H}}''_{\ell'}(\underline{r}) \\ \nabla \times \underline{\mathcal{H}}''_{\ell'}(\underline{r}) &= -j\omega\epsilon_0 \underset{\sim}{\epsilon}^{+*}(r) \cdot \underline{\mathcal{E}}''_{\ell'}(\underline{r})\end{aligned}\right\} \qquad (6)^{+}$$

where * denotes complex conjugate,

ϵ^{+} (or $\underset{\sim}{\mu}^{+}$) = adjoint of $\underset{\sim}{\epsilon}$ (or $\underset{\sim}{\mu}$) = complex conjugate of the transpose of $\underset{\sim}{\epsilon}$ (or $\underset{\sim}{\mu}$).

Now it can be shown (see the above mentioned references) that the required orthogonality relation is

$$\iint_s \mathcal{E}_{t\ell}(\underline{r}) \cdot \mathcal{H}''_{t\ell'}(\underline{r}) \times \underline{z}_0 \, ds = N_{\ell}\delta_{\ell\ell'} = \iint_s \mathcal{H}_{t\ell}(\underline{r}) \cdot \underline{z}_0 \times \underline{\mathcal{E}}''_{t\ell'}(\underline{r}) \, ds \qquad (7)$$

+In general the single index ℓ (or ℓ') is actually a double index ℓn (or $\ell' n'$) corresponding to radial and angular variations.

where ℓ and ℓ' correspond to $\mathcal{X}_\ell$-th and $\mathcal{X}_{\ell'}^*$ -th modes (eigenvalues) respectively.

$\underline{z}_o$ = unit vector in z-direction

N_ℓ = normalization constant

$\delta_{\ell\ell'}$ = 1, for $\mathcal{X}_\ell = \mathcal{X}_{\ell'}^*$
= 0, for $\mathcal{X}_\ell \neq \mathcal{X}_{\ell'}^*$

$\underline{\mathcal{E}}_{t\ell}$ and $\underline{\mathcal{E}}''_{t\ell'}$ are transverse components of $\underline{\mathcal{E}}_\ell$ and $\underline{\mathcal{E}}''_{\ell'}$ respectively. s is the cross section of the waveguide.

Since in the present problem the only source is a magnetic current, the total fields can be expressed in the following way, using the appropriate dyadic Green's functions (eqs. (16b) and (17a) in Appendix B):

$$\underline{E}(\underline{r}) = -\sum_{\ell} \int_0^{2\pi}\int_0^{b}\int_{-\infty}^{\infty} \frac{\underline{\mathcal{E}}_\ell(\underline{r})\underline{\mathcal{X}}''_\ell(\underline{r}') \cdot \underline{\theta}_0 I_m(\underline{r}')\, r'\, dz'\, dr'\, d\theta'}{2N_\ell} \tag{8}$$

and

$$\underline{H}(\underline{r}) = -\sum_{\ell} \int_0^{2\pi}\int_0^{b}\int_{-\infty}^{\infty} \frac{\underline{\mathcal{X}}_\ell(\underline{r})\underline{\mathcal{X}}''_\ell(\underline{r}') \cdot \underline{\theta}_0 I_m(\underline{r}')\, r'\, dz'\, dr'\, d\theta'}{2N_\ell} \tag{9}$$

$\underline{r}$ = observation position vector

$\underline{r}'$ = source position vector (i.e., primed coordinates refer to source) and the time dependence $e^{j\omega t}$ is suppressed everywhere.

Using eq. (1), the above two expressions can be reduced to the following forms

$$\underline{E}(\underline{r}) = -c \sum_{\ell} \frac{\mathcal{E}_{\ell}(\underline{r})}{2N_{\ell}} \int_0^{2\pi} \mathcal{H}''_{\theta_{\ell}}(c, \theta', 0)\, m(\theta')\, d\theta' \tag{10}$$

and

$$\underline{H}(\underline{r}) = -c \sum_{\ell} \frac{\mathcal{H}_{\ell}(\underline{r})}{2N_{\ell}} \int_0^{2\pi} \mathcal{H}''_{\theta_{\ell}}(c, \theta', 0)\, m(\theta')\, d\theta' \tag{11}$$

where $\mathcal{H}''_{\theta_{\ell}}(c, \theta', 0)$ is the θ-component of $\underline{\mathcal{H}}''_{\ell}(c, \theta', 0)$. If $\underline{\mathcal{E}}_{\ell}(\underline{r})$, $\underline{\mathcal{H}}_{\ell}(\underline{r})$ and $\underline{\mathcal{H}}''_{\ell}(\underline{r})$ can be expressed as (the single index ℓ is replaced by the double index in):

$$\left.\begin{aligned} \underline{\mathcal{E}}_{in}(\underline{r}) &= A_{1\,in}\, \underline{f}_{in}(\underline{r}) \\ \underline{\mathcal{H}}_{in}(\underline{r}) &= A_{1\,in}\, \underline{g}_{in}(\underline{r}) \\ \underline{\mathcal{H}}''_{in}(\underline{r}) &= A''_{in}\, \underline{g}''_{in}(\underline{r}) \end{aligned}\right\} \tag{12}$$

and moreover if

$$\underline{g}''_{in}(\underline{r}) = \underline{g}''_{in}(r, z)\, e^{-jn\theta}, \qquad \begin{aligned} i &= 1, 2, 3, \ldots \\ n &= 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{13}$$

then the above expressions (10) and (11) can be reduced further to the following

$$\underline{E}(\underline{r}) = -\frac{c}{2}\sum_{l,n}\left(\frac{A_{1\,ln}A''_{1ln}}{N_{ln}}\right)\underline{f}_{ln}(\underline{r})\,g''_{\theta ln}(c)\,\tilde{m}_n \tag{14}$$

$$\underline{H}(\underline{r}) = -\frac{c}{2}\sum_{l,n}\left(\frac{A_{1\,ln}A''_{1ln}}{N_{ln}}\right)\underline{g}_{ln}(\underline{r})\,g''_{\theta ln}(c)\,\tilde{m}_n \tag{15}$$

where

$$\tilde{m}_n = \int_0^{2\pi} e^{-jn\theta'}\,m(\theta')\,d\theta' \;, \tag{16a}$$

A_{1ln} and A''_{ln} are constants and $\underline{f}_{ln}(\underline{r})$, and $\underline{g}''_{ln}(\underline{r})$ are known functions.

If $m(\theta') = m$, a constant, then $n = 0$ $\left[\text{i.e.,}\ \frac{\partial}{\partial\theta} \equiv 0\right]$ and consequently

$$\tilde{m}_n = 2\pi m \tag{16b}$$

For $n = 0$, the above summation is no more than a single summation over l.

Now using (12) and the orthogonality relation

$$\iint_s \underline{\mathcal{E}}_{tln}(\underline{r}) \cdot \underline{\mathcal{H}}''_{tln}(\underline{r}) \times \underline{z}_o\,ds = N_{ln} \tag{17}$$

it can be shown easily that

$$\frac{A_{1ln}A''_{1ln}}{N_{ln}} = \frac{1}{\iint_s \underline{f}_{tln}(\underline{r}) \cdot \underline{g}''_{tln}(\underline{r}) \times \underline{z}_o\,ds} = \frac{1}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r})g''_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r})g''_{rln}(\underline{r})\right] r\,dr\,d\theta} \tag{18}$$

Now using the above relation (18) in the expressions (14) and (15), it can be shown that the complete fields have the following forms

$$\underline{E}(\underline{r}) = -\frac{c}{2} \sum_{l,n} \frac{\widetilde{m}_n \underline{f}_{ln}(\underline{r})\, g''_{\theta ln}(c)}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r}) g''_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r}) g''_{rln}(\underline{r}) \right] r dr d\theta} \tag{19}$$

$$\underline{H}(\underline{r}) = -\frac{c}{2} \sum_{l,n} \frac{\widetilde{m}_n \underline{g}_{ln}(\underline{r})\, g''_{\theta ln}(c)}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r}) g''_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r}) g''_{rln}(\underline{r}) \right] r dr d\theta} \tag{20}$$

Although the above two expressions (19) and (20) are valid for both dissipative and non-dissipative media, the relation between g_{ln} and g''_{ln} becomes simple in the case of non-dissipative medium. Thus in a non-dissipative medium (see Appendix B and [9]) it can be shown that

$$\begin{aligned} \underline{g}''_{ln} &= \underline{g}^*_{ln} = \text{complex conjugate of } \underline{g}_{ln} \\ \underline{f}''_{ln} &= \underline{f}^*_{ln} = \text{complex conjugate of } \underline{f}_{ln} \end{aligned} \tag{21}$$

Therefore, for non-dissipative medium, the total electromagnetic fields are given by the following expressions (using (21)):

$$\underline{E}(\underline{r}) = -\frac{c}{2}\sum_{l,n} \frac{\widetilde{m}_n \underline{f}_{ln}(\underline{r})\, g^*_{\theta ln}(c)}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r})g^*_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r})g^*_{rln}(\underline{r})\right] r\,dr\,d\theta} \tag{22}$$

$$\underline{H}(\underline{r}) = -\frac{c}{2}\sum_{l,n} \frac{\widetilde{m}_n \underline{g}_{ln}(\underline{r})\, g^*_{\theta ln}(c)}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r})g^*_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r})g^*_{rln}(\underline{r})\right] r\,dr\,d\theta} \tag{23}$$

It may be noted here that for non-dissipative medium the propagation wave number $\mathcal{X}_{ln}$ (eigenvalue) in the z-direction is a real number.

An alternative set of expressions for total electromagnetic fields which are particularly suitable for dissipative media (although valid for non-dissipative media also), can be obtained from (19) and (20) using the following transformations

$$\left.\begin{aligned}
\mathcal{X}^*_{ln} &= -\mathcal{X}_{ln}\\
\underline{f}''_{ln}(\underline{r}, \mathcal{X}_{ln}, \epsilon_{r\theta}, \mu_{r\theta}) &= -\underline{f}_{ln}(\underline{r}, -\mathcal{X}_{ln} - \epsilon_{r\theta}, -\mu_{r\theta}) = \underline{\widetilde{f}}_{ln}(\underline{r})\\
\underline{g}''_{ln}(\underline{r}, \mathcal{X}_{ln}, \epsilon_{r\theta}, \mu_{r\theta}) &= \underline{g}_{ln}(\underline{r}, -\mathcal{X}_{ln}, -\epsilon_{r\theta}, -\mu_{r\theta}) = \underline{\widetilde{g}}_{ln}(\underline{r})
\end{aligned}\right\} \tag{24}$$

Thus the total fields, which are particularly suitable for a dissipative medium, can be expressed in the following way (using (24)):

$$\underline{E}(\underline{r}) = \frac{c}{2} \sum_{l,n} \frac{\widetilde{m}_n \underline{f}_{ln}(\underline{r})\, \widetilde{g}_{\theta ln}(c)}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r})\widetilde{g}_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r})\widetilde{g}_{rln}(\underline{r}) \right] r dr d\theta} \tag{25}$$

$$\underline{H}(\underline{r}) = \frac{c}{2} \sum_{l,n} \frac{\widetilde{m}_n \underline{g}_{ln}(\underline{r})\, \widetilde{g}_{\theta ln}(c)}{\int_0^{2\pi}\int_0^b \left[f_{rln}(\underline{r})\widetilde{g}_{\theta ln}(\underline{r}) - f_{\theta ln}(\underline{r})\widetilde{g}_{rln}(\underline{r}) \right] r dr d\theta} \tag{26}$$

It should be pointed out here that for a dissipative medium, the propagation wave number χ_{ln} (eigenvalue) in the z-direction is complex.

Solutions of the Homogeneous (source-free) Maxwell's Equations

It has been demonstrated in Appendix A that for source-free and homogeneous (or for piecewise constant $\underset{\sim}{\mu}$ and $\underset{\sim}{\epsilon}$) medium the longitudinal (z-component) components of electric and magnetic fields obey the following two equations

$$\nabla_t^2 \mathcal{E}_z + \frac{\epsilon_z}{\epsilon_r \mu_r} a_1 \mathcal{E}_z = \frac{j\omega\mu_0 \mu_z \chi}{\epsilon_r \mu_r} a_3 \mathcal{H}_z \tag{27*}$$

$$\nabla_t^2 \mathcal{H}_z + \frac{\mu_z}{\epsilon_r \mu_r} a_1 \mathcal{H}_z = \frac{-j\omega\epsilon_0 \epsilon_z \chi}{\epsilon_r \mu_r} a_3 \mathcal{E}_z \tag{28}$$

* To simplify notation, indices are omitted from both field quantities and χ.

where $a_1' = k^2 \epsilon_r (\mu_r^2 - \mu'^2) - \mu_r \varkappa^2$

$$a_1 = k^2 \mu_r (\epsilon_r^2 - \epsilon'^2) - \epsilon_r \varkappa^2$$

$$a_3 = \mu_r \epsilon' + \mu' \epsilon_r$$

$$\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_r$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \epsilon'$$

$$\epsilon_{zz} = \epsilon_z$$

$$\mu_{rr} = \mu_{\theta\theta} = \mu_r$$

$$\mu_{r\theta} = \mu_{\theta r} = \mu'$$

$$\mu_{zz} = \mu_z$$

$$k^2 = \omega^2 \mu_o \epsilon_o$$

The transverse fields (i.e., r and θ components) can now be expressed (see Appendix A) in terms of $\mathcal{E}_z$ and $\mathcal{H}_z$ in the following manner

$$\underline{\mathcal{E}}_t = \frac{1}{p_1}\left[j\varkappa a_4 \nabla_t \mathcal{E}_z - \omega\mu_o a_2' \nabla_t \mathcal{H}_z\right] - \frac{z_o}{p_1} \times \left[k^2 \varkappa a_3 \nabla_t \mathcal{E}_z + j\omega\mu_o a_1' \nabla_t \mathcal{H}_z\right] \quad (29)$$

and

$$\underline{\mathcal{H}}_t = \frac{1}{p_1}\left[j\varkappa a_4 \nabla_t \mathcal{H}_z + \omega\epsilon_o a_2 \nabla_t \mathcal{E}_z\right] - \frac{z_o}{p_1} \times \left[k^2 \varkappa a_3 \nabla_t \mathcal{H}_z - j\omega\epsilon_o a_1 \nabla_t \mathcal{E}_z\right] \quad (30)$$

where $a_2' = k^2\epsilon'(\mu_r^2 - \mu'^2) + \mathcal{X}^2\mu'$

$a_2 = k^2\mu'(\epsilon_r^2 - \epsilon'^2) + \mathcal{X}^2\epsilon'$

$a_4 = k^2(\mu_r\epsilon_r + \mu'\epsilon') - \mathcal{X}^2$

$p_1 = k^4 a_3^2 - a_4^2$

It is observed here that if $a_3 = 0$, the two equations (27) and (28) become uncoupled.* Although this is a necessary condition that the conventional E-type and H-type modes separate, it is not sufficient. The sufficient condition depends on the boundary conditions. For example, if the medium completely fills a perfectly conducting waveguide, and if $a_3 = 0$, E-type and H-type modes can exist separately. But on the other hand, if there are two coaxial media, (the outer one may or may not be bounded by a perfect conductor) E-type and H-type modes can exist separately if and only if $\frac{\partial}{\partial\theta} = 0$ (with $a_3 = 0$). But if $\frac{\partial}{\partial\theta} \neq 0$, and even if $a_3 = 0$, in the above two coaxial media-system E-type and H-type modes cannot exist separately. A similar discussion for isotropic media where $a_3 = 0$, can be found in [16], sec. 11.6.

For the solution of (27) and (28) $\mathcal{E}_z$ and $\mathcal{H}_z$ can be eliminated yielding a single 4th degree equation in each of $\mathcal{E}_z$ and $\mathcal{H}_z$ which satisfy both of the second degree equations (27) and (28). If a choice is made such that $\phi = \mathcal{E}_z + j\alpha\mathcal{H}_z$,

* TE and TM modes also decouple, i.e., they can exist separately if there is a constant line source in the z-direction. In this case $\mathcal{X} = 0$.

then the equations (27) and (28) can be reduced to the following equation (for detail see Appendix A):

$$\nabla_t^2 \phi + \eta'^2 \phi = 0 \tag{31}$$

where

$$\frac{\epsilon_z}{\epsilon_r \mu_r}\left[a_1' - \omega \epsilon_o \chi a_3 \alpha\right] = \eta'^2 = \frac{\mu_z}{\epsilon_r \mu_r}\left[a_1 = \frac{\omega \mu_o \chi a_3}{\alpha}\right] \tag{32}$$

Solving equation (32) for α, one obtains

$$\alpha_{1,2} = \frac{\epsilon_z a_1' - \mu_z a_1 \mp \left[(a_1' \epsilon_z - a_1 \mu_z)^2 + 4k^2 \chi^2 a_3^2 \mu_z \epsilon_z\right]^{\frac{1}{2}}}{2 \omega \epsilon_o \epsilon_z a_3 \chi} \tag{33}$$

Therefore, the roots of η'^2 can also be expressed in the following way

$$\eta'^2_{1,2} = V \pm \sqrt{V^2 - U} \tag{34}$$

where

$$V = \frac{a_1' \epsilon_z + a_1 \mu_z}{2 \epsilon_r \mu_r} = \frac{\eta_1'^2 + \eta_2'^2}{2} \tag{35a}$$

and

$$\left.\begin{aligned} U &= \frac{\epsilon_z \mu_z}{\epsilon_r \mu_r}\left[a_4^2 - k^4 a_3^2\right] \\ &= -\frac{\epsilon_z \mu_z}{\mu_r \epsilon_r}\, p_1 = \eta_1'^2 \; \eta_2'^2 \end{aligned}\right\} \tag{35b}$$

Equation (31) has the following form in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \eta'^2 (r)\, \phi = 0 \tag{36}$$

where

$$\eta'^2 (r) = \eta'^2_{1,2} \quad \text{for } 0 \leqslant r \leqslant a \tag{37a}$$

$$= \eta^2 = k^2 \mu_2 \epsilon_2 - \chi^2, \quad \text{for } a \leqslant r \leqslant b \tag{37b}$$

The general solution of (36) in the region containing the origin can be written as

$$\phi_{1,2} = A_{1,2} J_n (\eta'_{1,2} r)\, e^{jn\theta}, \qquad n = 0, \pm 1, \pm 2 \ldots \quad \text{for } 0 \leqslant r \leqslant a \tag{38*}$$

and the solutions of (36) in the region $a \leqslant r \leqslant b$, which correspond to longitudinal fields, are given by

$$\mathcal{E}_z = \left[B_1'' J_n (\eta r) + C_1'' N_n (\eta r) \right] e^{jn\theta}, \quad a \leqslant r \leqslant b \tag{39*}$$

$$\mathcal{H}_z = \left[B_2'' J_n (\eta r) + C_2'' N_n (\eta r) \right] e^{jn\theta}, \quad a \leqslant r \leqslant b \tag{40*}$$

---

* A_1, A_2, B''_1, B''_2, C_1' and C''_2 are arbitrary constants which depend on n and $\eta'_{1,2}$ and η. If not clearly indicated, these constants and radial propagation wave numbers are understood to have the double index in.

where J_n and N_n are Bessel's functions of the 1 st kind and 2 nd kind of order n respectively.

Since $\phi_1 = \mathcal{E}_z + j\alpha_1 \mathcal{H}_z = A_1 J_n(\eta_1' r) e^{jn\theta}$, and

$$\phi_2 = \mathcal{E}_z + j\alpha_2 \mathcal{H}_z = A_2 J_n(\eta_2' r) e^{jn\theta},$$

it is easy to verify that

$$\mathcal{E}_z = \frac{e^{jn\theta}}{\alpha_1 - \alpha_2}\left[\alpha_1 A_2 J_n(\eta_2' r) - a_2 A_1 J_n(\eta_1' r)\right], \text{ for } 0 \leqslant r \leqslant a, \tag{41}$$

and

$$\mathcal{H}_z = \frac{je^{jn\theta}}{\alpha_1 - \alpha_2}\left[A_2 J_n(\eta_2' r) - A_1 J_n(\eta_1' r)\right], \text{ for } 0 \leqslant r \leqslant a, \tag{42}$$

Since the boundary condition requires that

$$\mathcal{E}_z(r)\Big|_{r=b} = 0 = \frac{\partial}{\partial r}\mathcal{H}_z(r)\Big|_{r=b},$$

the equations (39) and (40) can be rewritten in the following way

$$\mathcal{E}_z = B_1 \mathcal{J}_n(r) e^{jn\theta}, \qquad \text{for } a \leqslant r \leqslant b \tag{43}$$

$$\mathcal{H}_z = B_2 G_n(r) e^{jn\theta}, \qquad \text{for } a \leqslant r \leqslant b \tag{44}$$

where

$$\mathcal{J}_n(r) = J_n(\eta b)\, N_n(\eta r) - J_n(\eta r)\, N_n(\eta b) \tag{45a}$$

$$G_n(r) = J_n(\eta r)\, N_n'(\eta b) - J_n'(\eta b)\, N_n(\eta r) \tag{45b}$$

$$B_1 = -\frac{B_1''}{N_n(\eta b)} \tag{45c}$$

$$B_2 = \frac{B_2''}{N_n'(\eta b)} \tag{45d}$$

$$N_n'(\eta b) = \left.\frac{dN_n(\eta r)}{d(\eta r)}\right|_{r=b} \tag{45e}$$

$$J_n'(\eta b) = \left.\frac{dJ_n(\eta r)}{d(\eta r)}\right|_{r=b} \tag{45f}$$

Now using equations (41) to (44), in the relations (29) and (30), the transverse components of the source-free solutions of Maxwell's equation (5) can be expressed as

$$\begin{aligned}\mathcal{E}_r = \frac{-j\, e^{jn\theta}}{p_1(\alpha_1-\alpha_2)\,\epsilon_z} \Bigg[& A_1\left\{\epsilon_z \eta_1' R\, J_n'(\eta_1' r) - n\omega\mu_o \epsilon_r \mu_r \eta_2'^2 \frac{J_n(\eta_1' r)}{r}\right\} \\ & + A_2\left\{\epsilon_z \eta_2' T\, J_n'(\eta_2' r) + n\omega\mu_o \epsilon_r \mu_r \eta_1'^2 \frac{J_n(\eta_2' r)}{r}\right\}\Bigg],\end{aligned} \tag{46a}$$

for $0 \leqslant r \leqslant a$

$$\mathcal{E}_r = \frac{n\omega\mu_o\mu_2}{\eta^2 r} B_2 e^{jn\theta} G_n(r) - \frac{j\mathcal{K}}{\eta} B_1 e^{jn\theta} C_n(r), \qquad \text{for } a \leqslant r \leqslant b \tag{46b}$$

$$\mathcal{E}_\theta = \frac{e^{jn\theta}}{p_1(\alpha_1-\alpha_2)\epsilon_z}\Bigg[A_1\Bigg\{-\omega\mu_o\mu_r\epsilon_r\eta_1'\eta_2'^2 J_n'(\eta_1' r) + n\epsilon_z R\frac{J_n(\eta_1' r)}{r}\Bigg\}$$

$$+ A_2\Bigg\{\omega\mu_o\epsilon_r\mu_r\eta_1'^2\eta_2' J_n(\eta_2' r) + n\epsilon_z T\frac{J_n(\eta_2' r)}{r}\Bigg\}\Bigg] \text{for } 0 \leqslant r \leqslant a \tag{47a}$$

$$\mathcal{E}_\theta = \frac{-j\omega\mu_o\mu_2}{\eta} B_2 e^{jn\theta} S_n(r) + \frac{n\mathcal{K}}{\eta^2} B_1 e^{jn\theta}\frac{\mathcal{J}_n(r)}{r}, \qquad \text{for } a \leqslant r \leqslant b \tag{47b}$$

$$\mathcal{H}_r = \frac{-e^{jn\theta}}{p_1(\alpha_1-\alpha_2)}\Bigg[A_1\Bigg\{\eta_1' R' J_n'(\eta_1' r) + \frac{n\eta_2'^2 M\epsilon_r^2\mu_r^2 J_n(\eta_1' r)}{\mathcal{K}\epsilon_z\mu_z a_3 r}\Bigg\}$$

$$- A_2\Bigg\{\eta_2' T' J_n'(\eta_2' r) + \frac{n\eta_1'^2 S\epsilon_r^2\mu_r^2 J_n(\eta_2' r)}{\mathcal{K}\epsilon_z\mu_z a_3 r}\Bigg\}\Bigg] \text{for } 0 \leqslant r \leqslant a \tag{48a}$$

$$\mathcal{H}_r = \frac{-n\omega\epsilon_o\epsilon_2}{\eta^2 r} B_1 e^{jn\theta}\mathcal{J}_n(r) + \frac{j\mathcal{K}B_2 e^{jn\theta}}{\eta} S_n(r), \qquad \text{for } a \leqslant r \leqslant b \tag{48b}$$

$$\mathcal{H}_\theta = \frac{-je^{jn\theta}}{p_1(\alpha_1-\alpha_2)}\Bigg[A_1\Bigg\{\frac{\epsilon_r^2\mu_r^2\eta_1'\eta_2'^2 M}{\mathcal{K}\epsilon_z\mu_z a_3} J_n'(\eta_1' r) + \frac{nR'}{r} J_n(\eta_1' r)\Bigg\}$$

$$- A_2\Bigg\{\frac{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2' S}{\mathcal{K}\epsilon_z\mu_z a_3} J_n'(\eta_2' r) + \frac{nT'}{r} J_n(\eta_2' r)\Bigg\}\Bigg], \text{for } 0 \leqslant r \leqslant a \tag{49a}$$

$$\mathcal{H}_\theta = \frac{-j\omega\epsilon_o\epsilon_2 B_1}{\eta} e^{jn\theta} C_n(r) + \frac{n\mathcal{K}}{\eta^2 r} B_2 e^{jn\theta} G_n(r), \qquad \text{for } a \leqslant r \leqslant b \tag{49b}$$

where

$$S = \frac{\epsilon_z a_1'}{\mu_r \epsilon_r} - \eta_1'^2 \tag{50a}$$

$$M = \frac{\epsilon_z a_1'}{\mu_r \epsilon_r} - \eta_2'^2 \tag{50b}$$

$$R = \chi a_4 \alpha_2 - \omega \mu_o a_2' = \frac{a_4 \epsilon_r \mu_r M - k^2 \epsilon_z a_3 a_2'}{\omega \epsilon_o \epsilon_z a_3} \tag{50c}$$

$$T = \omega \mu_o a_2' - \chi a_4 \alpha_1 = \frac{k^2 \epsilon_z a_3 a_2' - a_4 \mu_r \epsilon_r S}{\omega \epsilon_o \epsilon_z a_3} \tag{50d}$$

$$R' = \omega \epsilon_o a_2 \alpha_2 - \chi a_4 = \frac{M a_2 \mu_r \epsilon_r - \chi^2 a_3 a_4 \epsilon_z}{\chi a_3 \epsilon_z} \tag{50e}$$

$$T' = \omega \epsilon_o a_2 \alpha_1 - \chi a_4 = \frac{S a_2 \mu_r \epsilon_r - \chi^2 a_3 a_4 \epsilon_z}{\chi a_3 \epsilon_z} \tag{50f}$$

$$S_n(r) = J_n'(\eta b) N_n'(\eta r) - J_n'(\eta r) N_n'(\eta b) = -\frac{1}{\eta} \frac{d}{dr} G_n(r) \tag{50g}$$

$$C_n(r) = J_n(\eta b) N_n'(\eta r) - J_n'(\eta r) N_n(\eta b) = \frac{1}{\eta} \frac{d}{dr} \mathcal{J}_n(r) \tag{50h}$$

Dispersion Relation

Since in the present problem the eigenvalues χ are discrete, the boundary conditions satisfied by the total fields (i.e., the fields due to the presence of the source) are the same as those satisfied by any individual fields (i.e., source-free field). Therefore, the following boundary conditions can be imposed upon the source-free fields, $\mathcal{E}_z$, $\mathcal{H}_z$, $\mathcal{E}_\theta$ and $\mathcal{H}_\theta$:

$$\mathcal{E}_z(a^-) = \mathcal{E}_z(a^+) \qquad \text{at } r = a \tag{51a}$$

$$\mathcal{E}_\theta(a^-) = \mathcal{E}_\theta(a^+) \qquad \text{at } r = a \tag{51b}$$

$$\mathcal{E}_z(b) = 0 \qquad \text{at } r = b \tag{51c}$$

$$\mathcal{E}_\theta(b) = 0 \qquad \text{at } r = b \tag{51d}$$

$$\mathcal{H}_z(a^-) = \mathcal{H}_z(a^+) \qquad \text{at } r = a \tag{51e}$$

$$\mathcal{H}_\theta(a^-) = \mathcal{H}_\theta(a^+) \qquad \text{at } r = a \tag{51f}$$

The constructions of $\mathcal{E}_z$ and $\mathcal{E}_\theta$ in $a \lessgtr r \lessgtr b$ are made in such a way that the boundary conditions (51c) and (51d) are now automatically satisfied, since $\mathcal{J}_n(b)=0=S_n(b)$. If the remaining boundary conditions in (51) are imposed upon the source-free fields expressed in the equations (41) to (44) and (47) and (49), the following relations among the arbitrary coefficients A_1, A_2, B_1 and B_2 are obtained:

$$\frac{1}{\alpha_1 - \alpha_2}\Big[\alpha_1 A_2 J_n(\eta_2' a) - \alpha_2 A_1 J_n(\eta_1' a)\Big] = B_1 \mathcal{J}_n(a), \tag{52a}$$

$$\frac{1}{p_1(\alpha_1 - \alpha_2)\epsilon_z}\Bigg[A_1\left\{-\omega\mu_0\mu_r\epsilon_r\eta_1'\eta_2'^2 J_n'(\eta_1' a) + n\epsilon_z R\,\frac{J_n(\eta_1' a)}{a}\right\}$$
$$+A_2\left\{\omega\mu_0\epsilon_r\mu_r\eta_1'^2\eta_2'\; J_n'(\eta_2' a) + n\epsilon_z\, T\,\frac{J_n(\eta_2' a)}{a}\right\}\Bigg] \tag{52b}$$
$$= \frac{-j\mu_0\omega\mu_2}{\eta}\, B_2\, S_n(a) + \frac{n\mathcal{K}}{\eta^2}\, B_1\,\frac{\mathcal{J}_n(a)}{a},$$

$$\frac{j}{\alpha_1 - \alpha_2}\left[A_2 J_n(\eta_2' a) - A_1 J_n(\eta_1' a)\right] = B_2 G_n(a), \qquad (52c)$$

$$\frac{-j}{p_1(\alpha_1-\alpha_2)}\left[A_1\left\{\frac{\epsilon_r^2\mu_r^2\eta_1'\eta_2'^2 M}{\chi\epsilon_z\mu_z a_3} J_n'(\eta_1' a) + \frac{nR'}{a} J_n(\eta_1' a)\right\} - A_2\left\{\frac{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2' S}{\chi\epsilon_z\mu_z a_3} J_n'(\eta_2' a) + \frac{nT'}{a} J_n(\eta_2' a)\right\}\right]$$

$$= \frac{-j\omega\epsilon_0\epsilon_2}{\eta} B_1 C_n(a) + \frac{n\chi}{\eta^2 a} B_2 G_n(a) \quad . \qquad (52d)$$

Non-trivial solutions for the constants A_1, A_2, B_1, and B_2 exist, if and only if the determinant of the coefficients of these constants appearing in equations (52) vanishes. The vanishing condition of the determinant gives the characteristic equation (or dispersion relation). Instead of calculating the determinant of the coefficients and then equating it to zero, one can also eliminate B_1 and B_2 from (52) and from the remaining equations it is easy to obtain two independent values of the ratio A_2/A_1. Now equating these two values of A_2/A_1, one obtains the desired dispersion relation expressed in the following:

$$\frac{J_n(\eta_1'a)\left\{n\varkappa^2k^2\eta_1'^2a_3^2\epsilon_z\mathcal{J}_n(a)+n\epsilon_r\mu_r\eta^2M\mathcal{J}_n(a)\left[a_4\epsilon_z-\epsilon_r\eta_1'^2\right]-k^2a\eta\epsilon_2\eta_1'^2a_3\mu_r\epsilon_rMC_n(a)\right\}+a\eta^2k^2a_3\epsilon_z\mu_r\epsilon_r\eta_1'M\mathcal{J}_n(a)J_n'(\eta_1'a)}{J_n(\eta_2'a)\left\{n\varkappa^2k^2\eta_2'^2a_3^2\epsilon_z\mathcal{J}_n(a)+n\epsilon_r\mu_r\eta^2S\mathcal{J}_n(a)\left[a_4\epsilon_z-\epsilon_r\eta_2'^2\right]-k^2a\eta\epsilon_2\eta_2'^2a_3\mu_r\epsilon_rSC_n(a)\right\}+a\eta^2k^2a_3\epsilon_z\mu_r\epsilon_r\eta_2'S\mathcal{J}_n(a)J_n'(\eta_2'a)} \tag{53}$$

$$=\frac{J_n(\eta_1'a)\left\{n\epsilon_r\mu_r\eta^2MG_n(a)\left[\mu_r\epsilon'\eta_1'^2-a_2'\epsilon_z\right]+n\eta_1'^2\varkappa^2a_3\epsilon_r\mu_rMG_n(a)+k^2a\eta\eta_1'^2\varkappa^2a_3^2\epsilon_z\mu_2S_n(a)\right\}+\eta^2ak^2\varkappa^2\eta_1'^2a_3^2\epsilon_z\mu_zG_n(a)J_n'(\eta_1'a)}{J_n(\eta_2'a)\left\{n\epsilon_r\mu_r\eta^2SG_n(a)\left[\mu_r\epsilon'\eta_2'^2-a_2'\epsilon_z\right]+n\eta_2'^2\varkappa^2a_3\epsilon_r\mu_rSG_n(a)+k^2a\eta\eta_2'^2\varkappa^2a_3^2\epsilon_z\mu_2S_n(a)\right\}+\eta^2ak^2\varkappa^2\eta_2'^2a_3^2\epsilon_z\mu_zG_n(a)J_n'(\eta_2'a)}$$

The above relation can also be rewritten in the following way

$$\frac{J_n(\eta_1'a)\left\{n\mu_z\eta^2\mathcal{J}_n(a)\left[a_4\epsilon_z-\epsilon_r\eta_1'^2\right]-n\epsilon_r\mu_r\eta_1'^2S\mathcal{J}_n(a)-k^2a\eta\epsilon_2\eta_1'^2a_3\mu_zC_n(a)\right\}+a\eta^2k^2a_3\epsilon_z\mu_z\eta_1'\mathcal{J}_n(a)J_n'(\eta_1'a)}{J_n(\eta_2'a)\left\{n\mu_z\eta^2\mathcal{J}_n(a)\left[a_4\epsilon_z-\epsilon_r\eta_2'^2\right]-n\epsilon_r\mu_r\eta_2'^2M\mathcal{J}_n(a)-k^2a\eta\epsilon_2\eta_2'^2a_3\mu_zC_n(a)\right\}+a\eta^2k^2a_3\epsilon_z\mu_z\eta_2'^2\mathcal{J}_n(a)J_n'(\eta_2'a)} \tag{54*}$$

$$=\frac{J_n(\eta_1'a)\left\{n\varkappa^2\eta_1'^2a_3\mu_zG_n(a)+n\mu_z\eta^2\left[\mu_r\epsilon'\eta_1'^2-a_2\epsilon_z\right]G_n(a)-\eta a\eta_1'^2S\mu_2\epsilon_r\mu_rS_n(a)\right\}-\eta^2a\eta_1'S\mu_z\epsilon_r\mu_rG_n(a)J_n'(\eta_1'a)}{J_n(\eta_2'a)\left\{n\varkappa^2\eta_2'^2a_3\mu_zG_n(a)+n\mu_z\eta^2\left[\mu_r\epsilon'\eta_2'^2-a_2'\epsilon_z\right]G_n(a)-\eta a\eta_2'^2M\mu_2\epsilon_r\mu_rS_n(a)\right\}-\eta^2a\eta_2'M\mu_z\epsilon_r\mu_rG_n(a)J_n'(\eta_2'a)}$$

* When $\mu_2 = 1 = \epsilon_2$, the dispersion relation (54) agrees with that of Van Trier [1].

It should be noted here that to obtain the expressions (53) and (54), some useful relations (tabulated in Appendix A) have been used. These relations will be used frequently in subsequent derivations.

When the ring-source $m(\theta)$ is constant, n=0 (i.e., $\frac{\partial}{\partial\theta} \equiv 0$). In this case it can be shown that the dispersion relation simplifies to

$$\frac{S\left\{\eta\mu_z G_0(a)J_1(\eta_1' a)-\eta_1'\mu_2 S_0(a)J_0(\eta_1' a)\right\}}{M\left\{\eta\mu_z G_0(a)J_1(\eta_2' a)-\eta_2'\mu_2 S_0(a)J_0(\eta_2' a)\right\}} = \frac{\eta\epsilon_z \mathcal{S}_0(a)J_1(\eta_1' a)+\epsilon_2\eta_1' C_0(a)J_0(\eta_1' a)}{\eta\epsilon_z \mathcal{S}_0(a)J_1(\eta_2' a)+\epsilon_2\eta_2' C_0(a)J_0(\eta_2' a)} \tag{55}$$

Alternatively, equation (55) can also be written in the following manner:

$$\begin{aligned}
&\frac{(\eta_2'^2-\eta_1'^2)}{\eta_1'\eta_2'}\, G_0(a)\mathcal{S}_0(a)J_1(\eta_1' a)\,J_1(\eta_2' a) - \frac{\mu_2\epsilon_2(\eta_2'^2-\eta_1'^2)}{\epsilon_z\mu_z\eta^2}\, C_0(a)S_0(a)J_0(\eta_1' a)J_0(\eta_2' a) \\
&\qquad - \frac{J_0(\eta_1' a)J_1(\eta_2' a)}{\epsilon_z\mu_z\eta\eta_2'}\left[\epsilon_2\mu_z M\, C_0(a)G_0(a)+\mu_2\epsilon_z S\mathcal{S}_0(a)S_0(a)\right] \\
&\qquad + \frac{J_0(\eta_2' a)J_1(\eta_1' a)}{\epsilon_z\mu_z\eta\eta_1'}\left[\epsilon_2\mu_z S\, C_0(a)G_0(a)+\mu_2\epsilon_z M\mathcal{S}_0(a)S_0(a)\right] = 0 \quad .
\end{aligned} \tag{56}$$

A number of dispersion relations for various special cases has been developed in Appendix C.

The solution of the dispersion relation (54) together with the relations (34) and (37b) gives an infinite number of discrete values of η_1', η_2', η (and hence $\varkappa$ also). The radial wave numbers η_1' and η_2' can also be called eigenvalues in the domain $0 \leq r \leq a$, and similarly η is the eigenvalue in the domain $a \leq r \leq b$, of the differential equation (36) together with the appropriate boundary conditions stated in (51).

Expressions for the Constants A_2, B_1, and B_2 in Terms of A_1

In the foregoing derivation of the dispersion relation (53), the following two expressions for the ratio A_2/A_1 were equated:

$$\frac{A_2}{A_1} = \frac{\eta_2'^2 M\left[J_n(\eta_1' a)\left\{n\mu_z \eta^2 \mathcal{J}_n(a)(a_4\epsilon_z - \epsilon_r \eta_1'^2) - n\epsilon_r \mu_r \eta_1'^2 S \mathcal{J}_n(a) - k^2 a\eta\epsilon_2 \eta_1'^2 a_3 \mu_z C_n(a)\right\} + a\eta^2 k^2 a_3 \epsilon_z \mu_z \eta_1' \mathcal{J}_n(a) J_n'(\eta_1' a)\right]}{\eta_1'^2 S\left[J_n(\eta_2' a)\left\{n\mu_z \eta^2 \mathcal{J}_n(a)(a_4\epsilon_z - \epsilon_r \eta_2'^2) - n\epsilon_r \mu_r \eta_2'^2 M \mathcal{J}_n(a) - k^2 a\eta\epsilon_2 \eta_2'^2 a_3 \mu_z C_n(a)\right\} + a\eta^2 k^2 a_3 \epsilon_z \mu_z \eta_2' \mathcal{J}_n(a) J_n'(\eta_2' a)\right]} \tag{57}$$

$$\frac{A_2}{A_1} = \frac{\eta_2'^2 M\left[J_n(\eta_1' a)\left\{n\kappa^2 \eta_1'^2 a_3 \mu_z G_n(a) + n\mu_z \eta^2(\mu_r \epsilon' \eta_1'^2 - a_2'\epsilon_z) G_n(a) - \eta a \eta_1'^2 S \mu_2 \epsilon_r \mu_r S_n(a)\right\} - \eta^2 a \eta_1' S \mu_z \epsilon_r \mu_r G_n(a) J_n'(\eta_1' a)\right]}{\eta_1'^2 S\left[J_n(\eta_2' a)\left\{n\kappa^2 \eta_2'^2 a_3 \mu_z G_n(a) + n\mu_z \eta^2(\mu_r \epsilon' \eta_2'^2 - a_2'\epsilon_z) G_n(a) - \eta a \eta_2'^2 M \mu_2 \epsilon_r \mu_r S_n(a)\right\} - \eta^2 a \eta_2' M \mu_z \epsilon_r \mu_r G_n(a) J_n'(\eta_2' a)\right]} \tag{58}$$

$$\text{Let } A_2/A_1 = \xi_1, \text{ or } A_2 = A_1 \xi_1 \tag{59}$$

where either the value (57) or (58) may be assigned to ξ_1, since they are equivalent through the dispersion relation (53).

Now using (59) and the relations (52a), (52c) of this chapter and (38(4)), (38(10)), and (38(11)) of Appendix A, one can express B_1 and B_2 in the following manner:

$$B_1 = A_1 \xi_2 \tag{60}$$

$$B_2 = A_1 \xi_3 \tag{61}$$

where

$$\xi_2 = \frac{S \xi_1 J_n(\eta_2' a) - M J_n(\eta_1' a)}{(\eta_2'^2 - \eta_1'^2) \mathcal{J}_n(a)} \tag{62}$$

$$\xi_3 = \frac{j\omega \epsilon_o \ell \epsilon_z a_3}{(\eta_2'^2 - \eta_1'^2)\epsilon_r \mu_r G_n(a)} \left[\xi_1 J_n(\eta_2' a) - J_n(\eta_1' a) \right] \tag{63}$$

<u>Expressions for Source-Free Fields in Terms of Only One Unknown Constant A_1</u>

Since all the unknown coefficients A_2, B_1 and B_2 are now expressible in terms of the only one unknown A_1, the source-free fields can be written in the following way ($e^{-j(\ell z - \omega t)}$ is assumed to multiply all the expressions for the fields):

$$\mathcal{E}_z = \frac{A_1 e^{jn\theta}}{(\eta_2'^2 - \eta_1'^2)} \left[S \xi_1 J_n(\eta_2' r) - M J_n(\eta_1' r) \right] = A_1 f_z, \text{ for } 0 \leq r \leq a \tag{64a}$$

$$\mathcal{E}_z = A_1 \xi_2 e^{jn\theta} \mathcal{J}_n(r) = A_1 f_z, \text{ for } a \leq r \leq b, \tag{64b}$$

$$\mathcal{E}_r = \frac{j\epsilon_z\mu_z\omega\epsilon_0\mathcal{K}a_3e^{jn\theta}A_1}{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2'^2(\eta_2'^2-\eta_1'^2)}\left[\left\{\epsilon_z\eta_1' R J_n'(\eta_1' r) - n\omega\mu_0\epsilon_r\mu_r\eta_2'^2\frac{J_n(\eta_1' r)}{r}\right\}\right.$$

(65a)

$$\left.+\xi_1\left\{\eta_2' T\epsilon_z J_n'(\eta_2' r) + n\omega\mu_0\epsilon_r\mu_r\eta_1'^2\frac{J_n(\eta_2' r)}{r}\right\}\right] = A_1 f_r, \ldots \text{ for } 0\leq r\leq a$$

$$\mathcal{E}_r = A_1 e^{jn\theta}\left[\frac{n\omega\mu_0\mu_2}{\eta^2 r}\xi_3(G_n(r) - \frac{j\mathcal{K}}{\eta}\xi_2 C_n(r)\right] = A_1 f_r, \ldots \text{ for } a\leq r\leq b \quad (65b)$$

$$\mathcal{E}_\theta = -\frac{\epsilon_z\mu_z\omega\epsilon_0\mathcal{K}a_3e^{jn\theta}A_1}{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2'^2(\eta_2'^2-\eta_1'^2)}\left[\left\{-\omega\mu_0\mu_r\epsilon_r\eta_1'\eta_2'^2 J_n'(\eta_1' r) + n\epsilon_z R\frac{J_n(\eta_1' r)}{r}\right\}\right.$$

(66a)

$$\left.+\xi_1\left\{\omega\mu_0\epsilon_r\mu_r\eta_1'^2\eta_2' J_n'(\eta_2' r) + n\epsilon_z T\frac{J_n(\eta_2' r)}{r}\right\}\right] = A_1 f_\theta, \text{ for } 0\leq r\leq a,$$

$$\mathcal{E}_\theta = A_1 e^{jn\theta}\left[\frac{-j\omega\mu_0\mu_2}{\eta}\xi_3 S_n(r) + \frac{n\mathcal{K}\xi_2}{\eta^2 r}\mathcal{J}_n(r)\right] = A_1 f_\theta, \text{ for } a\leq r\leq b, \quad (66b)$$

$$\mathcal{H}_z = \frac{j\omega\epsilon_0\mathcal{K}a_3\epsilon_z A_1 e^{jn\theta}}{\mu_r\epsilon_r(\eta_2'^2-\eta_1'^2)}\left[\xi_1 J_n(\eta_2' r) - J_n(\eta_1' r)\right] = A_1 g_z, \text{ for } 0\leq r\leq a, \quad (67a)$$

$$\mathcal{H}_z = A_1\xi_3 e^{jn\theta} G_n(r) = A_1 g_z, \text{ for } a\leq r\leq b, \quad (67b)$$

$$\mathcal{H}_r = \frac{\omega\epsilon_0\epsilon_z A_1 e^{jn\theta}}{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2'^2(\eta_2'^2-\eta_1'^2)}\left[\left\{\eta_1' R\varkappa\epsilon_z\mu_z a_3 J_n'(\eta_1' r) + n\eta_2'^2 M\epsilon_r^2\mu_r^2 \frac{J_n(\eta_1' r)}{r}\right\}\right. \tag{68a}$$

$$\left. - \xi_1\left\{\eta_2' T'\varkappa\epsilon_z\mu_z a_3 J_n'(\eta_2' r) + n\eta_1'^2 S\epsilon_r^2\mu_r^2 \frac{J_n(\eta_2' r)}{r}\right\}\right] = A_1 g_r, \quad \text{for } 0 \le r \le a.$$

$$\mathcal{H}_r = A_1 e^{jn\theta}\left[-\frac{n\omega\epsilon_0\epsilon_2}{\eta^2 r}\xi_2\mathcal{J}_n(r) + \frac{j\varkappa\xi_3}{\eta}S_n(r)\right] = A_1 g_r, \quad \text{for } a \le r \le b, \tag{68b}$$

$$\mathcal{H}_\theta = \frac{j\omega\epsilon_0\epsilon_z A_1 e^{jn\theta}}{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2'^2(\eta_2'^2-\eta_1'^2)}\left[\left\{\epsilon_r^2\mu_r^2\eta_1'\eta_2'^2 M J_n'(\eta_1' r) + nR'\varkappa\epsilon_z\mu_z a_3 \frac{J_n(\eta_1' r)}{r}\right\}\right. \tag{69a}$$

$$\left. - \xi_1\left\{\epsilon_r^2\mu_r^2\eta_1'^2\eta_2' S J_n'(\eta_2' r) + nT'\varkappa\epsilon_z\mu_z a_3 \frac{J_n(\eta_2' r)}{r}\right\}\right] = A_1 g_\theta, \quad \text{for } 0 \le r \le a$$

$$\mathcal{H}_\theta = A_1 e^{jn\theta}\left[\frac{-j\omega\epsilon_0\epsilon_2\xi_2}{\eta}C_n(r) + \frac{n\varkappa\xi_3}{\eta^2 r}G_n(r)\right] = A_1 g_\theta, \quad \text{for } a \le r \le b, \tag{69b}$$

Determination of the Constant A_1

Determination of the unknown constant A_1 depends on the orthogonality condition satisfied by the source-free fields. Since orthogonality conditions are different* for a dissipative and a non-dissipative medium, the constant A_1 will also

* Although the forms of the orthogonality relation given in (22) of Appendix B are valid for both dissipative and non-dissipative media, they are particularly suitable for dissipative media.

be different for the above-mentioned two media. Here A_1 will be calculated for non-dissipative medium only. For dissipative media the corresponding A_1 can be calculated using equation (18) together with the transformations given in (24). The primary aim here is to calculate the total fields with amplitudes due to a given source. Therefore, to calculate total fields it is only necessary to find the ratio $\frac{|A_{1\,\ell n}|^2}{N_{\ell n}}$ for any mode (eigenvalue) ℓn. This ratio can be determined from the knowledge of source-free fields, as suggested in equation (18) together with (21). In other words, the ratio $\frac{|A_{1\,\ell n}|^2}{N_{\ell n}}$ is given by

$$\frac{|A_{1\,\ell n}|^2}{N_{\ell n}} \qquad \frac{1}{\int_0^{2\pi} d\theta \int_0^b \left[f_{r\ell n}(\underline{r})\, g^*_{\theta\ell n}(\underline{r}) - f_{\theta\ell n}(\underline{r})\, g^*_{r\ell n}(\underline{r})\right] r \; dr} \tag{70}$$

The above integral in (70) can be expressed in the following compact form

$$\frac{N_{\ell n}}{|A_{1\,\ell n}|^2} = \int_0^{2\pi} d\theta \int_0^b \left[f_{r\ell n}(\underline{r})\, g^*_{\theta\ell n}(\underline{r}) - f_{\theta\ell n}(\underline{r})\, g^*_{r\ell n}(\underline{r})\right] r\, dr = 2\pi \sum_{\ell=1}^{32} F_{\ell n \ell} I_{\ell n \ell} \tag{71}$$

where

$$F_{\ell n1} = L_{\ell n}\epsilon_r\mu_r \left|\eta'_{1\ell n}\right|^2 \epsilon_z \left[\epsilon_r\mu_r \overset{*}{\eta}{}'^2_{2\ell n} \overset{*}{M}_{\ell n} \overset{*}{R}_{\ell n} - \omega\mu_o \eta'^2_{2\ell n} \overset{*}{R}{}'_{\ell n} \chi_{\ell n}\mu_z a_3\right] \tag{72-1}$$

$$F_{\ell n2} = L_{\ell n}\epsilon_r\mu_r \eta'_{1\ell n} \overset{*}{\eta}{}'_{2\ell n} \overset{*}{\xi}_{1\ell n} \epsilon_z \left[\omega\mu_o \eta'^2_{2\ell n} \overset{*}{T}{}'_{\ell n} \chi_{\ell n}\mu_z a_3 - R_{\ell n}\epsilon_r\mu_r \overset{*}{\eta}{}'^2_{1\ell n} \overset{*}{S}_{\ell n}\right] \tag{72-2}$$

$$F_{in3} = L_{in}\epsilon_r\mu_r\epsilon_z\overset{*}{\eta}'_{1in}\eta'_{2in}\xi_{1in}\left[T_{in}\epsilon_r\mu_r\overset{*}{\eta}'^2_{2in}\overset{*}{M}_{in} + \omega\mu_o\eta'^2_{1in}\overset{*}{R}'_{in}\mathcal{K}_{in}\mu_z a_3\right] \tag{72-3}$$

$$F_{in4} = -L_{in}\epsilon_r\mu_r\epsilon_z\left|\eta'_{2in}\right|^2\left|\xi_{1in}\right|^2\left[\omega\mu_o\eta'^2_{1in}\overset{*}{T}'_{in}\mathcal{K}_{in}\mu_z a_3 + \epsilon_r\mu_r T_{in}\overset{*}{\eta}'^2_{1in}\overset{*}{S}_{in}\right] \tag{72-4}$$

$$F_{in5} = \frac{\omega\mathcal{K}_{in}}{\left|\eta_{in}\right|^2}\left[\epsilon_o\epsilon_2\left|\xi_{2in}\right|^2\left|N_n(\eta_{in}b)\right|^2 + \mu_o\mu_2\left|\xi_{3in}\right|^2\left|N'_n(\eta_{in}b)\right|^2\right] \tag{72-5}$$

$$F_{in6} = -\frac{\omega\mathcal{K}_{in}}{\left|\eta_{in}\right|^2}\left[\epsilon_o\epsilon_2\left|\xi_{2in}\right|^2\overset{*}{J}_n(\eta_{in}b)\,N_n(\eta_{in}b) + \mu_o\mu_2\left|\xi_{3in}\right|^2 N'_n(\eta_{in}b)\overset{*}{J}'_n(\eta_{in}b)\right] \tag{72-6}$$

$$F_{in7} = -\frac{\omega\mathcal{K}_{in}}{\left|\eta_{in}\right|^2}\left[\epsilon_o\epsilon_2\left|\xi_{2in}\right|^2 J_n(\eta_{in}b)\overset{*}{N}_n(\eta_{in}b) + \mu_o\mu_2\left|\xi_{3in}\right|^2\overset{*}{N}'_n(\eta_{in}b)J'_n(\eta_{in}b)\right] \tag{72-7}$$

$$F_{in8} = \frac{\omega\mathcal{K}_{in}}{\left|\eta_{in}\right|^2}\left[\epsilon_o\epsilon_2\left|\xi_{2in}\right|^2\left|J_n(\eta_{in}b)\right|^2 + \mu_o\mu_2\left|\xi_{3in}\right|^2\left|J'_n(\eta_{in}b)\right|^2\right] \tag{72-8}$$

$$F_{in9} = nL_{in}\eta'_{1in}\left[\epsilon_z^2\mu_z a_3\mathcal{K}_{in}\overset{*}{R}'_{in}R_{in} - \omega\mu_o\mu_r^3\epsilon_r^3\left|\eta'^2_{2in}\right|^2\overset{*}{M}_{in}\right] \tag{72-9}$$

$$F_{in10} = nL_{in}\eta'_{1in}\overset{*}{\xi}_{in}\left[\omega\mu_o\epsilon_r^3\mu_r^3\,\eta'^2_{2in}\,\overset{*}{\eta}'^2_{1in}\overset{*}{S}_{in} - \epsilon_z^2 R_{in}\overset{*}{T}'_{in}\mathcal{K}_{in}\mu_z a_3\right] \tag{72-10}$$

$$F_{in11} = nL_{in}\overset{*}{\eta}'_{1in}\left[\epsilon_z^2\overset{*}{R}'_{in}R_{in}\mathcal{K}_{in}\mu_z a_3 - \omega\mu_o\epsilon_r^3\mu_r^3\left|\eta'^2_{2in}\right|^2\overset{*}{M}_{in}\right] \tag{72-11}$$

$$F_{in12} = n^2 L_{in}\epsilon_r\mu_r\epsilon_z\left[R_{in}\overset{*}{\eta}'^2_{2in}\overset{*}{M}_{in}\epsilon_r\mu_r - \omega\mu_o\eta'^2_{2in}\overset{*}{R}'_{in}\mathcal{K}_{in}\mu_z a_3\right] \tag{72-12}$$

$$F_{in13} = nL_{in}\overset{*}{\eta}'_{2in}\overset{*}{\xi}_{1in}\left[\omega\mu_o\epsilon_r^3\mu_r^3\overset{*}{\eta}'^2_{1in}\eta'^2_{2in}\overset{*}{S}_{in} - \epsilon_z^2 R_{in}\overset{*}{T}'_{in}\mathcal{K}_{in}\mu_z a_3\right] \tag{72-13}$$

$$F_{in14} = n^2 L_{in} \epsilon_r \mu_r \epsilon_z \overset{*}{\xi}_{1\,in} \left[\omega\mu_o \eta_{2\,in}'^{2} \overset{*}{T}'_{in} \chi_{in} \mu_z a_3 - R_{in} \overset{*}{\eta}{}'^{2}_{1\,in} \overset{*}{S}_{in} \epsilon_r \mu_r \right] \qquad (72\text{-}14)$$

$$F_{in15} = nL_{in} \eta'_{2\,in} \xi_{1\,in} \left[\omega\mu_o \epsilon_r^3 \mu_r^3 \eta_{1\,in}'^{2} \overset{*}{\eta}{}'^{2}_{2\,in} \overset{*}{M}_{in} + T_{in} \overset{*}{R}'_{in} \chi_{in} \epsilon_z \mu_z a_3 \right] \qquad (72\text{-}15)$$

$$F_{in16} = -nL_{in} \eta'_{2\,in} \left| \xi_{1\,in} \right|^2 \left[\epsilon_z^2 T_{in} \overset{*}{T}'_{in} \chi_{in} \mu_z a_3 + \omega\mu_o \epsilon_r^3 \mu_r^3 \left| \eta_{1\,in}'^{2} \right|^2 \overset{*}{S}_{in} \right] \qquad (72\text{-}16)$$

$$F_{in17} = nL_{in} \overset{*}{\eta}{}'_{1\,in} \xi_{1\,in} \left[\epsilon_z^2 T_{in} \overset{*}{R}'_{in} \chi_{in} \mu_z a_3 + \omega\mu_o \epsilon_r^3 \mu_r^3 \eta_{1\,in}'^{2} \overset{*}{\eta}{}'^{2}_{2\,in} \overset{*}{M}_{in} \right] \qquad (72\text{-}17)$$

$$F_{in18} = n^2 L_{in} \xi_{1\,in} \epsilon_z \mu_r \epsilon_r \left[T_{in} \overset{*}{\eta}{}'^{2}_{2\,in} \overset{*}{M}_{in} \epsilon_r \mu_r + \omega\mu_o \eta_{1\,in}'^{2} \overset{*}{R}'_{in} \chi_{in} \mu_z a_3 \right] \qquad (72\text{-}18)$$

$$F_{in19} = -nL_{in} \overset{*}{\eta}{}'_{2in} \left| \xi_{1in} \right|^2 \left[\epsilon_z^2 T_{in} \overset{*}{T}'_{in} \chi_{in} \mu_z a_3 + \omega\mu_o \epsilon_r^3 \mu_r^3 \left| \eta_{1\,in}'^{2} \right|^2 \overset{*}{S}_{in} \right] \qquad (72\text{-}19)$$

$$F_{in20} = -n^2 L_{in} \left| \xi_{1in} \right|^2 \epsilon_r \mu_r \epsilon_z \left[\omega\mu_o \overset{*}{T}'_{in} \chi_{in} \eta_{1\,in}'^{2} \mu_z a_3 + T_{in} \overset{*}{\eta}{}'^{2}_{1\,in} \overset{*}{S}_{in} \epsilon_r \mu_r \right] \qquad (72\text{-}20)$$

$$F_{in21} = \frac{jn}{\eta_{in}^2 \overset{*}{\eta}_{in}} \left[k^2 \mu_2 \epsilon_2 \xi_{3in} \overset{*}{\xi}_{2in} \overset{*}{J}_n(\eta_{in} b) \overset{*}{N}{}'_n(\eta_{in} b) - \chi_{in}^2 \xi_{2in} \overset{*}{\xi}_{3in} \overset{*}{J}{}'_n(\eta_{in} b) N_n(\eta_{in} b) \right]$$
$$(72\text{-}21)$$

$$F_{in22} = \frac{jn}{\eta_{in}^2 \overset{*}{\eta}_{in}} \left[\chi_{in}^2 \xi_{2in} \overset{*}{\xi}_{3in} N_n(\eta_{in} b) \overset{*}{N}{}'_n(\eta_{in} b) - k^2 \mu_2 \epsilon_2 \xi_{3in} \overset{*}{\xi}_{2in} N'_n(\eta_{in} b) \overset{*}{N}_n(\eta_{in} b) \right]$$
$$(72\text{-}22)$$

$$F_{in23} = \frac{jn}{\eta_{in}^2 \overset{*}{\eta}_{in}} \left[\chi_{in}^2 \xi_{2in} \overset{*}{\xi}_{3in} \overset{*}{J}{}'_n(\eta_{in} b) J_n(\eta_{in} b) - k^2 \mu_2 \epsilon_2 \xi_{3in} \overset{*}{\xi}_{2in} J'_n(\eta_{in} b) \overset{*}{J}_n(\eta_{in} b) \right]$$
$$(72\text{-}23)$$

$$F_{in24} = \frac{jn}{\eta_{in}^{2}\overset{*}{\eta}_{in}}\left[k^{2}\mu_{2}\epsilon_{2}\xi_{3in}\overset{*}{\xi}_{2in}J_{n}'(\eta_{in}b)\overset{*}{N}_{n}(\eta_{in}b)-\varkappa_{in}^{2}\xi_{2in}\overset{*}{\xi}_{3in}J_{n}(\eta_{in}b)\overset{*}{N}_{n}'(\eta_{in}b)\right] \tag{72-24}$$

$$F_{in25} = \frac{jn}{\eta_{in}\overset{*}{\eta}_{in}^{2}}\left[\varkappa_{in}^{2}\xi_{2in}\overset{*}{\xi}_{3in}\overset{*}{J}_{n}'(\eta_{in}b)J_{n}(\eta_{in}b)-k^{2}\mu_{2}\epsilon_{2}\xi_{3in}\overset{*}{\xi}_{2in}J_{n}'(\eta_{in}b)\overset{*}{J}_{n}(\eta_{in}b)\right] \tag{72-25}$$

$$F_{in26} = \frac{jn}{\eta_{in}\overset{*}{\eta}_{in}^{2}}\left[k^{2}\mu_{2}\epsilon_{2}\xi_{3in}\overset{*}{\xi}_{2in}J_{n}'(\eta_{in}b)\overset{*}{N}_{n}(\eta_{in}b)-\varkappa_{in}^{2}\xi_{2in}\overset{*}{\xi}_{3in}J_{n}(\eta_{in}b)\overset{*}{N}_{n}'(\eta_{in}b)\right] \tag{72-26}$$

$$F_{in27} = \frac{jn}{\eta_{in}\overset{*}{\eta}_{in}^{2}}\left[k^{2}\mu_{2}\epsilon_{2}\xi_{3in}\overset{*}{\xi}_{2in}\overset{*}{J}_{n}(\eta_{in}b)N_{n}'(\eta_{in}b)-\varkappa_{in}^{2}\xi_{2in}\overset{*}{\xi}_{3in}\overset{*}{J}_{n}'(\eta_{in}b)N_{n}(\eta_{in}b)\right] \tag{72-27}$$

$$F_{in28} = \frac{jn}{\eta_{in}\overset{*}{\eta}_{in}^{2}}\left[\varkappa_{in}^{2}\xi_{2in}\overset{*}{\xi}_{3in}\overset{*}{N}_{n}'(\eta_{in}b)N_{n}(\eta_{in}b)-k^{2}\mu_{2}\epsilon_{2}\xi_{3in}\overset{*}{\xi}_{2in}\overset{*}{N}_{n}(\eta_{in}b)N_{n}'(\eta_{in}b)\right] \tag{72-28}$$

$$F_{in29} = \frac{n^{2}\omega\varkappa_{in}}{\left|\eta_{in}^{2}\right|^{2}}\left[\epsilon_{o}\epsilon_{2}\left|\xi_{2in}\right|^{2}\left|J_{n}(\eta_{in}b)\right|^{2}+\mu_{o}\mu_{2}\left|\xi_{3in}\right|^{2}\left|J_{n}'(\eta_{in}b)\right|^{2}\right] \tag{72-29}$$

$$F_{in30} = \frac{n^{2}\omega\varkappa_{in}}{\left|\eta_{in}^{2}\right|^{2}}\left[\epsilon_{o}\epsilon_{2}\left|\xi_{2in}\right|^{2}\left|N_{n}(\eta_{in}b)\right|^{2}+\mu_{o}\mu_{2}\left|\xi_{3in}\right|^{2}\left|N_{n}'(\eta_{in}b)\right|^{2}\right] \tag{72-30}$$

$$F_{in31} = -\frac{n^{2}\omega\varkappa_{in}}{\left|\eta_{in}^{2}\right|^{2}}\left[\epsilon_{o}\epsilon_{2}\left|\xi_{2in}\right|^{2}\overset{*}{J}_{n}(\eta_{in}b)N_{n}(\eta_{in}b)+\mu_{o}\mu_{2}\left|\xi_{3in}\right|^{2}\overset{*}{J}_{n}'(\eta_{in}b)N_{n}'(\eta_{in}b)\right] \tag{72-31}$$

$$F_{in32} = -\frac{n^{2}\omega\varkappa_{in}}{\left|\eta_{in}^{2}\right|^{2}}\left[\epsilon_{o}\epsilon_{2}\left|\xi_{2in}\right|^{2}J_{n}(\eta_{in}b)\overset{*}{N}_{n}(\eta_{in}b)+\mu_{o}\mu_{2}\left|\xi_{3in}\right|^{2}J_{n}'(\eta_{in}b)\overset{*}{N}_{n}'(\eta_{in}b)\right] \tag{72-32}$$

$$L_{in} = \frac{\omega^2 \epsilon_o^2 \epsilon_z^2 \mu_z \varkappa_{in} a_3}{\epsilon_r^4 \mu_r^4 \left| \eta_{1in}'^2 \eta_{2in}'^2 (\eta_{2in}'^2 - \eta_{1in}'^2) \right|^2} \tag{73}$$

$$I_{in1} = \int_0^a \left[J_n'(\eta_{1in}' r) \overset{*}{J}{}_n'(\eta_{1in}' r) \right] rdr \tag{74-1}$$

$$I_{in2} = \int_0^a \left[J_n'(\eta_{1in}' r) \overset{*}{J}{}_n'(\eta_{2in}' r) \right] rdr \tag{74-2}$$

$$I_{in3} = \int_0^a \left[J_n'(\eta_{2in}' r) \overset{*}{J}{}_n'(\eta_{1in}' r) \right] rdr \tag{74-3}$$

$$I_{in4} = \int_0^a \left[J_n'(\eta_{2in}' r) \overset{*}{J}{}_n'(\eta_{2in}' r) \right] rdr \tag{74-4}$$

$$I_{in5} = \int_a^b \left[J_n'(\eta_{in} r) \overset{*}{J}{}_n'(\eta_{in} r) \right] rdr \tag{74-5}$$

$$I_{in6} = \int_a^b \left[J_n'(\eta_{in} r) \overset{*}{N}{}_n'(\eta_{in} r) \right] rdr \tag{74-6}$$

$$I_{in7} = \int_a^b \left[\overset{*}{J}{}_n'(\eta_{in} r) N_n'(\eta_{in} r) \right] rdr \tag{74-7}$$

$$I_{in8} = \int_a^b \left[N_n'(\eta_{in} r) \overset{*}{N}{}_n'(\eta_{in} r) \right] rdr \tag{74-8}$$

$$I_{in9} = \int_0^a \left[J_n'(\eta'_{1in} r) \, \overset{*}{J}_n(\eta'_{1in} r) \right] dr \tag{74-9}$$

$$I_{in10} = \int_0^a \left[J_n'(\eta'_{1in} r) \, \overset{*}{J}_n(\eta'_{2in} r) \right] dr \tag{74-10}$$

$$I_{in11} = \int_0^a \left[J_n(\eta'_{1in} r) \, \overset{*\prime}{J}_n(\eta'_{1in} r) \right] dr \tag{74-11}$$

$$I_{in12} = \int_0^a \left[\frac{J_n(\eta'_{1in} r) \, \overset{*}{J}_n(\eta'_{1in} r)}{r} \right] dr \tag{74-12}$$

$$I_{in13} = \int_0^a \left[J_n(\eta'_{1in} r) \, \overset{*\prime}{J}_n(\eta'_{2in} r) \right] dr \tag{74-13}$$

$$I_{in14} = \int_0^a \left[\frac{J_n(\eta'_{in} r) \, \overset{*}{J}_n(\eta'_{2in} r)}{r} \right] dr \tag{74-14}$$

$$I_{in15} = \int_0^a \left[J_n'(\eta'_{2in} r) \, \overset{*}{J}_n(\eta'_{1in} r) \right] dr \tag{74-15}$$

$$I_{in16} = \int_0^a \left[J_n'(\eta'_{2in} r) \, \overset{*}{J}_n(\eta'_{2in} r) \right] dr \tag{74-16}$$

$$I_{in17} = \int_0^a \left[J_n(\eta'_{2in} r) \, \overset{*\prime}{J}_n(\eta'_{1in} r) \right] dr \tag{74-17}$$

$$I_{in18} = \int_0^a \left[\frac{J_n(\eta'_{2in} r) \, \overset{*}{J}_n(\eta'_{1in} r)}{r} \right] dr \tag{74-18}$$

$$I_{in19} = \int_0^a \left[J_n(\eta'_{2in} r) \, \overset{*}{J}{}'_n(\eta'_{2in} r) \right] dr \tag{74-19}$$

$$I_{in20} = \int_0^a \left[\frac{J_n(\eta'_{2in} r) \, \overset{*}{J}_n(\eta'_{2in} r)}{r} \right] dr \tag{74-20}$$

$$I_{in21} = \int_a^b \left[J_n(\eta_{in} r) \, \overset{*}{N}{}'_n(\eta_{in} r) \right] dr \tag{74-21}$$

$$I_{in22} = \int_a^b \left[J_n(\eta_{in} r) \, \overset{*}{J}{}'_n(\eta_{in} r) \right] dr \tag{74-22}$$

$$I_{in23} = \int_a^b \left[N_n(\eta_{in} r) \, \overset{*}{N}{}'_n(\eta_{in} r) \right] dr \tag{74-23}$$

$$I_{in24} = \int_a^b \left[N_n(\eta_{in} r) \, \overset{*}{J}{}'_n(\eta_{in} r) \right] dr \tag{74-24}$$

$$I_{in25} = \int_a^b \left[N'_n(\eta_{in} r) \, \overset{*}{N}_n(\eta_{in} r) \right] dr \tag{74-25}$$

$$I_{in26} = \int_a^b \left[\overset{*}{J}_n(\eta_{in} r) \, N'_n(\eta_{in} r) \right] dr \tag{74-26}$$

$$I_{in27} = \int_a^b \left[J_n'(\eta_{in} r) \overset{*}{N}_n(\eta_{in} r) \right] dr \tag{74-27}$$

$$I_{in28} = \int_a^b \left[J_n'(\eta_{in} r) \overset{*}{J}_n(\eta_{in} r) \right] dr \tag{74-28}$$

$$I_{in29} = \int_a^b \left[\frac{N_n(\eta_{in} r) \overset{*}{N}_n(\eta_{in} r)}{r} \right] dr \tag{74-29}$$

$$I_{in30} = \int_a^b \left[\frac{J_n(\eta_{in} r) \overset{*}{J}_n(\eta_{in} r)}{r} \right] dr \tag{74-30}$$

$$I_{in31} = \int_a^b \left[\frac{J_n(\eta_{in} r) \overset{*}{N}_n(\eta_{in} r)}{r} \right] dr \tag{74-31}$$

$$I_{in32} = \int_a^b \left[\frac{\overset{*}{J}_n(\eta_{in} r) N_n(\eta_{in} r)}{r} \right] dr \tag{74-32}$$

It may be noted here that for $n = 0$ (i.e., when $\frac{\partial}{\partial\theta} \equiv 0$), $F_{in\ell} = 0$, for $\ell \geqslant 9$, and all the integrals $I_{in}\ell$, for $\ell \leqslant 8$ can be evaluated in closed form. But when $n \neq 0$, the above integrals can be expressed partly in closed form, partly in series, or they may be calculated numerically also.

Expressions for Total Fields (for loss-less media)

Now substituting the expressions (64) to (69) and (71), in the equations (22) and (23), the complete fields due to the magnetic current ring source can be expressed in the following way (suppressing the time dependence factor $e^{j\omega t}$):

$$E_z = -\frac{c}{4\pi}\sum_{i,n}\frac{\tilde{m}_n e^{\mp j\varkappa_{in}z + jn\theta}}{(\eta'^2_{2in} - \eta'^2_{1in})\sum_{\ell=1}^{\infty} F_{in\ell} I_{in\ell}}\left[S_{in}\xi_{1in}J_n(\eta'_{2in}r) - M_{in}J_n(\eta'_{1in}r)\right]\cdot \tag{75a}$$

$$\cdot\left[\frac{j\omega\epsilon_o\epsilon_2\overset{*}{\xi}_{2in}\overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\varkappa_{in}\overset{*}{\xi}_{3in}\overset{*}{G}_{in}(c)}{c\,\overset{*}{\eta}{}^2_{in}}\right], \quad \text{for } 0 \leq r \leq a$$

$$E_z = -\frac{c}{4\pi}\sum_{i,n}\frac{\tilde{m}_n e^{\mp j\varkappa_{in}z + jn\theta}}{\sum_{\ell=1}^{\infty} F_{in\ell} I_{in\ell}}\ \xi_{2in}\mathscr{J}_{in}(r)\,. \tag{75b}$$

$$\cdot\left[\frac{j\omega\epsilon_o\epsilon_2\overset{*}{\xi}_{2in}\overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\varkappa_{in}\overset{*}{\xi}_{3in}\overset{*}{G}_{in}(c)}{c\,\overset{*}{\eta}{}^2_{in}}\right], \quad \text{for } a \leq r \leq b$$

$$E_r = \mp \frac{j\omega\epsilon_o C\mu_z\epsilon_z a_3}{4\pi\epsilon_r^2\mu_r^2} \sum_{i,n} \frac{\widetilde{m}_n \mathcal{H}_{in} e^{\mp j\mathcal{H}_{in} z + jn\theta}}{\eta_{1in}'^2 \eta_{2in}'^2 (\eta_{2in}'^2 - \eta_{1in}'^2) \sum_{\ell=1}^{12} F_{in\ell} I_{in\ell}} \cdot$$

$$\cdot \left[\frac{j\omega\epsilon_o\epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\mathcal{H}_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\,\eta_{in}^{*2}} \right] \cdot$$

$$\cdot \left[\left\{ \epsilon_z \eta_{1in}' R_{in} J_n'(\eta_{1in}' r) - n\omega\mu_o\epsilon_r\mu_r \eta_{2in}'^2 \frac{J_n(\eta_{1in}' r)}{r} \right\} \right.$$

$$\left. + \xi_{2in} \left\{ \eta_{2in}' T_{in} \epsilon_2 J_n'(\eta_{2in}' r) + n\omega\mu_o\epsilon_r\mu_r \eta_{1in}'^2 \frac{J_n(\eta_{2in}' r)}{r} \right\} \right] \quad \text{for } 0 \leq r \leq a. \tag{76a}$$

$$E_r = \mp \frac{c}{4\pi} \sum_{i,n} \frac{\widetilde{m}_n e^{\mp j\mathcal{H}_{in} z + jn\theta}}{\sum_{\ell=1}^{12} F_{in\ell} I_{in\ell}} \left[\frac{j\omega\epsilon_o\epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\eta_{in}^*} + \frac{n\mathcal{H}_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\,\eta_{in}^{*2}} \right] \cdot$$

$$\cdot \left[\frac{n\omega\mu_o\mu_2 \xi_{3in} G_{in}(r)}{\eta_{in}^2 r} - \frac{j\mathcal{H}_{in} \xi_{2in} C_{in}(r)}{\eta_{in}} \right], \quad \text{for } a \leq r \leq b \tag{76b}$$

$$E_\theta = \pm\frac{c}{4\pi}\,\frac{\epsilon_z\mu_z\omega\epsilon_o a_3}{\epsilon_r^2\mu_r^2}\sum_{i,n}\left[\frac{\widetilde{m}_n e^{\mp j\alpha_{in}z+jn\theta}\,\alpha_{in}\left\{\frac{j\omega\epsilon_o\epsilon_2\overset{*}{\xi}_{2in}C_{in}(c)}{\eta^*_{in}}+\frac{n\alpha_{in}\overset{*}{\xi}_{3in}G_{in}(c)}{c\,\eta^{*2}_{in}}\right\}}{\eta'^2_{1in}\eta'^2_{2in}(\eta'^2_{2in}-\eta'^2_{1in})\sum_{\ell=1}^{4}F_{in\ell}I_{in\ell}}\right]$$

$$\cdot\left[\left\{n\epsilon_z R_{in}\frac{J_n(\eta'_{1in}r)}{r}-\omega\mu_o\mu_r\epsilon_r\eta'_{1in}\eta'^2_{2in}J'_n(\eta'_{1in}r)\right\}\right.$$

$$\left.+\,\xi_{1in}\left\{\omega\mu_o\epsilon_r\mu_r\eta'^2_{1in}\eta'_{2in}J'_n(\eta'_{2in}r)+n\,\epsilon_z T_{in}\frac{J_n(\eta'_{2in}r)}{r}\right\}\right],\quad \text{for } 0\leq r\leq a \tag{77a}$$

$$E_\theta = \mp\frac{c}{4\pi}\sum_{i,n}\frac{\widetilde{m}_n e^{\mp j\alpha_{in}z+jn\theta}}{\sum_{\ell=1}^{4}F_{in\ell}I_{in\ell}}\left[\frac{j\omega\epsilon_o\epsilon_2\overset{*}{\xi}_{2in}C_{in}(c)}{\overset{*}{\eta}_{in}}+\frac{n\alpha_{in}\overset{*}{\xi}_{3in}\overset{*}{G}_{in}(c)}{c\,\eta^{*2}_{in}}\right]\cdot$$

$$\cdot\left[\frac{n\alpha_{in}\xi_{2in}\mathcal{J}_{in}(r)}{\eta^2_{in}\,r}-\frac{j\omega\mu_o\mu_2\xi_{3in}S_{in}(r)}{\eta_{in}}\right],\quad\text{for } a\leq r\leq b \tag{77b}$$

$$H_z = \mp \frac{j\omega\epsilon_o\epsilon_z a_3}{4\pi\mu_r\epsilon_r} \sum_{i,n} \frac{\widetilde{m}_n e^{\mp j\varkappa_{in} z + jn\theta} \; \varkappa_{in}}{(\eta_{2in}'^2 - \eta_{1in}'^2) \sum_{\ell=1}^{32} F_{in\ell} I_{in\ell}} \cdot$$

$$\cdot \left[\frac{j\omega\epsilon_o\epsilon_2 \overset{*}{\xi}_{2in} C_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\varkappa_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\overset{*}{\eta}{}_{in}^{2}} \right] \cdot \left[\xi_{1in} J_n(\eta_{2in}' r) - J_n(\eta_{1in}' r) \right] \quad \text{for } 0 \leqslant r \leqslant a \tag{78a}$$

$$H_z = \mp \frac{c}{4\pi} \sum_{i,n} \frac{\widetilde{m}_n e^{\mp j\varkappa_{in} z + jn\theta}}{\sum_{\ell=1}^{32} F_{in\ell} I_{in\ell}} \left[\frac{j\omega\epsilon_o\epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\varkappa_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\overset{*}{\eta}{}_{in}^{2}} \right] \xi_{3in} G_{in}(r)$$

$$\text{for } a \leqslant r \leqslant b \tag{78b}$$

$$H_r = - \frac{c\omega\epsilon_o\epsilon_z}{4\pi\epsilon_r^2\mu_r^2} \sum \frac{\widetilde{m}_n e^{\mp j\varkappa_{in} z + jn\theta}}{\eta_{1in}'^2 \eta_{2in}'^2 (\eta_{2in}'^2 - \eta_{1in}'^2) \sum_{\ell=1}^{32} F_{in\ell} I_{in\ell}} \left[\frac{j\omega\epsilon_o\epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \right.$$

$$\left. \frac{n\varkappa_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\overset{*}{\eta}{}_{in}^{2}} \right] \left\{ \eta_{1in}' R_{in}' \varkappa_{in} \epsilon_z \mu_z a_3 J_n'(\eta_{1in}' r) + n\eta_{2in}'^2 \frac{M_{in} \epsilon_r^2 \mu_r^2 J_n(\eta_{1in}' r)}{r} \right\}$$

$$- \xi_{1_{in}} \left\{ \eta'_{2_{in}} T'_{in} \varkappa_{in} \epsilon_z \mu_z a_3 J'_n(\eta'_{2_{in}} r) + n \eta'^2_{1_{in}} S_{in} \epsilon_r^2 \mu_r^2 \frac{J_n(\eta'_{2_{in}} r)}{r} \right\} \Bigg] \quad \text{for } 0 \leqslant r \leqslant a$$

(79a)

$$H_r = - \frac{c}{4\pi} \sum \frac{\widetilde{m}_n e^{\mp j\varkappa_{in} z + jn\theta}}{\sum_{\ell=1}^{3\ell} F_{in\ell} I_{in\ell}} \left[\frac{j\omega\epsilon_o \epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\varkappa_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\, \overset{*}{\eta}{}^2_{in}} \right] \times$$

$$\times \left[\frac{-n\omega\epsilon_o \epsilon_2 \xi_{2in} \mathcal{J}_{in}(r)}{\eta^2_{in} r} + \frac{j\varkappa_{in} \xi_{3in} S_{in}(r)}{\eta_{in}} \right] \quad \text{for } a \leqslant r \leqslant b$$

(79b)

$$H_\theta = - \frac{j\omega\epsilon_o \epsilon_z}{4\pi \epsilon_r^2 \mu_r^2} \sum_{i,n} \frac{\widetilde{m}_n e^{\mp j\varkappa_{in} z + jn\theta}}{\eta'^2_{1in} \eta'^2_{2in} (\eta'^2_{2in} - \eta'^2_{1in}) \sum_{\ell=1}^{3\ell} F_{in\ell} I_{in\ell}} \left[\frac{j\omega\epsilon_o \epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \right.$$

$$\left. \frac{n\varkappa_{in} \overset{*}{\xi}_{3in} \overset{*}{G}_{in}(c)}{c\, \overset{*}{\eta}{}^2_{in}} \right] \left[\left\{ \epsilon_r^2 \mu_r^2 \eta'_{1in} \eta'^2_{2in} M_{in} J'_n(\eta'_{1in} r) + nR'_{in} \varkappa_{in} \epsilon_z \mu_z a_3 \frac{J_n(\eta'_{1in} r)}{r} \right\} \right.$$

$$- \xi_{1in} \left\{ \epsilon_r^2 \mu_r^2 \eta'^2_{1in} \eta'_{2in} S_{in} J'_n(\eta'_{2in} r) + n T'_{in} \varkappa_{in} \frac{\epsilon_z \mu_z}{r} a_3 J_n(\eta'_{2in} r) \right\} \Bigg] \quad \text{for } 0 \leqslant r \leqslant a$$

(80a)

$$H_\theta = -\frac{c}{4\pi}\sum_{i,n} \frac{\tilde{m}_n e^{\mp j\mathcal{H}_{in} z + jn\theta}}{\sum_{\ell=1}^{\mathcal{N}} F_{in\ell} I_{in\ell}} \left[\frac{j\omega\epsilon_o\epsilon_2 \overset{*}{\xi}_{2in} \overset{*}{C}_{in}(c)}{\overset{*}{\eta}_{in}} + \frac{n\mathcal{H}_{in}\overset{*}{\xi}_{3in}\overset{*}{G}_{in}(c)}{c\overset{*}{\eta}{}^2_{in}} \right] \times$$

$$\times \left[\frac{-j\omega\epsilon_o\epsilon_2\xi_{2in}C_{in}(r)}{\eta_{in}} + \frac{n\mathcal{H}_{in}\xi_{3in}G_{in}(r)}{\eta^2_{in} r} \right] \quad \text{for } 0 \leqslant r \leqslant b. \tag{80b}$$

Wherever the sign $\pm$ or $\mp$ appears, the upper sign corresponds to the propagation in the positive z-direction and the lower sign for the negative z-direction.

<u>Expression for Average Power-Flow Due to a Magnetic Current Ring Source</u>

The average power flow is defined as

$$\begin{aligned} P_{av} &= \frac{1}{2}\,\text{Re} \iint_S \underline{E} \times \overset{*}{\underline{H}} \cdot \underline{z}_o \, dS \\ &= \frac{1}{2}\,\text{Re} \iint_S \underline{E} \cdot \overset{*}{\underline{H}} \times \underline{z}_o \, dS \end{aligned} \tag{81}$$

where Re means real part of

$\underline{E}$ = total electric field due to the source at any point

$\underline{H}$ = total magnetic field due to the source at any point

Now using equations (18), (21), (22) and (23) in (81), the expression for P_{av} can be written in the following manner

$$P_{av} = \frac{c^2}{16\pi^2} \operatorname{Re}\left[\sum_{in} \frac{\left|\tilde{m}_n\right|^2 \left|g_{\theta in}(c)\right|^2}{\sum_{\ell=1}^{3\ell} F_{in\ell} I_{in\ell}}\right] \tag{82}$$

$$= \frac{c^2}{16\pi^2} \operatorname{Re}\left[\sum_{in} \frac{\left|\tilde{m}_n\right|^2 \left|\frac{n\mathcal{H}_{in}\xi_{3in}}{\eta_{in}^2 c} G_{in}(c) - \frac{j\omega\epsilon_0\epsilon_2\xi_{2in}}{\eta_{in}} C_{in}(c)\right|^2}{\sum_{\ell=1}^{3\ell} F_{in\ell} I_{in\ell}}\right] \tag{83}$$

In particular when n = 0, i.e., when the ring source is of constant amplitude, the expression for the power flow reduces to the following form:

$$P_{av} = \frac{c^2 m}{4} \operatorname{Re}\left[\sum_{i} \frac{\left|\omega\epsilon_0\epsilon_2 \frac{\xi_{2i}}{\eta_i} C_{io}(c)\right|^2}{\sum_{\ell=1}^{\ell} F_{io\ell} I_{io\ell}}\right] \tag{84}$$

It may be noted that for non-dissipative media the quantity inside the square bracket is real. Any individual term in the series in (82) or (83) represents average power flow corresponding to that particular mode in (or i when n = 0).

II

WAVE PROPAGATION IN AN ANISOTROPIC PLASMA: SLOW SURFACE WAVES

Introduction

In this chapter the general results of the previous chapter will be applied to the study of propagation of electromagnetic waves in an infinitely long anisotropic plasma column enclosed by a dielectric cylinder, which is also enclosed by a perfectly conducting metallic cylindrical waveguide; i.e. the geometry and the source of excitation are the same as those of the general probelm except that in the present situation the anisotropic medium is represented by a plasma column with a uniform static magnetic field in the axial direction z. The relative permeability μ_2 of the dielectric medium which encloses the plasma column is assumed to be unity.

The plasma is considered to be fully ionized (i.e. macroscopically neutral) and there is no drift velocity (d. ~) of electrons or of ions, i.e. the plasma is also stationary. If one also assumes ..at the illuminating electromagnetic waves are weak, then it is possible to describe a plasma as a dielectric medium. In this analysis it will be assumed that the plasma is homogeneous, i.e. its density (and hence dielectric constant) is not a function of space.

In the presence of a static magnetic field, the dielectric constant of the plasma becomes a tensor, which means it is an anisotropic medium. If the static magnetic field is applied in the axial direction, it can be shown [2][3][5] that the plasma has the following dielectric tensor

$$\underset{\sim}{\epsilon} \rightarrow \begin{Vmatrix} \epsilon_{rr} & j\epsilon_{r\theta} & 0 \\ -j\epsilon_{\theta r} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{Vmatrix} \tag{1}$$

where

$$\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_r = 1 + \frac{\omega_p^2 \left(1 - \frac{j\nu}{\omega}\right)}{\omega_c^2 - \omega^2 \left(1 - \frac{j\nu}{\omega}\right)^2} \tag{2a}$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \epsilon' = \frac{\omega_c}{\omega} \cdot \frac{\omega_p^2}{\omega_c^2 - \omega^2 \left(1 - \frac{j\nu}{\omega}\right)^2} \tag{2b}$$

and

$$\epsilon_{zz} = \epsilon_z = 1 - \frac{\omega_p^2}{\omega^2 \left(1 - \frac{j\nu}{\omega}\right)} \tag{2c}$$

ν = collision frequency (radian) (3a)

$\omega_c = \frac{q_e B_o}{m_e}$ = cyclotron frequency (radian) (3b)

q_e = charge of an electron (3c)

m_e = mass of an electron, (3d)

B_o = d.c. magnetic induction (3e)

$\omega_p = \left(\frac{q_e^2 N_e}{\epsilon_o m_e}\right)^{1/2}$ = electron-plasma frequency (radian) (3f)

N_e = electron density (3g)

ϵ_o = free-space dielectric constant (3h)

In the above analysis the motion of an ion due to a disturbance is neglected in comparison to that of an electron. The relative permeability of the plasma is assumed to be unity.

It can be shown from the relations in (2) that the components of $\underset{\sim}{\epsilon}$ satisfy the following relation

$$\epsilon'^2 = (1 - \epsilon_r)(\epsilon_z - \epsilon_r) \tag{4}$$

An interesting conclusion can be made from the relation (4), namely $\epsilon' = 0$, for either $\epsilon_z = \epsilon_r$ or $\epsilon_r = 1$. The physical interpretation of these results can be given in the following way. For isotropic plasma (i.e. when $B_o = 0$), $\epsilon_z = \epsilon_r$, and $\epsilon' = 0$. On the other hand when $B_o \to \infty$ (i.e. $\omega_c \to \infty$), $\epsilon_r \to 1$ and $\epsilon' \to 0$. The above statements can also be verified directly from (2). It may be noted here that the collision term ν in the expression for $\underset{\sim}{\epsilon}$ in (2), represents loss in the plasma. Although the various dispersion relations developed in Chapter I and in Appendix C are valid for a lossy plasma, the complete field expressions obtained in the previous chapter are not. On the other hand if it is desirable to find the expressions for a dissipative medium (i.e. plasma), one must use the appropriate orthogonality condition and the resulting Green's functions which have been discussed in Chapter I as well as in Appendix B. But here only the loss-free ($\nu = 0$) plasma will be considered.

Although the results obtained in the previous chapter are valid for all possible modes of propagation in the structure, in the present chapter attention will be directed to the analysis connected with slow surface waves. The slow surface waves are those waves which decay radially in the dielectric region and propagate along the interface of the plasma column and the dielectric, in other words for such slow waves one finds $\frac{\varkappa}{k\sqrt{\epsilon_2}} > 1$, where ϵ_2 is the relative dielectric constant of the medium surrounding the plasma column. This medium may represent a glass tube. Various passbands for slow-wave propagation will be obtained in the following investigation. Some of the passbands depend on the range of the values of the ratio $\frac{\varkappa}{k\sqrt{\epsilon_2}} > 1$, and some passbands do not depend on the particular values of this ratio, provided $\frac{\varkappa}{k\sqrt{\epsilon_2}} > 1$, the condition for the existence of slow surface waves.

Finally numerical computation will be made for a special case.

<u>Conditions for Slow-Wave Propagation and Determination of Various Passbands</u>

In the following analysis only the expressions for the radial wave numbers, η_1' and η_2', will be considered. These quantities do not depend explicitly on any particular boundary, except that the geometry is cylindrical and uniform in the z-direction, thus the results will be true for any such structure, closed or open, which can support slow waves. For a particular structure, one must consider the solutions of the respective dispersion relation together with the following analysis. Since the solution of any dispersion relation, determines only a particular set of values of η_1' and η_2', the limitation imposed on the values of η_1' and η_2', which are obtained from the expressions for $\eta_1'^2$ and $\eta_2'^2$ (valid for unbounded medium also) is furthermore narrowed. Therefore the requirements which are obtained from a study of the expressions for $\eta_1'^2$ and $\eta_2'^2$ alone, are nothing but necessary conditions of wave propagation. The sufficient conditions for propagation of waves are provided only by the simultaneous solutions of the respective dispersion relation and the expressions for $\eta_1'^2$ and $\eta_2'^2$.

The expressions for radial propagation wave numbers η_1' and η_2' can be written here in the following way

$$\eta_{1,2}'^2 = v \pm \sqrt{v^2 - u} \tag{5a}$$

$$\left.\begin{aligned} &= \frac{k^2(1-y)\left\{(\beta^2-1)(1-x)+y\right\}}{x+y-1} - \frac{k^2\,xy(\beta^2-1)}{2(x+y-1)} \\ &\pm k^2 y\,\frac{\sqrt{(\beta^2-1)^2x^2+4\beta^2x(1-y)}}{2(x+y-1)} \end{aligned}\right\} \tag{5b}$$

where

$$v = \frac{k^2}{2(x+y-1)} \left[(\beta^2 - 1) \left\{ xy - 2(x+y-1) \right\} + 2y(1-y) \right] \qquad (6a)$$

$$u = -\frac{k^4(1-y)}{x+y-1} \left[(\beta^2 - 1)^2 (1-x) + 2y(\beta^2 - 1) + y^2 \right] \qquad (6b)$$

$$x = \frac{\omega_c^2}{\omega^2} \qquad (7a)$$

$$y = \frac{\omega_p^2}{\omega^2} \qquad (7b)$$

$$\epsilon_z = 1 - y \qquad (7c)$$

$$\epsilon_r = \frac{x+y-1}{x-1} \qquad (7d)$$

$$\epsilon' = \frac{y\sqrt{x}}{x-1} \qquad (7e)$$

$$\beta = \frac{\varkappa}{k} \qquad (7f)$$

In the above definitions of ϵ_z, ϵ_r, and ϵ', the collision frequency ν is neglected.

It can also be shown that

$$v^2 - u = \left[\frac{k^2 y}{2(x+y-1)} \right]^2 \cdot \left[(\beta^2 - 1)^2 x^2 + 4\beta^2 x(1-y) \right] \qquad (8)$$

In the dielectric region

$$\eta^2 = k^2 \epsilon_2 - \varkappa^2 \tag{9}$$

since $\mu_2 = 1.$

Therefore for slow surface waves, $\eta^2 < 0$, and

$$\frac{\delta^2}{k^2 \epsilon_2} = \frac{\varkappa^2}{k^2 \epsilon_2} - 1 > 0$$

i.e.

$$\frac{\varkappa^2}{k^2 \epsilon_2} > 1$$

or

$$\frac{\varkappa^2}{k^2} = \beta^2 > \epsilon_2 \tag{10}$$

where $\eta = -j\delta \quad , \delta > 0.$

It will also be assumed that $\epsilon_2 \geq 1$.

Since it is assumed that both plasma and dielectric are non-dissipative, the propagation wave number $\varkappa$ is always real. Although the expressions for $\eta_1'^2$ and $\eta_2'^2$ show that these radial wave numbers may be complex, on physical grounds only the real values of $\eta_1'^2$ and $\eta_2'^2$ will be allowed. For example* if complex values of $\eta_1'^2$ and $\eta_2'^2$ (which are complex conjugates of one another) are allowed, this means that there exist growing waves showing instability of the plasma in the

* The primary reason for allowing only the real values of $\eta_1'^2$ and $\eta_2'^2$ in a non-dissipative medium, is that the power flow, the characteristic impedance, etc. must be real for such a medium.

radial direction. Therefore this apparent inconsistant situation will be avoided by allowing only the real values of $\eta_1'^2$ and $\eta_2'^2$ in the following analysis.

The following three cases specify the conditions for which $\eta_1'^2$ and $\eta_2'^2$ are real.

<u>Case I</u>

$$\left.\begin{array}{l} u < 0 \end{array}\right\} \begin{array}{l}\text{in this case}\\ \eta_1'^2 > 0 \text{ and}\\ \eta_2'^2 < 0.\end{array} \tag{11a}$$

<u>Case II</u>

$$\left.\begin{array}{rl} & u > 0\\ & v > 0,\\ \text{and} & v^2 - u > 0 \end{array}\right\} \begin{array}{l}\text{in this case}\\ \eta_1'^2 > 0, \text{ and}\\ \eta_2'^2 > 0.\end{array} \tag{11b}$$

<u>Case III</u>

$$\left.\begin{array}{rl} & u > 0,\\ & v < 0,\\ \text{and} & v^2 - u > 0 \end{array}\right\} \begin{array}{l}\text{in this case}\\ \eta_1'^2 < 0, \text{ and}\\ \eta_2'^2 < 0.\end{array} \tag{11c}$$

The regions of u and v in the above three cases are shown in Figure 2. $\eta_1'^2$ and $\eta_2'^2$ become equal on the parabola $v^2 = u$ and both of them assume complex values inside parabola, which contains the positive axis of u. Therefore this is the forbidden zone for u and v. When $y < 1$ (i.e. when $\omega_p < \omega$), it is easy to show from (5) that $\eta_1'^2$ and $\eta_2'^2$ are both real. So even if one considers instability in plasma, this does not occur for $\omega_p < \omega$, provided the medium is loss-free.

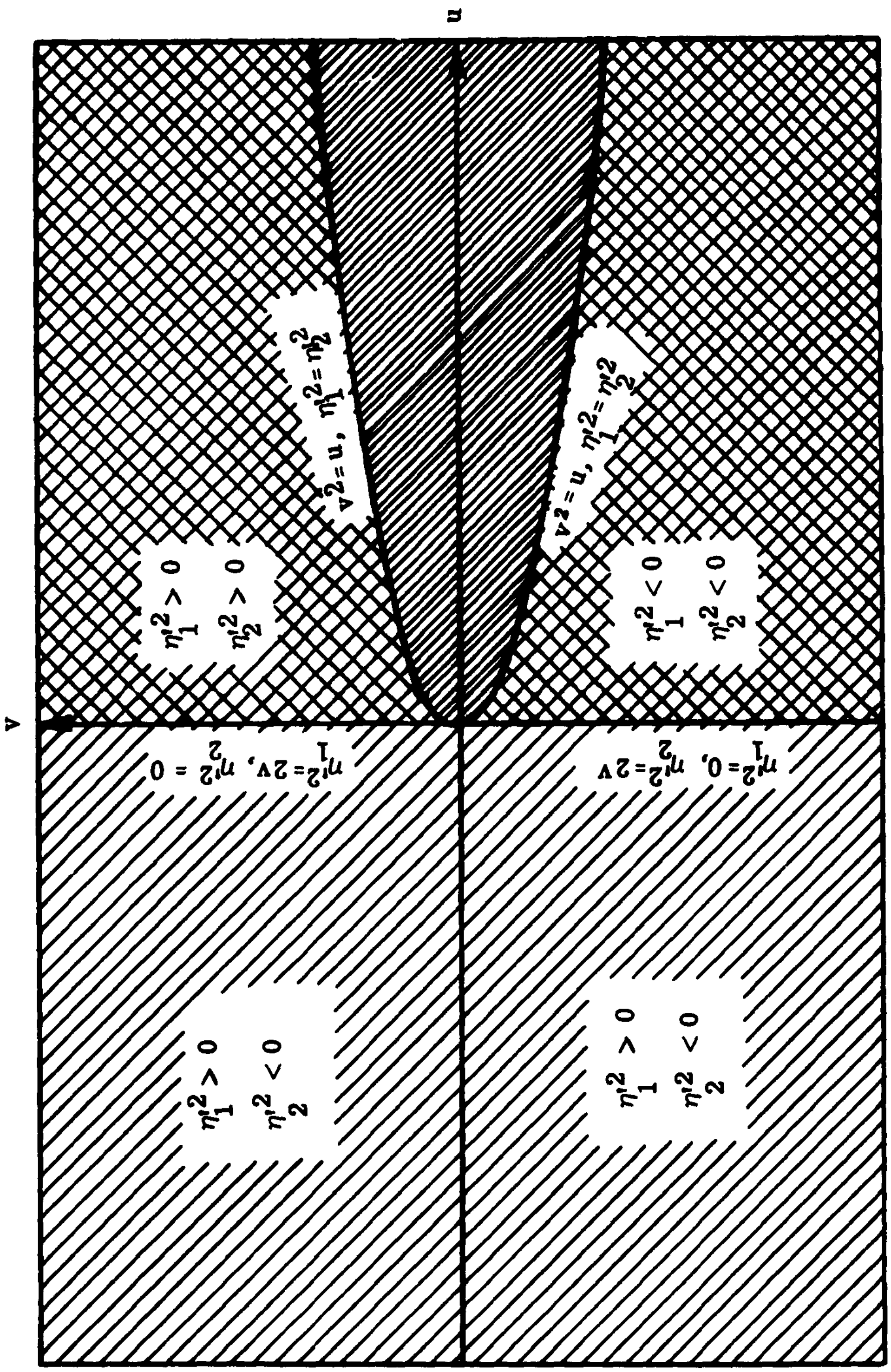

u
v
$v^2 = u,\ \eta_1'^2 = \eta_2'^2$
$v^2 = u,\ \eta_1'^2 = \eta_2'^2$
$\eta_1'^2 > 0$
$\eta_2'^2 > 0$
$\eta_1'^2 < 0$
$\eta_2'^2 < 0$
$\eta_1'^2 = 2v,\ \eta_2'^2 = 0$
$\eta_1'^2 = 0,\ \eta_2'^2 = 2v$
$\eta_1'^2 > 0$
$\eta_2'^2 < 0$
$\eta_1'^2 > 0$
$\eta_2'^2 < 0$

FIGURE 2

In the following detailed analysis the possibility of the above three individual cases, i.e. the conditions under which the parameters x, y, β, etc. have to be chosen, will be shown. These conditions on the parameters, which are necessary, will give the maximum passbands of slow wave propagation. Before starting the actual analysis, it will be convenient to introduce $\psi = \beta^2 - 1$ in the equations (6a), (6b) and (8) which can be rewritten in the following manner (for slow wave $\beta^2 > \epsilon_2 \geq 1$):

$$u = -\frac{k^2(1-y)}{x+y-1}\left[\psi^2(1-x) + 2\psi y + y^2\right] \tag{12a}$$

$$v = \frac{k^2}{2(x+y-1)}\left[\psi\left\{xy - 2(x+y-1)\right\} + 2y(1-y)\right] \tag{12b}$$

$$v^2 - u = \left[\frac{k^2 y}{2(x+y-1)}\right]^2 \cdot \left[\psi^2 x^2 + 4(\psi+1)x(1-y)\right] \tag{12c}$$

$$\psi + 1 = \beta^2 > \epsilon_2 \tag{13}$$

<u>Case I</u>

$$u < 0$$

$$\eta_1'^2 > 0, \qquad \eta_2'^2 < 0$$

This situation can be satisfied in the following three ways.

$$\left.\begin{array}{l} 1 - x > 0 \\ 1 - y > 0 \\ 1 > x + y - 1 > 0 \end{array}\right\} \qquad (14a)$$

$$\left.\begin{array}{l} 1 - x < 0 \\ 1 - y < 0 \\ \psi > \dfrac{y}{x-1}(1 + \sqrt{x}) \end{array}\right\} \qquad (14b)$$

$$\left.\begin{array}{l} 1 - x < 0 \\ 1 - y > 0 \\ \psi < \dfrac{y}{x-1}(1 + \sqrt{x}) \end{array}\right\} \qquad (14c)$$

Although the condition in (14a) does not explicitly depend on ψ, for slow wave ψ must satisfy the condition (13). On the other hand the conditions in (14b) and (14c) show that besides the restriction imposed on x and y, ψ must satisfy two simultaneous conditions. More precisely, the inequalitites in (14b) suggest that ψ must be greater than a certain value, i.e. the wave propagation which satisfies (14b) is possible for a value of ψ above a certain value. In other words this passband of wave propagation depends on the degree of slowness of the waves. On the other hand, the condition in (14c) can be met only for a definite range of ψ, namely

$$\epsilon_2 - 1 < \psi < \frac{y}{x-1}(1 + \sqrt{x})$$

where $1 - x < 0$, and $1 - y > 0$.

Before investigating Case II and Case III, it is desirable to find conditions under which $u > 0$, $v^2 - u > 0$, $v > 0$, and $v^2 < 0$, respectively.

For $\underline{u > 0}$, one finds the following possibilities

$$\left.\begin{array}{l} 1 - y < 0 \\ 1 - x > 0 \end{array}\right\} \tag{15a}$$

$$\left.\begin{array}{l} 1 - y < 0 \\ 1 - x < 0 \\ \psi < \dfrac{y}{x-1}\left(1 + \sqrt{x}\right) \end{array}\right\} \tag{15b}$$

$$\left.\begin{array}{l} 1 - y > 0 \\ 1 - x < 0 \\ \psi > \dfrac{y}{x-1}\left(1 + \sqrt{x}\right) \end{array}\right\} \tag{15c}$$

$$\left.\begin{array}{l} 1 - x > 0 \\ 1 - y > 0 \\ x + y - 1 < 0 \end{array}\right\} \tag{15d}$$

For $\underline{v^2 - u > 0}$, the following conditions must be satisfied

$$1 - y > 0 \tag{16a}$$

(here one may choose either $x + y - 1 > 0$, or $x + y - 1 < 0$)

$$\left.\begin{aligned} & 1 - y < 0 \\ & \Psi > \frac{2(y-1)}{x} + \frac{2}{x}\sqrt{(y-1)(x+y-1)} \end{aligned}\right\} \tag{16b}$$

The following three situations satisfy $\underline{V > 0}$:

$$\left.\begin{aligned} & x + y - 1 > 0 \\ & 1 - y > 0 \\ & xy < 2(x + y - 1) \\ & \Psi < \frac{2y(1-y)}{2(x+y-1) - xy} \end{aligned}\right\} \tag{17a}$$

$$\left.\begin{aligned} & x + y - 1 > 0 \\ & 1 - y > 0 \\ & xy > 2(x + y - 1) \end{aligned}\right\} \tag{17b}$$

$$\left.\begin{aligned} & 1 - y < 0 \\ & xy > 2(x + y - 1) \\ & \Psi > \frac{2y(y-1)}{xy - 2(x+y-1)} \end{aligned}\right\} \tag{17c}$$

Finally one obtains the following four possibilities for V < 0:

$$\left.\begin{array}{l} 1 - y < 0 \\ 2(x + y - 1) > xy \end{array}\right\} \qquad (18a)$$

$$\left.\begin{array}{l} 1 - y < 0 \\ xy > 2(x + y - 1) \\ \psi < \dfrac{2y(1 - y)}{xy - 2(x+y-1)} \end{array}\right\} \qquad (18b)$$

$$\left.\begin{array}{l} x + y - 1 > 0 \\ 1 - y > 0 \\ 2(x + y - 1) > xy \\ \psi > \dfrac{2y(1 - y)}{2(x + y - 1) - xy} \end{array}\right\} \qquad (18c)$$

$$\left.\begin{array}{l} x + y - 1 < 0 \\ 1 - y > 0 \end{array}\right\} \qquad (18d)$$

To satisfy the requirements for Case II and Case III, it is necessary to satisfy inequalities (11b) and (11c) respectively. It can be shown by a little analysis that the following are the conditions by which Case II and Case III can be realized.

Case II

This situation can be obtained if the following conditions are met:

$$\left.\begin{aligned} &1 - y < 0 \\ &1 - x < 0 \\ &2(x + y - 1) + x \left\{xy - 2(x + y - 1)\right\} > xy\sqrt{x} + 2(x + y - 1) \\ &\qquad > xy + 2(x - 1) \cdot \sqrt{(y - 1)(x + y - 1)} \end{aligned}\right\} \tag{19}$$

The inequalities in (19) are obtained from (15b), (16b) and (17c). There is no other possibility which can satisfy Case II.

Case III

In this case one can show that there are only four possible conditions as follows:

$$\begin{aligned} &1 - y > 0 \\ &1 - x < 0 \\ &\psi > \frac{y}{x - 1}\left(1 + \sqrt{x}\right) \end{aligned} \tag{20a}$$

Note that this condition is the same as (15c), which also satisfies conditions (16a) and (18c) automatically.

$$\left.\begin{array}{l} 1 - x > 0 \\ 1 - y > 0 \\ x + y - 1 < 0 \end{array}\right\} \qquad (20b)$$

The condition (20b) is the same as (15d) which also satisfies conditions (16a) and (18d).

$$\left.\begin{array}{l} 1 - y < 0 \\ 1 - x > 0 \\ \psi > \frac{2(y-1)}{x} + \frac{2}{x}\sqrt{(y-1)(x+y-1)} \end{array}\right\} \qquad (20c)$$

This condition (20c) is obtained by combining the conditions (15a), (16b) and (18a). It may be noted that condition (15a) automatically satisfies (18a) also.

The fourth possibility of realizing Case III is obtained by combining the conditions (15b), (16b) and (18b). Since it is not obvious that these three conditions can be satisfied simultaneously, it is necessary that they must meet the following requirements (a detailed analysis is omitted for the sake of brevity).

$$\left.\begin{array}{l} 1 - y < 0 \\ 1 - x < 0 \\ xy > 2(x + y - 1) \\ 2(x + y - 1) + xy\sqrt{x} > xy + 2(x - 1)\sqrt{(y-1)(x+y-1)} \\ \qquad > 2(x + y - 1) + x\left\{xy - 2(x + y - 1)\right\} \end{array}\right\} \qquad (20d)$$

Thus it is found that when the condition (10) and any of the following conditions (14), (19) and (29) are satisfied simultaneously, one obtains maximum pass-band of slow wave propagation. These conditions are necessary for slow waves.

Study of Dispersion Relation for Slow Wave Propagation Under Various Special Situations

Although the various results together with dispersion relations obtained in the preceding chapter are valid for any arbitrary angular variation of the magnetic current ring source, in this section only those dispersion relations which are independent of angular variation (i.e., n = 0, for constant amplitude of the ring source) will be considered.

Since the solution of the dispersion relation, appropriate for any particular case, together with the expressions in (5) for η'^2_1 and η'^2_2 gives exact information of the propagation of waves, and as this solution cannot be obtained analytically in general, the information obtained here without actual solutions will give only necessary conditions for slow wave propagation. In general, the actual solution can be obtained only by numerical computation.

+Static limit: This static limit is a good approximation in the following situations:

1) circumference of the plasma column is much shorter than the wavelength of the operating frequency

+Trivel piece in his work [8] discusses this problem in detail and his method of solving this problem is different. Here his results are obtained as a limiting case.

2) for extremely slow wave, i.e., $\frac{\varkappa^2}{k^2\epsilon_2} >> 1$.

This dispersion relation in this case for n = 0, can be obtained from the relation (3) of Appendix C, and it reduces to the following form:

$$\frac{\epsilon_r \eta'_1}{\epsilon_2 \varkappa} \frac{J_1(\eta'_1 a)}{J_o(\eta'_1 a)} = \frac{I_1(\varkappa a) K_o(\varkappa b) + I_o(\varkappa b) K_1(\varkappa a)}{I_o(\varkappa b) K_o(\varkappa a) - I_o(\varkappa a) K_o(\varkappa b)} \tag{21}$$

where

$$\left.\begin{aligned} \eta'^2_1 &\approx -(\epsilon_z/\epsilon_r)\varkappa^2 \\ \eta^2 &\approx -\varkappa^2 \end{aligned}\right\} \tag{22}$$

The above relations show that the dispersion relation and hence fields do not depend on η'_2 in this limit. Moreover, in this limit the magnetic current ring source excites only E-type mode. But in a general anisotropic medium characterized by $\underset{\sim}{\epsilon}$ of the form shown in (1), pure E-type and H-type modes do not exist, i.e., they are coupled to each other.

From the properties of modified Bessel's functions it can be shown that the right-hand side of (21) is positive and greater than unity (it approaches unity as $\varkappa \to \infty$). If $\epsilon_z/\epsilon_r < 0$, η'_1 is real, a solution of (21) is possible.

Since $\epsilon_z/\epsilon_r = \frac{(1-y)(x-1)}{x+y-1} < 0$

either

$$\begin{aligned} x &> 1 \\ y &> 1 \end{aligned} \tag{23a}$$

or

$$\left.\begin{array}{l} x < 1 \\ y < 1 \\ 1 < x + y < 2 \end{array}\right\} \qquad (23b)$$

If $\epsilon_z/\epsilon_r > 0$, $\eta_1'^2 < 0$, and η_1' is purely imaginary, the dispersion relation (21) becomes

$$-\epsilon_r \frac{\sqrt{\epsilon_z/\epsilon_r}\, I_1(\kappa a\sqrt{\epsilon_z/\epsilon_r})}{\epsilon_2\, I_o(\kappa a\sqrt{\epsilon_z/\epsilon_r})} = \frac{I_1(\kappa a)K_o(\kappa b) + I_o(\kappa b)K_1(\kappa a)}{I_o(\kappa b)K_o(\kappa a) - I_o(\kappa a)K_o(\kappa b)} > 1 \qquad (24)$$

A solution of (24) is possible if $\epsilon_r < 0$, $\epsilon_z < 0$ (since $\epsilon_2 > 0$). As it is also known that $I_1(z) < I_o(z)$, for any $z > 0$, it is necessary that

$$-\epsilon_r \cdot \sqrt{\epsilon_z/\epsilon_r} > \epsilon_2$$

or

$$\frac{x + y - 1}{1 - x} \cdot \sqrt{\frac{(1 - y)(x - 1)}{x + y - 1}} > \epsilon_2 \qquad (25)$$

In particular if $\epsilon_2 = 1$ (i.e., when the plasma column is surrounded by air), the inequality (25) reduces to the following

$$\frac{(x + y - 1)(y - 1)}{1 - x} > 1 \qquad (26)$$

Finally the inequality (26) can be shown to be equivalent to the following two passbands for slow waves when the dielectric surrounding the plasma column is air:

$$\left.\begin{aligned} y - 1 &> 0 \\ 1 - x &> 0 \\ x + y &> 2 \end{aligned}\right\} \tag{27a}$$

$$\left.\begin{aligned} y - 1 &< 0 \\ 1 - x &< 0 \\ x + y &< 2 \end{aligned}\right\} \tag{27b}$$

It may be noted here also that when the plasma column is surrounded by a dielectric $\epsilon_2 > 1$, the passbands are reduced further.

When $b = a$, i.e., when the plasma completely fills the waveguide, it can be shown that the corresponding dispersion relation (in the static limit) reduces to the following

$$J_o(\eta'_1 a) = 0 \tag{28}$$

It is easy to show that only real values of η'_1 can satisfy the above equation (28). Therefore, in this case also it is necessary that $\epsilon_z/\epsilon_r < 0$.

A study of the relations (21) and (22) reveals that for $x = 1$, $y = 1$ or $x + y = 1$, the dispersion relation (21) does not possess any solution, which is equivalent to saying that these points represent cut-off for the slow wave propagation.

It may be remarked here that the passbands for slow wave propagation when the plasma completely fills the waveguide, give maximum range for the case when the plasma column partially fills the waveguide (provided η'_1 is real) in the static limit. These maximum passbands are depicted in the following Figures 3 and 4, subject to the conditions (23a) and (23b).

$$\frac{\chi}{\eta'_1} = \pm\sqrt{-\epsilon_r/\epsilon_z} = \pm\sqrt{\frac{(\omega_p^2+\omega_c^2-\omega^2)\omega^2}{(\omega_p^2-\omega^2)(\omega_c^2-\omega^2)}} \qquad (29)^{+}$$

Since group velocity is defined as $d\omega/d\chi$, Figures 3 and 4 show that this value can also assume negative values — which proves the existence of backward wave in such a structure.

With a few more remarks, the discussion of the static limit case will be concluded. The static limit results are reasonably valid for extremely slow wave propagation, as pointed out in the beginning of this section. In this limit η'_2 does not appear in the dispersion relation, showing that the field components for extremely slow wave propagation do not depend on η'_2, when the source of excitation is a magnetic ring current. In other words in this limit an H-type mode is not excited. This does not mean, however, that in this limit $\eta'_2 = 0$. In fact, $\eta'_2 = -j\chi$, a large imaginary number, which shows that a wave dependent on η'_2 decays away very rapidly. So it may be conceived that in this limiting condition the H-type mode is very weakly coupled with the E-type mode and the components representing H-type mode are also very highly attenuated. It may be noted here

+ In [8], Trivelpiece has also obtained similar results.

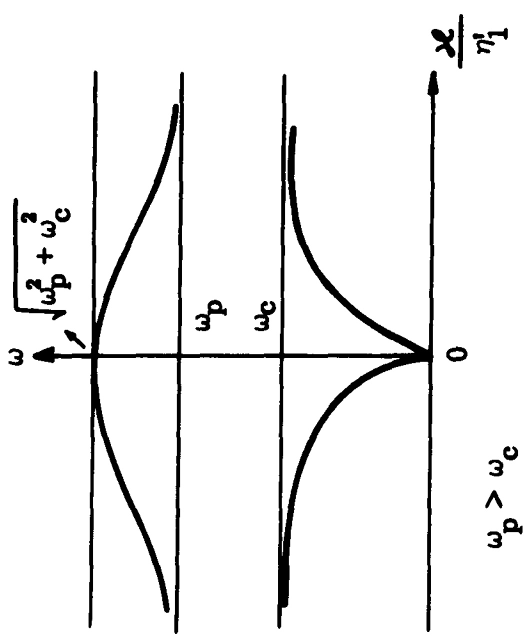


FIGURE 3

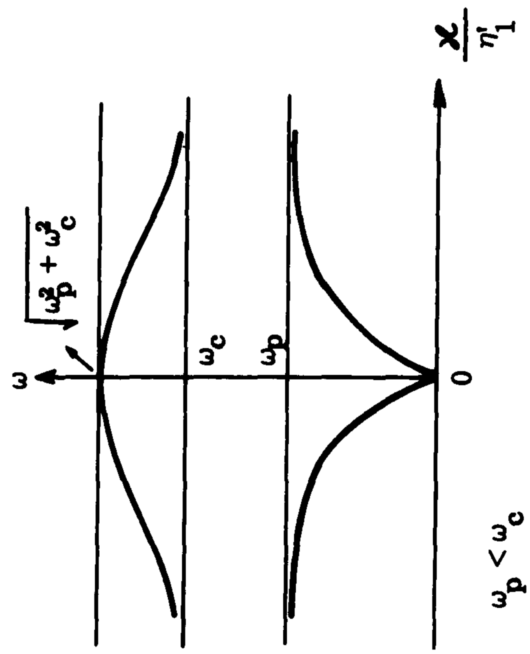


FIGURE 4

also that the results for $\beta^2/\epsilon_2 >> 1$, essentially correspond to those for static limit.

<u>General Dispersion Relation (when n = 0) for Slow Wave Propagation</u>

It can be shown from equation (56I) of Chapter I, that for slow waves (when $\eta = -j\delta$, $\delta > 0$), the dispersion relation becomes

$$\frac{\epsilon_z(\eta_2'^2 - \eta_1'^2)\bar{\mathcal{J}}_o(a)\bar{G}_o(a)J_1(\eta_1' a)J_1(\eta_2' a)}{\eta_1' \eta_2'} + \frac{\epsilon_2(\eta_2'^2 - \eta_1'^2)\bar{C}_o(a)\bar{S}_o(a)J_o(\eta_1' a)J_o(\eta_2' a)}{k^2(\beta^2 - \epsilon_2)}$$

$$\frac{-J_o(\eta_2' a)J_1(\eta_2' a)}{k\eta_2'\sqrt{\beta^2 - \epsilon_2}}\left[\epsilon_2 M\bar{C}_o(a)\bar{G}_o(a) - \epsilon_z S\bar{\mathcal{J}}_o(a)\bar{S}_o(a)\right]$$

$$+\frac{J_o(\eta_2' a)J_1(\eta_1' a)}{k\eta_1'\sqrt{\beta^2 - \epsilon_2}}\left[\epsilon_2 S\bar{C}_o(a)\bar{G}_o(a) - \epsilon_z M\mathcal{J}_o(a)\bar{S}_o(a)\right] = 0 \tag{30}$$

where

$$\eta^2 = -\delta^2 = k^2\epsilon_2 - \varkappa^2 \tag{31a}$$

$$\varkappa/k = \beta \tag{31b}$$

$$M = k^2(\epsilon_z/\epsilon_r)(\epsilon_r - \beta^2) - \eta_2'^2 \tag{31c}$$

$$S = \frac{k^2\epsilon_z}{\epsilon_r}(\epsilon_r - \beta^2) - \eta_1'^2 \tag{31d}$$

$$\bar{\mathcal{J}}_o(a) = I_o(\delta b)K_o(\delta a) - I_o(\delta a)K_o(\delta b) \tag{31e}$$

$$\bar{G}_o(a) = I_1(\delta b)K_o(\delta a) + I_o(\delta a)K_1(\delta b) \tag{31f}$$

$$\overline{S_o(a)} = I_1(\delta b)K_1(\delta a) - I_1(\delta a)K_1(\delta b) \tag{31g}$$

$$\overline{C_o(a)} = I_1(\delta a)K_o(\delta b) + I_o(\delta b)K_1(\delta a) \tag{31h}$$

In connection with the solution of the dispersion relation (30), appropriate for slow waves, no general discussion can be made. Only a numerical solution subject to the expressions for $\eta_1'^2$ and $\eta_2'^2$ given in (5), can give the actual nature of slow wave propagation.

The dispersion relation (30) will be solved subsequently for a special case stated in (14a), for which $u < 0$, η'_1 is real and η'_2 is imaginary.

For Zero Magnetic Field (with n = 0)

It has been shown in Appendix C that in this special case the dispersion relation has the following particular form (for surface waves)

$$\frac{-\delta\,\epsilon_z J_1(\eta'_1 a)}{\eta'_1\,\epsilon_2 J_o(\eta'_1 a)} = \frac{I_1(\delta a)K_o(\delta b) + I_o(\delta b)K_1(\delta a)}{I_o(\delta b)K_o(\delta a) - I_o(\delta a)K_o(\delta b)} \tag{32$^+$}$$

where

$$\eta_1'^2 = k^2\epsilon_z - \varkappa^2 = -\rho^2 \tag{33a}$$

$$\eta^2 = k^2\epsilon_2 - \varkappa^2 = \delta^2 \tag{33b}$$

$^+$This result agrees with that obtained in [7].

In this case of isotropic plasma only an E-type mode is excited, due to the type of the source of excitation chosen. Since ϵ_z is always less than 1 and $\epsilon_2 > 1$, it can be shown from (33a) and (33b) that when η is imaginary, η'_1 is also imaginary. In this circumstance a surface wave is possible if $\epsilon_z < 0$, moreover, since the left hand side of (32) is greater than unity, $\rho/\delta < 1$, and $\frac{I_1(\rho a)}{I_0(\rho a)} < 1$, it is also necessary that $|\epsilon_z|/\epsilon_2 > 1$, i.e., $\omega < \frac{\omega_p}{\sqrt{1+\epsilon_2}}$.

For Infinite D.C. Magnetic Field in the z-Direction

In this case the dispersion relation for slow waves has the same form as (32), with $\eta'^2_1 = \epsilon_z \eta^2 = -\epsilon_z \delta^2$. It can be shown, [7], that slow wave is possible if $\epsilon_z < 0$, which makes η'_1 real. For zero or infinite d.c. magnetic field in the axial direction, an elaborate investigation has been made in [7] .

Propagation of Slow Waves in an Infinitely Long Column of Plasma Embedded in an Unbounded Medium, with an Axial Uniform Static Magnetic Field

To investigate all related results a' the nature of slow wave propagation, in this case, it is only necessary to let $b \to \infty$ in the corresponding results of the waveguide problem, with $\eta^2 = -\delta^2 = k^2\epsilon_2 - \mathcal{X}^2$. In this case the dispersion relation takes the following form which is equivalent to equation (22b) of Appendix C (with $\mu_r = \mu_z = 1$, $\mu' = 0$, $\mu_2 = 1$)

$$\frac{\epsilon_z}{\eta'_1\eta'_2}\;\frac{J_1(\eta'_1 a)J_1(\eta'_2 a)}{J_o(\eta'_1 a)J_o(\eta'_2 a)} + \frac{\epsilon_2}{\delta^2}\cdot\frac{K_1^2(\delta a)}{K_o^2(\delta a)}$$

$$+ \frac{\epsilon_2}{\eta'_1\eta'_2\,\delta(\eta'^2_2-\eta'^2_1)}\;\frac{K_1(\delta a)}{K_o(\delta a)}\left[\eta'_2 S\,\frac{J_1(\eta'_1 a)}{J_o(\eta'_1 a)} - \eta'_1 M\,\frac{J_1(\eta'_2 a)}{J_o(\eta'_2 a)}\right]$$

$$+ \frac{\epsilon_z}{n'_1 n'_2\,\delta(\eta'^2_2-\eta'^2_1)}\;\frac{K_1(\delta a)}{K_o(\delta a)}\left[\eta'_1 S\,\frac{J_1(\eta'_2 a)}{J_o(\eta'_2 a)} - \eta'_2 M\,\frac{J_1(\eta'_1 a)}{J_o(\eta'_1 a)}\right] = 0 \qquad (34)^+$$

The following identities are found useful

$$\epsilon_r^2(1-\epsilon_z)(S\eta'^2_2 + M\eta'^2_1) = \epsilon'^2\epsilon_z(k^2+\varkappa^2)(k^2\epsilon_z+\varkappa^2) \qquad (35a)$$

and

$$\epsilon_r^2(1-\epsilon_z)(S\eta'^2_2 - M\eta'^2_1) = \epsilon_r\epsilon_z(S-M)\left[(k^2\epsilon_r-\varkappa^2)(1+\epsilon_r) - k^2\epsilon'^2\right] \qquad (35b)$$

To derive relation (34) it also has been assumed that $\frac{\partial}{\partial\theta} \equiv 0$, which may be interpreted as following from taking the excitation to be a constant magnetic current ring source.

When the d.c. magnetic field is either zero or infinity, only the E-type mode is excited due to the type of source chosen here. In this particular case the dispersion relation takes the following form

---

+ This result agrees with the corresponding result in [4], when the identities (35a) and (35b) are used (with $\epsilon_2 = 1$). It should be noted, however, that to obtain radiated fields in this configuration the present limiting process is not valid.

$$\frac{\delta\epsilon_z J_1(\eta'_1 a)}{\eta'_1\epsilon_2 J_o(\eta'_1 a)} = -\frac{K_1(\delta a)}{K_o(\delta a)} \tag{36}$$

If the axial static magnetic field is zero, $\epsilon_r = \epsilon_z$, and $\eta'^2_1 = k^2\epsilon_z - \mathcal{K}^2$, the condition of slow wave is exactly the same as stated in connection with the similar situation in the metallic wave guide, namely, $\epsilon_z < 0$, and $|\epsilon_z| / \epsilon_2 > 1$.

On the other hand, if the magnetic field is infinity, a slow wave is possible if $\epsilon_z < 0$ and $\eta'^2_1 = \epsilon_z\, \delta^2$.

<u>Static limit</u>: It can be shown that in the static limit with $b \to \infty$, the dispersion relation (21) reduces to the following expression

$$\frac{\epsilon_r \eta'_1}{\epsilon_2 \mathcal{K}} \quad \frac{J_1(\eta'_1 a)}{J_o(\eta'_1 a)} = \frac{K_1(\mathcal{K}a)}{K_o(\mathcal{K}a)} \tag{37}$$

where

$$\left.\begin{aligned} \eta'^2_1 &\approx -\frac{\epsilon_z}{\epsilon_r}\, \mathcal{K}^2 \\ \delta &\approx \mathcal{K} \end{aligned}\right\} \tag{38}$$

If $\epsilon_z/\epsilon_r < 0$, then η'_1 is real and a solution to (37) is possible for a slow wave.

If $\epsilon_z/\epsilon_r > 0$, then η'_1 is imaginary, i.e., $\eta'_1 = -j\rho$, where $\rho > 0$. In this case (37) transforms to the following form

$$-\frac{\epsilon_r \rho I_1(a\rho)}{\epsilon_2 \varkappa I_0(a\rho)} = \frac{K_1(\varkappa a)}{K_0(\varkappa a)} \tag{39}$$

Since $\frac{K_1(\varkappa a)}{K_0(\varkappa a)} > 1$ and $\frac{I_1(a\rho)}{I_0(a\rho)} < 1$, it is necessary in order for (39) to have any solution that $\epsilon_r < 0$ and also $\frac{|\epsilon_r| \rho}{\epsilon_2} > 1$. It has already been stated above that $\epsilon_z/\epsilon_r > 0$, therefore, $\epsilon_z < 0$ and $\epsilon_r < 0$.

If one writes $\epsilon_r = \frac{x+y-1}{x-1}$, $\epsilon_z = 1-y$, as defined in (7), then the conditions $\epsilon_z < 0$, $\epsilon_r < 0$ and $\frac{|\epsilon_r|\rho}{\epsilon_2} > 1$, become equivalent to

$$\left.\begin{array}{l} \frac{(y-1)(x+y-1)}{1-x} > \epsilon_2^2 \\ y - 1 > 0 \\ 1 - x > 0 \end{array}\right\} \tag{40}$$

Note: The discussion on page 497 of [4] of the situation where u = 0, i.e., $\eta'_2 = 0$ seems to be inconsistent. The first reason is that when $\eta'_2 = 0$, it can be shown from the general expressions for $\underline{E}_t$ and $\underline{H}_t$ appearing in Chapter I, as well as in Appendix A, that electromagnetic waves which can exist under such a situation are TEM only. In this statement it is also assumed that the diagonal components of $\underset{\sim}{\epsilon}$ are finite and non-zero. But a TEM wave cannot exist in a structure considered by the authors of [4]. A similar inconsistency appears on pp 183 - 185 of [6], discussed by Agdur.

Secondly, the authors of [4] consider a case when $y \to \infty$ (and $\eta'_2 = 0$), and simplify the dispersion relation to an expression containing a logarithmic term, although the modified Bessel's functions $K_1(z)$ and $K_o(z)$ do not behave as logarithmic functions for large arguments.

Any other interesting situation can be studied by considering the corresponding dispersion relation given in Appendix C.

Expressions for E_z Which is Independent of Angular Variation (i.e., n = 0)

Since E_z plays an important role in a plasma, its expression will be given here for n = 0. $|E_z|$ will also be calculated numerically as a function of r for a special case of slow wave.

$$E_z = -j\omega\epsilon_o\epsilon_2 \frac{cm}{2} \sum_i \frac{e^{\mp j\mathcal{X}_i z}\, \xi^*_{2i} C^*_{io}(c)}{\eta^*_i (\eta'^2_2 - \eta'^2_1) \sum_{\ell=1} F_{i\ell} I_{i\ell}} \left[S_i \xi_{1i} J_o(\eta'_{2i} r) - M_i J_o(\eta'_1 r) \right] \quad \text{for } 0 \leq r \leq a \tag{41}$$

$$\xi_{2i} = \frac{S_i \xi_{1i} J_o(\eta'_{2i} a) - M_i J_o(\eta'_{1i} a)}{(\eta'^2_{2i} - \eta'^2_{1i})\, \mathcal{J}_{io}(a)} \tag{42a}$$

$$S_i = \frac{\epsilon_z}{\epsilon_r} \cdot (k^2\epsilon_r - \mathcal{X}^2_i) - \eta'^2_{1i} \tag{42b}$$

$$M_i = \frac{\epsilon_z}{\epsilon_r} \cdot (k^2\epsilon_r - \mathcal{X}^2_i) - \eta'^2_{2i} \tag{42c}$$

$$F_{i1} = L_i \epsilon_r \epsilon_z \left|\eta'_{1i}\right|^2 \left[\epsilon_r \overset{*}{\eta}{}'^2_{2i} \overset{*}{M}_i \overset{*}{R}_i - \omega \mu_o \eta'^2_{2i} \overset{*}{R}{}'_i \mathscr{H}_i \epsilon'\right] \tag{42d}$$

$$F_{i2} = L_i \epsilon_r \epsilon_z \overset{*}{\eta}{}'_{1i} \overset{*}{\eta}{}'_{2i} \overset{*}{\xi}_{1i} \left[\omega \mu_o \eta'^2_{2i} \overset{*}{T}{}'_i \mathscr{H}_i \epsilon' - R_i \epsilon_r \overset{*}{\eta}{}'^2_{1i} \overset{*}{S}_i\right] \tag{42e}$$

$$F_{i3} = L_i \epsilon_r \epsilon_z \overset{*}{\eta}{}'_{1i} \eta'_{2i} \xi_{1i} \left[\epsilon_r \overset{*}{\eta}{}'^2_{2i} \overset{*}{M}_i T_i + \omega \mu_o \mathscr{H}_i \epsilon' \eta'^2_{1i} \overset{*}{R}{}'_i\right] \tag{42f}$$

$$F_{i4} = - L_i \epsilon_r \epsilon_z \left|\eta'_{2i}\right|^2 \left|\xi_{1i}\right|^2 \left[\omega \mu_o \eta'^2_{1i} \overset{*}{T}{}'_i \mathscr{H}_i \epsilon' + \epsilon_r T_i \overset{*}{\eta}{}'^2_{1i} S_i\right] \tag{42g}$$

$$F_{i5} = \frac{\omega \mathscr{H}_i}{\left|\eta_i\right|^2} \left[\epsilon_o \epsilon_2 \left|\xi_{2i}\right|^2 \left|N_o(\eta_i b)\right|^2 + \mu_o \left|\xi_{3i}\right|^2 \left|N_1(\eta_i b)\right|^2\right] \tag{42h}$$

$$F_{i6} = - \frac{\omega \mathscr{H}_i}{\left|\eta_i\right|^2} \left[\epsilon_o \epsilon_2 \left|\xi_{2i}\right|^2 \overset{*}{J}_o(\eta_i b) N_o(\eta_i b) + \mu_o \left|\xi_{3i}\right|^2 N_1(\eta_i b) \overset{*}{J}_1(\eta_i b)\right] \tag{42i}$$

$$F_{i7} = - \frac{\omega \mathscr{H}_i}{\left|\eta_i\right|^2} \left[\epsilon_o \epsilon_2 \left|\xi_{2i}\right|^2 J_o(\eta_i b) \overset{*}{N}_o(\eta_i b) + \mu_o \left|\xi_{3i}\right|^2 \overset{*}{N}_1(\eta_i b) J_1(\eta_i b)\right] \tag{42j}$$

$$F_{i8} = \frac{\omega \mathscr{H}_i}{\left|\eta_i\right|^2} \left[\epsilon_o \epsilon_2 \left|\xi_{2i}\right|^2 \left|J_o(\eta_i b)\right|^2 + \mu_o \left|\xi_{3i}\right|^2 \left|J_1(\eta_i b)\right|^2\right] \tag{42k}$$

$$\xi_{3i} = \frac{j\omega \epsilon_o \epsilon_z \epsilon' \mathscr{H}_i}{(\eta'^2_{2i} - \eta'^2_{1i}) \epsilon_r G_o(a)} \left[\xi_{1i} J_o(\eta'_{2i} a) - J_o(\eta'_{1i} a)\right] \tag{42l}$$

$$L_i = \frac{\omega^2 \epsilon_o^2 \epsilon_z^2 \mathscr{H}_i \epsilon'}{\epsilon_r^4 \left|\eta'^2_{1i} \eta'^2_{2i} (\eta'^2_{2i} - \eta'^2_{1i})\right|^2} \tag{42m}$$

$$R_i = \frac{\epsilon_r M_i (k^2 \epsilon_r - \mathscr{H}_i^2) - k^4 \epsilon_z \epsilon'^2}{\omega \epsilon_o \epsilon_z \epsilon'} \tag{43a}$$

$$R_i' = \frac{\epsilon_r M_i \mathscr{K}_i - \mathscr{K}_i \epsilon_z (k^2 \epsilon_r - \mathscr{K}_i^2)}{\epsilon_z} \tag{43b}$$

$$\mathscr{S}_{io}(r) = J_o(\eta_i b) N_o(\eta_i r) - J_o(\eta_i r) N_o(\eta_i b) \tag{43c}$$

$$S_{io}(a) = J_1(\eta_i b) N_1(\eta_i a) - J_1(\eta_i a) N_1(\eta_i b) \tag{43d}$$

$$G_{io}(a) = J_1(\eta_i b) N_o(\eta_i a) - J_o(\eta_i a) N_1(\eta_i b) \tag{43e}$$

$$C_{io}(c) = J_1(\eta_i c) N_o(\eta_i b) - J_o(\eta_i b) N_1(\eta_i c) \tag{43f}$$

$$I_{11} = \int_0^a J_1(\eta'_{1i} r) \overset{*}{J}_1(\eta'_{1i} r)\, r dr \tag{44a}$$

$$I_{12} = \int_0^a J_1(\eta'_{1i} r) \overset{*}{J}_1(\eta'_{2i} r)\, r dr \tag{44b}$$

$$I_{13} = \int_0^a J_1(\eta'_{2i} r) \overset{*}{J}_1(\eta'_{1i} r)\, r dr \tag{44c}$$

$$I_{14} = \int_0^a J_1(\eta'_{2} r) \overset{*}{J}_1(\eta'_{2i} r)\, r dr \tag{44d}$$

$$I_{15} = \int_a^b J_1(\eta_i r) \overset{*}{J}_1(\eta_i r)\, r dr \tag{44e}$$

$$I_{16} = \int_a^b J_1(\eta_i r) \overset{*}{N}_1(\eta_i r)\, r dr \tag{44f}$$

$$I_{i7} = \int_a^b \overset{*}{J}_1(\eta_i r) N_1(\eta_i r)\, r dr \tag{44g}$$

$$I_{i8} = \int_a^b N_1(\eta_i r) \overset{*}{N}_1(\eta_i r)\, r dr \tag{44h}$$

III

PROPAGATION OF SLOW WAVES IN AN ANISOTROPIC FERRITE

In this chapter the general problem solved in Chapter I will be applied to the study of wave propagation in an infinitely long anisotropic ferrite column enclosed in a dielectric medium, which is again enclosed by a perfectly conducting metallic cylindrical waveguide. All other conditions are similar to those described in the previous chapters. In a ferrite medium with an axial static magnetic field, anisotropy is exhibited by the following dyadic form [2] of $\underset{\sim}{\mu}$

$$\underset{\sim}{\mu} \rightarrow \begin{Vmatrix} \mu_{rr} & j\mu_{r\theta} & 0 \\ -j\mu_{\theta r} & \mu_{\theta\theta} & 0 \\ 0 & 0 & \mu_{zz} \end{Vmatrix} \tag{1}$$

where $\mu_{rr} = \mu_{\theta\theta} = \mu_r = 1 - \dfrac{\sigma P}{1-\sigma^2}$ (2a)

$$\mu_{r\theta} = \mu_{\theta r} = \mu' = -\frac{P}{1-\sigma^2} \tag{2b}$$

$$\mu_{zz} = 1 \tag{2c}$$

$$P = |\gamma| \frac{M_o}{\omega\mu_o} \tag{3a}$$

$$\sigma = |\gamma| \frac{H_o}{\omega} \tag{3b}$$

M_o = d.c. magnetization

H_o = d.c. magnetic intensity

γ = gyromagnetic ratio for electron

It can be shown from (2) that μ_r and μ' are related by the following relation

$$\mu_r - 1 = \sigma\mu' \tag{4}$$

It will be assumed that the relative dielectric constant of the ferrite is ϵ_1, a scalar quantity and the medium surrounding the ferrite has relative dielectric constant ϵ_2 and relative permeability 1.

With the above assumptions, the magnetic fields, dispersion relations etc. for this problem under any limiting conditions can be easily derived from the corresponding results given in Chapter I and Appendix C. Therefore, no detailed discussion will be given in this chapter.

It may be noted here that when a ferrite column or a plasma column is situated in an unbounded dielectric medium, the boundary conditions for E and H in both cases are identical, consequently any general expression for one situation can be derived from the other, using the duality, provided μ_z is not replaced by unity in any general expression.

Expressions for Transverse Wave Numbers

If the values for $\underset{\sim}{\mu}$ and $\underset{\sim}{\epsilon}$ given above are substituted in equations (34) and (35) of Chapter I, one obtains the following expressions

$$\eta'^2_{1,2} = V \pm \sqrt{V^2 - U} \tag{5a}+$$

$$= \frac{k^2\epsilon_1(\mu_r^2 - \mu'^2) - \mu_r\varkappa^2}{\mu_r} - \frac{\mu'}{2\mu_r}\left\{k^2\epsilon_1(\sigma\mu_r - \mu') - \varkappa^2\sigma\right\}$$

$$\pm \frac{\mu'}{2\mu_r}\left[\left\{k^2\epsilon_1(\sigma\mu_r - \mu') - \varkappa^2\sigma\right\}^2 + 4k^2\varkappa^2\epsilon_1\right]^{1/2} \tag{5b}$$

$$V = \frac{k^2\epsilon_1(\mu_r^2 - \mu'^2 + \mu_r) - \varkappa^2(\mu_r + 1)}{2\mu_r} \tag{6a}$$

$$U = \frac{1}{\mu_r}\left[(k^2\epsilon_1\mu_r - \varkappa^2)^2 - k^4\epsilon_1^2\mu'^2\right] \tag{6b}$$

For slow surface waves the following condition should be satisfied

$$\left.\begin{aligned} \frac{\varkappa^2}{k^2} &= \beta^2 > \epsilon_2 \\ \eta &= -j\delta, \quad \delta > 0 \end{aligned}\right\} \tag{7}$$

+ For ferrite problem the relation $\eta_1' = \eta_2'$ cannot be satisfied, since $V^2 - U > 0$.

Since $\mu_z = 1$, equations (5) show that $V^2 - U > 0$ and $\eta_1'^2$ and $\eta_2'^2$ are real for real $\underset{\sim}{\mu}$, ϵ_1 and ϵ_2. Therefore, no instability phenomena appears in the case of a ferrite column. It may be noted that η_1' and η_2' may assume purely imaginary values as follows:

<u>Case I</u>
$$\left.\begin{array}{l}\eta_1'^2 > 0 \\ \eta_2'^2 < 0\end{array}\right\} \text{ if } U < 0,\ V > 0, \text{ or } V < 0 \tag{8a}$$

<u>Case II</u>
$$\left.\begin{array}{l}\eta_1'^2 > 0 \\ \eta_2'^2 > 0\end{array}\right\} \text{ if } U > 0, \text{ and } V > 0 \tag{8b}$$

<u>Case III</u>
$$\left.\begin{array}{l}\eta_1'^2 < 0 \\ \eta_2'^2 < 0\end{array}\right\} \text{ if } U > 0, \text{ and } V < 0 \tag{8c}$$

The above information and the general results given in Chapter I and Appendix C are sufficient to obtain any particular result for a ferrite column.

It may be noted that an electric current dipole source is more appropriate for a ferrite problem than a magnetic current ring source. The field expressions for an electric dipole source can be obtained easily by using the appropriate dyadic Green's functions developed in Appendix B.

IV CONCLUSIONS

In conclusion it may be mentioned that the work described here gives a systematic rigorous approach of solving a source problem (not necessarily a ring source) involving a homogeneous anisotropic cylindrical structure bounded by conductors. Since the source free solutions are capable of representing all possible modes for the structure of the problem, total fields due to any arbitrary source of any kind (namely electric or magnetic current source) can be calculated by using the appropriate dyadic Green's function. If there is more than one source, the total fields can be obtained by using the superposition theorem, provided there are no interactions among the individual sources. As the present analysis considers a magnetic current ring source of arbitrary angular variation, the results can be used for any given angular variation of the source.

From the general dispersion relation which is an eigenvalue equation and independent of source, various interesting special cases, some of which are already known, have been studied. The limiting proce :es used in obtaining the dispersion relations for these special cases can also be used to obtain the expressions for the fields in the corresponding situations.

The analysis for the plasma problem which is a special case of a general anisotropic medium characterized by dyadics $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$, emphasizes the slow wave propagation. Here the necessary conditions for slow wave propagation, including a number of special cases, have been obtained from the expressions for the transverse wave numbers. These necessary conditions also give maximum passbands. The sufficient conditions and hence the actual passbands can be obtained from the solution of the dispersion relation. For slow wave propagation, some passbands depend on the degree of slowness of the waves. The degree of slowness of the waves depends on the relative dielectric constant, ϵ_2, of the medium surrounding the plasma column, when all other parameters are kept constant. It has been shown that the higher the value of ϵ_2, the slower the phase velocity of the wave. In other words, for a given ϵ_2 there is a minimum phase velocity for which a corresponding slow-surface wave can propagate. Most of the energy of the slow waves considered here is confined into the anisotropic plasma. In general, the lower the phase velocity of the surface wave, the lower the amplitude. Not all the various passbands for slow wave propagation in an anisotropic plasma column, mentioned above, are known in the literature, at least to the best knowledge of the author. Although these passbands could be obtained without any consideration of the presence of a source.

In the case of an unbounded homogeneous anisotropic medium where a TEM wave can propagate, conditions of TEM wave propagation in the direction parallel to or perpendicular to the static magnetic field are obtained from the general expressions of the transverse wave numbers. It is also shown that the condition of

TEM wave propagation in the direction of the static magnetic field is equivalent to the vanishing condition of the product of the transverse wave numbers. This also establishes the fact that the consideration of the situation under which the product of the transverse wave numbers vanishes is not justified in connection with the wave propagation in a bounded medium which cannot support TEM waves. In other words, if a bounded isotropic medium cannot support TEM waves, so also is the case for a bounded anisotropic medium. For not being able to recognize the fact that the condition of TEM wave propagation in the direction of the static magnetic field, is equivalent to the zero-value of the product of the two transverse wave numbers, some authors+ discussed the possibility of wave propagations in a bounded anisotropic plasma column, under the situation for which the product of the two transverse wave numbers vanishes.

+ Agdur, pages 183-185 of Ref. [6] and the authors of Ref. [4], page 497.

APPENDIX A

MAXWELL'S EQUATIONS FOR ANISOTROPIC MEDIUM

The medium to be described here is characterized by a relative dielectric (permittivity) tensor $\underset{\sim}{\epsilon}$ and a relative permeability tensor $\underset{\sim}{\mu}$ having the following particular form

$$\underset{\sim}{\epsilon} = \begin{Vmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{Vmatrix}, \quad \epsilon_{11} = \epsilon_{22} \tag{1}$$

$$\underset{\sim}{\mu} = \begin{Vmatrix} \mu_{11} & j\mu_{12} & 0 \\ -j\mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{Vmatrix}, \quad \mu_{11} = \mu_{22} \tag{2}$$

The above representations show, in both cases, that the transverse components and longitudinal components of the tensors are uncoupled, where ϵ_{33} and μ_{33} correspond to the longitudinal, a preferred direction (say z-direction) components of the tensors $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ respectively. Therefore, $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ can be written in the following manner also

$$\underset{\sim}{\epsilon} = \underset{\sim}{\epsilon}_{tt} + \underline{z}_o \underline{z}_o \epsilon_{33} \tag{3}$$

$$\underset{\sim}{\mu} = \mu_{tt} + \underline{z}_o \underline{z}_o \mu_{33} \tag{4}$$

where ϵ_{tt} and μ_{tt} are tensors (transverse to z-direction) and $\underline{z}_o$ is a unit vector in the z-direction.

The Maxwell's equations for anisotropic media with sources have the following form (the time dependence being $e^{j\omega t}$).

$$\nabla \times \underline{E}(\underline{r}) = -j\omega\mu_o \underset{\sim}{\mu}(\underline{r}) \cdot \underline{H}(\underline{r}) - \underline{I}_m(\underline{r}') \tag{5}$$

$$\nabla \times \underline{H}(\underline{r}) = j\omega\epsilon_o \epsilon(\underline{r}) \cdot \underline{E}(\underline{r}) + \underline{I}_e(\underline{r}'') \tag{6}$$

$$\nabla \cdot \underset{\sim}{\mu}(r) \cdot \underline{H}(\underline{r}) = 0, \qquad \text{for } \underline{r} \neq \underline{r}' \tag{7}$$

$$\nabla \cdot \epsilon(\underline{r}) \cdot \underline{E}(\underline{r}) = 0, \qquad \text{for } \underline{r} \neq \underline{r}'' \tag{8}$$

where,

$\underline{r}$ = observation position vector (3-dimensional)

$\underline{I}_m(\underline{r}')$ = magnetic current source at $\underline{r} = \underline{r}'$

$\underline{I}_e(\underline{r}'')$ = electric current source at $\underline{r} = \underline{r}''$

In the following discussion the transverse vector and transverse operators (which are designated by the subscript t) correspond to any plane transverse to the z-direction, and the transverse plane may have any arbitrary cross section. That is, the following derivations are suitable for any cylindrical geometry having the z-direction as its axis.

First of all, it will be shown (see [10], [11], [12], and [14]) that the longitudinal fields, E_z and H_z, can be expressed in terms of the transverse fields, $\underline{E}_t$ and $\underline{H}_t$. Secondly, it will be demonstrated that the transverse fields can also be expressed from the knowledge of the longitudinal fields. In the particular problem discussed in the text, the latter method has been adopted for the solutions of Maxwell's equations.

If equations (5) and (6) are multiplied by $\underline{z}_o$ in a scalar product fashion, E_z and H_z can be expressed in the following way

$$E_z = \frac{1}{j\omega\epsilon_o\epsilon_{33}} \nabla_t \cdot \underline{H}_t \times \underline{z}_o - \frac{I_{ez}}{j\omega\epsilon_o\epsilon_{33}} \qquad (9)$$

$$H_z = \frac{1}{j\omega\epsilon_o\mu_{33}} \nabla_t \cdot \underline{z}_o \times \underline{E}_t - \frac{I_{mz}}{j\omega\mu_o\mu_{33}} \qquad (10)$$

where I_{ez} and I_{mz} are the z-components of $\underline{I}_e$ and $\underline{I}_m$ respectively.

Now taking the vector product of (5) and (6) with $\underline{z}_o$, one obtains

$$\underline{z}_o \times (\nabla \times \underline{E}) = -j\omega\mu_o\underline{z}_o \times \underset{\sim}{\mu} \cdot \underline{H} + \underline{I}_m \times \underline{z}_o \tag{11}$$

$$\underline{z}_o \times (\nabla \times \underline{H}) = j\omega\epsilon_o\underline{z}_o \times \underset{\sim}{\epsilon} \cdot \underline{E} + \underline{z}_o \times \underline{I}_e \tag{12}$$

Introducing

$$\nabla = \nabla_t + \underline{z}_o \frac{\partial}{\partial z} ,$$

$$\underline{E} = \underline{E}_t + \underline{z}_o E_z$$

and

$$\underline{H} = \underline{H}_t + \underline{z}_o H_z ,$$

one obtains

$$\underline{z}_o \times (\nabla \times \underline{E}) = \nabla_t E_z - \frac{\partial}{\partial z} \underline{E}_t$$

and

$$\underline{z}_o \times (\nabla \times \underline{H}) = \nabla_t H_z - \frac{\partial}{\partial z} \underline{H}_t$$

Now equations (11) and (12) can be rewritten as

$$\nabla_t E_z - \frac{\partial}{\partial z} \underline{E}_t = -j\omega\mu_o\underline{z}_o \times \underset{\sim}{\mu}_t \cdot \underline{H}_t + \underline{I}_{mt} \times \underline{z}_o \tag{13}$$

and

$$\nabla_t H_z - \frac{\partial}{\partial z} \underline{H}_t = j\omega\epsilon_o\underline{z}_o \times \underset{\sim}{\epsilon}_t \cdot \underline{E}_t + \underline{z}_o \times \underline{I}_{et} \tag{14}$$

where $\underline{I}_{mt}$ and $\underline{I}_{et}$ are transverse components of $\underline{I}_m$ and $\underline{I}_e$ respectively.

Again operating (14) by $j\omega\mu_o\underline{z}_o \times \underset{\sim}{\mu}_t$ and (13) by $\frac{\partial}{\partial z}$ from the left one obtains the following expressions

$$\frac{\partial}{\partial z} \nabla_t E_z - \frac{\partial^2}{\partial z^2} \underline{E}_t = -j\omega\mu_o \frac{\partial}{\partial z} (\underline{z}_o \times \underset{\sim}{\mu}_t \cdot \underline{H}_t) + \frac{\partial}{\partial z} \underline{I}_{mt} \times \underline{z}_o \tag{15}$$

and
$$j\omega\mu_o \underline{z}_o \times (\mu_t \cdot \nabla_t H_z) - j\omega\mu_o \frac{\partial}{\partial z} (\underline{z}_o \times \underset{\sim}{\mu}_t \cdot \underline{H}_t)$$

$$= -k^2 \underline{z}_o \times \left[\underset{\sim}{\mu}_t \cdot \underline{z}_o \times (\underset{\sim}{\epsilon}_t \cdot \underline{E}_t)\right] + j\omega\mu_o \underline{z}_o \times (\underset{\sim}{\mu}_t \cdot \underline{z}_o \times \underline{I}_{et}) \tag{16}$$

where $k^2 = \omega^2 \mu_o \epsilon_o$.

Adding (15) and (16), one can show that

$$\frac{\partial}{\partial z} \nabla_t E_z + j\omega\mu_o \underline{z}_o \times (\underset{\sim}{\mu}_t \cdot \nabla_t H_z) + \frac{\partial}{\partial z} \underline{z}_o \times \underline{I}_{mt} + (\mu_{11} \underset{\sim}{1}_t - j\mu_{12} \underline{z}_o \times \underset{\sim}{1}_t) \cdot \underline{I}_{et}$$

$$= a_4 \underline{E}_t - j k^2 a_3 \underline{z}_o \times \underline{E}_t \tag{17}$$

where $\underset{\sim}{1}_t$ = transverse unit dyadic (18a)

$$a_3 = \mu_{11}\epsilon_{12} + \mu_{12}\epsilon_{11} \tag{18b}$$

$$a_4 = k^2(\epsilon_{11}\mu_{11} + \epsilon_{12}\mu_{12}) + \frac{\partial^2}{\partial z^2} \tag{18c}$$

$$(\mu_{11} \underset{\sim}{1}_t - j\mu_{12} \underline{z}_o \times \underset{\sim}{1}_t) \cdot \underline{I}_{et} = -\underline{z}_o \times (\underset{\sim}{\mu}_t \cdot \underline{z}_o \times \underline{I}_{et}) \tag{18d}$$

Taking the vector product of (17) with $\underline{z}_o$, one obtains the following independent equation:

$$\frac{\partial}{\partial z}\, \underline{z}_o \times \nabla_t E_z - j\omega\mu_o \underset{\sim}{\mu}_t \cdot \nabla_t H_z - \frac{\partial}{\partial z}\, \underline{I}_{mt} + (j\mu_{12} \underset{\sim}{1}_t + \mu_{11} \underline{z}_o \times \underset{\sim}{1}_t) \cdot \underline{I}_{et}$$

$$= a_4 \underline{z}_o \times \underline{E}_t + j k^2 a_3 \underline{E}_t \tag{19}$$

Eliminating $\underline{z}_o \times \underline{E}_t$ from (17) and (19) by multiplying (17) by a_4 from the left and (19) by $j k^2 a_3$ and then adding the results, $\underline{E}_t$ can be expressed in terms of E_z, H_z, $\underline{I}_{mt}$ and $\underline{I}_{et}$ in the following way:

$$p_1 \underline{E}_t = \left[- a_4 \frac{\partial}{\partial z} \nabla_t E_z - \omega\mu_o a_2' \nabla_t H_z \right] - \underline{z}_o \times \left[j k^2 a_3 \frac{\partial}{\partial z} \nabla_t E_z + j\omega\mu_o a_1' \nabla_t H_z \right]$$

$$- \left[a_1' \underset{\sim}{1}_t + j a_2' \underline{z}_o \times \underset{\sim}{1}_t \right] \cdot \underline{I}_{et}$$

$$- \left[a_4 \frac{\partial}{\partial z} \underline{z}_o \times \underset{\sim}{1}_t - j k^2 a_3 \frac{\partial}{\partial z} \underset{\sim}{1}_t \right] \cdot \underline{I}_{mt} \tag{20}$$

where the following relations have been used.

$$p_1 = k^4 a_3^2 - a_4^2 \tag{21a}$$

$$a_1' = a_4 \mu_{11} - k^2 a_3 \mu_{12} = k^2 \epsilon_{11} (\mu_{11}^2 - \mu_{12}^2) + \mu_{11} \frac{\partial^2}{\partial z^2} \tag{21b}$$

$$a_2' = k^2 a_3 \mu_{11} - a_4 \mu_{12} = k^2 \epsilon_{12} (\mu_{11}^2 - \mu_{12}^2) - \mu_{12} \frac{\partial^2}{\partial z^2} \tag{21c}$$

and the identities

$$\mu_t \cdot \nabla_t H_z = \mu_{11} \nabla_t H_z - j\mu_{12} \underline{z}_o \times \nabla_t H_z \tag{22a}$$

$$\underline{z}_o \times \underset{\sim}{\mu}_t \cdot \nabla_t H_z = \mu_{11} \underline{z}_o \times \nabla_t H_z + j\mu_{12} \nabla_t H_z \tag{22b}$$

By a similar method (or by duality of (20)), one can obtain the expressions for $\underline{H}_t$ in terms of H_z, E_z, $\underline{I}_{mt}$ and $\underline{I}_{et}$ in the following form:

$$\begin{aligned} p_1 \underline{H}_t = & \left[- a_4 \frac{\partial}{\partial z} \nabla_t H_z + \omega \epsilon_o a_2 \nabla_t E_z \right] \\ & - \underline{z}_o \times \left[j k^2 a_3 \frac{\partial}{\partial z} \nabla_t H_z - j\omega \epsilon_o a_1 \nabla_t E_z \right] \\ & - \left[a_1 \underset{\sim}{1}_t + j a_2 \underline{z}_o \times \underset{\sim}{1}_t \right] \cdot \underline{I}_{mt} \\ & + \left[a_4 \frac{\partial}{\partial z} \underline{z}_o \times \underset{\sim}{1}_t - j k^2 a_3 \frac{\partial}{\partial z} \underset{\sim}{1}_t \right] \cdot \underline{I}_{et} \end{aligned} \tag{23}$$

where

$$a_1 = k^2 \mu_{11} (\epsilon_{11}^2 - \epsilon_{12}^2) + \epsilon_{11} \frac{\partial^2}{\partial z^2} \tag{24a}$$

$$a_2 = k^2 \mu_{12} (\epsilon_{11}^2 - \epsilon_{12}^2) - \epsilon_{12} \frac{\partial^2}{\partial z^2} \tag{24b}$$

Although the medium is anisotropic, if it is homogeneous (i.e., components of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are not functions of position, although they may be piecewise constants) and source free (i.e., $\underline{I}_m = 0 = \underline{I}_e$), one can show that $\mathcal{E}_z$ and $\mathcal{H}_z$

satisfy the following equations [using (20) and (23) in (9) and (10)],

$$\nabla_t^2 \mathcal{E}_z + \frac{\epsilon_{33}}{\epsilon_{11}\mu_{11}} a_1' \mathcal{E}_z = \frac{j\omega\mu_o\mu_{33}\mathcal{K}}{\epsilon_{11}\mu_{11}} a_3 \mathcal{H}_z \tag{25}$$

$$\nabla_t^2 \mathcal{H}_z + \frac{\mu_{33}}{\epsilon_{11}\mu_{11}} a_1 \mathcal{H}_z = - \frac{j\omega\epsilon_o\epsilon_{33}\mathcal{K}}{\epsilon_{11}\mu_{11}} a_3 \mathcal{E}_z \tag{26}$$

where $\mathcal{E}_z$ and $\mathcal{H}_z$ are solutions of homogeneous (source-free) Maxwell's equations.

To obtain the above two expressions, it has been assumed that $\frac{\partial}{\partial z} = -j\mathcal{K}$, where $\mathcal{K}$ is the propagation wave number in the z-direction. This assumption is permissible in the situations where both the $\underset{\sim}{\epsilon}$, $\underset{\sim}{\mu}$ and the geometry of the problem are independent of z, subject to another restriction, that tne transverse anisotropy of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are not coupled to the longitudinal anisotropy of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ respectively.

To obtain solutions for $\mathcal{E}_z$ and $\mathcal{H}_z$, it is possible to have 4th degree equations in $\mathcal{E}_z$ and $\mathcal{H}_z$ from (25) and (26) by elimination. Since it is a tedious task to solve such equations, one can alternatively find a function ϕ which is a linear combination of $\mathcal{E}_z$ and $\mathcal{H}_z$, satisfying a two dimensional wave equation. Let such a choice be

$$\phi = \mathcal{E}_z + j\alpha \mathcal{H}_z \tag{27}$$

Now multiplying (26) by $j\alpha$ and then adding the result to (25), one obtains the following relations

$$\nabla_t^2(\mathcal{E}_z + j\alpha\mathcal{H}_z) + \frac{\epsilon_{33}}{\epsilon_{11}\mu_{11}}\left[a_1' - \omega\epsilon_o\mathcal{K}a_3\alpha\right]\mathcal{E}_z + \frac{\mu_{33}}{\epsilon_{11}\mu_{11}}\left[a_1 - \frac{\omega\mu_o\mathcal{K}a_3}{\alpha}\right]j\alpha\mathcal{H}_z = 0 \tag{28}$$

The above equation can be represented as a two dimensional wave equation in ϕ of the following form

$$\nabla_t^2\phi + \eta'^2\phi = 0 \tag{29}$$

where

$$\frac{\epsilon_{33}}{\epsilon_{11}\mu_{11}}\left[a_1' - \omega\epsilon_o\mathcal{K}a_3\alpha\right] = \eta'^2 = \frac{\mu_{33}}{\epsilon_{11}\mu_{11}}\left[a_1 - \frac{\omega\mu_o\mathcal{K}a_3}{\alpha}\right] \tag{30}$$

Solving equation (30) for α, one obtains

$$\alpha_{1,2} = \frac{\epsilon_{33}a_1' - \mu_{zz}a_1 \mp\left[(a_1'\epsilon_{33} - a_1\mu_{33})^2 + 4k^2\mathcal{K}^2a_3^2\epsilon_{33}\mu_{33}\right]^{1/2}}{2\omega\epsilon_o\epsilon_z\mathcal{K}a_3} \tag{31}$$

Therefore, the roots η'^2 can also be expressed in the following form:

$$\eta'^2_{1,2} = \frac{\epsilon_{33}a_1'}{\epsilon_{11}\mu_{11}} - \frac{\epsilon_{33}a_1' - \mu_{33}a_1}{2\epsilon_{11}\mu_{11}} \pm \frac{\left[(a_1'\epsilon_{33} - a_1\mu_{33})^2 + 4k^2\mathcal{K}^2a_3^2\epsilon_{33}\mu_{33}\right]^{1/2}}{2\epsilon_{11}\mu_{11}} \tag{32}$$

The equation (32) can also be rewritten in the following form

$$\eta'^2_{1,2} = V \pm \sqrt{V^2 - U} \tag{33}$$

where

$$V = \frac{a_1' \epsilon_{33} + a_1 \mu_{33}}{2\epsilon_{11}\mu_{11}} = \frac{\eta_1'^2 + \eta_2'^2}{2} \tag{34a}$$

$$U = \frac{\epsilon_{33}\mu_{33}}{\epsilon_{11}\mu_{11}} \left[a_4^2 - k^4 a_3^2\right] = -\frac{\epsilon_{33}\mu_{33}}{\epsilon_{11}\mu_{11}}\, p_1 = \eta_1'^2 \eta_2'^2 \tag{34b}$$

<u>**Relations Between Various Parameters Introduced in the Above Analysis**</u>

First of all, the following new parameters are introduced and defined:

$$S = \frac{\epsilon_{33} a_1'}{\mu_{11}\epsilon_{11}} - \eta_1'^2 = \frac{\omega\epsilon_o \epsilon_{33} \mathcal{X} a_3 \alpha_1}{\mu_{11}\epsilon_{11}} \tag{35a}$$

$$M = \frac{\epsilon_{33} a_1'}{\mu_{11}\epsilon_{11}} - \eta_2'^2 = \frac{\omega\epsilon_o \epsilon_{33} \mathcal{X} a_3 \alpha_2}{\mu_{11}\epsilon_{11}} \tag{35b}$$

$$R = a_4 \mathcal{X} \alpha_2 - \omega\mu_o a_2' \tag{36a}$$

$$T = \omega\mu_o a_2' - \mathcal{X} a_4 \alpha_1 \tag{36b}$$

$$R' = \omega\epsilon_o a_2 \alpha_2 - \mathcal{X} a_4 \tag{37a}$$

$$T' = \omega\epsilon_o a_2 \alpha_1 - \mathcal{X} a_4 \tag{37b}$$

It will be found convenient to carry out algebraic operations in the sequel using the relations listed in the following table. In tabulating the following results, no attempt has been made to write them in such a way that any result is a consequence of those preceding it.

$$\frac{\alpha_1}{\alpha_2} = \frac{S}{M} \tag{38-1}$$

$$\eta_1'^2 = \frac{\epsilon_{33} a_1'}{\mu_{11}\epsilon_{11}} - \frac{\omega\epsilon_o\epsilon_{33}\mathcal{K} a_3\alpha_1}{\mu_{11}\epsilon_{11}} \tag{38-2}$$

$$\eta_2'^2 = \frac{\epsilon_{33} a_1'}{\mu_{11}\epsilon_{11}} - \frac{\omega\epsilon_o\epsilon_{33}\mathcal{K} a_3\alpha_2}{\mu_{11}\epsilon_{11}} \tag{38-3}$$

$$\alpha_1 - \alpha_2 = \frac{\mu_{11}\epsilon_{11}(\eta_2'^2 - \eta_1'^2)}{\omega\epsilon_o\mathcal{K} a_3\epsilon_{33}} \tag{38-4}$$

$$\alpha_1\alpha_2 = -\frac{\mu_o\mu_{33}}{\epsilon_o\epsilon_{33}} \tag{38-5}$$

$$S = \frac{\epsilon_{33} a_1'}{\mu_{11}\epsilon_{11}} - \eta_1'^2 = \frac{\omega\epsilon_o\epsilon_{33}\mathcal{K} a_3\alpha_1}{\mu_{11}\epsilon_{11}} \tag{38-6}$$

$$\alpha_1 = \frac{\epsilon_{11}\mu_{11} S}{\omega\epsilon_o a_3\mathcal{K}\epsilon_{33}} \tag{38-7}$$

$$M = \frac{\epsilon_{33} a_1'}{\mu_{11}\epsilon_{11}} - \eta_2'^2 = \frac{\omega\epsilon_o \epsilon_{33} \mathcal{L} a_3 \alpha_2}{\mu_{11}\epsilon_{11}} \tag{38-8}$$

$$\alpha_2 = \frac{\epsilon_{11}\mu_{11} M}{\omega\epsilon_o \mathcal{L} a_3 \epsilon_{33}} \tag{38-9}$$

$$\frac{\alpha_1}{\alpha_1 - \alpha_2} = \frac{S}{\eta_2'^2 - \eta_1'^2} \tag{38-10}$$

$$\frac{\alpha_2}{\alpha_1 - \alpha_2} = \frac{M}{\eta_2'^2 - \eta_1'^2} \tag{38-11}$$

$$\eta_1'^2 + \eta_2'^2 = \frac{a_1' \epsilon_{33} + a_1 \mu_{33}}{\epsilon_{11}\mu_{11}} \tag{38-12}$$

$$S - M = \eta_2'^2 - \eta_1'^2 \tag{38-13}$$

$$MS = - \frac{k^2 \mathcal{L}^2 a_3^2 \mu_{33} \epsilon_{33}}{\epsilon_{11}^2 \mu_{11}^2} \tag{38-14}$$

$$p_1 = k^4 a_3^2 - a_4^2 = - \frac{\epsilon_{11}\mu_{11}}{\epsilon_{33}\mu_{33}} \quad \eta_1'^2 \quad \eta_2'^2 \tag{38-15}$$

$$SR + MT = - \omega\mu_o a_2' \left(\eta_2'^2 - \eta_1'^2\right) \tag{38-16}$$

$$R = \mathcal{L} a_4 \alpha_2 - \omega\mu_o a_2' = \frac{a_4 M \epsilon_{11}\mu_{11} - k^2 a_3 a_2' \epsilon_{33}}{\omega\epsilon_o a_3 \epsilon_{33}} \tag{38-17}$$

$$T = \omega\mu_o a_2' - \mathcal{H}a_4\alpha_1 = \frac{k^2\epsilon_{33}a_3a_2' - a_4S\epsilon_{11}\mu_{11}}{\omega\epsilon_o a_3\epsilon_{33}} \tag{38-18}$$

$$R' = \omega\epsilon_o a_2\alpha_2 - \mathcal{H}a_4 = \frac{Ma_2\epsilon_{11}\mu_{11} - \mathcal{H}^2a_3a_4\epsilon_{33}}{\mathcal{H}a_3\epsilon_{33}} \tag{38-19}$$

$$T' = \omega\epsilon_o a_2\alpha_1 - \mathcal{H}a_4 = \frac{Sa_2\epsilon_{11}\mu_{11} - \mathcal{H}^2a_3a_4\epsilon_{33}}{\mathcal{H}a_3\epsilon_{33}} \tag{38-20}$$

$$\epsilon_{33}a_1' - \epsilon_{11}\mu_{11}\eta_2'^2 = \epsilon_{11}\mu_{11}M \tag{38-21}$$

$$\epsilon_{33}a_1' - \epsilon_{11}\mu_{11}\eta_1'^2 = \epsilon_{11}\mu_{11}S \tag{38-22}$$

$$\mu_{33}a_1 - \epsilon_{11}\mu_{11}\eta_1'^2 = \frac{k^2\mathcal{H}^2a_3^2\epsilon_{33}\mu_{33}}{S\epsilon_{11}\mu_{11}} = -M\mu_{11}\epsilon_{11} \tag{38-23}$$

$$\mu_{33}a_1 - \epsilon_{11}\mu_{11}\eta_2'^2 = \frac{k^2\mathcal{H}^2a_3^2\epsilon_{33}\mu_{33}}{M\epsilon_{11}\mu_{11}} = -S\epsilon_{11}\mu_{11} \tag{38-24}$$

$$\mathcal{H}^2a_3 = a_2\mu_{11} - a_1\mu_{12} \tag{38-25}$$

$$a_1 = a_4\epsilon_{11} - k^2a_3\epsilon_{12} \tag{38-26}$$

$$\mathcal{H}^2a_3 = a_2'\epsilon_{11} - a_1'\epsilon_{12} \tag{38-27}$$

$$a_1' = a_4\mu_{11} - k^2a_3\mu_{12} \tag{38-28}$$

$$a_2' = k^2 a_3 \mu_{11} - a_4 \mu_{12} \tag{38-29}$$

$$a_2 = k^2 a_3 \epsilon_{11} - a_4 \epsilon_{12} \tag{38-30}$$

$$k^2 a_3 a_2' - a_1' a_4 = \mu_{11} p_1 \tag{38-31}$$

$$\varkappa^2 a_4 a_3 - a_2' a_1 = p_1 \mu_{11} \epsilon_{12} \tag{38-32}$$

$$k^2 a_3 a_2 - a_4 a_1 = \epsilon_{11} \, p_1 \tag{38-33}$$

$$k^2 \varkappa^2 a_3^2 - a_1 a_1' = p_1 \mu_{11} \epsilon_{11} \tag{38-34}$$

$$a_1 = k^2 \mu_{11} (\epsilon_{11}^2 - \epsilon_{12}^2) - \varkappa^2 \epsilon_{11} \tag{38-35}$$

$$a_1' = k^2 \epsilon_{11} (\mu_{11}^2 - \mu_{12}^2) - \varkappa^2 \mu_{11} \tag{38-36}$$

$$a_2 = k^2 \mu_{12} (\epsilon_{11}^2 - \epsilon_{12}^2) + \varkappa^2 \epsilon_{12} \tag{38-37}$$

$$a_2' = k^2 \epsilon_{12} (\mu_{11}^2 - \mu_{12}^2) + \varkappa^2 \mu_{12} \tag{38-38}$$

$$a_3 = \epsilon_{11} \mu_{12} + \mu_{11} \epsilon_{12} \tag{38-39}$$

$$a_4 = k^2 (\mu_{11} \epsilon_{11} + \mu_{12} \epsilon_{12}) - \varkappa^2 \tag{38-40}$$

It may be noted that all of the analyses and results obtained in this Appendix are based on the assumption that the problem has a cylindrical geometry with axis in the z-direction and having an arbitrary cross section which is independent of the coordinate z.

Propagation of TEM Waves in an Unbounded Homogeneous Anisotropic Medium

Since the foregoing analysis does not include any particular boundary, it is valid for an unbounded medium also. Hence, it is possible to obtain conditions for TEM wave propagation in a direction parallel or perpendicular to the z-axis. It may be mentioned here that a plasma and a ferrite with a static uniform magnetic field in the z-direction will have tensor permittivity and tensor permeability respectively. The forms of these tensors are given in equations (1) and (2). As mentioned earlier, here also the medium considered will have both ϵ and μ as tensors with constant elements.

TEM Wave Parallel to the Magnetic Field

For a TEM wave in the z-direction both $E_z = 0$ and $H_z = 0$. If the source terms in equations (20) and (23) are equated to zero, non-vanishing values of $\underline{E}_t$ and $\underline{H}_t$ are possible if and only if $p_1 = 0$, when $E_z = 0 = H_z$, provided the elements ϵ_{11}, ϵ_{33}, μ_{11} and μ_{33} are finite and non-zero. The condition $p_1 = 0$, gives two TEM waves propagating in the z-direction. These two waves are characterized by the following expression of the propagation wave number $\mathcal{K}$ in the z-direction

$$\frac{\mathcal{K}^2}{k^2} = (\epsilon_{11} \pm \epsilon_{12})(\mu_{11} \pm \mu_{12}) , \tag{39}$$

Since p is proportional to $\eta_1'^2 \eta_2'^2$, the condition $\eta_1'^2 \eta_2'^2 = 0$, is equivalent to TEM wave propagation in the z-direction, provided the diagonal elements of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are finite and non-zero.

TEM Wave in the Direction Perpendicular to the Static Magnetic Field

The conditions for a TEM wave propagating in the direction perpendicular (i.e., perpendicular to the z-axis) to the static magnetic field can be obtained upon substitution of $\mathcal{K} = 0$ (i.e., $\frac{\partial}{\partial z} = 0$) in the expression (32). This substitution gives two propagation wave numbers $\eta'_{1,2}$ which represent two TEM waves in the transverse plane of the z-axis

$$\frac{\eta_1'^2}{k^2} = \frac{\epsilon_{33}}{\mu_{11}} (\mu_{11}^2 - \mu_{12}^2) , \tag{40a}$$

and

$$\frac{\eta_2'^2}{k^2} = \frac{\mu_{33}}{\epsilon_{11}} (\epsilon_{11}^2 - \epsilon_{12}^2) \tag{40b}$$

The results obtained in (39) and (40) agree with those obtained by Van Trier

[1] by an entirely different approach.

When $p_1 = 0$, a study of the expressions (33) and (34) shows that $\eta_2' = 0$, provided the diagonal components of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are finite and non-zero. It can be shown also that such TEM modes in an unbounded homogeneous anisotropic medium do not vary in the transverse plane, but in a coaxial waveguide TEM waves behave as $1/r$ in the transverse plane. The above statement follows from the fact that for a TE M wave the transversely varying part of the transverse fields can be derived from $-\nabla_t \phi(\underline{\rho})$, where $\phi(\underline{\rho})$ is a scalar potential dependent on the transverse coordinate $\underline{\rho}$.

The two waves given by (39) are known as ordinary and extraordinary waves in the literatures of the ionospheric wave propagation. The permeability of the ionosphere is a scalar quantity and equal to that of free space. These two equations also explain the phenomena known as Faraday Rotation. The two waves represented by (40a) and (40b) can be said to explain [1] the phenomena known as magnetic and electric Cotton-Mouten effect.

APPENDIX B

CONSTRUCTION OF DYADIC GREEN'S FUNCTIONS

Although a construction of Dyadic Green's functions from the source-free solutions of Maxwell's equations for inhomogeneous anisotropic non-dissipative media in a uniform waveguide of arbitrary cross section bounded by a perfect conductor, has been discussed in [13], they will be also briefly presented here for the sake of completeness of this work. In this appendix the corresponding results for anisotropic dissipative medium will also be obtained.

The most important technique involved in the construction of dyadic Green's functions is the determination of an appropriate orthogonality condition among the source-free solutions (i.e., eigenfunctions) of Maxwell's equations. Methods of finding such orthogonality conditions have been discussed elaborately by the authors in [9], under different situations.

Here an indirect method will be presented for the construction of dyadic Green's functions [12]. In this method it will be assumed that the sources are due to some discontinuities, which causes discontinuities in the fields also.

Dyadic Green's functions $\underset{\sim}{Z}(\underline{r}, \underline{r}')$, $\underset{\sim}{T}_{em}(\underline{r}, \underline{r}')$, $\underset{\sim}{Y}(\underline{r}, \underline{r}')$ and $\underset{\sim}{T}_{me}(\underline{r}, \underline{r}')$ are defined by the following expressions:

$$\underline{E}(\underline{r}) = -\iiint\limits_V \underset{\sim}{Z}(\underline{r}, \underline{r}') \cdot \underline{I}_e(\underline{r}')dV' - \iiint\limits_V \underset{\sim}{T}_{em}(\underline{r}, \underline{r}') \cdot \underline{I}_m(\underline{r}')dV' \qquad (1)$$

$$\underline{H}(\underline{r}) = -\iiint_V \underset{\sim}{Y}(\underline{r}, \underline{r}') \cdot \underline{I}_m(\underline{r}')\,dV' - \iiint_V \underset{\sim}{T}_{me}(\underline{r}, \underline{r}') \cdot \underline{I}_e(\underline{r}')\,dV' \qquad (2)$$

Where the $\underline{E}(\underline{r})$, $\underline{H}(\underline{r})$, $\underline{I}_e$, $\underline{I}_m$ have the same significance as given by the Maxwell's equations (5) to (8) of Appendix A. Instead of volume currents, if $\underline{I}_e$ and $\underline{I}_m$ represent surface currents, the volume integrals in (1) and (2) should be replaced by surface integrals (over the regions of surface currents).

In the following are given the physical meanings of dyadic Green's functions:

- $\underset{\sim}{Z}(\underline{r}, \underline{r}') \cdot \underline{u}$ = electric field at $\underline{r}$ due to a point electric current source at $\underline{r}'$, directed along the unit vector $\underline{u}$.

- $\underset{\sim}{T}_{em}(\underline{r}, \underline{r}') \cdot \underline{v}$ = electric field at $\underline{r}$ due to a point magnetic current source at $\underline{r}'$, directed along the unit vector $\underline{v}$.

- $\underset{\sim}{Y}(\underline{r}, \underline{r}') \cdot \underline{v}$ = magnetic field at $\underline{r}$ due to a point magnetic current source at $\underline{r}'$ directed along the unit vector $\underline{v}$.

- $\underset{\sim}{T}_{me}(\underline{r}, \underline{r}') \cdot \underline{u}$ = magnetic field at $\underline{r}$ due to a point electric current source at $\underline{r}'$ directed along the unit vector $\underline{u}$.

In the above statements the point source means a source which has spatial variation as a Dirac delta function $\delta(\underline{r} - \underline{r}')$.

Let $\mathcal{E}(\underline{r})$ and $\mathcal{H}(\underline{r})$ be the solutions of the homogeneous (source-free) Maxwell's equations (3)

$$\nabla \times \mathcal{E}(\underline{r}) = -j\omega\mu_o \underset{\sim}{\mu}(\rho) \cdot \mathcal{H}(\underline{r}) \qquad (3a)$$

$$\nabla \times \underline{\mathcal{H}}(\underline{r}) = j\omega\epsilon_o \underset{\sim}{\epsilon}(\underline{\rho}) \cdot \underline{\mathcal{E}}(\underline{r}) \tag{3b}$$

where $\underset{\sim}{\mu}(\underline{\rho})$ and $\underset{\sim}{\epsilon}(\underline{\rho})$ are functions of transverse coordinate $\underline{\rho}$ only.

Since under appropriate boundary conditions, $\underline{\mathcal{E}}(\underline{r})$ and $\underline{\mathcal{H}}(\underline{r})$ form a complete orthogonal set, the total fields $\underline{E}(\underline{r})$ and $\underline{H}(\underline{r})$ due to any arbitrary source can be expressed as a superposition of $\underline{\mathcal{E}}(\underline{r})$ and $\underline{\mathcal{H}}(\underline{r})$ in the following way:

$$\underline{E}(\underline{r}) = \sum_{\alpha} A_\alpha \underline{\mathcal{E}}_\alpha(\underline{r}) \tag{4a}$$

and

$$\underline{H}(\underline{r}) = \sum_{\alpha} A_\alpha \underline{\mathcal{H}}_\alpha(\underline{r}) \tag{4b}$$

where A_α is the coefficient of expansion corresponding to α-th-mode (eigenvalue).

Reciprocity Relations for Homogeneous Maxwell's Equations

To establish Lorentz's reciprocity relation and hence an orthogonality condition it is desirable to consider another set of Maxwell's equations. This new set of equations is some time s called the Adjoint-Maxwell's equations [9]. After taking complex-conjugates of these so-called adjoint equations, the resulting Maxwell's equations have the following forms:

$$\nabla \times \underline{\mathcal{E}}''_\beta = j\omega\mu_o \underset{\sim}{\mu}^{+*} \cdot \underline{\mathcal{H}}''_\beta \tag{5a}$$

$$\nabla \times \mathcal{H}''_{\beta} = -j\omega\epsilon_0 \underset{\sim}{\epsilon}^{+*} \cdot \mathcal{E}''_{\beta} \tag{5b}$$

where $\underset{\sim}{\mu}^{+}$, $\underset{\sim}{\epsilon}^{+}$ = adjoint of $\underset{\sim}{\mu}$, and $\underset{\sim}{\epsilon}$ respectively

= complex conjugate of the transpose of $\underset{\sim}{\mu}$ and $\underset{\sim}{\epsilon}$ respectively.

$\therefore$ $\underset{\sim}{\mu}^{+*} = \tilde{\underset{\sim}{\mu}}$ = transpose of $\underset{\sim}{\mu}$

$$= \begin{vmatrix} \mu_{11} & -j\mu_{12} & 0 \\ j\mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix} \tag{6}$$

and $\underset{\sim}{\epsilon}^{+*} = \tilde{\underset{\sim}{\epsilon}}$ = transpose of $\underset{\sim}{\epsilon}$

$$= \begin{vmatrix} \epsilon_{11} & -j\epsilon_{12} & 0 \\ j\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{vmatrix} \tag{7}$$

Although the authors in [9] have used the symbols $\mathcal{E}^{+*}_{\beta}$ for $\mathcal{E}''_{\beta}$ and $\mathcal{H}^{+*}_{\beta}$ for $\mathcal{H}''_{\beta}$, in general $\mathcal{E}^{+*}_{\beta}$ and $\mathcal{H}^{+*}_{\beta}$ have no simple relations with the solutions of equations (3a) and (3b). Thus, to avoid confusion, the symbols

$\underline{\mathcal{E}}''_\beta$ and $\underline{\mathcal{H}}''_\beta$ have been used here. Here $\underline{\mathcal{E}}''_\beta$ and $\underline{\mathcal{H}}''_\beta$ only mean the solutions of (5a) and (5b), subject to some appropriate boundary conditions.

Now multiplying (3a) by $\underline{\mathcal{H}}''_\beta$ and (3b) by $\underline{\mathcal{E}}''_\beta$ and (5a) by $\underline{\mathcal{H}}_\alpha$ and (5b) by $\underline{\mathcal{E}}_\alpha$ in a scalar product fashion from the left and then subtracting, one can show that

$$\nabla \cdot \left[\underline{\mathcal{E}}_\alpha \times \underline{\mathcal{H}}''_\beta - \underline{\mathcal{H}}_\alpha \times \underline{\mathcal{E}}''_\beta\right] = 0 \qquad (8)$$

Since

$$\underline{\mathcal{E}}_\alpha \cdot \underset{\sim}{\epsilon}^{+*} \cdot \underline{\mathcal{E}}''_\beta = \underline{\mathcal{E}}''_\beta \cdot \underset{\sim}{\epsilon} \cdot \underline{\mathcal{E}}_\alpha \qquad (9a)$$

and

$$\underline{\mathcal{H}}_\alpha \cdot \underset{\sim}{\mu}^{+*} \cdot \underline{\mathcal{H}}''_\beta = \underline{\mathcal{H}}''_\beta \cdot \underset{\sim}{\mu}^{+*} \cdot \underline{\mathcal{H}}_\beta \qquad (9b)$$

Let $\nabla = \nabla_t + \underline{z}_o \frac{\partial}{\partial z}$, where $\underline{z}_o$ is the unit vector in the z-direction.

Now using $\frac{\partial}{\partial z} \underline{\mathcal{E}}_\alpha = - j\mathcal{K}_\alpha \underline{\mathcal{E}}_\alpha$, $\frac{\partial}{\partial z} \underline{\mathcal{H}}_\alpha = - j\mathcal{K}_\alpha \underline{\mathcal{H}}_\alpha$

and

$$\frac{\partial}{\partial z} \underline{\mathcal{E}}''_\beta = j\mathcal{K}^*_\beta \underline{\mathcal{E}}''_\beta, \quad \frac{\partial}{\partial z} \underline{\mathcal{H}}''_\beta = j\mathcal{K}^*_\beta \underline{\mathcal{H}}''_\beta,$$

equation (8) can be rewritten in the following way

$$\nabla_t \cdot \left[\underline{\mathcal{E}}_\alpha \times \underline{\mathcal{H}}''_\beta - \underline{\mathcal{H}}_\alpha \times \underline{\mathcal{E}}''_\beta\right] = j(\mathcal{K}_\alpha - \mathcal{K}^*_\beta) \left[\underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}''_\beta \times \underline{z}_o + \underline{\mathcal{H}}_\alpha \cdot \underline{z}_o \times \underline{\mathcal{E}}''_\beta\right] \qquad (10)$$

Now integrating over the cross-section of the waveguide one obtains (using the two dimensional divergence theorem)

$$\oint_s \left[\underline{\nu} \times \underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}''_\beta + \underline{\nu} \times \underline{\mathcal{E}}''_\beta \cdot \underline{\mathcal{H}}_\alpha\right] ds = j(\mathcal{K}_\alpha - \overset{*}{\mathcal{K}}_\beta) \iint_S \left[\underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}''_\beta \times \underline{z}_o + \underline{\mathcal{H}}_\alpha \cdot \underline{z}_o \times \underline{\mathcal{E}}''_\beta\right] dS$$

where $\underline{\nu}$ is a unit outward normal vector on the boundary curve s of the waveguide cross section S.

The left-hand side of the above expression vanishes on the boundary of a perfect conductor

$\therefore$ the orthogonality relation becomes

$$\iint_S \left[\underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}''_\beta \times \underline{z}_o + \underline{\mathcal{H}}_\alpha \cdot \underline{z}_o \times \underline{\mathcal{E}}''_\beta\right] dS = 2N_\alpha \, \delta_{\alpha\beta}, \tag{11}$$

where N_α is a normalization constant and

$$\delta_{\alpha\beta} = 1, \quad \text{for } \mathcal{K}_\alpha = \overset{*}{\mathcal{K}}_\beta$$

$$= 0, \quad \text{for } \mathcal{K}_\alpha \neq \overset{*}{\mathcal{K}}_\beta$$

When the waveguide has a reflection symmetry, i.e., when the properties of the waveguide are independent of the coordinate z, the orthogonality relation can be rewritten as

$$\iint_S \underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}''_\beta \times \underline{z}_o \, dS = N_\alpha \delta_{\alpha\beta} = \iint_S \underline{\mathcal{H}}_\alpha \cdot \underline{z}_o \times \underline{\mathcal{E}}''_\beta \, dS \tag{12}$$

or

$$\iint_S \underline{\mathcal{E}}_{t\alpha} \cdot \underline{\mathcal{H}}''_{t\beta} \times \underline{z}_o \, dS = N_\alpha \delta_{\alpha\beta} = \iint_S \underline{\mathcal{H}}_{t\alpha} \cdot \underline{z}_o \times \underline{\mathcal{E}}''_{t\beta} \, dS \tag{13}$$

where the suffix t represents the transverse components of the fields.

Construction of Dyadic Green's Functions (see [11] and [12])

Knowing the total fields $\underline{E}(\underline{r}')$ and $\underline{H}(\underline{r}')$ which one may consider are due to discontinuity at some cross section S_z of the waveguide, one can write, using equation (4)

$$\underline{\mathcal{H}}''_\beta(\underline{r}') \times \underline{z}_o \cdot \underline{E}(\underline{r}') = \underline{\mathcal{H}}''_\beta(\underline{r}') \times \underline{z}_o \cdot \sum_\alpha A_\alpha \underline{\mathcal{E}}_\alpha(\underline{r}')$$

and

$$\underline{z}_o \times \underline{\mathcal{E}}''_\beta(\underline{r}') \cdot \underline{H}(\underline{r}') = \underline{z}_o \times \underline{\mathcal{E}}''_\beta(\underline{r}') \cdot \sum_\alpha A_\alpha \underline{\mathcal{H}}_\alpha(\underline{r}')$$

Integrating the sum of the above two equations over the cross section at S_z and using (11), one obtains the value of the coefficient A_α as

$$A_\alpha = \frac{\iint_{S_z} \left[\underline{E}(\underline{r}') \cdot \underline{\mathcal{H}}''_\alpha(\underline{r}') \times \underline{z}_o + \underline{H}(\underline{r}') \cdot \underline{z}_o \times \underline{\mathcal{E}}''_\alpha(\underline{r}') \right] dS'}{2N_\alpha} \tag{14}$$

Therefore, the total electric field at any point $\underline{r}$ due to $\underline{E}(\underline{r}')$ and $\underline{H}(\underline{r}')$ is given by

$$\underline{E}(\underline{r}) = \sum_\alpha A_\alpha \underline{\mathcal{E}}_\alpha(\underline{r}) = \sum_\alpha \frac{\iint_{S_z} dS' \left[\underline{E}(\underline{r}') \cdot \underline{\mathcal{H}}''_\alpha(\underline{r}') \times \underline{z}_o + \underline{H}(\underline{r}') \cdot \underline{z}_o \times \underline{\mathcal{E}}''_\alpha(\underline{r}')\right] \underline{\mathcal{E}}_\alpha(\underline{r})}{2N_\alpha}$$

which can also be written

$$\underline{E}(\underline{r}) = -\iint_{S_z} \left[\sum_\alpha \frac{\underline{\mathcal{E}}_\alpha(\underline{r})\, \underline{\mathcal{E}}''_\alpha(\underline{r}')}{2N_\alpha}\right] \cdot \underline{z}_o \times \underline{H}(\underline{r}')\, dS'$$

$$-\iint_{S_z} \left[\sum_\alpha \frac{\underline{\mathcal{E}}_\alpha(\underline{r})\, \underline{\mathcal{H}}''_\alpha(\underline{r}')}{2N_\alpha}\right] \cdot \underline{E}(\underline{r}') \times \underline{z}_o\, dS' \qquad (15)$$

Since $\underline{z}_o \times \underline{H}(\underline{r}')$ and $\underline{E}(\underline{r}') \times \underline{z}_o$ represent sources due to discontinuities in fields at $\underline{r}'$, one can represent

$$\underline{z}_o \times \underline{H}(\underline{r}') = \underline{I}_{et}(\underline{r}') \quad \text{and} \quad \underline{E}(\underline{r}') \times \underline{z}_o = \underline{I}_{mt}(\underline{r}')$$

Alternatively one can consider the discontinuities in $\underline{z}_o \times \underline{H}(\underline{r}')$ and $\underline{E}(\underline{r}') \times \underline{z}_o$ at $\underline{r}'$ are due to actual sources $\underline{I}_{et}(\underline{r}')$ and $\underline{I}_{mt}(\underline{r}')$ respectively. Now with the above assumption if one compares equation (15) with equation (1), one finds

$$\underset{\sim}{Z}(\underline{r}, \underline{r}') = \sum_\alpha \frac{\underline{\mathcal{E}}_\alpha(\underline{r})\ \ \underline{\mathcal{E}}''_\alpha(\underline{r}')}{2N_\alpha} \qquad (16a)$$

$$\underset{\sim}{T}_{em}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{E}}_{\alpha}(\underline{r})\, \underline{\mathcal{H}}''_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{16b}$$

Similarly if one expresses the total magnetic field as $\underline{H}(\underline{r}) = \sum_{\alpha} A_{\alpha} \underline{\mathcal{H}}_{\alpha}(\underline{r})$ and uses equation (14), following the above procedure, it can be shown that

$$\underset{\sim}{Y}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{H}}_{\alpha}(\underline{r})\, \underline{\mathcal{H}}''_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{17a}$$

$$\underset{\sim}{T}_{me}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{H}}_{\alpha}(\underline{r})\, \underline{\mathcal{E}}''_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{17b}$$

<u>Special Cases</u>

It has been pointed out before that in general there are no simple relations between $\underline{\mathcal{E}}_{\alpha}$ and $\underline{\mathcal{E}}''_{\alpha}$ and $\underline{\mathcal{H}}_{\alpha}$ and $\underline{\mathcal{H}}''_{\alpha}$, but in some special cases these relations simplify. For example, in non-dissipative anisotropic media $\underset{\sim}{\epsilon}^{+} = \underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}^{+} = \underset{\sim}{\mu}$, i.e., $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are hermitean (self-adjoint) dyadics. In this case

$$\underline{\mathcal{E}}''_{\alpha} = \underline{\mathcal{E}}^{*}_{\alpha}, \quad \underline{\mathcal{H}}''_{\alpha} = \underline{\mathcal{H}}^{*}_{\alpha}, \text{ and } \mathcal{K}_{\alpha} \text{ is real.}$$

Therefore, for non-dissipative medium, the dyadic Green's functions can be expressed in the following way

$$\underset{\sim}{Z}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{E}}_{\alpha}(\underline{r})\, \underline{\mathcal{E}}^{*}_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{18a}$$

$$\underset{\sim}{T}_{em}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{E}}_{\alpha}(\underline{r})\, \overset{*}{\underline{\mathcal{H}}}_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{18b}$$

$$\underset{\sim}{Y}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{H}}_{\alpha}(\underline{r})\, \overset{*}{\underline{\mathcal{H}}}_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{18c}$$

$$T_{me}(\underline{r}, \underline{r}') = \sum_{\alpha} \frac{\underline{\mathcal{H}}_{\alpha}(\underline{r})\, \overset{*}{\underline{\mathcal{E}}}_{\alpha}(\underline{r}')}{2N_{\alpha}} \tag{18d}$$

Another kind of orthogonality relation and hence dyadic Green's function can be constructed in the following way (see [9]). These results are particularly suitable for dissipative medium.

In this method the following replacement is made

$$\underset{\sim}{\mu}^{+} \rightarrow \breve{\underset{\sim}{\mu}} = \text{transpose of } \underset{\sim}{\mu} \tag{19a}$$

$$\underset{\sim}{\epsilon}^{+} \rightarrow \tilde{\underset{\sim}{\epsilon}} = \text{transpose of } \underset{\sim}{\epsilon} \tag{19b}$$

$$\mathcal{H}_{\alpha}^{+} = \overset{*}{\mathcal{H}}_{\alpha} \rightarrow - \mathcal{H}_{\alpha} \tag{19c}$$

$$\underline{\mathcal{E}}_{\alpha}'' \rightarrow \mp \underline{\mathcal{E}}_{\alpha}' \tag{19d}$$

$$\underline{\mathcal{H}}_{\alpha}'' \rightarrow \pm \underline{\mathcal{H}}_{\alpha}' \tag{19e}$$

Due to the transformations given in (19) and the Maxwell's equations (3) and (5), the following simple relations can be established

$$\underline{\mathcal{E}}'_\alpha(\mathcal{X}_\alpha, \epsilon_{12}, \mu_{12}, \underline{r}) = \mp \underline{\mathcal{E}}_\alpha(-\mathcal{X}_\alpha, -\epsilon_{12}, -\mu_{12}, \underline{r}) \tag{20a}$$

and

$$\underline{\mathcal{H}}'_\alpha(\mathcal{X}_\alpha, \epsilon_{12}, \mu_{12}, \underline{r}) = \pm \underline{\mathcal{H}}_\alpha(-\mathcal{X}_\alpha, -\epsilon_{12}, -\mu_{12}, \underline{r}) \tag{20b}$$

Therefore, for the dissipative - anisotropic medium the orthogonality relation and dyadic Green's functions can be expressed in the following way

$$\iint_S \left[\underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}'_\beta \times \underline{z}_o - \underline{\mathcal{H}}_\alpha \cdot \underline{z}_o \times \underline{\mathcal{E}}'_\beta\right] dS = 2N_\alpha \delta_{\alpha\beta} \tag{21}$$

If there is reflection symmetry in the waveguide then

$$\iint_S \underline{\mathcal{E}}_\alpha \cdot \underline{\mathcal{H}}'_\beta \times \underline{z}_o \, dS = N_\alpha \delta_{\alpha\beta} = - \iint_S \underline{\mathcal{H}}_\alpha \cdot \underline{z}_o \times \underline{\mathcal{E}}'_\beta \, dS \tag{22}$$

$$\underset{\sim}{Z}(\underline{r}, \underline{r}') = - \sum_\alpha \frac{\underline{\mathcal{E}}_\alpha(\underline{r}, \mathcal{X}_\alpha, \epsilon_{12}, \mu_{12}) \, \underline{\mathcal{E}}_\alpha(\underline{r}', -\mathcal{X}_\alpha, -\epsilon_{12}, -\mu_{12})}{2N_\alpha} \tag{23a}$$

$$\underset{\sim}{T}_{em}(\underline{r}, \underline{r}') = \sum_\alpha \frac{\underline{\mathcal{E}}_\alpha(\underline{r}, \mathcal{X}_\alpha, \epsilon_{12}, \mu_{12}) \underline{\mathcal{H}}_\alpha(\underline{r}', -\mathcal{X}_\alpha, -\epsilon_{12}, -\mu_{12})}{2N_\alpha} \tag{23b}$$

$$\underset{\sim}{Y}(\underline{r}, \underline{r}') = \sum_\alpha \frac{\underline{\mathcal{H}}_\alpha(\underline{r}, \mathcal{X}_\alpha, \epsilon_{12}, \mu_{12}) \underline{\mathcal{H}}_\alpha(\underline{r}', -\mathcal{X}_\alpha, -\epsilon_{12}, \mu_{12})}{2N_\alpha} \tag{23c}$$

$$\underset{\sim}{T}_{me}(\underline{r}, \underline{r}') = - \sum_{\alpha} \frac{\underline{\mathcal{H}}_{\alpha}(\underline{r}, \mathcal{K}_{\alpha}, \epsilon_{12}, \mu_{12}) \; \underline{\mathcal{E}}_{\alpha}(\underline{r}', -\mathcal{K}_{\alpha}, -\epsilon_{12}, -\mu_{12})}{2N_{\alpha}} \tag{23d}$$

For dissipative medium $\mathcal{K}_{\alpha}$ and N_{α} are complex.

APPENDIX C

DISPERSION RELATIONS FOR VARIOUS SPECIAL CASES

Another alternative form of the general dispersion relation can be obtained from the relations in (52-I) by eliminating A_1 and A_2 and then equating two different expressions of the ratio B_1/B_2. It may be noted that the dispersion relation thus obtained is equivalent to those given in (53-I) and (54-I). This form which is shown in the following will be found useful in a few cases.

$$\frac{\mathcal{K}^2 G_n(a)\mathcal{J}_n(a)\left[k^2\eta^2 a\, a_3 \eta'_1\eta'_2\epsilon_z\mu_z\left\{\eta'_1 J'_n(\eta'_2 a)J_n(\eta'_1 a)-\eta'_2 J_n(\eta'_2 a)J'_n(\eta'_1 a)\right\}+n(\eta'^2_2-\eta'^2_1)\left\{\epsilon_r\mu_r\eta'^2_1\eta'^2_2-\eta^2 a_4\epsilon_z\mu_z\right\}J_n(\eta'_1 a)J_n(\eta'_2 a)\right]^2}{a\eta\epsilon_r\mu_r\epsilon_z\eta'_1\eta'_2\mathcal{J}_n(a)\left\{\eta'_2 MJ'_n(\eta'_1 a)J_n(\eta'_2 a)-\eta'_1 SJ_n(\eta'_1 a)J'_n(\eta'_2 a)\right\}+(\eta'^2_2-\eta'^2_1)\left\{a\epsilon_2\mu_r\epsilon_r\eta'^2_1\eta'^2_2 C_n(a)-n\eta a_2\epsilon_z\mu_z\mathcal{J}_n(a)\right\}J_n(\eta'_1 a)J_n(\eta'_2 a)}$$

$$= k^2\eta^2\left[a\eta\mu_r\epsilon_r\mu_z\eta'_1\eta'_2 G_n(a)\left\{n'_1 MJ'_n(\eta'_2 a)J_n(\eta'_1 a)-\eta'_2 SJ'_n(\eta'_1 a)J_n(\eta'_2 a)\right\}-(\eta'^2_2-\eta'^2_1)\left\{a\mu_2\mu_r\epsilon_r\eta'^2_1\eta'^2_2 S_n(a)+n\eta\epsilon_z\mu_z a'_2 G_n(a)\right\}J_n(\eta'_1 a)J_n(\eta'_2 a)\right] \tag{1}$$

For an extremely slow wave propagation one can [8] introduce static limit approximation for which $k/\mathcal{K} << 1$ and also $k\mathcal{K} << 1$. In this case one can show (equations (25) and (26) of Appendix A) that E-type modes and H-type modes can exist independently. This approximation is also valid when (ka) << 1, where 2a is the diameter of the anisotropic column.

For static approximations it is easy to show that

$$
\left.\begin{aligned}
\eta_1'^2 &\approx \frac{\epsilon_z a_1'}{\epsilon_r \mu_r} \approx -\frac{\epsilon_z \varkappa^2}{\epsilon_r} \\
\eta_2'^2 &\approx \frac{\mu_z a_1}{\mu_r \epsilon_r} \approx -\frac{\mu_z \varkappa^2}{\mu_r} \\
S &\approx 0 \\
M &\approx -(\eta_2'^2 - \eta_1'^2) \\
\eta &\approx -j\varkappa
\end{aligned}\right\} \qquad (2)
$$

Now using these relations and the assumption $k/\varkappa << 1$, it can be shown from the dispersion relation (1) that for the H-type mode the right-hand side of the expression (1) vanishes and for the E-type mode the denominator of the left-hand side of (1) vanishes. For a magnetic current ring source an E-type mode will be excited in this static-limit situation. Whereas for an electric dipole source an H-type mode can be excited in the static limit. It is of practical interest to consider the E-type modes in a plasma and the H-type modes in a ferrite. The following dispersion relations for these two limiting cases can be expressed in the following way:

$$(\eta'_1 a)\epsilon_r \frac{J'_n(\eta'_1 a)}{J_n(\eta'_1 a)} - n\epsilon' = -(\varkappa a)\epsilon_2 \frac{\left[I'_n(\varkappa a)K_n(\varkappa b) - I_n(\varkappa b)K'_n(\varkappa a)\right]}{\left[I_n(\varkappa b)K_n(\varkappa a) - I_n(\varkappa a)K_n(\varkappa b)\right]} \qquad (3)^+$$

for E-type modes

and

$$(a\eta'_2)\mu_r \frac{J'_n(\eta'_2 a)}{J_n(\eta'_2 a)} - n\mu' = (\varkappa a)\mu_2 \frac{\left[I'_n(\varkappa b)K'_n(\varkappa a) - I'_n(\varkappa a)K'_n(\varkappa b)\right]}{\left[I'_n(\varkappa b)K_n(\varkappa a) - I_n(\varkappa a)K'_n(\varkappa b)\right]} \qquad (4)^+$$

for H-type modes

When the anisotropic medium completely fills the waveguide, i.e., when a = b, the above two relations reduce to the following:

$$\left.\begin{array}{ll} & J_n(\eta'_1 a) = 0\,, \quad \text{for } n \neq 0 \\ \text{and} \quad \text{for } n = 0\,, & J_o(\eta'_1 a) = 0 \end{array}\right\} \text{ for E-type modes} \qquad (5)$$

It should be noted also that in this case a change in sign of n does not effect any result.

$$\left.\begin{array}{ll} & (\eta'_2 a)\mu_r \dfrac{J'_n(\eta'_2 a)}{J_n(\eta'_2 a)} = n\mu' \\ \text{and} \quad \text{for } n = 0\,, & J_1(\eta'_2 a) = 0 \end{array}\right\} \text{ for H-type modes} \qquad (6)$$

When the radius of the waveguide $b \to \infty$, the above equations (3) and (4) reduce to the following simple forms:

+These results agree with those obtained by Trivelpiece [8] except for a change in sign in the term $n\mu'$ of equation (4).

$$(\eta'_1 a)\epsilon_r \frac{J'_n(\eta'_1 a)}{J_n(\eta'_1 a)} - n\epsilon' = (\mathcal{X}a)\epsilon_2 \frac{K'_n(\mathcal{X}a)}{K_n(\mathcal{X}a)}$$

for E-type modes (7)

and

$$(\eta'_2 a)\mu_r \frac{J'_n(\eta'_2 a)}{J_n(\eta'_2 a)} - n\mu' = (\mathcal{X}a)\mu_2 \frac{K'_n(\mathcal{X}a)}{K_n(\mathcal{X}a)}$$

for H-type modes (8)

It may be noted here that for $b \to \infty$, the result in (7) is the dual of that in (8). The reason for this duality is that as $b \to \infty$, the boundary conditions on $\underline{E}$ and $\underline{H}$ become identical in both cases.

For $\eta'_1 = \eta'_2$ +, the dispersion relations (1) assumes a form 0/0, so using l'Hospital's rule the following results can be obtained:

$$\frac{\mathcal{X}^2 G_n(a)\mathcal{J}_n(a)\left[k^2\eta^2 a a_3 \eta'_1 \epsilon_z \mu_z \left\{a\eta'_1 J_n(\eta'_1 a)J''_n(\eta'_1 a) - J_n(\eta'_1 a)J'_n(\eta'_1 a) - a\eta'_1 J'^2_n(\eta'_1 a)\right\} + 2n\, J^2_n(\eta'_1 a)\left\{\epsilon_r \mu_r \eta'^4_1 - \eta^2 a_4 \mu_z \epsilon_z\right\}\right]^2}{\epsilon_r \mu_r \epsilon_z a \eta \eta'_1 \mathcal{J}_n(a)\left\{S J'_n(\eta'_1 a) J_n(\eta'_1 a) - 2\eta'^2_1 J'_n(\eta'_1 a) J_n(\eta'_1 a) + a\eta'_1 S J'^2_n(\eta'_1 a) - a\eta'_1 S J_n(\eta'_1 a) J''_n(\eta'_1 a)\right\} + 2J^2_n(\eta'_1 a)\left\{a\epsilon_2 \mu_r \epsilon_r \eta'^4_1 C_n(a) - n\eta a_2 \epsilon_z \mu_z \mathcal{J}_n(a)\right\}}$$

$$= k^2\eta^2\left[a n \mu_r \epsilon_r \mu_z \eta'_1 G_n(a)\left\{a\eta'_1 S J''_n(\eta'_1 a) J_n(\eta'_1 a) - 2\eta'^2_1 J'_n(\eta'_1 a) J_n(\eta'_1 a) - S J'_n(\eta'_1 a) J_n(\eta'_1 a) - a\eta'_1 S J'^2_n(\eta'_1 a)\right\}\right.$$
$$\left. - 2J^2_n(\eta'_1 a)\left\{a\mu_2 \mu_r \epsilon_r \eta'^4_1 S_n(a) + n\eta \epsilon_z \mu_z a'_2 G_n(a)\right\}\right] \qquad (9)$$

+For a ferrite problem only, the relation $\eta'_1 = \eta'_2$ cannot be satisfied.

For n = 0, the above relation becomes

$$\frac{k^2\eta^2\mathscr{K}^2 a_3^2\epsilon_z^2\mu_z^2\left[2J_o(\eta_1' a)J_1(\eta_1' a)-a\eta_1'\left\{J_o^2(\eta_1' a)+J_1^2(\eta_1' a)\right\}\right]^2 G_o(a)\mathscr{J}_o(a)}{\epsilon_r^2\mu_r^2\left[\epsilon_z\eta\mathscr{J}_o(a)\left\{2J_1(\eta_1' a)J_o(\eta_1' a)\left[\eta_1'^2-S\right]+a\eta_1' S\left(J_o^2(\eta_1' a)+J_1^2(\eta_1' a)\right)\right\}+2\epsilon_2\eta_1'^3 C_o(a)J_o^2(\eta_1' a)\right]}$$

$$= \eta\mu_z G_o(a)\left\{2J_o(\eta_1' a)J_1(\eta_1' a)\left[\eta_1'^2+S\right]-a\eta_1' S\left(J_1^2(\eta_1' a)+J_o^2(\eta_1' a)\right)\right\}-2\mu_2\eta_1'^3 S_o(a)J_o^2(\eta_1' a) \tag{10}$$

For n = 0, the dispersion relations (55I) and (56I) can also be derived from the relation (1).

It should be mentioned here that all the following dispersion relations for various special cases can be derived from any of the general relations (55I), (56I), and (1) of this Appendix. The purpose of writing these various forms of the general dispersion relation is that it is found more convenient to use one particular form rather than another for some special cases.

When the anisotropic medium completely fills the w ·eguide (i.e., when a = b), the corresponding dispersion relation (without using any approximation) becomes (since $G_n(b) = C_n(b) = \frac{2}{(\pi\eta b)}$, $\mathscr{J}_n(b) = 0 = S_n(b)$, at a = b):

$$a\epsilon_r\mu_r M\eta_1'^2\eta_2'\left[\frac{J_n'(\eta_2' a)}{J_n(\eta_2' a)}\right]-a\mu_r\epsilon_r S\eta_1'\eta_2'^2\left[\frac{J_n'(\eta_1' a)}{J_n(\eta_1' a)}\right]=n\epsilon_z a_2'(\eta_2'^2-\eta_1'^2),\ \text{for } n\neq 0 \tag{11}$$

and
$$\frac{\eta'_2 S}{\eta'_1 M} = \frac{J_1(\eta'_2 a)J_o(\eta'_1 a)}{J_o(\eta'_2 a)J_1(\eta'_1 a)}, \qquad \text{for } n = 0 \tag{12}$$

If one analyzes equations (34I) and (35I) or equation (32a) of Appendix A, it is easy to show that in the limit $\epsilon' = \mu' = a_3 \longrightarrow 0$

$$\left.\begin{aligned}
\eta_1'^2 &= \frac{\epsilon_z a'_1}{\epsilon_r \mu_r} = \frac{\epsilon_z a_1}{\epsilon_r} \\
\eta_2'^2 &= \frac{\mu_z a_1}{\epsilon_r \mu_r} = \frac{\mu_z a_4}{\mu_r} \\
S &= 0 \\
M &= -(\eta_2'^2 - \eta_1'^2), \quad \epsilon_z a_4 - \eta_1'^2 \epsilon_r = \epsilon_r S \\
\frac{S}{a_3} &= 0, \quad \epsilon_z a_4 - \epsilon_r \eta_1'^2 = \epsilon_r M \\
\frac{\mu_r \epsilon' \eta_1'^2 - a'_2 \epsilon_z}{a_3} &= -\mathcal{X}^2 \frac{\epsilon_z}{\epsilon_r}
\end{aligned}\right\} \tag{13}$$

Now if one divides the numerators of both sides of (54I) by a_3 and uses relations (13), the dispersion relation can be shown to have the following form, after rearranging terms:

$$\left[\frac{\epsilon_r m\eta'_1 J'_n(\eta'_1 a)}{J_n(\eta'_1 a)} - \frac{\epsilon_2 a_4 C_n(a)}{\mathcal{J}_n(a)}\right]\left[\frac{\mu_r m\eta'_2 J'_n(\eta'_2 a)}{J_n(\eta'_2 a)} + \frac{\mu_2 a_4 S_n(a)}{G_n(a)}\right] = \left[\frac{n\mathcal{X}(a_4 - \eta^2)}{ka\eta}\right]^2 \tag{14}$$

For an isotropic medium, the dispersion relation can be obtained from (14) letting $\eta_1'^2 = \eta_2'^2 = a_4 = k^2\epsilon_r - \mathcal{X}^2$, and $\epsilon_r = \epsilon_z$, $\mu_r = \mu_z$.

For axially symmetric fields (i.e., when the ring source is of constant strength, $\frac{\partial}{\partial\theta} = 0$), n = 0, and one obtains from equation (14) the following dispersion relations

$$\frac{\epsilon_r m\eta'_1 J_1(\eta'_1 a)}{J_o(\eta'_1 a)} = - \frac{\epsilon_2 a_4 C_o(a)}{\mathcal{J}_o(a)}, \quad \text{for E-type mode} \tag{15a}$$

and

$$\frac{\mu_r m\eta'_2 J_1(\eta'_2 a)}{J_o(\eta'_2 a)} = \frac{\mu_2 a_4 S_o(a)}{G_o(a)}, \quad \text{for H-type mode} \tag{15b}$$

The relations (15a) and (15b) can also be obtained from (55I) with appropriate limiting procedures.

It should be noted here that for axially symmetric fields and $\epsilon' = 0 = \mu'$, E-type modes and H-type modes can exist separately. But in the present problem where the source is a magnetic current ring source, only E-type modes will be excited for $\epsilon' = 0 = \mu'$. Further it may be stated that even in an isotropic medium (i.e., $\epsilon' = 0 = \mu'$ and $\epsilon_r = \epsilon_z$ and $\mu_r = \mu_z$) if $\frac{\partial}{\partial\theta} \neq 0$, E-type and H-type modes cannot exist independently.

If $n \neq 0$, but $a = b$ and $\epsilon' = 0 = \mu'$, it can be shown either from (11) or from (14) that the dispersion relations become

$$J_n(\eta'_1 a) = 0 \quad , \quad \text{for E-type modes} \tag{16a}$$

and

$$J'_n(\eta'_2 a) = 0 \quad , \quad \text{for H-type modes} \tag{16b}$$

It is now trivial to see that for $n = 0$, $a = b$ and $\epsilon' = 0 = \mu'$, one obtains the following dispersion relations

$$J_o(\eta'_1 a) = 0 \quad , \quad \text{for E-type modes} \tag{17a}$$

$$J_1(\eta'_2 a) = 0 \quad , \quad \text{for H-type modes} \tag{17b}$$

Relations (16) and (17) are valid for isotropic media as well as diagonally anisotropic media. However, in our present problem, we consider only (16a) and (17a), restricting our consideration to E-type modes (due to choice of the source).

+When an infinite column of an anisotropic medium is placed in another unbounded isotropic medium, the corresponding dispersion relation for surface

+It should be noted, however, that to obtain radiated fields in the present situation, this limiting process (namely $b \to \infty$) is not valid.

waves can be obtained either from (53I) or from (54I) with $b \to \infty$. To study surface wave propagation (which is also slow wave), for which $\varkappa/k > 1$, one can show that η is purely imaginary. So putting $\eta = -j\delta$, $\delta > 0$, one obtains

$$\mathcal{J}_n(a) = -(2/\pi)\cdot\left[I_n(\delta b)K_n(\delta a) - I_n(\delta a)K_n(\delta b)\right] \tag{18a}$$

$$S_n(a) = -(2/\pi)\cdot\left[I'_n(\delta a)K'_n(\delta b) - I'_n(\delta b)K'_n(\delta a)\right] \tag{18b}$$

$$C_n(a) = (j2/\pi)\cdot\left[I'_n(\delta a)K_n(\delta b) - I_n(\delta b)K'_n(\delta a)\right] \tag{18c}$$

$$G_n(a) = (j2/\pi)\cdot\left[I'_n(\delta b)K_n(\delta a) - I_n(\delta a)K'_n(\delta b)\right] \tag{18d}$$

for convenience let

$$\overline{\mathcal{J}_n(r)} = I_n(\delta b)K_n(\delta r) - I_n(\delta r)K_n(\delta b) \tag{19a}$$

$$\overline{S_n(r)} = I'_n(\delta r)K'_n(\delta b) - I'_n(\delta b)K'_n(\delta r) \tag{19b}$$

$$\overline{C_n(r)} = I'_n(\delta r)K_n(\delta b) - I_n(\delta b)K'_n(\delta r) \tag{19c}$$

$$\overline{G_n(r)} = I'_n(\delta b)K_n(\delta r) - I_n(\delta r)K'_n(\delta b) \tag{19d}$$

Also it is easy to show that (if $n < \delta b$, δ may be finite or very large),

$$\overline{\mathcal{J}}_n \sim \frac{K_n(\delta a)e^{\delta b}}{\sqrt{2\pi\delta b}} , \quad \text{as } b \to \infty \tag{20a}$$

$$\overline{S_n(a)} \sim -\frac{K_n'(\delta a)e^{\delta b}}{\sqrt{2\pi\delta b}}, \quad \text{as} \quad b \to \infty \tag{20b}$$

$$\overline{C_n(a)} \sim -\frac{K_n'(\delta a)e^{\delta b}}{\sqrt{2\pi\delta b}}, \quad \text{as} \quad b \to \infty \tag{20c}$$

$$\overline{G_n(a)} \sim \frac{K_n(\delta a)e^{\delta b}}{\sqrt{2\pi\delta b}}, \quad \text{as} \quad b \to \infty \tag{20d}$$

Now using the above relations in equation (54I), one obtains the following dispersion relation for slow wave propagation along an anisotropic column situated in another unbounded isotropic medium:

$$\frac{J_n(\eta_1'a)\left\{n\mu_z\delta^2K_n(\delta a)\left[a_4\epsilon_z-\epsilon_r\eta_1'^2\right]+n\mu_r\epsilon_r\eta_1'^2SK_n(\delta a)+k^2a\delta\epsilon_2\eta_1'^2a_3\mu_z\,K_n'(\delta a)\right\}+a\delta^2k^2a_3\epsilon_z\mu_z\eta_1'K_n(\delta a)J_n'(\eta_1'a)}{J_n(\eta_2'a)\left\{n\mu_z\delta^2K_n(\delta a)\left[a_4\epsilon_z-\epsilon_r\eta_2'^2\right]+n\mu_r\epsilon_r\eta_2'^2MK_n(\delta a)+k^2a\delta\epsilon_z\eta_2'^2a_3\mu_zK_n'(\delta a)\right\}+a\delta^2k^2a_3\epsilon_z\mu_z\eta_2'K_n(\delta a)J_n'(\eta_2'a)}$$

$$=\frac{J_n(\eta_1'a)\left\{nk^2\eta_1'^2a_3\mu_zK_n(\delta a)-n\mu_z\delta^2\left[\mu_r\epsilon'\eta_1'^2-a_2'\epsilon_z\right]K_n(\delta a)+\delta a\eta_1'^2S\mu_2\epsilon_r\mu_rK_n'(\delta a)\right\}+\delta^2a\eta_1'S\mu_z\epsilon_r\mu_rK_n(\delta a)J_n'(\eta_1'a)}{J_n(\eta_2'a)\left\{nk^2\eta_2'^2a_3\mu_zK_n(\delta a)-n\mu_z\delta^2\left[\mu_r\epsilon'\eta_2'^2-a_2'\epsilon_z\right]K_n(\delta a)+\delta a\eta_2'^2M\mu_2\epsilon_r\mu_rK_n'(\delta a)\right\}+\delta^2a\eta_2'M\mu_z\epsilon_r\mu_rK_n(\delta a)J_n'(\eta_2'a)} \tag{21}$$

When $n = 0$, the above relation becomes

$$\frac{S\left\{\delta\mu_z K_o(\delta a)J_1(\eta'_1 a)+\eta'_1\mu_2 K_1(\delta a)J_o(\eta'_1 a)\right\}}{M\left\{\delta\mu_z K_o(\delta a)J_1(\eta'_2 a)+\eta'_2\mu_2 K_1(\delta a)J_o(\eta'_2 a)\right\}} = \frac{\delta\epsilon_z K_o(\delta a)J_1(\eta'_1 a)+\epsilon_2\eta'_1 K_1(\delta a)J_o(\eta'_1 a)}{\delta\epsilon_z K_o(\delta a)J_1(\eta'_2 a)+\epsilon_2\eta'_2 K_1(\delta a)J_o(\eta'_2 a)} \tag{22a}$$

The equation (22a) may also be written in the following form:

$$\frac{\epsilon_z\mu_z(\eta_2'^2-\eta_1'^2)}{\eta'_1\ \eta'_2}\ K_o^2(\delta a)J_1(\eta'_1 a)J_1(\eta'_2 a) + \frac{\epsilon_2\mu_2(\eta_2'^2-\eta_1'^2)}{\delta^2} K_1^2(\delta a)J_o(\eta'_1 a)J_o(\eta'_2 a)$$
$$-\frac{\left[\epsilon_2\mu_z M-\mu_2\epsilon_z S\right]}{\delta\eta'_2} K_o(\delta a)K_1(\delta a)J_o(\eta'_1 a)J_1(\eta'_2 a)+\frac{\left[\epsilon_2\mu_z S-\mu_2\epsilon_z M\right]}{\delta\eta'_1} K_o(\delta a)K_1(\delta a)J_o(\eta'_2 a)J_1(\eta'_1 a)=0 \tag{22b}$$

When the anisotropic medium completely fills the waveguide in such a way that $a = b \gg 1$, the corresponding dispersion relation[+] can be obtained from (11) letting $a \gg 1$. It should be noted here that if $a \gg 1$ in equations (21) where the limit $b \to \infty$ has been taken the result will be different from that obtained from (11) with $a \gg 1$. The reason is that if $a \gg 1$ in (11), it also means $a = b \gg 1$, but if $a \gg 1$ in (21), it means that $b \gg 1$, $a \gg 1$, yet $a \neq b$. The difference between the above two results can be shown rather easily for $n = 0$ as follows: either from (11) or (12) one obtains the following dispersion relation

[+]When $a = b \to \infty$, there is no dispersion due to the boundary.

$$\frac{\eta'_2 S}{\eta'_1 M} = \frac{\tan(\eta'_2 a - \pi/4)}{\tan(\eta'_1 a - \pi/4)}, \quad \text{as} \quad a = b >> 1 \tag{23}$$

and from (22a) one obtains

$$\frac{S\left\{\delta\mu_z \operatorname{Sin}(\eta'_1 a-\pi/4)+\eta'_1\mu_2\cos(\eta'_1 a-\pi/4)\right\}}{M\left\{\delta\mu_z \operatorname{Sin}(\eta'_2 a-\pi/4)+\eta'_2\mu_2\cos(\eta'_2 a-\pi/4)\right\}} = \frac{\delta\epsilon_z \operatorname{Sin}(\eta'_1 a-\pi/4)+\epsilon_2\eta'_1\cos(\eta'_1 a-\pi/4)}{\delta\epsilon_z \operatorname{Sin}(\eta'_2 a-\pi/4)+\epsilon_2\eta'_2\cos(\eta'_2 a-\pi/4)}$$

$$\text{for } a >> 1,\ b >> 1, \text{ but } a \neq b. \tag{24}$$

For $\epsilon' = 0 = \mu' = a_3$, the dispersion relation (21) becomes

$$\left[\frac{\epsilon_r\delta\eta'_1 J'_n(\eta'_1 a)}{J_n(\eta'_1 a)} + \frac{\epsilon_2 a_1 K'_n(\delta a)}{K_n(\delta a)}\right]\left[\frac{\mu_r\delta\eta'_2 J'_n(\eta'_2 a)}{J_n(\eta'_2 a)} + \frac{\mu_2 a_4 K'_n(\delta a)}{K_n(\delta a)}\right] = \left[\frac{n\mathcal{K}(a_4+\delta^2)}{ka\,\delta}\right]^2 \tag{25}$$

Equation (25) can also be obtained from (14) directly using $\eta = -j\delta$, and letting $b \longrightarrow \infty$.

For $n = 0$, the E-type and H-type modes separate and one obtains the following two relations from (25)

$$\frac{\epsilon_r\delta\eta'_1 J_1(\eta'_1 a)}{J_o(\eta'_1 a)} = -\frac{\epsilon_2 a_4 K_1(\delta a)}{K_o(\delta a)} \quad \text{for E-type modes} \tag{26a}$$

and

$$\frac{\mu_r\delta\eta'_2 J_1(\eta'_2 a)}{J_o(\eta'_2 a)} = -\frac{\mu_2 a_4 K_1(\delta a)}{K_o(\delta a)} \quad \text{for H-type modes} \tag{26b}$$

It should be noted here that for isotropic media, η'_1 and η'_2 in (25) and (26) are given by

$$\eta'^2 = \eta'^2_1 = \eta'^2_2 = a_4 = k^2\epsilon_r\mu_r - \chi^2, \text{ where } \epsilon_r = \epsilon_z \cdot \mu_r = \mu_z.$$

For $\epsilon' = 0 = \mu'$, $n = 0$ and also $a = b >> 1$, it can be shown either from (16) or from (23) that the dispersion relation reduces to

$$\cos(\eta'_1 a - \pi/4) = 0, \text{ for E-type modes, } a >> 1 \tag{27a}$$

and

$$\sin(\eta'_2 a - \pi/4) = 0, \text{ for H-type modes, } a >> 1 \tag{27b}$$

Now it is also natural to discuss the dispersion relation for $a << 1$ and b finite. To do this the following approximations will be used (see [17]):

$$J_n(x) \sim \frac{(x/2)^n}{n!}, \quad \text{for} \quad 0 < x << 1 \tag{28a}$$

$$N_o(x) \sim (2/\pi)\log\left(\frac{\alpha x}{2}\right), \quad \text{for} \quad 0 < x << 1 \tag{28b}$$

where $\alpha = 1.781072$

$$N_n(x) \sim -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n, \quad n \neq 0, \quad 0 < x << 1 \tag{28c}$$

$$K_o(x) \sim -\log\left(\frac{\alpha x}{2}\right), \quad 0 < x << 1 \tag{28d}$$

$$K_n(x) \sim \frac{(n-1)!}{2}\left(\frac{2}{x}\right)^n \quad n \neq 0, \quad 0 < x << 1 \tag{28e}$$

$$J'_n(x) \sim \frac{x^{n-1}}{2^n(n-1)!}, \quad n \geq 1, \quad 0 < x << 1 \tag{28f}$$

$$N_n'(x) \sim \frac{n!\,2^n}{\pi x^{n+1}}\,, \quad \text{for } n \geqslant 1, \quad 0 < x << 1 \tag{28g}$$

$$K_n'(x) \sim -\frac{n!\,2^{n-1}}{x^{n+1}}\,, \quad \text{for } n \geqslant 1, \quad 0 < x << 1 \tag{28h}$$

It will be assumed that $\eta a << 1$, $\eta'_1 a << 1$, $\eta'_2 a << 1$, in addition to $a << 1$.

Then, one obtains the following expressions:

$$\left.\begin{aligned}
G_n(a) &\sim \frac{(\eta a)^n}{2^n n!} N_n'(\eta b) + \frac{(n-1)!}{\pi}\left(\frac{2}{\eta a}\right)^n J_n'(\eta b)\\
\mathcal{J}_n(a) &\sim -\frac{(n-1)!}{\pi}\left(\frac{2}{\eta a}\right)^n J_n(\eta b) - \frac{(\eta a)^n}{2^n n!} N_n(\eta b)\\
S_n(a) &\quad \frac{n!\,2^n}{\pi(\eta a)^{n+1}} J_n'(\eta b) - \frac{(\eta a)^{n-1}}{2^n (n-1)!} N_n'(\eta b)\\
C_n(a) &\sim \frac{n!\,2^n}{\pi(\eta a)^{n+1}} J_n(\eta b) - \frac{(\eta a)^{n-1}}{2^n (n-1)!} N_n(\eta b)
\end{aligned}\right\} \quad \text{for } n \geqslant 1 \tag{29}$$

$$\left.\begin{aligned}
G_o(a) &\sim -N_1(\eta b) + \frac{2}{\pi} J_1(\eta b) \log\left(\frac{\alpha \eta a}{2}\right)\\
\mathcal{J}_o(a) &\sim \frac{2}{\pi} J_o(\eta b) \log\left(\frac{\alpha \eta a}{2}\right) - N_o(\eta b)\\
S_o(a) &\sim -\frac{2}{\pi \eta b} J_1(\eta b) - \frac{\eta b}{2} N_1(\eta b)\\
C_o(a) &\sim \frac{2}{\pi \eta b} J_o(\eta b) + \frac{\eta b}{2} N_o(\eta b)
\end{aligned}\right\} \quad \text{for } n = 0 \tag{30}$$

Thus, for $a << 1$, $\eta a << 1$, $\eta_1' a << 1$ and $\eta_2' a << 1$, various forms of the dispersion relation can be obtained merely by substituting these expressions in (54I), (55I) etc.

APPENDIX D

A NUMERICAL EXAMPLE

In this appendix, results of numerical calculations will be presented for a special case of the physical situation described in (14a) of Chapter II.

A normalized value of the electric field, E_z, will be computed as a function of $\frac{r}{a}$ for the smallest eigenvalue, η'_1, where we are treating the case

$$\left.\begin{array}{l} \eta'^2_1 > 0 \\ \eta'^2_2 < 0 \\ \frac{\partial}{\partial\theta} = 0 \end{array}\right\} \qquad (1a)$$

$$\left.\begin{array}{l} 1 - x > 0 \\ 1 - y > 0 \\ 2 > x + y > 1 \end{array}\right\} \qquad (1b)$$

with the specific parameter choices

$$\left.\begin{array}{l} ka \geq 10^2 \\ \epsilon_2 = 6 \text{ (which represents a glass tube)} \\ \mu_2 = 1 \\ \mathcal{X}/k = \beta = 2.5 \,(> \sqrt{6\,\mu_2}\,) \end{array}\right\} \qquad (2)$$

$$\left.\begin{aligned} a\delta &= ka\sqrt{\beta^2 - \epsilon_2\mu_2} = ka\sqrt{\beta^2 - 6} \\ a &= 10^{-3} \\ c &= b = 2 \times 10^{-3} \end{aligned}\right\} \qquad (2)$$

First the lowest eigenvalue must be determined. To do this, first some necessary constants must be computed as follows:

$$\left.\begin{aligned} \bar{\mathcal{J}}_o &= c_1 = I_o(\delta b)\, K_o(\delta a) - I_o(\delta a)\, K_o(\delta b) > 0 \\ \bar{G}_o &= c_2 = I_1(\delta b)\, K_o(\delta a) + I_o(\delta a)\, K_1(\delta b) > 0 \\ \bar{S}_o &= c_3 = I_1(\delta b)\, K_1(\delta a) - I_1(\delta a)\, K_1(\delta b) > 0 \\ \bar{C}_o &= c_4 = I_1(\delta a)\, K_o(\delta b) + I_o(\delta b)\, K_1(\delta a) > 0 \end{aligned}\right\} \qquad (3)$$

$\eta'_1(y)$ will then be computed from the following expression

$$\frac{\eta'^2_1}{k^2} = \frac{1}{2(x + y - 1)} \left[2(1 - y) \left\{ \psi(1 - x) + y \right\} - \psi x y + y f(y) \right], \qquad (4a)$$

where $$\psi = \beta^2 - 1, \qquad (4b)$$

and $$f^2(y) = \psi^2 x^2 + 4\beta^2 x(1 - y), \qquad (4c)$$

For a range of values of y (where $1 - x < y < 1$).

Starting with y corresponding to the smallest value of η'_1 obtained from (4a), the function $G(\eta'_1, y)$ given below will be computed for neighboring y until a change in sign is obtained. [i.e., we want to find lowest value of η'_1/k which satisfies $G(\eta'_1, y) = 0$]

$$G(\eta'_1, y) = (\rho^2+\eta'^2_1)\left[\frac{(1-y)c_1 c_2 J_1(\eta'_1 a) I_1(\rho a)}{\eta'_1 \rho} + \frac{\epsilon_2 c_3 c_4 J_0(\eta'_1 a) I_0(\rho a)}{\delta^2}\right]$$

$$+ \frac{J_0(\eta'_1 a) I_1(\rho a)}{\delta \rho}\left[\epsilon_2 M c_4 c_2 - (1-y) S c_1 c_3\right]$$

$$+ \left[(1-y) M c_1 c_3 - \epsilon_2 S c_4 c_2\right]\frac{I_0(\rho a) J_1(\eta'_1 a)}{\delta \eta'_1} = 0 \tag{5}$$

where

$$\frac{\rho^2(y)}{k^2} = -\frac{\eta'^2_2}{k^2} = \frac{y\, f(y)}{x+y-1} - \frac{\eta'^2_1(y)}{k^2}, \tag{6a}$$

$$M = \frac{k^2(1-y)(1-x)}{x+y-1} \cdot \left[\frac{x+y-1}{1-x} + \beta^2\right] + \rho^2 \tag{6b}$$

$$S = M - (\rho^2 + \eta'^2_1)$$

It should be noted that neither $\eta'_1 = 0$, nor $\rho = 0$, can be a solution to (5).

When the lowest eigenvalue η'_1 is determined, $|E_z|$ will be calculated as a function of r from the following expression where all relevant parameters are evaluated at the previously determined lowest η'_1.

$$\frac{c\,|E_z|}{m} = \frac{\omega\epsilon_0\epsilon_2 c}{\pi}\left|\frac{\xi_2\left[S\xi_1 I_0(\rho r) - M J_0(\eta'_1 r)\right]}{\delta^2(\rho^2+\eta'^2_1)\cdot\sum_{\ell=1}^{8} F_\ell I_\ell}\right| \quad \text{for } 0 \le r \le a \tag{7a}$$

$$= \frac{2\omega\epsilon_o\epsilon_2 c}{\pi^2} \left| \frac{|\xi_2|^2 \bar{\mathscr{I}}_o(r)}{\delta^2 \sum_{\ell=1}^{8} F_\ell \, I_\ell} \right| , \quad \text{for } a \leq r \leq b \tag{7b}$$

where

$$\xi_1 = \frac{\rho \left[\delta c_2 J_1(\eta'_1 a) + \eta'_1 c_3 J_o(\eta'_1 a)\right]}{\eta'_1 \left[\delta c_2 I_1(\rho a) + \rho c_3 I_o(\rho a)\right]} \tag{8a}$$

$$\xi_2 = \frac{\pi \left[S \xi_1 I_o(\rho a) - M J_o(\eta'_1 a)\right]}{2 c_1 (\rho^2 + \eta'^2_1)} \tag{8b}$$

$$F_1 = L \epsilon_r \epsilon_z \eta'^2_1 \left[\epsilon_r \rho^2 \overset{*}{M}\overset{*}{R} + \omega\mu_o \rho^2 \overset{*}{R}' k \beta \epsilon'\right] \tag{8c}$$

$$F_2 = -j L \epsilon_r \epsilon_z \eta'_1 \rho \overset{*}{\xi_1} \left[\omega \mu_o \rho^2 T' k \beta \epsilon' + \epsilon_r R \eta'^2_1 \overset{*}{S}\right] \tag{8d}$$

$$F_3 = -j L \epsilon_r \epsilon_z \rho \eta'_1 \xi_1 \left[\omega \mu_o \eta'^2_1 \overset{*}{R}' k \beta \epsilon' - \epsilon_r \rho^2 \overset{*}{M} T\right], \tag{8e}$$

$$F_4 = -L \epsilon_r \epsilon_z \rho^2 \xi_1^2 \left[\omega \mu_o \eta'^2_1 \overset{*}{T}' k \beta \epsilon' + \epsilon_r T \eta'^2_1 S\right], \tag{8f}$$

$$F_5 = \frac{\omega k \beta}{\delta^2} \left[\epsilon_o \epsilon_2 \xi_2^2 \left\{I_o^2(\delta b) + \frac{4}{\pi^2} K_o^2(\delta b)\right\} + \mu_o \xi_3^2 \left\{I_1^2(\delta b) + \frac{4}{\pi^2} K_1^2(\delta b)\right\}\right] \tag{8g}$$

$$F_6 = \frac{\omega k \beta}{\delta^2} \left[\epsilon_o \epsilon_2 \xi_2^2 I_o(\delta b) \left\{ j I_o(\delta b) + \frac{2}{\pi} K_o(\delta b)\right\} - j \mu_o \xi_3^2 I_1(\delta b) \left\{ j \frac{2}{\pi} K_1(\delta b) + I_1(\delta b)\right\}\right] \tag{8h}$$

$$F_7 = \overset{*}{F}_6 , \tag{8i}$$

$$F_8 = \frac{\omega k\beta}{\delta^2}\left[\epsilon_o\epsilon_2\xi_2^2 I_o^2(\delta b) + \mu_o\ \xi_3^2 I_1^2(\delta b)\right], \tag{8j}$$

$$\xi_3 = \frac{\pi\omega\epsilon_o\epsilon_z\epsilon' k\beta}{2\epsilon_r c_2(\rho^2+\eta_1'^2)}\left[J_o(\eta'_1 a) - \xi_1 I_o(\rho a)\right], \tag{8k}$$

$$L = \frac{\omega^2\epsilon_o^2\epsilon_z^2\epsilon' k\beta}{\epsilon_r^4\left|\rho^2\eta_1'^2(\rho^2+\eta_1'^2)\right|^2} \tag{9a}$$

$$R = \frac{k^2\epsilon_r M(\epsilon_r - \beta^2) - k^4\epsilon_z\epsilon'^2}{\omega\epsilon_o\epsilon_z\epsilon'}, \tag{9b}$$

$$R' = \frac{k\beta\left[\epsilon_r M - \epsilon_z k^2(\epsilon_r - \beta^2)\right]}{\epsilon_z}, \tag{9c}$$

$$T = \frac{k^4\epsilon_z\epsilon'^2 - k^2\epsilon_r S(\epsilon_r - \beta^2)}{\omega\epsilon_o\epsilon_z\epsilon'}, \tag{9d}$$

$$T' = \frac{k\beta\left[\epsilon_r S - k^2\epsilon_z(\epsilon_r - \beta^2)\right]}{\epsilon_z}, \tag{9e}$$

$$\mathcal{J}_o(r) = I_o(\delta b)\, K_o(\delta r) - I_o(\delta r)\, K_o(\delta b), \tag{9f}$$

$$\omega = \frac{k}{\sqrt{\epsilon_o\mu_o}}, \tag{9g}$$

$$\epsilon_o = \frac{10^{-9}}{36\pi} \quad \text{(in M.K.S. unit)} \tag{9h}$$

$$\mu_o = 4\pi \times 10^{-7} \text{ (in M.K.S. unit)} \tag{9i}$$

$$\epsilon_z = 1 - y \,, \tag{9j}$$

$$\epsilon_r = \frac{x + y - 1}{x - 1} \,, \tag{9k}$$

$$\epsilon' = \frac{y\sqrt{x}}{x-1} \,, \tag{9l}$$

$$I_1 = \frac{a^2}{2}\left[J_1^2(\eta'_1 a) - \frac{2}{\eta'_1 a} J_o(\eta'_1 a) J_1(\eta'_1 a) + J_o^2(\eta'_1 a)\right], \tag{10a}$$

$$I_2 = \overset{*}{I_3} = \frac{ja}{(\rho^2 + \eta'^2_1)}\left[\rho J_1(\eta'_1 a) I_2(\rho a) + \eta'_1 I_1(\rho a) J_2(\eta'_1 a)\right], \tag{10b}$$

$$I_4 = \frac{a^2}{2}\left[I_1^2(\rho a) + \frac{2I_o(\rho a)\, I_1(\rho a)}{\rho a} - I_o^2(\rho a)\right], \tag{10c}$$

$$\left.\begin{aligned} I_5 &= \frac{b^2}{2}\left[I_1^2(\delta b) + \frac{2I_o(\delta b)\, I_1(\delta b)}{\delta b} - I_o^2(\delta b)\right] \\ &\quad - \frac{a^2}{2}\left[I_1^2(\delta a) + \frac{2I_o(\delta a)\, I_1(\delta a)}{\delta a} - I_o^2(\delta a)\right] \end{aligned}\right\} \tag{10d}$$

$$\left.\begin{aligned} I_6 &= \overset{*}{I_7} = jI_5 + \frac{b^2}{\pi}\left[I_1(\delta b)K_1(\delta b) + I_o(\delta b)K_o(\delta b) + \frac{I_o(\delta b)K_1(\delta b) - I_1(\delta b)K_o(\delta b)}{\delta b}\right] \\ &\quad - \frac{a^2}{\pi}\left[I_1(\delta a)K_1(\delta a) + I_o(\delta a)K_o(\delta a) + \frac{I_o(\delta a)K_1(\delta a) - I_1(\delta a)K_o(\delta a)}{\delta a}\right], \end{aligned}\right\} \tag{10e}$$

$$I_8 = I_5 + \frac{2b^2}{\pi^2}\left[K_1^2(\delta b) - \frac{2K_o(\delta b)K_1(\delta b)}{\delta b} - K_o^2(\delta b)\right]$$

$$- \frac{2a^2}{\pi^2}\left[K_1^2(\delta a) - \frac{2K_o(\delta a)K_1(\delta a)}{\delta a} - K_o^2(\delta a)\right] \tag{10f}$$

Discussion of the Computation Procedure and Results

Starting with y values near 1.0, it is found that η'_1 increases very rapidly as $1 - y$ increases. An analysis of the expression for G shows that the first zero of G will occur shortly after $J_o(\eta'_1 a)$ changes sign, i.e., when $\eta'_1 a$ is slightly greater than 2.4, i.e., $\eta'_1 > 2.4 \times 10^3$ (with $a = 10^{-3}$). With the given parameters it is found y is very near 1.0.

After finding η'_1, and the corresponding x and y, for which G = 0, various expressions in equations (8), (9) and (10) have been computed. Since the parameters δa, δb are large in this calculation, the asymptotic formulas for evaluating $I_o(z)$, $I_1(z)$, $K_o(z)$ and $K_1(z)$ have been used (where z stands for either δa or δb). With the parameter used, $K_o(\delta b)$ and $K_1(\delta b)$ are very small and nearly all terms involving them were insignificant. However, the form used included all operations and can be used, without alteration, for any set of parameters.

It should be noted that, although some of the F_ℓ and I_ℓ in (8) and (10) are real, some pure imaginary, and some complex, all imaginary terms in the summation cancel out, and hence $\sum_{\ell=1}^{8} F_\ell \; I_\ell$ is real. This fact is in agreement with the theory as discussed in connection with the equations (82) - (84) of Chapter I, where it is stated that the power-flow in a non-dissipative medium is real.

Computation has been completed for the following cases:

Case I) $x = 0.7$ and $x = 0.5$, with $ka = 2 \times 10^2$ and $\beta = 2.5$(or $\beta^2 = 6.25$)

Case II) $x = 0.7$ and $x = 0.5$, with $ka = 10^2$, $\beta^2 = 6.000025$

Case III) $x = 0.7$ and $x = 0.5$, with $ka = 7$, $\beta^2 = 6.005102041$

The following pages show tables of values of many of the variables involved, and the values of $\frac{c|E_z|}{m}$, for $0 < a < b$. Also graphs of $\frac{c|E_z|}{m}$ in the range $0 < r/a < 1.0$ corresponding to the above cases have been shown. For the range $r \geq a$, the values of $\frac{c|E_z|}{m}$ are too small to show on the graph.

The behavior of all the graphs plotted here is more or less the same. It is the nature of the slow waves. The higher the value of $\beta = \frac{\varkappa}{k}$, the slower the wave. Moreover, the higher the value of ϵ_2, there is a minimum value of β, for which a corresponding slow-surface wave can propagate. Since in the above computation ϵ_2 is chosen to be 6, the minimum value of β is greater than 2.45. On the other hand a larger value of ϵ_2 will permit a lesser slow wave to propagate. Although the above statements show that the degree of slowness of the surface waves is markedly influenced by the value of ϵ_2 and hence β, the strength or amplitude, however, of these waves depends on various other parameters. For example, in the Cases I and III the graphs show that higher the value of $x = \frac{\omega_c^2}{\omega^2}$, the lower the amplitudes of $\frac{c\,|E_z|}{m}$. On the other hand, in the Case II the graphs show that the higher the value of x, the larger the amplitudes of $\frac{c\,|E_z|}{m}$, although the values of β in all of the above cases are of the same order. Moreover, the graphs of the Case II show that the amplitudes of $\frac{c\,|E_z|}{m}$ is about 10^{299} times higher than that of the Case I and is about 10 times higher than that of the Case III. Therefore, the above discussions of the numerical results suggest that for any practical purposes the results of Case II will be of greater significance.

All the graphs plotted here change monotomically, because of the higher value of β. On the other hand, if β is small (and hence the smaller value of the parameters (ρa), δ a etc.), it is expected that there will be a few oscillations of $\frac{c\,|E_z|}{m}$ in the range $0 < r \leq a$. In support of this statement reference may be made elsewhere [7] .

Finally, it should be noted here that is is the value of δ a and δ b which played the significant role in producing tremendous difference of amplitudes of $\frac{c\,|E_z|}{m}$ in Case I and either Case II or Case III. In the former case, the value of δ is much higher than that of either of the later cases.

CASE I

Summary of Values for $ak = 2 \times 10^2$, $\beta = 2.5$

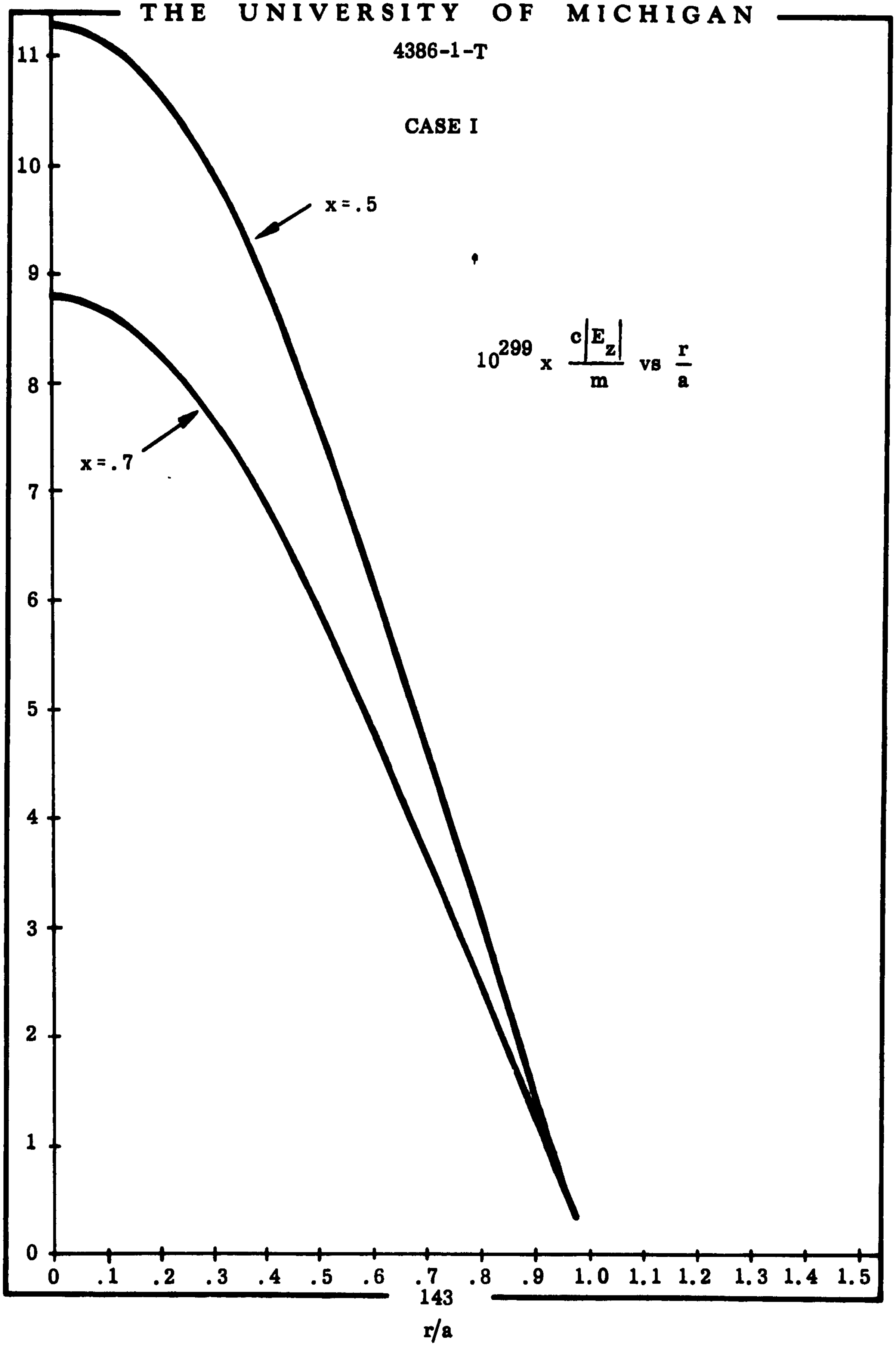
Values of $\frac{c|E_z|}{m}$

x	.7	.5	r/a	For x = .7	For x = .5
y	.9999731125	.9999849842	0	$8.7559211 \times 10^{-299}$	$1.1286428 \times 10^{-298}$
f	3.675064017	2.625035752	.1	$8.6297346 \times 10^{-299}$	$1.1123792 \times 10^{-298}$
η'_1	$2.405305063 \times 10^{3}$	$2.405160330 \times 10^{3}$	.2	$8.2566279 \times 10^{-296}$	$1.0642912 \times 10^{-298}$
P	$4.582578887 \times 10^{5}$	$4.582578191 \times 10^{5}$	.3	$7.6526916 \times 10^{-299}$	$9.8645246 \times 10^{-299}$
M	$2.100042490 \times 10^{11}$	$2.100045835 \times 10^{11}$	.4	$6.8438881 \times 10^{-299}$	$8.8220867 \times 10^{-297}$
S	-1.8291×10^{6}	-1.4301×10^{6}	.5	$5.8648043 \times 10^{-299}$	$7.5601706 \times 10^{-299}$
ξ_1	$9.062918575 \times 10^{-197}$	$9.064671742 \times 10^{-197}$	.6	$4.7570032 \times 10^{-299}$	$6.1323402 \times 10^{-299}$
ξ_2	$1.574477758 \times 10^{-45}$	$8.863836389 \times 10^{-46}$	.7	$3.5670449 \times 10^{-299}$	$4.5985988 \times 10^{-299}$
ϵ_r	-2.333243708	- .9999699684	.8	$2.3442840 \times 10^{-299}$	$3.0225528 \times 10^{-299}$
ϵ'	-2.788791769	-1.414192327	.9	$1.1385489 \times 10^{-299}$	$1.4684199 \times 10^{-299}$
R	$-4.228947963 \times 10^{24}$	$-5.405747531 \times 10^{24}$	1.0	$8.3359942 \times 10^{-303}$	$6.0491791 \times 10^{-303}$
T	$-4.524434966 \times 10^{19}$	$-4.107746983 \times 10^{19}$	1.1	$3.6088193 \times 10^{-307}$	$2.6188111 \times 10^{-307}$
R'	$-9.111703604 \times 10^{21}$	$-6.992432040 \times 10^{21}$	1.2	$1.5687972 \times 10^{-311}$	$1.1384287 \times 10^{-311}$
T'	$2.510277019 \times 10^{17}$	$1.926178097 \times 10^{17}$	1.3	$6.843\ 557 \times 10^{-316}$	$4.9660890 \times 10^{-316}$
ξ_3	$-1.187508895 \times 10^{-47}$	$-7.847883607 \times 10^{-48}$	1.4	$2.9941111 \times 10^{-320}$	$2.1727359 \times 10^{-320}$
L	$-1.470483272 \times 10^{-58}$	$-6.895189885 \times 10^{-58}$	1.5	$1.3133097 \times 10^{-324}$	$9.5302916 \times 10^{-325}$
F_1	$3.396169086 \times 10^{-8}$	$1.896654271 \times 10^{-8}$	1.6	$5.7733840 \times 10^{-329}$	$4.1895703 \times 10^{-329}$
F_2	$5.107586708j \times 10^{-207}$	$2.230758055j \times 10^{-207}$	1.7	$2.5429686 \times 10^{-333}$	$1.8453555 \times 10^{-333}$
F_3	$-8.171358617j \times 10^{-208}$	$-3.568967890j \times 10^{-208}$	1.8	$1.1220024 \times 10^{-337}$	$8.1421868 \times 10^{-338}$
F_4	$2.429623822 \times 10^{-405}$	$1.061285373 \times 10^{-405}$	1.9	$4.9582959 \times 10^{-342}$	$3.5980855 \times 10^{-342}$
I_1	$1.347570276 \times 10^{-7}$	$1.347570418 \times 10^{-7}$	2.0	0	0
I_2	$2.205105178j \times 10^{188}$	$2.205083423j \times 10^{188}$			
I_3	$-2.205105178j \times 10^{188}$	$-2.205083423j \times 10^{188}$			
I_4	$4.126696590 \times 10^{385}$	$4.126122615 \times 10^{385}$			
F_5	$3.842092989 \times 10^{80}$	$1.481363914 \times 10^{80}$			
$R_e F_6$	$1.474033506 \times 10^{-93}$	$5.68546845 \times 10^{-94}$			
$I_m F_6$	$-5.592484518 \times 10^{79}$	$-4.409151006 \times 10^{79}$			
F_8	$3.842092989 \times 10^{80}$	$1.481363914 \times 10^{30}$			
I_5	$4.139482976 \times 10^{162}$	$4.139482976 \times 10^{162}$			
$R_e I_6$	$3.183039180 \times 10^{-9}$	$3.183039180 \times 10^{-9}$			
I_8	$4.139482976 \times 10^{162}$	$4.139482976 \times 10^{162}$			
$F_1 I_1$	$4.576576513 \times 10^{-15}$	$2.555875189 \times 10^{-15}$			
$F_2 I_2$	$-1.126276590 \times 10^{-18}$	$-4.919007608 \times 10^{-19}$			
$F_3 I_3$	$-1.801870520 \times 10^{-19}$	$-7.869871931 \times 10^{-20}$			
$F_4 I_4$	$1.002632034 \times 10^{-19}$	$4.378993579 \times 10^{-20}$			
$F_5 I_5$	$1.590427852 \times 10^{243}$	$6.132080703 \times 10^{242}$			
$R_e F_6 I_6$	$2.314999446 \times 10^{242}$	$1.825160553 \times 10^{242}$			
$F_8 I_8$	$1.590427852 \times 10^{243}$	$6.132080703 \times 10^{242}$			
$\sum F_e I_8$	$3.643855593 \times 10^{243}$	$1.591448251 \times 10^{243}$			
Coeff 1	$4.169401884 \times 10^{-310}$	$5.374372065 \times 10^{-310}$			
Coeff 2	$8.776518558 \times 10^{-344}$	$6.368854307 \times 10^{-344}$			

$$\text{Coeff}_1 = \frac{\omega \epsilon_o \epsilon_2 c}{\pi} \left| \frac{\xi_2}{S^2(\rho^2+\eta_1'^2)\Sigma} \right|$$

$$\text{Coeff}_2 = \frac{2\omega \epsilon_o \epsilon_2 c}{\pi^2} \left| \frac{\xi_2^2}{S^2 \Sigma} \right|$$

$$\text{where } \sum = \sum_{\ell=1}^{8} F_\ell I_\ell$$


CASE I
10^{299} x $\frac{c|E_z|}{m}$ vs $\frac{r}{a}$
x = .5
x = .7
11
10
9
8
7
6
5
4
3
2
1
0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
r/a

CASE II

Summary of Values for $ak = 10^2$, $\beta^2 = 6.000025$

$\delta = 5 \times 10^2$

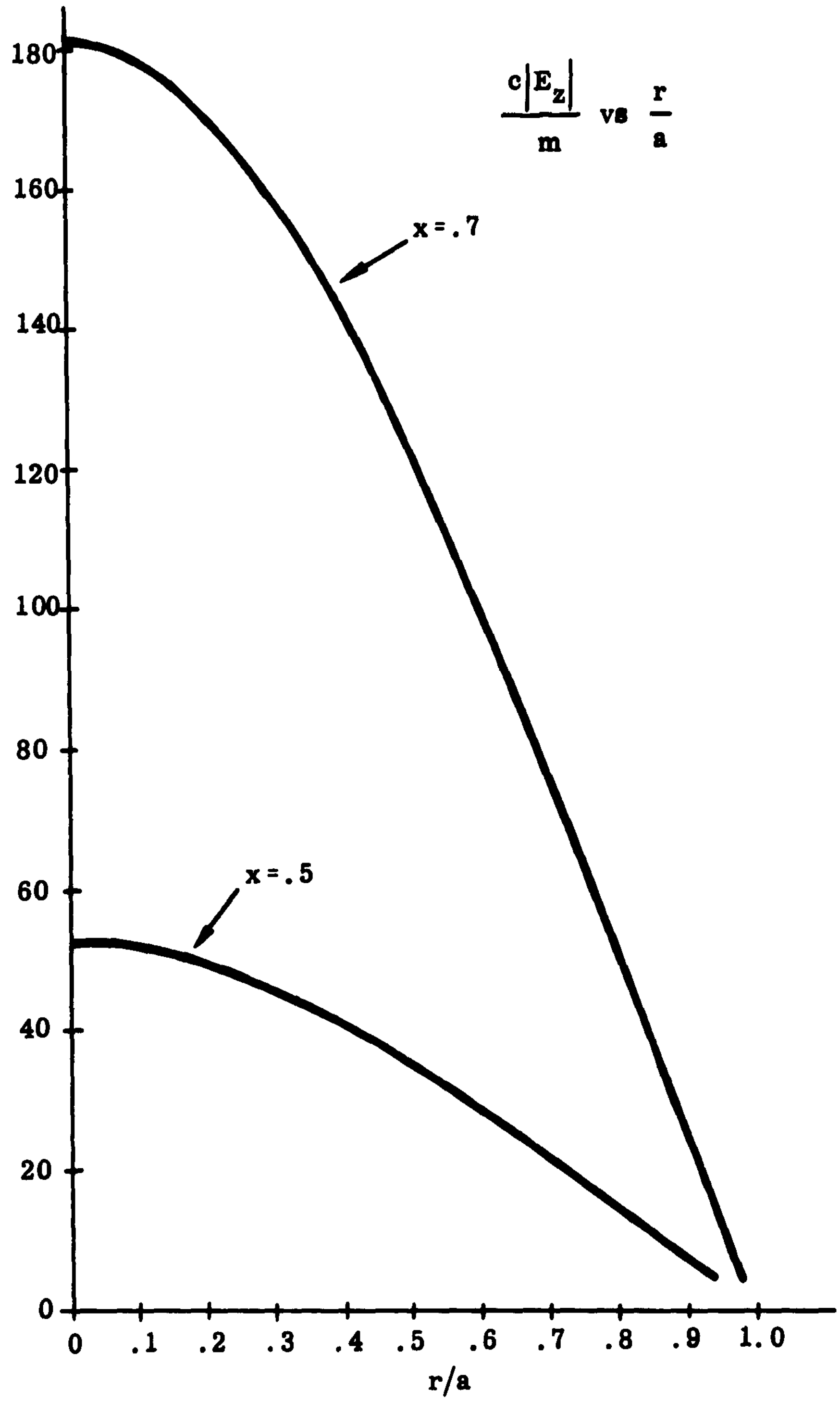
Values of $\frac{c\,|E_z|}{m}$

x	.7	.5	r/a	For x = .7	For x = .5
y	.9998905959	.9999384817	0	1.8096432×10^{2}	5.2136187×10
f	3.500280060	2.500160139	.1	1.7835734×10^{2}	5.1385115×10
η_1'	$2.404840387 \times 10^{3}$	$2.404835008 \times 10^{3}$	.2	1.7064903×10^{2}	4.9164345×10
ρ	$2.236080556 \times 10^{5}$	$2.236079069 \times 10^{5}$	.3	1.5817168×10^{2}	4.5569609×10
M	$5.000447026 \times 10^{10}$	$5.000480279 \times 10^{10}$	.4	1.4146147×10^{2}	4.0755384×10
s	-1.87552×10^{6}	-1.47648×10^{6}	.5	1.2123270×10^{2}	3.4927458×10
ξ_1	$4.638354547 \times 10^{-97}$	$4.639144141 \times 10^{-97}$	.6	9.8343796×10	2.8333140×10
ξ_2	$3.646241238 \times 10^{-6}$	$3.532368146 \times 10^{-7}$	.7	7.3756387×10	2.1249478×10
ϵ_r	-2.332968653	- .9998769634	.8	4.8489892×10	1.3970166×10
ϵ'	-2.788561642	-1.414126562	.9	2.3573572×10	6.7917354
R	$-1.201246835 \times 10^{23}$	$-1.516656482 \times 10^{23}$	1.0	3.0356136×10^{-4}	8.4724674×10^{-6}
T	$-5.556824512 \times 10^{18}$	$-5.011324661 \times 10^{18}$	1.1	2.5978231×10^{-4}	7.2505840×10^{-6}
R'	$-2.611720668 \times 10^{20}$	$-1.990641619 \times 10^{20}$	1.2	2.2038224×10^{-4}	6.1509190×10^{-6}
T'	$3.020818349 \times 10^{16}$	$2.302445054 \times 10^{16}$	1.3	1.8470813×10^{-4}	5.1552463×10^{-6}
ξ_3	$-3.676783183 \times 10^{-8}$	$-2.446310694 \times 10^{-8}$	1.4	1.5209538×10^{-4}	4.2450169×10^{-6}
L	$-9.287120418 \times 10^{-56}$	$-4.413530397 \times 10^{-55}$	1.5	1.2211892×10^{-4}	3.4083672×10^{-6}
F_1	$1.421535942 \times 10^{-7}$	$7.993544921 \times 10^{-8}$	1.6	9.4365436×10^{-5}	2.6337610×10^{-6}
F_2	$2.299760241j \times 10^{-106}$	$1.018226137j \times 10^{-106}$	1.7	6.8530496×10^{-5}	1.9127019×10^{-6}
F_3	$-3.832039965j \times 10^{-107}$	$-1.696764786j \times 10^{-107}$	1.8	4.4333925×10^{-5}	1.2373700×10^{-6}
F_4	$1.147162520 \times 10^{-204}$	$5.079880743 \times 10^{-205}$	1.9	2.1556305×10^{-5}	6.0164144×10^{-7}
I_1	$1.347570470 \times 10^{-7}$	$1.347570466 \times 10^{-7}$	2.0	0	0
I_2	$8.015023185j \times 10^{86}$	$8.013856901j \times 10^{86}$			
I_3	$-8.015023185j \times 10^{86}$	$-8.013856901j \times 10^{86}$			
I_4	$2.653635970 \times 10^{182}$	$2.652850350 \times 10^{182}$			
F_5	$5.800314707 \times 10^{-8}$	1.06320303[illegible] $\times 10^{-8}$	.91	2.1139941×10	6.0906023
R_eF_6	$1.784938139 \times 10^{-8}$	$4.853114861 \times 10^{-9}$	.92	1.8720750×10	5.3936289
I_mF_6	$1.728284623 \times 10^{-8}$	$-6.748558216 \times 10^{-9}$	.93	1.6316944×10	4.7010883
F_8	$4.918169934 \times 10^{-8}$	$7.372337806 \times 10^{-9}$	.94	1.3929462×10	4.0132505
I_5	$2.787895257 \times 10^{-7}$	$2.787895257 \times 10^{-7}$	.95	1.1559233×10	3.3303830
R_eI_6	$3.605254500 \times 10^{-7}$	$3.605254500 \times 10^{-7}$	.96	9.2071729	2.6527501
I_8	$8.762842367 \times 10^{-7}$	$8.762842367 \times 10^{-7}$	.97	6.8741914	1.9806136
F_1I_1	$1.915619857 \times 10^{-14}$	$1.077186505 \times 10^{-14}$	.98	4.5611931	1.3142336
F_2I_2	$-1.843263165 \times 10^{-19}$	$-8.159918555 \times 10^{-20}$	.99	2.2691461	.65388399
F_3I_3	$-3.071388917 \times 10^{-20}$	$-1.359763019 \times 10^{-20}$	.995	1.1314051	.32608387
F_4I_4	$3.044151727 \times 10^{-22}$	$1.347616341 \times 10^{-22}$	.998	4.5160701×10^{-1}	.13021180
F_5I_5	$1.617066986 \times 10^{-14}$	$2.964098699 \times 10^{-15}$	.999	2.2552059×10^{-1}	6.5065232×10^{-2}
$R_eF_6I_6$	$1.616879755 \times 10^{-15}$	$3.631098763 \times 10^{-15}$			
F_8I_8	$4.309714787 \times 10^{-14}$	$6.460263407 \times 10^{-15}$			
ΣF_eI_e	$8.165756107 \times 10^{-14}$	$2.745832962 \times 10^{-14}$			
Coeff_1	$3.618962810 \times 10^{-9}$	$1.042623596 \times 10^{-9}$			
Coeff_2	$4.200826659 \times 10^{-4}$	$1.172460375 \times 10^{-5}$			

$$\mathrm{Coeff}_1 = \frac{\omega \epsilon_o \epsilon_2 c}{\pi} \left| \frac{\xi_2}{s^2(\rho^2 + \eta_1'^2)\sum} \right|$$

$$\mathrm{Coeff}_2 = \frac{2\omega \epsilon_o \epsilon_2 c}{\pi^2} \left| \frac{\xi_2^2}{s^2 \sum} \right|$$

CASE II



$\frac{c|E_z|}{m}$ vs $\frac{r}{a}$
x = .7
x = .5
180
160
140
120
100
80
60
40
20
0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
r/a

CASE III

$ka = 7 \times 10^3$, $\beta^2 = 6.005102041$

$\delta = 5 \times 10^2$

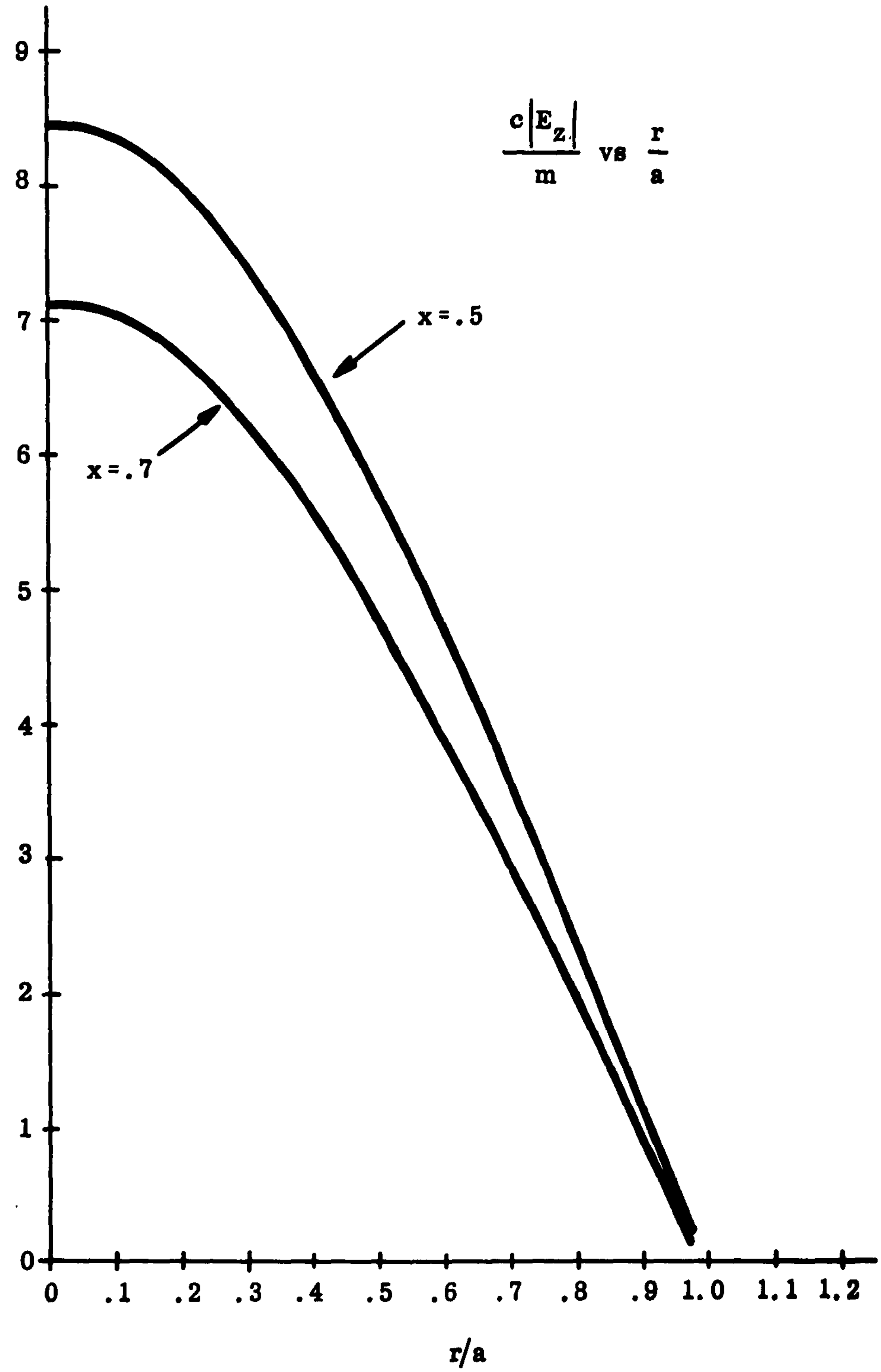
Values of $\frac{c\,|E_z|}{m}$

x	.7	.5	r/a	For x = .7	For x = .5
y	.9779975594	.9876519579	0	7.125499252	8.421984918
f	3.555976389	2.532007921	.1	7.022644030	8.300472142
η_1'	2.407251260×10^3	2.406684664×10^3	.2	6.718529845	7.941190526
P	1.566993507×10^4	1.566800030×10^4	.3	6.226294257	7.359653509
M	2.494896933×10^8	2.498167021×10^8	.4	5.567130934	6.580887224
S	-1.8520304×10^6	-1.4616625×10^6	.5	4.769275220	5.638235455
ξ_1	$2.343388642 \times 10^{-7}$	$2.353220927 \times 10^{-7}$	.6	3.866659658	4.571767509
ξ_2	$2.793267998 \times 10^{-4}$	$1.570659319 \times 10^{-4}$	.7	2.897321346	3.426394461
ϵ_r	-2.259991865	-.9753039158	.8	1.901712998	2.249848394
ϵ'	-2.727504880	-1.396750794	.9	$9.213164880 \times 10^{-1}$	1.090918066
R	$-2.045747178 \times 10^{17}$	$-2.600475240 \times 10^{17}$	1.0	$9.224264276 \times 10^{-4}$	$6.120993715 \times 10^{-4}$
T	$-1.873920396 \times 10^{15}$	$-1.703190155 \times 10^{-5}$	1.1	$7.893958174 \times 10^{-4}$	$5.238235476 \times 10^{-4}$
R'	$-4.326419710 \times 10^{14}$	$-3.326038291 \times 10^{14}$	1.2	$6.696715401 \times 10^{-4}$	$4.443774772 \times 10^{-4}$
T'	$1.021026696 \times 10^{13}$	$7.847629400 \times 10^{12}$	1.3	$5.612692559 \times 10^{-4}$	$3.724444016 \times 10^{-4}$
ξ_3	$-6.973370110 \times 10^{-6}$	$-4.646927041 \times 10^{-6}$	1.4	$4.621694767 \times 10^{-4}$	$3.066842382 \times 10^{-4}$
L	$-2.340389651 \times 10^{-45}$	$-1.090450853 \times 10^{-44}$	1.5	$3.710805671 \times 10^{-4}$	$2.462398898 \times 10^{-4}$
F_1	$2.794648081 \times 10^{-5}$	$1.575790576 \times 10^{-5}$	1.6	$2.867465463 \times 10^{-4}$	$1.902779187 \times 10^{-4}$
F_2	$3.235412563j \times 10^{-13}$	$1.444443912j \times 10^{-13}$	1.7	$2.082423802 \times 10^{-4}$	$1.381844950 \times 10^{-4}$
F_3	$-5.153970917j \times 10^{-14}$	$-2.323110925j \times 10^{-14}$	1.8	$1.347166981 \times 10^{-4}$	$8.939467016 \times 10^{-5}$
F_4	$1.153494515 \times 10^{-20}$	$5.155845188 \times 10^{-21}$	1.9	$6.550275578 \times 10^{-5}$	$4.346600927 \times 10^{-5}$
I_1	$1.347567668 \times 10^{-7}$	$1.347568769 \times 10^{-7}$	2.0	0	0
I_2	$2.099601048j \times 10^{-2}$	$2.096263246j \times 10^{-2}$			
I_3	$-2.099601048j \times 10^{-2}$	$-2.096263246j \times 10^{-2}$			
I_4	1.256431006×10^4	1.251876324×10^4			
F_5	$5.104127906 \times 10^{-6}$	$2.138824275 \times 10^{-6}$			
R_eF_6	$2.109232969 \times 10^{-6}$	$9.107530442 \times 10^{-7}$			
I_mF_6	$-1.856345443 \times 10^{-6}$	$-9.465968673 \times 10^{-7}$			
F_8	$3.768417259 \times 10^{-6}$	$1.551161837 \times 10^{-6}$			
I_5	$2.787895257 \times 10^{-7}$	$2.787895257 \times 10^{-7}$			
R_eI_6	$3.60525450 \times 10^{-7}$	$3.60525450 \times 10^{-7}$			
I_8	$8.762842367 \times 10^{-7}$	$8.762842367 \times 10^{-7}$			
F_1I_1	$3.765977397 \times 10^{-12}$	$2.123486167 \times 10^{-12}$			
F_2I_2	$-6.793075608 \times 10^{-15}$	$-3.027934684 \times 10^{-15}$			
F_3I_3	$-1.082128274 \times 10^{-15}$	$-4.869852048 \times 10^{-16}$			
F_4I_4	$1.449286274 \times 10^{-16}$	$6.454480521 \times 10^{-17}$			
F_5I_5	$1.422977398 \times 10^{-12}$	$5.962818052 \times 10^{-13}$			
$R_eF_6I_6$	$1.277961831 \times 10^{-12}$	$5.922509428 \times 10^{-13}$			
F_8I_8	$3.302204641 \times 10^{-12}$	$1.359258666 \times 10^{-12}$			
F_eI_e	$1.103935282 \times 10^{-11}$	$5.260078149 \times 10^{-12}$			
$Coeff_1$	$2.856029500 \times 10^{-8}$	$3.371265747 \times 10^{-8}$			
$Coeff_2$	$1.276497612 \times 10^{-3}$	$8.470522556 \times 10^{-4}$			

$$Coeff_1 = \frac{\omega \epsilon_o \epsilon_2 c}{\pi} \left| \frac{\xi_2}{\delta^2(\rho^2 + \eta_1^2)\sum} \right|$$

$$Coeff_2 = \frac{2\omega \epsilon_o \epsilon_2 c}{\pi^2} \left| \frac{\xi_2^2}{\delta^2 \sum} \right|$$

CASE III



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