

WAVE PROPAGATION IN AN ELASTIC SOLID WITH A LINE OF DISCONTINUITY OR FINITE CRACK*

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Abstract. With the aid of integral transforms, a method is presented for solving the problem of scattering of plane harmonic compression and shear waves by a line of discontinuity or crack of finite width embedded in an elastic medium of infinite extent. When the incoming waves are applied in an arbitrary direction, the scattered-wave field may be determined by separating the crack-surface boundary conditions into functions even and odd with respect to the variable along the line crack. The problem is reduced to the evaluation of a system of coupled Fredholm integral equations with special emphasis placed on finding the *near-field* solution which consists of a knowledge of the detailed structure of the displacements and stresses in a small region around the crack vertex. Dynamic stress-intensity factors, the critical values of which govern the condition of crack propagation, are defined and found to be dependent on the incident wave length and Poisson's ratio of the medium. At certain wave lengths, they are larger than those encountered under static loading. Such information is of particular importance in predicting the fracture strength of structures subjected to oscillating loads.

Introduction. Although the scattering of waves by obstacles of different shapes has been the subject of many past investigations in various branches of physics [1]-[3], to the authors' knowledge none of these investigations analyzed, in detail, the singular behavior of the stresses near a scatterer in the form of a line of discontinuity or finite crack. The main reason for this omission is the lack of an effective mathematical method for obtaining the near-field solution, which is of considerable theoretical interest and has innumerable applications in the field of fracture mechanics as well as in electromagnetic and acoustic theory.

A popular approach to the diffraction of waves from obstacles has been that of separation of variables, where the formal solution of the wave equation is given by an infinite series of orthogonal functions. Such an approach, however, is effective only for obstacle shapes adapted to those coordinate systems in which the wave equation is separable. For this reason, the dynamic stress concentrations around circular and parabolic obstacles have received considerable attention in the past. A comprehensive survey of the literature in a field as wide and diversified as the propagation of elastic waves is clearly beyond the scope of this paper. In recent years, the Mow-Pao-Thau school [4]-[6] has

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published a number of papers on this subject. References to other work can be found in [4]–[6].

It is well known that problems involving diffraction of plane harmonic, horizontally polarized shear waves (SH-waves) by a semi-infinite crack can be formulated in terms of integral equations, and solved by the Wiener–Hopf technique [7]. As pointed out by Sih [8], however, since the static limit of the semi-infinite crack solution is zero, it is not possible to estimate the precise magnification of the stresses due to dynamic effects. To overcome this shortcoming, Loeber and Sih [9] proposed to add another characteristic dimension into the problem, namely the crack width, and managed to obtain the exact behavior of the crack-front displacement and stress fields for the case of SH-waves diffracted by a finite or internal crack. Ang and Knopoff [10] have attempted to solve the internal crack problem earlier but their method yields results which are restricted to low frequencies and to distances far away from the crack. In elastodynamics, the far-field crack solution is not useful in the sense that it offers no information to the development of the theories of crack propagation. Generally speaking, the far-field solution can always be determined by the standard method of Wiener–Hopf [7] in a straightforward manner. On the other hand, considerable difficulty is encountered when the Wiener–Hopf method is applied to find the near-field solution. One of the difficulties arises from the factorization of certain functions into functions analytic in the upper and lower half planes. The problem of the diffraction of electromagnetic waves¹ incident upon a slit has also been treated by Schmeltzer and Lewin [11] using the function-theoretic approach. Their results are left in terms of several complicated integrals the evaluation of which becomes a problem in itself, particularly in seeking the analytical form of the solution in the vicinity of the slit.

Having discussed the previous work related to crack problems of SH-waves, it is natural to follow the discussion with a few remarks concerning the diffractions of plane harmonic compression waves (P-waves) and vertically polarized shear waves (SV-waves) by a line crack. Although both Miles [12] and Papadopoulos [13] have investigated crack problems of this type, their work discusses only the qualitative character of the displacement potentials without any explicit information given as to the nature of the local stress distribution. The mathematical description of these problems is somewhat complex because the scattered waves, caused by the line crack, are composed of both compression and shear waves even though the input wave may be of one type, either the P- or SV-waves. For this and other reasons, the near-field solution of waves scattered by a crack with finite width is yet to be found.

The purpose of this paper, aside from obtaining the stress solution close to the crack point, is to offer a method of solution for solving diffraction problems involving P- and SV-waves incident upon a line of discontinuity. The method can handle different types of boundary conditions² on the line of discontinuity. For illustration, only the case of a traction-free crack will be considered. An important conclusion is that within certain ranges of wave lengths the dynamic stress distribution around the crack is quite sensitive to changes in the wave number. This is displayed graphically for different values of the

¹The scattering of plane-polarized electromagnetic waves by a screen in a fluid medium is mathematically analogous to the SH-wave crack problem in elastodynamics.

²By following the steps outlined in this paper, it is clear that the problems of a rigid and rigid-smooth strip can be solved in the same way.

Poisson's ratio. The knowledge gained in this investigation is believed to add further impetus to the understanding of the propagation of cracks under fluctuating loads.

Field equations and input waves. Consider the propagation of elastic waves, produced by the action of oscillating compressional and shear forces, which vary harmonically in time and are applied in the xy -plane containing a through crack. In the plane, there arise both compressional and shear waves, and the resulting displacements can be expressed in terms of two scalar functions ϕ and ψ each of which depend upon x , y , and t . The rectangular components of the displacement vector are

$$u_x = \partial\phi/\partial x + \partial\psi/\partial y, \quad u_y = \partial\phi/\partial y - \partial\psi/\partial x, \tag{1}$$

Substituting Eq. (1) into the equations of motion under the conditions of plane strain, the following wave equations on ϕ and ψ are obtained:

$$c_1^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial^2 \phi}{\partial t^2}, \quad c_2^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = \frac{\partial^2 \psi}{\partial t^2}. \tag{2}$$

In Eq. (2), c_1 and c_2 stand, respectively, for the velocities of compression (irrotational) and shear (equivoluminal) waves in an infinitely extended elastic medium; they are given by

$$c_1 = [(\lambda + 2\mu)/\rho]^{1/2}, \quad c_2 = (\mu/\rho)^{1/2} \tag{3}$$

with ρ being the mass density. As usual, in the case of generalized plane stress the Lamé constant λ in Eq. (3) is to be replaced by $2\lambda\mu/(\lambda + 2\mu)$, while the shear modulus of elasticity μ remains unchanged. From the stress and displacement relations, it is found that

$$\begin{aligned} \sigma_{xx} &= \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right), \\ \sigma_{yy} &= \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right), \\ \sigma_{xy} &= \mu \left(2 \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right). \end{aligned} \tag{4}$$

With reference to the rectangular coordinate system x , y of Fig. 1, let a line crack be

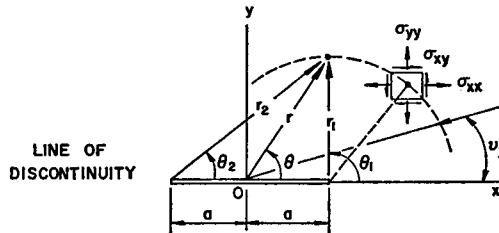


FIG. 1. Polar coordinates on line of discontinuity.

placed along the x -axis from $-a$ to $+a$, and an incident plane harmonic compressional wave (P-wave) be directed at an angle v_1 with the x -axis so that

$$\begin{aligned}\phi^{(i)} &= \phi_0 \exp \{-i [\alpha_1 (x \cos v_1 + y \sin v_1) + \omega t]\}, \\ \psi^{(i)} &= 0\end{aligned}\quad (5)$$

in which ϕ_0 is a measure of the wave amplitude, ω is the circular frequency, and $\alpha_1 = \omega/c_1$ is the wave number for compression waves.

In preparation for subsequent work, the stresses of the incident P-waves will be computed by inserting Eq. (5) into (4); the results are

$$\begin{aligned}\sigma_{xx}^{(i)} &= -\mu\alpha_1^2\phi^{(i)}(1 - 2\kappa^2 \sin^2 v_1), \\ \sigma_{yy}^{(i)} &= -\mu\alpha_1^2\phi^{(i)}(1 - 2\kappa^2 \cos^2 v_1), \\ \sigma_{xy}^{(i)} &= -\mu\alpha_1^2\phi^{(i)}\kappa^2 \sin 2v_1,\end{aligned}\quad (6)$$

where the elastic constant $\kappa^2 = (c_2/c_1)^2$ takes the value $(1 - 2\nu)/2(1 - \nu)$ for plane strain and $(1 - \nu)/2$ for generalized plane stress. The maximum value of the normal stress acting on the incident plane compression wave front is $-\mu\alpha_1^2\phi_0$ which is assumed to remain finite as $\alpha_1 \rightarrow 0$.

Similarly, if an incident plane harmonic shear wave (SV-wave) impinges on the crack at an angle v_2 with the x -axis, then

$$\begin{aligned}\phi^{(i)} &= 0, \\ \psi^{(i)} &= \psi_0 \exp \{-i [\alpha_2 (x \cos v_2 + y \sin v_2) + \omega t]\},\end{aligned}\quad (7)$$

and the corresponding stress field is

$$\begin{aligned}\sigma_{xx}^{(i)} &= -\mu\alpha_2^2\psi^{(i)} \sin 2v_2, \\ \sigma_{yy}^{(i)} &= -\sigma_{xx}^{(i)}, \\ \sigma_{xy}^{(i)} &= \mu\alpha_2^2\psi^{(i)} \cos 2v_2,\end{aligned}\quad (8)$$

in which $\mu\alpha_2^2\psi_0$ is the maximum shear stress acting on the shear wave front and is nonzero as $\alpha_2 \rightarrow 0$. The quantity ψ_0 is a constant, and $\alpha_2 = \omega/c_2$ is the shear-wave number.

In view of the harmonic time-variation of the input waves given by Eqs. (5) and (7), the displacements, stresses, and potentials ϕ and ψ will all contain the time factor $\exp(-i\omega t)$ which will henceforth be dropped.

Since both Eqs. (5) and (7) are already solutions of the wave equations, the main burden of the analysis is to determine the potentials $\phi^{(r)}$ and $\psi^{(r)}$ of the scattered-wave field governed by the Helmholtz equations

$$\begin{aligned}\frac{\partial^2 \phi^{(r)}}{\partial x^2} + \frac{\partial^2 \phi^{(r)}}{\partial y^2} + \alpha_1^2 \phi^{(r)} &= 0, \\ \frac{\partial^2 \psi^{(r)}}{\partial x^2} + \frac{\partial^2 \psi^{(r)}}{\partial y^2} + \alpha_2^2 \psi^{(r)} &= 0\end{aligned}\quad (9)$$

for problems involving steady-state motion. The total wave field may then be taken as the linear sum of the displacement potentials of the incident and diffracted waves as

$$\phi = \phi^{(i)} + \phi^{(r)}, \quad \psi = \psi^{(i)} + \psi^{(r)},$$

where

$$\phi^{(r)}, \psi^{(r)} \rightarrow 0 \quad \text{as} \quad (x^2 + y^2)^{1/2} \rightarrow \infty.$$

The same general relationships apply to the total displacement and stress fields.

For a traction-free crack surface, the total normal and shear stresses at $y = 0$ and $|x| < a$ must vanish, i.e.,

$$\begin{aligned} \sigma_{yy}^{(i)}(x, 0) + \sigma_{yy}^{(r)}(x, 0) &= 0, & |x| < a, \\ \sigma_{xy}^{(i)}(x, 0) + \sigma_{xy}^{(r)}(x, 0) &= 0, & |x| < a \end{aligned}$$

from which the boundary conditions of the scattered-wave problem may be established. For convenience, the problem will be split into two parts, namely, that of finding the solution of Eqs. (9) for the following conditions:

Case A.

$$\begin{aligned} u_y^{(r)}(x, 0) = \sigma_{xy}^{(r)}(x, 0) &= 0, & |x| > a, \\ \sigma_{yy}^{(r)}(x, 0) = -\sigma_{yy}^{(i)}(x, 0); & \quad \sigma_{xy}^{(r)}(x, 0) = 0, & |x| < a. \end{aligned} \quad (10)$$

Case B.

$$\begin{aligned} u_x^{(r)}(x, 0) = \sigma_{xy}^{(r)}(x, 0) &= 0, & |x| > a, \\ \sigma_{yy}^{(r)}(x, 0) = 0; & \quad \sigma_{xy}^{(r)}(x, 0) = -\sigma_{xy}^{(i)}(x, 0), & |x| < a. \end{aligned} \quad (11)$$

Clearly, the stress solutions for Cases A and B are even and odd in y , respectively; consequently, the former case will be called the symmetric problem and the latter the skew-symmetric problem. The desired solution of the original problem can be obtained by superposition of the solutions for the two Cases. Because of the symmetry conditions in Eqs. (10) and (11), it is possible, in each case, to lay down certain conditions on the plane $y = 0$, and consider the problem for the half-plane, $y \geq 0$, only.

Case A—symmetric problem. The resulting stresses of the scattered waves are said to be symmetric with respect to the crack plane $y = 0$, if the crack is opened³ by oppositely acting normal tractions. These normal tractions are taken to be the negative of those shown in equations (6) and (8), i.e.,⁴

$$\sigma_{yy}^{(r)}(x, 0) = -\sigma_{yy}^{(i)}(x, 0) = (-1)^j 2\mu P_j \exp(-i\alpha_j x \cos \nu_j), \quad |x| < a, \quad (12)$$

where $j = 1, 2$ such that

$$P_1 = -\frac{1}{2}\alpha_2^2 \phi_0 (1 - 2\kappa^2 \cos \nu_1)$$

corresponds to the incident P-wave and

$$P_2 = \frac{1}{2}\alpha_2^2 \psi_0 \sin 2\nu_2$$

to the incident SV-wave. To make use of the Fourier sine and cosine transforms, the problem will be further divided into parts that are even and odd in x by separating the real and imaginary parts of Eq. (12).

³When the tractions change in sign, it is obvious that on the compression cycle the two sides of the crack will come in contact with each other. Such a violation of the free-surface condition, however, is not serious, and can be easily corrected. A detailed account of this point is reserved for later discussion.

⁴The factor $(-1)^j$ ($j = 1, 2$) is defined for values of $0 \leq \nu_2 \leq \pi/4$. Appropriate changes should be made for other values of ν_2 , say $\pi/4 < \nu_2 \leq \pi/2$.

1. *Normal Traction Even in x.* Taking into account only the real part of $\sigma_{yy}^{(r)}(x, 0)$, which is even in x , the potentials $\phi^{(r)}$ and $\psi^{(r)}$ may be represented by the cosine and sine integrals

$$\begin{aligned} \phi^{(r)}(x, y) &= \frac{2}{\pi} \int_0^\infty A_1(s) \exp(-\beta_1 y) \cos(sx) ds, & y \geq 0, \\ \psi^{(r)}(x, y) &= \frac{2}{\pi} \int_0^\infty A_2(s) \exp(-\beta_2 y) \sin(sx) ds, & y \geq 0. \end{aligned} \tag{13}$$

The branch cuts of the functions

$$\beta_j = (s^2 - \alpha_j^2)^{1/2} = -i(\alpha_j^2 - s^2)^{1/2}, \quad j = 1, 2$$

have already been discussed in [7] and thus no additional comments are needed. Applying the boundary conditions in Eqs. (10) to the appropriate expressions of displacement and stress obtained from putting Eqs. (13) into Eqs. (1) and (2), the results may be written in the forms

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty [-\beta_1 A_1(s) - sA_2(s)] \cos(sx) ds &= 0, & |x| > a \\ \frac{2}{\pi} \int_0^\infty \left[\frac{1}{s} \left(s^2 - \frac{1}{2} \alpha_2^2 \right)^2 A_1(s) + \beta_2 A_2(s) \right] \sin(sx) ds & \\ &= \frac{(-1)^i P_i \sin(\alpha_i x \cos \nu_i)}{\alpha_i \cos \nu_i}, & |x| < a \end{aligned} \tag{14}$$

and the condition of $\sigma_{xy}^{(r)}(x, 0) = 0$ for all values of x is satisfied by setting

$$s\beta_1 A_1(s) + (s^2 - \frac{1}{2} \alpha_2^2) A_2(s) = 0.$$

Now, define a function $A(s)$ through

$$A(s) = -\beta_1 A_1(s) - sA_2(s)$$

so that

$$\begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} = \frac{2A(s)}{\alpha_2^2 \beta_1} \begin{bmatrix} (s^2 - \frac{1}{2} \alpha_2^2) \\ -s\beta_1 \end{bmatrix}.$$

Under these considerations, it can be confirmed from Eqs. (14) that $A(s)$ is governed by the pair of dual integral equations

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty A(s) \cos(sx) ds &= 0, & |x| > a \\ \frac{2}{\pi} \int_0^\infty f(s) A(s) \sin(sx) ds &= \frac{(-1)^i P_i \sin(\alpha_i x \cos \nu_i)}{\alpha_i \cos \nu_i}, & |x| < a \end{aligned} \tag{15}$$

in which

$$f(s) = \frac{2}{\alpha_2^2 \beta_1 s} [(s_2 - \frac{1}{2} \alpha_2^2)^2 - s^2 \beta_1 \beta_2].$$

The objective is to reduce the problem to an integral equation of a standard type.

Introduce the function

$$a(x) = \frac{2}{\pi} \int_0^{\infty} A(s) \cos (sx) ds$$

and by the Fourier inversion theorem, it is obvious that

$$A(s) = \int_0^{\infty} a(x) \cos (sx) dx.$$

According to the first of Eqs. (15), $a(x)$ vanishes for $|x| > a$ and thus

$$A(s) = \int_0^a a(x) \cos (sx) dx.$$

Elimination of $A(s)$ from the second of Eqs. (15) then gives

$$\frac{2}{\pi} \int_0^{\infty} f(s) \sin (sx) ds \int_0^a a(\tau) \cos (s\tau) d\tau = \frac{(-1)^i P_i \sin (\alpha_i x \cos v_i)}{\alpha_i \cos v_i}, \quad |x| < a \quad (16)$$

which involves $a(\tau)$ as the only unknown function.

From the physics of the crack problem [14], the displacements at the end points $x = \pm a$ of the crack must be bounded and are of the order $(x \mp a)^{1/2}$ as $|x| \rightarrow a$, i.e.,

$$a(x) = \lambda(a)(a^2 - x^2)^{1/2} + O(1), \quad \text{as } |x| \rightarrow a. \quad (17)$$

Guided by Eq. (17), it is assumed that $a(x)$ admits the representation

$$a(x) = \int_x^a \lambda(\tau) \frac{\tau d\tau}{(\tau^2 - x^2)^{1/2}}, \quad 0 \leq x \leq a \quad (18)$$

where $\lambda(\tau)$ is required to be continuous on the interval $[0, a]$. Equation (18) may be integrated by parts to give

$$a(x) = \lambda(a) (a^2 - x^2)^{1/2} - \int_x^a \lambda'(\tau) (\tau^2 - x^2)^{1/2} d\tau, \quad 0 \leq x \leq a$$

which agrees with the preassigned form of $a(x)$ in Eq. (17). In what follows, the function $\lambda(\tau)$ is permitted to depend on the physical parameters in the problem.

Substituting Eq. (18) into (16), and applying the Dirichlet formula for interchanging the order of integration [15] render

$$\int_0^{\infty} f(s) \sin (sx) ds \int_0^a \tau \lambda(\tau) J_0(s\tau) d\tau = \frac{(-1)^i P_i \sin (\alpha_i x \cos v_i)}{\alpha_i \cos v_i}, \quad |x| < a \quad (19)$$

where use was made of the identity [16]

$$\int_0^{\tau} \frac{\cos (sx) dx}{(\tau^2 - x^2)^{1/2}} = \frac{\pi}{2} J_0(s\tau).$$

Here, J_0 is the zero-order Bessel function of the first kind. For the purpose of reducing Eq. (19) to a standard Fredholm equation, let

$$g(s) = f(s) + (1 - \kappa^2) \rightarrow O(s^{-2}), \quad \text{as } s \rightarrow \infty$$

and recall the discontinuous integral

$$\int_0^{\infty} J_0(s\tau) \sin (sx) ds = \begin{cases} 0, & 0 \leq x < \tau \\ 1/(x^2 - \tau^2)^{1/2}, & 0 < \tau < x. \end{cases}$$

After a permissible reversal of the order of integrations with respect to s and τ , Eq. (19) takes the form of Abel's equation [15]

$$\int_0^x \lambda(\tau) \frac{\tau d\tau}{(x^2 - \tau^2)^{1/2}} = h(x) \tag{20}$$

provided

$$h(x) = (1 - \kappa^2)^{-1} \left[\frac{(-1)^{i-1} P_i \sin(\alpha_i x \cos v_i)}{\alpha_i \cos v_i} + \int_0^a \tau \lambda(\tau) d\tau \int_0^\infty g(s) J_0(s\tau) \sin(sx) ds \right].$$

The Abel equation may be inverted to free $\lambda(\tau)$ from the integration, i.e.,

$$\lambda(\tau) = \frac{2}{\pi} \int_0^\tau \frac{h'(x) dx}{(\tau^2 - x^2)^{1/2}}, \quad 0 \leq \tau \leq a. \tag{21}$$

The result of inserting the derivative of $h(x)$ into Eq. (21) is

$$\lambda(\tau) = (1 - \kappa^2)^{-1} \left[(-1)^{i-1} P_i J_0(\alpha_i \tau \cos v_i) + \int_0^a \zeta \lambda(\zeta) d\zeta \int_0^\infty sg(s) J_0(s\tau) J_0(s\zeta) ds \right], \quad 0 \leq \tau \leq a$$

whose kernel may be symmetrized by the introduction of the dimensionless quantities

$$\xi = \frac{\tau}{a}, \quad \eta = \frac{\zeta}{a}, \quad \Lambda(\xi) = \frac{(-1)^i (1 - \kappa^2) \xi^{1/2} \lambda(a\xi)}{P_i}, \quad 0 < \xi \leq 1.$$

This leads to a regular integral equation of Fredholm's second kind

$$\Lambda(\xi) - \int_0^1 \Lambda(\eta) F(\xi, \eta) d\eta = (-1)^i \xi^{1/2} J_0(\alpha_i a \xi \cos v_i), \quad 0 < \xi \leq 1, \tag{22}$$

where

$$F(\xi, \eta) = (1 - \kappa^2)^{-1} (\xi\eta)^{1/2} \int_0^\infty sg\left(\frac{s}{a}\right) J_0(s\xi) J_0(s\eta) ds, \quad 0 < \xi \leq 1; \quad 0 < \eta \leq 1. \tag{23}$$

In view of $\beta_j = (s^2 - \alpha_j^2)^{1/2}$ ($j = 1, 2$) (connected with the out-going waves) being complex, both $\Lambda(\xi)$ and $F(\xi, \eta)$ in equation (22) are also complex functions. Separating these functions into their real and imaginary parts in accordance with

$$\Lambda(\xi) = \Lambda_1(\xi) + i\Lambda_2(\xi), \quad F(\xi, \eta) = F_1(\xi, \eta) + iF_2(\xi, \eta)$$

the following system of coupled integral equations are obtained:

$$\begin{aligned} \Lambda_1(\xi) - \int_0^1 [\Lambda_1(\eta) F_1(\xi, \eta) - \Lambda_2(\eta) F_2(\xi, \eta)] d\eta &= (-1)^i \xi^{1/2} J_0(\alpha_i a \xi \cos v_i), \\ \Lambda_2(\xi) - \int_0^1 [\Lambda_1(\eta) F_2(\xi, \eta) + \Lambda_2(\eta) F_1(\xi, \eta)] d\eta &= 0, \quad 0 < \xi \leq 1. \end{aligned} \tag{24}$$

The kernels $F_j(\xi, \eta)$ ($j = 1, 2$) are given by

$$F_1(\xi, \eta) = \frac{2}{\alpha_2^2(1 - \kappa^2)} (\xi\eta)^{1/2} \int_0^\infty s \left[f_1\left(\frac{s}{a}\right) - g_2\left(\frac{s}{a}\right) + \frac{1}{2} \alpha_2^2 (1 - \kappa^2) \right] \cdot J_0(s\xi) J_0(s\eta) ds, \tag{25}$$

$$F_2(\xi, \eta) = \frac{2}{\alpha_2^2(1 - \kappa^2)} (\xi\eta)^{1/2} \int_0^\infty s \left[m_1\left(\frac{s}{a}\right) + n_2\left(\frac{s}{a}\right) \right] J_0(s\xi) J_0(s\eta) ds,$$

where $0 < \xi \leq 1, 0 < \eta \leq 1$, and

$$\begin{aligned}
 f_k(s) &= s^{-1}(s^2 - \frac{1}{2}\alpha_2^2)^2 \begin{cases} 0, & s < \alpha_k; \\ 1/(s^2 - \alpha_k^2), & s > \alpha_k \end{cases} \\
 g_l(s) &= s \begin{cases} 0, & s < \alpha_l \\ (s^2 - \alpha_l^2)^{1/2}, & s > \alpha_l \end{cases} \\
 m_k(s) &= s^{-1}(s^2 - \frac{1}{2}\alpha_2^2)^2 \begin{cases} 1/(\alpha_k^2 - s^2)^{1/2} & s < \alpha_k; \\ 0, & s > \alpha_k \end{cases} \\
 n_l(s) &= s \begin{cases} (\alpha_l^2 - s^2)^{1/2}, & s < \alpha_l \\ 0, & s > \alpha_l. \end{cases}
 \end{aligned}$$

The above format is employed with the knowledge that the solutions for the scattering problem of P- and SV-waves are obtained, respectively, by setting $k = 1, l = 2$ and $k = 2, l = 1$.

To improve the rate of convergence of the numerical calculations, it is desirable to evaluate the singular parts of the integrals in Eqs. (25) in closed form by letting

$$p(s) = 1 + 2 \frac{f_1(s) - g_2(s)}{\alpha_2^2(1 - \kappa^2)} - \frac{M}{s^2 + m^2}$$

in which m and M stand for

$$m^2 = \frac{1}{2}\alpha_2^2 \cdot \left(\frac{1 + 2\kappa^2 - 6\kappa^4 + 5\kappa^6}{3 - 4\kappa^2 + 3\kappa^4} \right), \quad M = \frac{1}{4}\alpha_2^2 \cdot \left(\frac{3 - 4\kappa^2 + 3\kappa^4}{1 - \kappa^2} \right).$$

With the aid of the identity [16]

$$\int_0^\infty \frac{s}{s^2 + m^2} J_0(\xi s) J_0(\eta s) ds = I_0(m\xi) K_0(m\eta), \quad 0 < \xi \leq \eta,$$

where I_0 and K_0 are the modified zero-order Bessel functions of the first and second kind, respectively, $F_1(\xi, \eta)$ becomes

$$\begin{aligned}
 F_1(\xi, \eta) &= (\xi\eta)^{1/2} [a^2 M I_0(am\xi) K_0(am\eta) \\
 &\quad + \int_0^\infty sp \left(\frac{s}{a} \right) J_0(s\xi) J_0(s\eta) ds], \quad 0 < \xi \leq \eta \leq 1 \quad (26)
 \end{aligned}$$

and its continuity on $[0 \leq \xi \leq 1, 0 \leq \eta \leq 1]$ is assured by the behavior of I_0 and K_0 in a neighborhood of the origin. The advantage of this alternative representation of $F_1(\xi, \eta)$, as compared with the first of Eqs. (25), is apparent from

$$p(s) \rightarrow O(s^{-6}), \quad \text{as } s \rightarrow \infty,$$

The system of integral equations for $\Lambda_j(\xi)$ ($j = 1, 2$) may be solved numerically on an electronic computer. With the knowledge of $\Lambda(\xi)$, the only unknown $A(s)$ in the scattering problem with normal tractions even in x can be evaluated:

$$A(s) = (-1)^j \frac{\pi P_j a}{2(1 - \kappa^2)s} \left\{ \Lambda(1) J_1(sa) - \int_0^1 \xi J_1(a\xi s) \left[\frac{\Lambda(\xi)}{\xi^{1/2}} \right]' d\xi \right\}. \quad (27)$$

The displacements and stresses everywhere in the elastic medium with a crack can then be calculated without difficulty. As mentioned in the Introduction, this method of solution is particularly suited for obtaining the explicit form of the crack-tip stress field, which together with the numerical results for $\Lambda(\xi)$ will be presented subsequently.

2. *Normal Traction Odd in x .* Suppose that the normal tractions on the crack are given by the imaginary part of $\sigma_{yy}^{(r)}(x, 0)$ in Eq. (12). Then the correct evenness and oddness properties in x for the potentials of the scattered waves may be represented by

$$\begin{aligned} \phi^{(r)}(x, y) &= \frac{2}{\pi} \int_0^\infty B_1(s) \exp(-\beta_1 y) \sin(sx) ds, & y \geq 0, \\ \psi^{(r)}(x, y) &= \frac{2}{\pi} \int_0^\infty B_2(s) \exp(-\beta_2 y) \cos(sx) ds, & y \geq 0. \end{aligned} \tag{28}$$

As before, the boundary conditions in Eqs. (10) lead to a pair of dual integral equations

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty B(s) \cos(sx) ds &= 0, & |x| > a, \\ \frac{2}{\pi} \int_0^\infty f(s)B(s) \sin(sx) ds &= i(-1)^{j-1} P_j \sin(\alpha_j x \cos \nu_j), & |x| < a \end{aligned} \tag{29}$$

in which $B(s)$ is connected with $B_j(s)$ ($j = 1, 2$) by

$$\begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} = \frac{2B(s)}{\alpha_2^2 \beta_1 s} \begin{bmatrix} (s^2 - \frac{1}{2}\alpha_2^2) \\ s\beta_1 \end{bmatrix}.$$

Making use of the first of Eqs. (29) and the Fourier inversion theorem, $B(s)$ may be linked to a function $b(x)$, i.e.,

$$B(s) = \int_0^a b(x) \cos(sx) dx.$$

The singularities to be expected in the function $b(x)$ will be pre-assigned. Knowing the nature of the stress singularities at the endpoints of the crack [14], it will be assumed that

$$b(x) = \frac{a\delta^*}{(a^2 - x^2)^{1/2}} + \int_x^a \frac{\gamma(\tau) d\tau}{(\tau^2 - x^2)^{3/2}}, \quad 0 \leq x < a. \tag{30}$$

Here δ^* is a constant and $\gamma(\tau)$ a function continuous on the interval $[0, a]$. This implies

$$b(s) = \frac{a\delta^*}{(a^2 - x^2)^{1/2}} + O(1), \quad \text{as } x \rightarrow a; \quad 0 \leq x < a$$

which preserves the $(x \mp a)^{-1/2}$ -type of stress singularities at the crack tips, $x = \pm a$, respectively. The function $B(s)$ may now be expressed in terms of the newly introduced function $\gamma(\tau)$ as

$$B(x) = \frac{\pi}{2} \left[a\delta^* J_0(sa) + \int_0^a J_0(s\tau) \gamma(\tau) d\tau \right]. \tag{31}$$

The constant δ^* can be evaluated from the condition that the displacements at the crack tips are bounded, i.e.,

$$\lim_{x \rightarrow a} \int_0^x b(x) dx = 0$$

which gives

$$\delta^* = -\frac{1}{a} \int_0^a \gamma(\tau) d\tau.$$

Inserting Eq. (31) into the second of Eqs. (29), and rearranging the subsequent result yield an integral equation of the Abel type:

$$\int_0^x \frac{\gamma(\tau) d\tau}{(x^2 - \tau^2)^{1/2}} = (1 - \kappa^{-2}) \{i(-1)^i P_i \sin(\alpha_i x \cos \nu_i) + \int_0^\infty g(s) \left[a\delta^* J_0(sa) + \int_0^a J_0(s\tau)\gamma(\tau) d\tau \right] \sin(sx) ds\}, \quad 0 \leq x < a. \tag{32}$$

The inverted form of the Abel equation is

$$\gamma(\tau) = (1 - \kappa^2)^{-1} \tau \{i(-1)^i P_i \alpha_i \cos \nu_i J_0(\alpha_i \tau \cos \nu_i) + \int_0^\infty sg(s) \left[a\delta^* J_0(sa) + \int_0^a J_0(s\zeta)\gamma(\zeta) d\zeta \right] J_0(s\tau) ds\}, \quad 0 < \tau \leq a. \tag{33}$$

With the aim of symmetrizing the kernel in Eq. (33), introduce the change of variables $\xi = \tau/a$, $\eta = \zeta/a$, and set

$$\begin{bmatrix} \delta^* \\ \gamma(\tau) \end{bmatrix} = \frac{(-1)^i P_i \cos \nu_i}{1 - \kappa^2} \begin{bmatrix} \delta \\ \xi^{1/2} \Gamma(\xi) \end{bmatrix}, \quad 0 < \xi \leq 1.$$

Thus, Eq. (33) assumes the form of a Fredholm integral equation

$$\Gamma(\xi) - \int_0^1 \Gamma(\eta) F(\xi, \eta) d\eta = i(-1)^{i-1} \alpha_i a \xi^{1/2} J_0(\alpha_i a \xi \cos \nu_i) + \delta F(\xi, 1), \quad 0 < \xi \leq 1, \tag{34}$$

where the kernel $F(\xi, \eta)$ is the same as that shown in equation (23), and

$$\delta = - \int_0^1 \xi^{1/2} \Gamma(\xi) d\xi. \tag{35}$$

Since $F(\xi, \eta)$ and

$$\Gamma(\xi) = \Gamma_1(\xi) + i\Gamma_2(\xi)$$

are complex, Eq. (34) may be further decomposed into real and imaginary parts:

$$\begin{aligned} \Gamma_1(\xi) - \int_0^1 [\Gamma_1(\eta) F_1(\xi, \eta) - \Gamma_2(\eta) F_2(\xi, \eta)] d\eta \\ = - \int_0^1 \left[\Gamma_1(\eta) F_1(\xi, 1) - \Gamma_2(\eta) F_2(\xi, 1) \right] \eta^{1/2} d\eta, \quad 0 < \xi \leq 1 \end{aligned} \tag{36}$$

$$\begin{aligned} \Gamma_2(\xi) - \int_0^1 [\Gamma_1(\eta) F_2(\xi, \eta) + \Gamma_2(\eta) F_1(\xi, \eta)] d\eta \\ = (-1)^{i-1} \alpha_i a \xi^{1/2} J_0(\alpha_i a \xi \cos \nu_i) - \int_0^1 [\Gamma_1(\eta) F_2(\xi, 1) + \Gamma_2(\eta) F_1(\xi, 1)] \eta^{1/2} d\eta. \end{aligned}$$

This constitutes a system of two Fredholm integral equations for the determination of the two functions $\Gamma_j(\xi)$ ($j = 1, 2$). The problem is essentially solved as the original

unknown may be calculated from

$$B(s) = \frac{(-1)^{j-1} \pi P_j a}{2(1 - \kappa^2) s} \left[\Upsilon(1) J_0(sa) - \cos v_j \int_0^1 \xi^{1/2} J_0(s a \xi) \Gamma(\xi) d\xi \right], \quad 0 < \xi \leq 1, \quad (37)$$

where

$$\Upsilon(1) = -\delta \cos v_j = \cos v_j \int_0^1 \xi^{1/2} \Gamma(\xi) d\xi. \quad (38)$$

Case B—skew-symmetric problem. The scattering problem, in which tangential tractions

$$\sigma_{xy}^{(r)}(x, 0) = -\sigma_{xy}^{(i)}(x, 0) = (-1)^j 2\mu Q_j \exp(-i\alpha_j x \cos v_j), \quad |x| < a \quad (39)$$

are applied to the crack surface, is said to be skew-symmetric with respect to the crack plane $y = 0$. With $j = 1, 2$, the quantities Q_1 and Q_2 in Eq. (39) are related, respectively, to the amplitudes of the P- and SV-waves:

$$Q_1 = -\frac{1}{2} \alpha_2^2 \phi_0 \kappa^2 \sin 2v_1, \quad Q_2 = \frac{1}{2} \alpha_2^2 \psi_0 \cos 2v_2.$$

Since the skew-symmetric problem may also be reduced to the solution of a system of coupled integrations in the same way as that of Case A, many of the detailed derivations will be omitted and only the essential steps will be given.

1. *Tangential Tractions Even in x.* In this case, $\phi^{(r)}$ is odd in x and $\psi^{(r)}$ even in x . Replacing $B_j(s)$ in Eqs. (28) by $C_j(s)$ ($j = 1, 2$), the vanishing of $\sigma_{xy}^{(r)}(x, 0)$ along the entire x -axis requires

$$(s^2 - \frac{1}{2} \alpha_2^2) C_1(s) - s\beta_2 C_2(s) = 0.$$

By having

$$C(s) = sC_1(s) - \beta_2 C_2(s)$$

such that

$$\begin{bmatrix} C_1(s) \\ C_2(s) \end{bmatrix} = \frac{2C(s)}{\alpha_2^2 \beta_2} \begin{bmatrix} s\beta_2 \\ (s^2 - \frac{1}{2} \alpha_2^2) \end{bmatrix}$$

the boundary conditions in Eqs. (11) lead to

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty C(s) \cos(sx) ds &= 0, \quad |x| > a, \\ \frac{2}{\pi} \int_0^\infty m(s) C(s) \sin(sx) ds &= \frac{(-1)^j Q_j \sin(\alpha_j x \cos v_j)}{\alpha_j \cos v_j}, \quad |x| < a \end{aligned} \quad (40)$$

for the determination of $C(s)$. The function $m(s)$ is equal to $(\beta_1/\beta_2)f(s)$ with $f(s)$ being given previously by Eq. (15). The solution of Eqs. (40) is

$$C(s) = \frac{\pi}{2} \int_0^a \tau \phi(\tau) J_0(s\tau) d\tau, \quad (41)$$

where

$$\begin{aligned} \phi(\tau) = (1 - \kappa^2)^{-1} \left[(-1)^{j-1} Q_j J_0(\alpha_j \tau \cos v_j) \right. \\ \left. + \int_0^a \xi \phi(\xi) d\xi \int_0^\infty sn(s) J_0(s\tau) J_0(s\xi) ds \right], \quad 0 < \tau \leq a \end{aligned} \quad (42)$$

is continuous on the interval $[0, a]$, and

$$n(s) = m(s) + (1 - \kappa^2) \rightarrow O(s^{-2}), \quad \text{as } s \rightarrow \infty.$$

Performing the substitutions

$$\xi = \frac{\tau}{a}, \quad \eta = \frac{\zeta}{a}, \quad \Phi(\xi) = \frac{(-1)^i (1 - \kappa^2) \xi^{1/2} \phi(a\xi)}{Q_i}, \quad 0 < \xi \leq 1$$

on Eq. (42), there results a Fredholm integral equation

$$\Phi(\xi) - \int_0^1 \Phi(\eta) G(\xi, \eta) d\eta = (-1)^i \xi^{1/2} J_0(\alpha_i a \xi \cos \nu_i), \quad 0 < \xi \leq 1 \tag{43}$$

whose kernel $G(\xi, \eta)$ is

$$G(\xi, \eta) = (1 - \kappa^2)^{-1} (\xi\eta)^{1/2} \int_0^\infty sn\left(\frac{s}{a}\right) J_0(s\xi) J_0(s\eta) ds, \quad 0 < \xi \leq 1; \quad 0 < \eta \leq 1 \tag{44}$$

For the purpose of computing the real and imaginary parts of $\Phi(\xi)$ numerically, a system of coupled integral equations can be deduced from Eq. (43):

$$\begin{aligned} \Phi_1(\xi) - \int_0^1 [\Phi_1(\eta) G_1(\xi, \eta) - \Phi_2(\eta) G_2(\xi, \eta)] d\eta &= (-1)^i \xi^{1/2} J_0(\alpha_i a \xi \cos \nu_i), \\ \Phi_2(\xi) - \int_0^1 [\Phi_1(\eta) G_2(\xi, \eta) + \Phi_2(\eta) G_1(\xi, \eta)] d\eta &= 0, \end{aligned} \tag{45}$$

In Eq. (45), $G_j(\xi, \eta)$ ($j = 1, 2$) represent

$$\begin{aligned} G_1(\xi, \eta) = (\xi\eta)^{1/2} \left[a^2 N I_0(na\xi) K_0(na\eta) \right. \\ \left. + \int_0^\infty sq\left(\frac{s}{a}\right) J_0(s\xi) J_0(s\eta) ds \right], \quad 0 < \xi \leq \eta \leq 1 \end{aligned} \tag{46}$$

$$G_2(\xi, \eta) = \frac{2}{\alpha_2^2 (1 - \kappa^2)} (\xi\eta)^{1/2} \int_0^\infty s \left[m_2\left(\frac{s}{a}\right) + n_1\left(\frac{s}{a}\right) \right] J_0(s\xi) J_0(s\eta) ds,$$

where

$$q(s) = 1 + 2 \frac{f_2(s) - g_1(s)}{\alpha_2^2 (1 - \kappa^2)} - \frac{N}{s^2 + n^2},$$

and

$$n^2 = \frac{1}{2} \alpha_2^2 \left(\frac{1 + \kappa^6}{1 + \kappa^4} \right), \quad N = \frac{1}{4} \alpha_2^2 \left(\frac{1 + \kappa^4}{1 - \kappa^2} \right).$$

Finally, $C(s)$ is found:

$$C(s) = \frac{(-1)^i \pi Q_i a}{2(1 - \kappa^2) s} \left\{ \Phi(1) J_1(sa) - \int_0^1 \xi J_1(sa\xi) \left[\frac{\Phi(\xi)}{\xi^{1/2}} \right]' d\xi \right\}. \tag{47}$$

2. *Tangential Traction Odd In x.* When the tangential surface tractions on the crack are odd in x , Eqs. (13) apply provided that $A_j(s)$ are to be replaced by $D_j(s)$ ($j = 1, 2$). From the defined relationship

$$D(s) = -s[sD_1(s) + \beta_2 D_2(s)]$$

and symmetry consideration, $D_i(s)$ are expressible in terms of $D(s)$ as

$$\begin{bmatrix} D_1(s) \\ D_2(s) \end{bmatrix} = \frac{2D(s)}{\alpha_2^2 \beta_2 s} \begin{bmatrix} -s\beta_2 \\ (s^2 - \frac{1}{2}\alpha_2^2) \end{bmatrix}.$$

Now, the boundary conditions in Eqs. (11) require that $D(s)$ satisfies the dual integral equations

$$\frac{2}{\pi} \int_0^\infty D(s) \cos (sx) ds = 0, \quad |x| > a \tag{48}$$

$$\frac{2}{\pi} \int_0^\infty m(s)D(s) \sin (sx) ds = i(-1)^{i-1}Q_i \sin (\alpha_i x \cos \nu_i), \quad |x| < a.$$

Let $D(s)$ be connected with another function $\psi(\tau)$ through the relation

$$D(s) = \frac{\pi}{2} \left[a\epsilon^* J_0(sa) + \int_0^a \psi(\tau) J_0(s\tau) d\tau \right]$$

where

$$\epsilon^* = -\frac{1}{a} \int_0^a \psi(\tau) d\tau.$$

Then, the second of equations (48) renders

$$\begin{aligned} \psi(\tau) = & (1 - \kappa^2)^{-1} \tau \left\{ i(-1)^i Q_i \cos \nu_i J_0(\alpha_i \tau \cos \nu_i) \right. \\ & \left. + \int_0^\infty sn(s) \left[a\epsilon^* J_0(sa) + \int_0^a \psi(\zeta) J_0(s\zeta) d\zeta \right] J_0(s\tau) ds \right\}, \quad 0 < \tau \leq a. \end{aligned} \tag{49}$$

Further, set $\xi = \tau/a$, $\eta = \zeta/a$ and adopt the notations

$$\begin{bmatrix} \epsilon \\ \xi^{1/2} \Psi(\xi) \end{bmatrix} = \frac{(-1)^i (1 - \kappa^2)}{Q_i \cos \nu_i} \begin{bmatrix} \epsilon^* \\ \Psi(a\xi) \end{bmatrix}.$$

This carries Eq. (49) into

$$\Psi(\xi) - \int_0^1 \Psi(\eta) G(\xi, \eta) d\eta = i(-1)^{i-1} \alpha_i a \xi^{1/2} J_0(\alpha_i a \xi \cos \nu_i) + \epsilon G(\xi, 1), \quad 0 < \xi \leq 1 \tag{50}$$

which can be separated into the following system of equations:

$$\begin{aligned} \Psi_1(\xi) - \int_0^1 [\Psi_1(\eta) G_1(\xi, \eta) - \Psi_2(\eta) G_2(\xi, \eta)] d\eta \\ = - \int_0^1 [\Psi_1(\eta) G_1(\xi, 1) - \Psi_2(\eta) G_2(\xi, \eta)] \eta^{1/2} d\eta, \quad 0 < \xi \leq 1 \end{aligned} \tag{51}$$

$$\begin{aligned} \Psi_2(\xi) - \int_0^1 [\Psi_1(\eta) G_2(\xi, \eta) + \Psi_2(\eta) G_1(\xi, \eta)] d\eta = & (-1)^{i-1} \alpha_i a \xi^{1/2} J_0(\alpha_i a \xi \cos \nu_i) \\ & - \int_0^1 [\Psi_1(\eta) G_2(\xi, 1) + \Psi_2(\eta) G_1(\xi, 1)] \eta^{1/2} d\eta. \end{aligned}$$

The kernels $G_j(\xi, \eta)$ ($j = 1, 2$) are given by Eqs. (46). Once $\Psi(\xi)$ is known, $D(s)$ follows from

$$D(s) = \frac{(-1)^{j-1} \pi Q_j a}{2(1 - \kappa^2)s} \left[\Omega(1) J_0(sa) - \cos v_j \int_0^1 \xi^{1/2} J_0(sa\xi) \Psi(\xi) d\xi \right], \quad (52)$$

where

$$\Omega(1) = -\epsilon \cos v_j = \cos v_j \int_0^1 \xi^{1/2} \Psi(\xi) d\xi.$$

Singular behavior of the near-field solution. The chief interest in the solution to the crack problem concerns its singular behavior at the crack points since it is the stresses and/or displacements in the vicinity of the crack that control the condition of crack propagation [17]. The singular character of these stresses for Cases A and B will be exhibited individually.

1. *Case A.* The symmetric crack-tip stresses of the scattered waves can be derived from Eqs. (13) and (28) which, when substituted into Eqs. (4), yield the expressions

$$\begin{aligned} \sigma_{xx}^{(r)}(x, y) &= -\frac{4\mu}{\pi} \int_0^\infty \left\{ (\beta_1^2 + \frac{1}{2}\alpha_2^2) \exp(-\beta_1 y) \begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix} \right. \\ &\quad \left. + s\beta_2 \exp(-\beta_2 y) \begin{bmatrix} A_2(s) \\ -B_2(s) \end{bmatrix} \right\} \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} ds, \\ \sigma_{yy}^{(r)}(x, y) &= \frac{4\mu}{\pi} \int_0^\infty \left\{ (s^2 - \frac{1}{2}\alpha_2^2) \exp(-\beta_1 y) \begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix} \right. \\ &\quad \left. + s\beta_2 \exp(-\beta_2 y) \begin{bmatrix} A_2(s) \\ -B_2(s) \end{bmatrix} \right\} \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} ds, \\ \sigma_{xy}^{(r)}(x, y) &= \frac{4\mu}{\pi} \int_0^\infty \left\{ s\beta_1 \exp(-\beta_1 y) \begin{bmatrix} A_1(s) \\ -B_1(s) \end{bmatrix} \right. \\ &\quad \left. + (s^2 - \frac{1}{2}\alpha_2^2) \exp(-\beta_2 y) \begin{bmatrix} A_2(s) \\ B_2(s) \end{bmatrix} \right\} \begin{bmatrix} \sin(sx) \\ \cos(sx) \end{bmatrix} ds. \end{aligned} \quad (53)$$

For an examination of the singular behavior of the stress components, it suffices to consider in $A_j(s)$ and $B_j(s)$ ($j = 1, 2$) the dominating terms

$$\begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - (s/\alpha_2)^2 \\ s\beta_1/\alpha_2^2 \end{bmatrix} \frac{(-1)^{j-1} \pi P_j a}{\beta_1(1 - \kappa^2)s} \Omega(1) J_1(sa) + \dots,$$

and

$$\begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + (s/\alpha_2)^2 \\ s\beta_1/\alpha_2^2 \end{bmatrix} \frac{(-1)^{j-1} \pi P_j a}{\beta_1(1 - \kappa^2)s^2} \Upsilon(1) J_0(sa) + \dots$$

which contribute to the divergence of the improper integrals in Eqs. (53) at $x = \pm a$, $y = 0$. This divergence is due to the behavior of the corresponding integrands as $s \rightarrow \infty$. Hence, asymptotic expansions of the integrands appearing in Eqs. (53) for large values of s lead to

$$\begin{aligned}
 \sigma_{xx}^{(r)}(x, y) &= \sigma_i a \int_0^\infty (sy - 1) \exp(-sy) \begin{bmatrix} \Lambda(1) \\ -\Upsilon(1) \end{bmatrix} \begin{bmatrix} \cos(sx)J_1(sa) \\ \sin(sx)J_0(sa) \end{bmatrix} ds, \\
 \sigma_{yy}^{(r)}(x, y) &= \sigma_i a \int_0^\infty (sy + 1) \exp(-sy) \begin{bmatrix} -\Lambda(1) \\ \Upsilon(1) \end{bmatrix} \begin{bmatrix} \cos(sx)J_1(sa) \\ \sin(sx)J_0(sa) \end{bmatrix} ds, \\
 \sigma_{xy}^{(r)}(x, y) &= -\sigma_i a \int_0^\infty sy \exp(-sy) \begin{bmatrix} \Lambda(1) \\ \Upsilon(1) \end{bmatrix} \begin{bmatrix} \sin(sx)J_1(sa) \\ \cos(sx)J_0(sa) \end{bmatrix} ds
 \end{aligned}
 \tag{54}$$

where $\sigma_j = (-1)^j 2\mu P_j$ ($j = 1, 2$). The foregoing integrals can then be evaluated in closed elementary form by means of the well-known Bessel integral identities [16]

$$\begin{aligned}
 \int_0^\infty \exp(-sy) \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} J_0(sa) ds &= \frac{1}{(r_1 r_2)^{1/2}} \begin{bmatrix} \sin \frac{1}{2}(\theta_1 + \theta_2) \\ \cos \frac{1}{2}(\theta_1 + \theta_2) \end{bmatrix} \\
 \int_0^\infty s \exp(-sy) \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} J_0(sa) ds &= -\frac{r}{(r_1 r_2)^{3/2}} \begin{bmatrix} \cos[\theta - \frac{3}{2}(\theta_1 + \theta_2)] \\ \sin[\theta - \frac{3}{2}(\theta_1 + \theta_2)] \end{bmatrix}
 \end{aligned}$$

etc. The polar coordinates (r, θ) , (r_1, θ_1) , and (r_2, θ_2) have the meaning shown in Figure 1. In the limit as $r_2 \rightarrow 2a$, $r \rightarrow a$, and $\theta_2 = \theta \rightarrow 0$, the unbounded contributions to the total stress field near $x = +a$, $y = 0$ are precisely equal to those of the scattered waves since the stress field of the incident waves are nonsingular. Carrying out the limiting process just mentioned yields

$$\begin{aligned}
 \sigma_{xx} &= \frac{k_1}{(2r_1)^{1/2}} \cos(\frac{1}{2}\theta_1)[1 - \sin(\frac{1}{2}\theta_1) \sin(\frac{3}{2}\theta_1)] + O(1) \\
 \sigma_{yy} &= \frac{k_1}{(2r_1)^{1/2}} \cos(\frac{1}{2}\theta_1)[1 + \sin(\frac{1}{2}\theta_1) \sin(\frac{3}{2}\theta_1)] + O(1) \\
 \sigma_{xy} &= \frac{k_1}{(2r_1)^{1/2}} \cos(\frac{1}{2}\theta_1) \sin(\frac{1}{2}\theta_1) \cos(\frac{3}{2}\theta_1) + O(1),
 \end{aligned}
 \tag{55}$$

where

$$k_1 = \begin{bmatrix} \Lambda(1) \\ \Upsilon(1) \end{bmatrix} \sigma_i a^{1/2}.
 \tag{56}$$

It is apparent that the same angular distribution in θ_1 and the characteristic inverse square-root singularity, $r_1^{-1/2}$, at the crack tip are observed in both the static and dynamical stress solutions. In the dynamic case, however, the strength of this stress singularity, measured by k_1 , fluctuates in time and is proportional to the circular frequency, amplitude of input wave, angle of incidence, etc. Adopting the criterion in the theory of brittle fracture [17], k_1 may be regarded as a dynamic stress-intensity factor the critical value of which is a function of material property and can be measured experimentally to determine the point of incipient fracture.

The influence of fluctuating loads upon the k_1 -factor can be brought out more clearly by a plot of $|k_1/\sigma_1 a^{1/2}|$ versus normalized wave number $\alpha_1 a$ at normal incidence, $\nu_1 = \pi/2$, such that the crack is excited by P-waves only. In this case, Υ in Eq. (56) is zero. Referring to Fig. 2, the static solution is given by $k_1 = \sigma_1 a^{1/2}$ which corresponds to $\omega = 0$. Here, σ_1 represents the magnitude of the static tensile stress applied in the direction normal to the crack plane. As ω increases from zero, the requisite numerical results are obtained by solving the integral equation for Λ on an electronic computer for values of

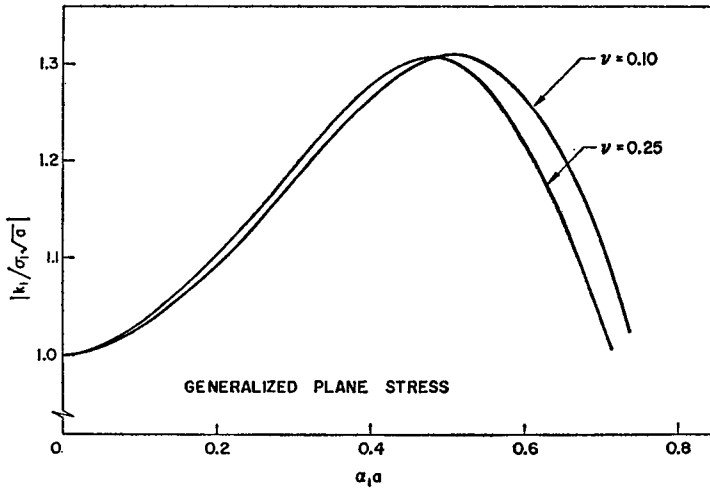


FIG. 2. P-waves at normal incidence ($\nu_1 = 90^\circ$).

Poisson's ratio $\nu = 0.10$ and $\nu = 0.25$ under the conditions of generalized plane stress. Note that for $\nu = 0.10$ the maximum value of $|k_1/\sigma_1 a^{1/2}|$ is 1.31 at $\alpha_1 a \simeq 0.51$ and beyond this point the curve declines quickly. The peak of the $|k_1/\sigma_1 a^{1/2}|$ -curve for $\nu = 0.25$ is slightly lower and occurs at $\alpha_1 a \simeq 0.48$.

2. Case B. By a similar procedure, the local stress distribution for the skew-symmetric case may be obtained by taking

$$\begin{bmatrix} C_1(s) \\ C_2(s) \end{bmatrix} = \begin{bmatrix} -s\beta_2/\alpha_2^2 \\ \frac{1}{2} - (s/\alpha_2)^2 \end{bmatrix} \frac{(-1)^{i-1} \pi Q_i a}{\beta_2(1 - \kappa^2)s} \Phi(1) J_1(sa) + \dots$$

and

$$\begin{bmatrix} D_1(s) \\ D_2(s) \end{bmatrix} = \begin{bmatrix} -s\beta_2/\alpha_2^2 \\ -\frac{1}{2} + (s/\alpha_2)^2 \end{bmatrix} \frac{(-1)^{i-1} \pi Q_i a}{\beta_2(1 - \kappa^2)s^2} \Omega(1) J_0(sa) + \dots$$

Without repeating the details, the singular stresses are found to be

$$\begin{aligned} \sigma_{xx} &= -\frac{k_2}{(2r_1)^{1/2}} \sin(\frac{1}{2}\theta_1)[2 + \cos(\frac{1}{2}\theta_1) \cos(\frac{3}{2}\theta_1)] + O(1), \\ \sigma_{yy} &= \frac{k_2}{(2r_1)^{1/2}} \sin(\frac{1}{2}\theta_1) \cos(\frac{1}{2}\theta_1) \cos(\frac{3}{2}\theta_1) + O(1), \\ \sigma_{xy} &= \frac{k_2}{(2r_1)^{1/2}} \cos(\frac{1}{2}\theta_1)[1 - \sin(\frac{1}{2}\theta_1) \sin(\frac{3}{2}\theta_1)] + O(1). \end{aligned} \tag{57}$$

The parameter k_2 is

$$k_2 = \begin{bmatrix} \Phi(1) \\ \Omega(1) \end{bmatrix} \tau_i a^{1/2}, \tag{58}$$

where $\tau_i = (-1)^i 2\mu Q_i$, and it has the same physical meaning as k_1 . At normal incidence, $\nu_i = \pi/2$, and $\Omega = 0$, the static solution of $k_2 = \tau_2 a^{1/2}$ is recovered as $\omega \rightarrow 0$, and τ_2 may be interpreted as the uniform shear stress at infinity.

Let the crack be excited by SV-waves and consider a state of plane strain. As is evident

from Fig. 3, the variation of $|k_2/\tau_2 a^{1/2}|$ with $\alpha_2 a$, the normalized wave number for shear waves, follows the same trend as that of the $|k_1/\sigma_1 a^{1/2}|$ -curves in Fig. 2. It is seen that the peaks of the curves in Fig. 3 become progressively lower as the Poisson's ratio is increased

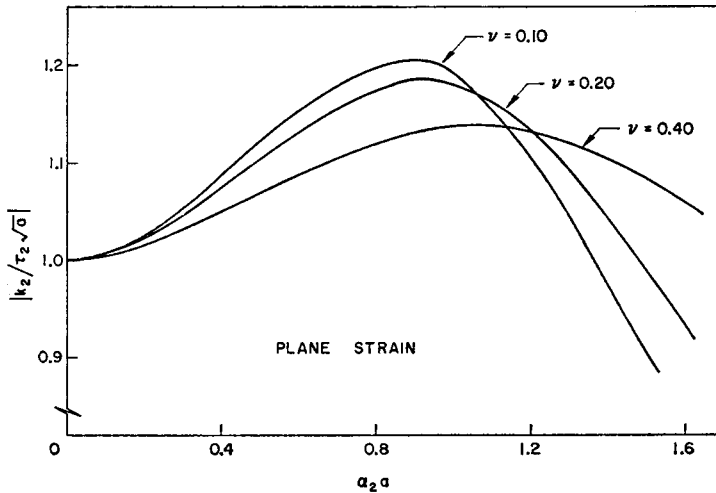


FIG. 3. SV-waves at normal incidence ($\nu_2 = 90^\circ$).

from $\nu = 0.10$ to $\nu = 0.40$. At $\nu = 0.10$, the peak occurs at $\alpha_2 a \simeq 0.9$ and is approximately 20.5% greater than the static value.

Further insight into the effects of different types of input waves on the intensity of the dynamic stresses may be gained from Fig. 4, which depicts the absolute value of the

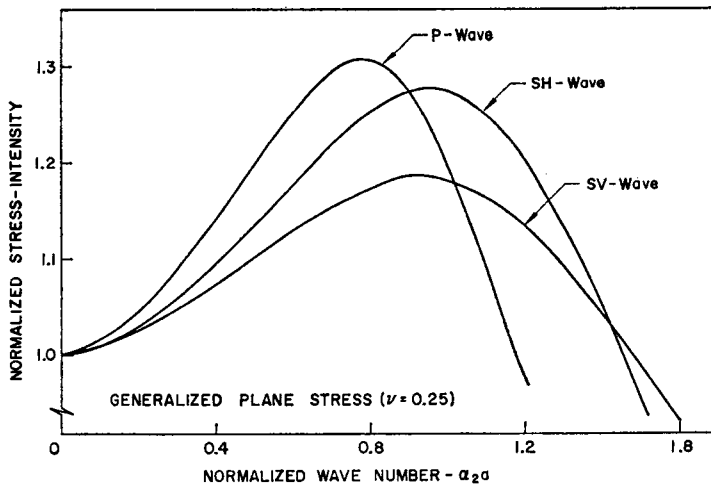


FIG. 4. Near-field stress solutions for three types of input waves.

dimensionless stress-intensity factor as a function of $\alpha_2 a$ for $\nu = 0.25$. The greatest amplification of the stress-intensity factor corresponds to the P-waves, and the lowest to the SV-waves. Between these two lies the curve for the SH-waves producing anti-plane

type of deformation at the crack tip. The data for this curve were taken from the work of Loeber and Sih [9] for the purpose of comparison.

Discussions. As remarked earlier, if a medium containing a crack is subjected to fluctuating loads, the crack-surface displacement may alternate in sign depending upon whether the incident field is tensile or compressive. Consequently, the relative displacement of the upper and lower surfaces of the crack can be negative in the compressive case which causes the opposing crack surfaces to come into contact. When this happens, the tractions and displacements are continuous across the line crack invalidating the original assumption of a traction-free crack. To avoid this situation, an additional tensile field must be added onto the crack to keep the crack surfaces separated. The requisite tensile field can be estimated from the displacement solution of the original problem of scattered waves.

For tractions that are applied symmetrically to the crack plane, the quantity of interest is the normal displacement

$$\begin{aligned} \frac{(-1)^i a(1 - \kappa^2)u_v(x, 0)}{P_i} &= [1 - (x/a)^2]^{1/2} \Lambda(1) - \int_{x/a}^1 [\xi^2 - (x/a)^2] \left[\frac{\Lambda(\xi)}{\xi^{1/2}} \right]' d\xi \\ &+ \cos v_i \left[\frac{\pi}{2} \int_0^{x/a} \xi^{1/2} \Gamma(\xi) d\xi + \int_{x/a}^1 \sin^{-1} \left(\frac{x}{a\xi} \right) \xi^{1/2} \Gamma(\xi) d\xi \right. \\ &\left. - \sin^{-1} \left(\frac{x}{a} \right) \int_0^1 \xi^{1/2} \Gamma(\xi) d\xi \right], \quad |x| < a. \end{aligned} \tag{59}$$

The numerical calculation of Eq. (59) will be carried out for the case of normal incidence, $v_1 = \pi/2$. Fig. 5 displays the deformed shapes of the crack for $\alpha_1 a = 0.0, 0.3, 0.5$, and 0.7 . In order to satisfy the zero traction condition, the crack opening shown by the curves in Fig. 5 should at least be doubled so that sufficient clearance is provided to prevent the crack surfaces from closing on the compression cycle. As expected, the maximum value of $|a(1 - \kappa^2)u_v(x, 0)/P_i| \simeq 1.25$ occurs at $\alpha_1 a \simeq 0.5$, the normalized wave number at which the dimensionless crack-tip stress-intensity factor $|k_1/\sigma_1 a^{1/2}|$ is also a maximum.

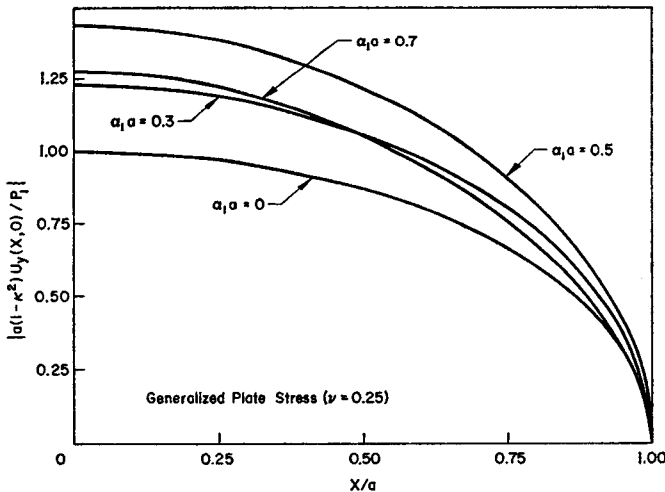


FIG. 5. Transverse displacement of crack surface.

To recapitulate, it should be mentioned that except for certain angles of incidence, the general input wave of the type in Eq. (5) or Eq. (7) will produce scattered waves composing both P- and SV-waves. A complete solution of the diffraction problem, therefore, necessitates both the symmetric (Case A) and skew-symmetric (Case B) treatments outlined in the paper. Only in the particular situations of normally incident waves does it suffice to consider only one part of the solution, namely Case A or Case B. To be noted further is that by superimposing a number of input waves, it is possible to generate other initial states of stresses which reduce, for low wave numbers, to familiar static problems. For instance, the sum of two compressional waves at normal and grazing incidence given by

$$\phi^{(i)} = \phi_0 [\exp(-i\alpha_1 y) + \exp(-i\alpha_1 x)] \exp(-i\omega t), \quad \psi^{(i)} = 0$$

and

$$\phi^{(i)} = \phi_0 [\exp(-i\alpha_1 y) - \nu \exp(-i\alpha_1 x)] \exp(-i\omega t), \quad \psi^{(i)} = 0$$

produces dynamic stress-states which reduce to the biaxial stress state

$$\sigma_{xx} = \sigma_{yy} = (1 + \nu)\sigma_1$$

and the uniaxial stress state

$$\sigma_{xx} = 0, \quad \sigma_{yy} = (1 - \nu^2)\sigma_1$$

respectively in the static limit.

Although the diffraction problem heretofore considered was that of a single incident wave which gave rise to sinusoidal type of boundary conditions on the line of discontinuity, i.e.,

$$\begin{bmatrix} \sigma_{yy}^{(r)}(x, 0) \\ \sigma_{xy}^{(r)}(x, 0) \end{bmatrix} \sim \begin{bmatrix} \cos(\alpha_1 x \cos \nu_1) \\ \sin(\alpha_1 x \cos \nu_1) \end{bmatrix}, \quad |x| < a,$$

the method of solution may be generalized to cover response to an arbitrary periodic disturbance by expanding the boundary function (s) in Fourier series as follows:

$$\sigma_{yy}^{(r)}(x, 0) = \sum_{n=0}^{\infty} \epsilon_n \left[A_n^* \cos\left(\frac{n\pi x}{a}\right) + B_n^* \sin\left(\frac{n\pi x}{a}\right) \right], \quad |x| < a.$$

The symbol ϵ_n takes the value unity for $n = 0$ and two for $n \geq 1$. The coefficients in the Fourier series are obtainable from the familiar integrals

$$A_n^* = \frac{1}{2a} \int_{-a}^a \sigma_{yy}^{(r)}(x, 0) \cos\left(\frac{n\pi x}{a}\right) dx, \quad n = 0, 1, 2, \dots$$

$$B_n^* = \frac{1}{2a} \int_{-a}^a \sigma_{yy}^{(r)}(x, 0) \sin\left(\frac{n\pi x}{a}\right) dx, \quad n = 1, 2, \dots$$

and $B_0^* = 0$. Similar expressions for the tangential traction $\sigma_{xy}^{(r)}(x, 0)$ can be written down.

It is evident that the present method can also be employed to solve a number of other problems dealing with the reflection of electromagnetic or acoustic waves by a finite line of discontinuity.

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