

# Wave Propagation in Fractured Porous Media

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**Abstract.** A theory of wave propagation in fractured porous media is presented based on the double-porosity concept. The macroscopic constitutive relations and mass and momentum balance equations are obtained by volume averaging the microscale balance and constitutive equations and assuming small deformations. In microscale, the grains are assumed to be linearly elastic and the fluids are Newtonian. Momentum transfer terms are expressed in terms of intrinsic and relative permeabilities assuming the validity of Darcy's law in fractured porous media. The macroscopic constitutive relations of elastic porous media saturated by one or two fluids and saturated fractured porous media can be obtained from the constitutive relations developed in the paper. In the simplest case, the final set of governing equations reduce to Biot's equations containing the same parameters as of Biot and Willis.

**Key words:** wave propagation, fractured porous media, balance equations, double porosity, Biot's theory.

## 1. Introduction

Although wave propagation in porous media has been studied for quite some time, Biot's (1956a,b) work appears to be the first one employing the fundamentals of transport phenomena in porous media. Biot's theory is an extension of a consolidation theory developed earlier (Biot, 1941). It is still well accepted and forms a basis for wave propagation in porous media. The theory predicts an additional compressional wave which was first confirmed experimentally by Plona (1980) (also see Berryman, 1980). Because of its highly dissipative behavior, this wave is very difficult to observe but contributes the energy losses which effect the characteristics of other types of body waves. The physical interpretations of the elastic constants in Biot's theory are given by Biot and Willis (1957). Fatt (1959) calculated Biot's constants for sandstone. The elastic coefficients were also studied by Geertsma and Smith (1961) and Pride *et al.* (1992). We refer to Corapcioglu (1991) for an extensive review of Biot's theory.

The single-porosity models are shown to be fairly successful to describe the behavior of porous materials. However, they are not suitable for fractured (or fissured) porous materials (Figure 1). In such systems, although most of the fluid mass is stored in the pores, the fracture permeability is much higher than the permeability of the pores. This leads to two distinct pressure fields: one in the

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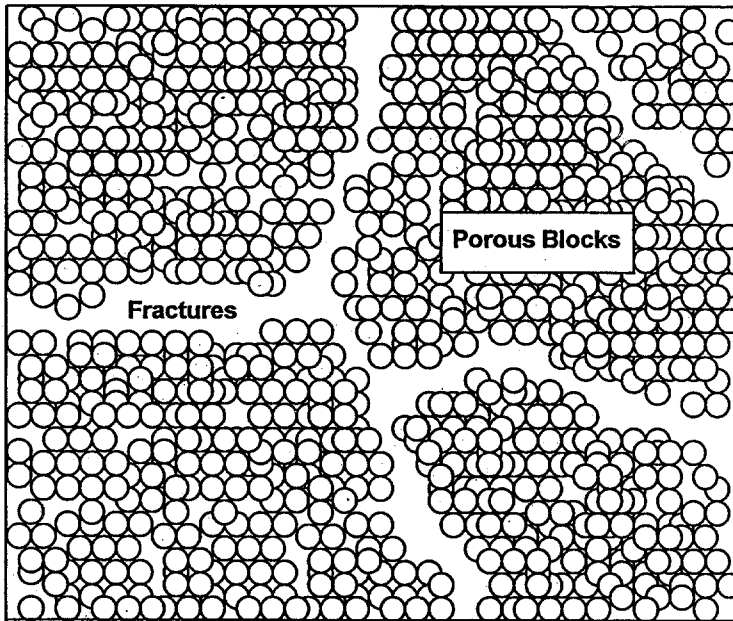


Figure 1. Conceptual model of a fractured porous medium

fractures and the other in the pores. Barenblatt *et al.* (1960) appear to be the first researchers proposing a double-porosity model to represent naturally fractured porous media. A double-porosity model can be considered as a three phase system, i.e., solid phase, fluid phase in the pores and fluid phase in the fractures, with fluid mass exchange between the pores and fractures.

Although flow in fractured porous media has been studied extensively (Barenblatt *et al.*, 1960; Barenblatt, 1963; Warren and Root, 1963; Kazemi, 1969; Shapiro, 1987; Douglas and Arbogast, 1990; Bear and Berkowitz, 1987; Torsaeter *et al.*, 1987), there is limited work in deformable fractured porous media. Duguid and Lee (1977) considered incompressible solid grains and used double-porosity concept in the formulation. They simplified the governing equations by neglecting solid displacement from the flow equations and used the finite element method for numerical analysis. Aifantis and his co-workers published a series of papers on consolidation of saturated fractured porous media (Wilson and Aifantis, 1982; Beskos and Aifantis, 1986; Khaled *et al.* 1984). The final set of equations is a direct generalization of Biot's consolidation theory. The phenomenological coefficients of the theory were expressed in terms of measurable quantities by Wilson and Aifantis (1982). Uniqueness and some general solutions were presented by Beskos and Aifantis (1986). Khaled *et al.* (1984) employed the finite element method to solve the governing equations for some practical problems. They reduced the number of coefficients from fifteen to nine by simply 'physically motivated arguments'. Wilson and Aifantis (1984) extended Aifantis' work and studied wave propagation

in saturated fractured porous media without detailed derivations. Their analysis showed three compressional waves. Similar results were obtained by Beskos who published a series of papers on the dynamics of fissured rocks (Beskos, 1989; Beskos *et al.* 1989a; Beskos *et al.*, 1989b). Beskos (1989) assumed that the medium is linearly elastic. However, the effect of the fluid pressure on the deformation of matrix was not considered. Beskos connected this to the definition of partial stresses. But when we assume that there is no relative movement between solid and fluid phases, Beskos' equation for the deformation of solid phase uncouples from fluid pressures. Bear and Berkowitz (1987) suggested a set of constitutive relations to model quasistatic behavior of fractured porous media. They assumed that the changes in the volume fraction of pores and fractures are linear functions of incremental pressures.

As noted earlier, flow in fractured porous media has been studied extensively by a number of researchers and formulated starting from the mass balance equations of the individual phases. Since generally the solid matrix was assumed to be rigid, the mass balance equations were stated for two phases, i.e., fluid in the fractures and fluid in the pores by employing the dual-porosity idealization. In flow problems, the inertial effects were neglected by assuming slow phenomena. The assumptions of rigid solid matrix and incompressible fluids simplify the macroscopic momentum balance equations considerably (Bear and Berkowitz, 1987). In this study, we will obtain macroscopic mass and momentum balance equations with compressible solid grains, solid matrix and fluid phase. Furthermore, we will incorporate the inertial effects in the formulation. Since dual-porosity approach is an accepted technique to express entity balance equations in fractured porous media, the same idealization can be used to study the wave propagation problems. Consequently, the resulting equations are obtained from the momentum balance equations.

In this study, we use the volume averaging technique to investigate the wave propagation in fractured porous media saturated by two immiscible fluids based on the double-porosity approach. Mass and momentum balance equations as well as the constitutive relations are obtained by volume averaging the equations and relations expressed at the microscopic scale. The volume averaging technique has been employed after the development of the theorem for volume average of a gradient (Slattery, 1967; Anderson and Jackson, 1967; Marle, 1967; Whitaker, 1967). We refer to Bear and Bachmat (1984) for an overview of the volume averaging technique. De la Cruz and Spanos (1985) made an attempt to formulate the constitutive relations and balance equations of wave propagation in saturated porous media. In a subsequent paper, de la Cruz and Spanos (1989) extended their theory to include the thermodynamic considerations. Pride *et al.* (1992) obtained Biot's (1941, 1956a,b) equations for saturated porous media by employing the volume averaging technique. The resulting constitutive relations of Pride *et al.* (1992) contained the same parameters as of Biot and Willis (1957).

The paper starts with a brief review of volume averaging theorems. The macro-scale mass and momentum balance equations and constitutive relations are obtained

by volume averaging the corresponding micro-scale equations. In the micro-scale, the grains are assumed to be linearly elastic and the fluids are Newtonian. The coefficients of macroscopic constitutive relations are expressed in terms of measurable quantities in a novel way. When the volume fraction of fractures vanishes, these constitutive relations reduce to those given by Tuncay and Corapcioglu (1996) for porous media saturated by two immiscible Newtonian fluids. Momentum transfer terms are formulated in terms of intrinsic and relative permeabilities assuming the validity of Darcy's law in fractured porous media.

## 2. Volume Averaging Theorems

A fractured porous medium is composed of two subsystems: an interconnected network of fractures and porous blocks. In such a medium, there are four different scales, i.e., pore, fracture, averaging volume and macroscopic scales. The pore scale is the pore opening of nonfractured solid matrix. Fractures are assumed to have openings greater than the pore scale. In their literature survey, Wilson and Aifantis (1982) reported fracture openings varying from 0.0025 to 0.75 cm which justifies this assumption. The scale associated with the fractures is the spacing between two fractures. Fracture spacings range from a couple of centimeters to the order of meters (Wilson and Aifantis, 1982). The averaging volume should be large enough to consider both porous blocks and fractures. Hence, the averaging volume scale in double-porosity approach is much greater than the scale in single-porosity approach. Since the characteristic length of a wave propagation problem is the wavelength, double-porosity approach restricts our study to low frequency wave propagation. We continue with the definitions used in volume averaging literature. Let  $B_i$  be a field quantity of phase  $i$ , then volume average of  $B_i$  is defined as

$$\langle B_i \rangle = \frac{1}{V} \int_{R_i} B_i dV, \quad (1)$$

where  $V$  is the averaging volume,  $R_i$  is the region occupied by phase  $i$ . The intrinsic volume average of  $B_i$ , i.e., the mean volume of  $B_i$  in  $R_i$ , is given by

$$\bar{B}_i = \frac{1}{V_i} \int_{R_i} B_i dV, \quad (2)$$

where  $V_i$  is the volume of phase  $i$  in the averaging volume. These two averages are related by

$$\langle B_i \rangle = \alpha_i \bar{B}_i, \quad (3)$$

where  $\alpha_i$  is the volume fraction of phase  $i$ . Now, we set the volume average theorem for a gradient and a time derivative (Slattery, 1967, 1981)

$$\langle \nabla B_i \rangle = \nabla \langle B_i \rangle + \frac{1}{V} \int_{S_{ij}} B_i n_i dA, \quad i \neq j, j = 1, \dots, N, \quad (4)$$

$$\left\langle \frac{\partial B_i}{\partial t} \right\rangle = \frac{\partial \langle B_i \rangle}{\partial t} - \frac{1}{V} \int_{S_{ij}} B_i u \cdot n_i \, dA, \quad i \neq j, j = 1, \dots, N, \quad (5)$$

where  $S_{ij}$  is the interface between phase  $i$  and phase  $j$ ,  $n_i$  is the outward normal of  $S_{ij}$  and  $u \cdot n_i$  is the speed of displacement of  $S_{ij}$  into other phases. The theorem of volume average of a divergence is stated as

$$\langle \nabla \cdot B_i \rangle = \nabla \cdot \langle B_i \rangle + \frac{1}{V} \int_{S_{ij}} B_i \cdot n_i \, dA, \quad i \neq j, j = 1, \dots, N. \quad (6)$$

If  $B_i$  is taken to be a constant, Equations (4) and (5) take the following forms

$$\nabla \alpha_i = -\frac{1}{V} \int_{S_{ij}} n_i \, dA, \quad i \neq j, j = 1, \dots, N, \quad (7)$$

$$\frac{\partial \alpha_i}{\partial t} = +\frac{1}{V} \int_{S_{ij}} u \cdot n_i \, dA, \quad i \neq j, j = 1, \dots, N. \quad (8)$$

### 3. Microscopic Constitutive Relations, and Mass and Momentum Balance Equations

In this study, the compressible porous medium consists of compressible solid grains, and two immiscible Newtonian fluids. The porous medium is assumed to have fractures which are referred as secondary pores. The pores in the nonfractured part of the porous medium are referred as primary pores. *The secondary pores are assumed to be saturated by the wetting fluid, whereas the primary pores are assumed to be saturated by the wetting and nonwetting fluids.* Therefore, there are four phases in the system: solid phase, wetting fluid phase in the secondary pores, wetting and nonwetting fluid phases in the primary pores. The solid phase is assumed to be initially at rest, linearly elastic, isotropic, and experiencing small deformations. Then the microscopic constitutive relations are given by

$$\tau_s = K_s \nabla \cdot u_s I + G_s \left( \nabla u_s + (\nabla u_s)^T - \frac{2}{3} \nabla \cdot u_s I \right), \quad (9)$$

where  $u_s$ ,  $\tau_s$ ,  $K_s$ ,  $G_s$ ,  $I$  are the displacement, incremental stress tensor, bulk modulus, shear modulus of the solid phase, and the unit tensor, respectively. The superscript  $T$  denotes the transpose of a tensor. We assume that both fluid phases are Newtonian with microscopic constitutive relations

$$\tau_i = P_i I + \mu_i \left( \nabla \nu_i + (\nabla \nu_i)^T - \frac{2}{3} \nabla \cdot \nu_i I \right), \quad i = 1, 2, f, \quad (10)$$

where  $\nu_i$ ,  $\tau_i$ ,  $P_i$  and  $\mu_i$  are the velocity, incremental stress tensor, incremental pore fluid pressure, and shear viscosity of fluid phase  $i$ , respectively. From now on,

subscripts 1, 2 and  $f$  will refer to the nonwetting fluid phase in the primary pores, wetting fluid phase in the primary pores, and fluid phase in the secondary pores (fractures). In Equation (10), the bulk viscosity of fluids is assumed to be negligible. The state equation of fluid phases is assumed to be in the form of

$$\frac{1}{K_i} \frac{dP_i^*}{dt} = \frac{1}{\rho_i} \frac{d\rho_i}{dt}, \quad i = 1, 2, f, \quad (11)$$

where  $K_i$  is the bulk modulus,  $\rho_i$  is the mass density of phase  $i$  and  $P_i^*$  is the pressure of phase  $i$ . We should note that the material properties of fluid phase 2 and fluid phase  $f$  are identical, i.e.,  $K_2 = K_f, \mu_2 = \mu_f$  and  $\rho_2 = \rho_f$ . The mass balance equations are expressed as

$$\frac{1}{\rho_i} \frac{d\rho_i}{dt} = -\nabla \cdot \nu_i, \quad i = 1, 2, f. \quad (12)$$

By combining Equations (11) and (12), we obtain

$$\frac{1}{K_i} \frac{dP_i^*}{dt} = -\nabla \cdot \nu_i, \quad i = 1, 2, f. \quad (13)$$

The pressure increment  $P_i$  can be written as

$$-P_i = K_i \nabla \cdot u_i, \quad i = 1, 2, f, \quad (14)$$

where  $u_i$  is the displacement of the fluid phase  $i$  from a reference position, i.e., incremental displacement. We continue with the microscopic momentum balance equation in terms of incremental stresses and velocities

$$\nabla \cdot \tau_j = \rho_j \frac{\partial \nu_j}{\partial t}, \quad j = s, 1, 2, f. \quad (15)$$

We neglect the convective acceleration since the displacements are assumed to be small. We note that the body forces do not appear in Equations (15) because equations are expressed in terms of incremental stresses. The boundary conditions at the solid-fluid interfaces are expressed as

$$\nu_s = \nu_i \quad \text{and} \quad \tau_s \cdot n_s + \tau_i \cdot n_i = 0 \quad \text{on} \quad S_{si}, \quad i = 1, 2, f, \quad (16)$$

where the subscript ( $si$ ) denotes the interface between the solid phase and fluid phase  $i$  and  $n_j$  is the unit outward vector normal to the interface. The boundary conditions at the fluid-fluid interfaces are (Slattery, 1981)

$$\nu_1 = \nu_2 \quad \text{and} \quad \tau_1 \cdot n_1 + \tau_2 \cdot n_2 = \nabla_\sigma \gamma - 2H\gamma n, \quad \text{on} \quad S_{12}, \quad (17)$$

$$\nu_2 = \nu_f \quad \text{and} \quad \tau_2 \cdot n_2 + \tau_f \cdot n_f = 0, \quad \text{on} \quad S_{2f}, \quad (18)$$

where  $\nabla_\sigma$ ,  $\gamma$ ,  $H$  are the surface gradient operator, interfacial tension, and mean curvature of the interface, respectively. The terms on the right-hand side of Equation (17) may be interpreted as the rate of momentum production per unit area of the phase interface. The first term on right-hand side incorporates the position dependency of surface tension upon the interface. Therefore the surface gradient operator incorporates the spatial variation of the surface tension along the phase interface.

#### 4. Macroscopic Mass Balance Equations

Employing the averaging theorems (Equations (4)–(6)), the volume average of Equation (12) is obtained as

$$\frac{\partial \langle \rho_j \rangle}{\partial t} + \nabla \cdot (\bar{\rho}_j \langle \nu_j \rangle + \Delta) = \frac{1}{V} \int_{S_{ji}} \rho_j (u - \nu_j) \cdot n_j \, dA,$$

$$j \neq i, i = s, 1, 2, f, \quad (19)$$

where

$$\Delta = \langle \rho_j \nu_j \rangle - \bar{\rho}_j \langle \nu_j \rangle. \quad (20)$$

For slightly compressible materials  $\Delta$  can be neglected. The right-hand side of Equation (19) corresponds to the mass transfer between the phases. The only mass exchange allowed is between the fluid phase in the fractures and the wetting fluid phase in the primary pores. This is usually approximated by (Barenblatt *et al.*, 1960; Bear and Berkowitz, 1987; Beskos, 1989).

$$\frac{1}{V} \int_{S_{2f}} \rho_2 (u - \nu_2) \cdot n_2 \, dA = -\frac{1}{V} \int_{S_{f2}} \rho_2 (u - \nu_f) \cdot n_f \, dA$$

$$= R \rho_2 (\bar{P}_f - \bar{P}_2), \quad (21)$$

where  $R$  is a material property of the porous medium and the wetting fluid,  $\bar{P}_2$  and  $\bar{P}_f$  are the pressures in fluid phase 2 and phase  $f$ , respectively. Barenblatt *et al.* (1960) proposed  $R$  as

$$R = \frac{c K_f A^2}{\mu_2}, \quad (22)$$

where  $A$  is the area of fracture-block contact per unit volume,  $c$  denotes a dimensionless shape factor of the fractured medium, and  $K_f$  is the intrinsic permeability of the fractures.

### 5. Macroscopic Constitutive Relations

Our next step is to obtain macroscopic constitutive relations by averaging the microscopic relations over a representative elementary volume (Bear, 1972). Volume averaging of Equation (9) yields

$$\begin{aligned} \frac{1}{V} \int_{R_s} \tau_s \, dV &= K_s \left( \nabla \cdot (\alpha_s \bar{u}_s) + \frac{1}{V} \int_{S_{si}} u_s \cdot n \, dA \right) I + \\ &+ G_s \left( \nabla(\alpha_s \bar{u}_s) + (\nabla(\alpha_s \bar{u}_s))^T - \frac{2}{3} \nabla \cdot (\alpha_s \bar{u}_s) I + K_{si} \right), \quad i = 1, 2, f, \end{aligned} \quad (23)$$

where

$$K_{si} = \frac{1}{V} \int_{S_{si}} \left( u_s n_s + n_s u_s - \frac{2}{3} (u_s \cdot n_s) I \right) \, dA, \quad i = 1, 2, f. \quad (24)$$

is a second-order tensor with zero trace. Since there is no mass exchange between the solid phase and fluid phases, the velocity of the interface is equal to the velocity of a point at the interface, i.e., material surface. Hence, by employing Equation (8), the integral in Equation (23) can be expressed as

$$\frac{1}{V} \int_{S_{si}} u_s \cdot n_s \, dA = (\alpha_s - \alpha_s^0) = \Delta \alpha_s, \quad i = 1, 2, f, \quad (25)$$

where the superscript (0) refers to the reference configuration. Since the displacements are assumed to be small, by definition  $\bar{u}_j \cdot \nabla \alpha_j \approx 0$ . Then volume averaged constitutive relations for the solid phase can be expressed as

$$\begin{aligned} \alpha_s \bar{\tau}_s &= K_s (\alpha_s \nabla \cdot \bar{u}_s + \Delta \alpha_s) I + G_s (\alpha_s \nabla \bar{u}_s + \alpha_s (\nabla \bar{u}_s)^T - \\ &- \frac{2}{3} \alpha_s \nabla \cdot \bar{u}_s I + K_{si}), \quad i = 1, 2, f, \end{aligned} \quad (26)$$

where  $\bar{\tau}_s$  is the intrinsic averaged incremental stress of the solid phase. Similarly, the volume averaged constitutive relations for nonwetting fluid phase are

$$\begin{aligned} \alpha_1 \bar{\tau}_1 &= K_1 (\alpha_1 \nabla \cdot \bar{u}_1 + \Delta \alpha_1) I + \mu_1 (\alpha_1 \nabla \bar{v}_1 + \alpha_1 (\nabla \bar{v}_1)^T - \\ &- \frac{2}{3} \alpha_1 \nabla \cdot \bar{v}_1 I + J_{1i}), \quad i = s, 2, f, \end{aligned} \quad (27)$$

where

$$J_{ij} = \frac{1}{V} \int_{S_{ij}} (v_i n_i + n_i v_i - \frac{2}{3} v_i \cdot n_i I) \, dA. \quad (28)$$

Since there is mass transfer between the primary pores and secondary pores, by employing Equation (8), we write

$$\frac{1}{V} \int_{S_{2j}} u_2 \cdot n_2 \, dA = \Delta \alpha_2 - \frac{1}{V} \int_{S_{2f}} (u_d - u_2) \cdot n_2 \, dA = \Delta \alpha_2 - M, \quad (29)$$



$$\frac{1}{V} \int_{S_{fj}} u_f \cdot n_f \, dA = \Delta\alpha_f - \frac{1}{V} \int_{S_{f2}} (u_d - u_f) \cdot n_f \, dA = \Delta\alpha_f + M, \quad (30)$$

where  $u_d$  is the displacement of the interface. Comparison of Equations (29), (30) and (21), for slightly compressible fluids experiencing small deformations, yields

$$\frac{\partial M}{\partial t} = R(\bar{P}_f - \bar{P}_2). \quad (31)$$

We note that  $\rho_2 \partial M / \partial t$  is equal to the mass transfer rate of wetting fluid phase between the primary pores and fractures. Then the volume averaged macroscopic constitutive relations of phase 2 and phase  $f$  become

$$\begin{aligned} \alpha_2 \bar{\tau}_2 = & K_2(\alpha_2 \nabla \cdot \bar{u}_2 + \Delta\alpha_2 - M)I + \mu_2(\alpha_2 \nabla \bar{v}_2 + \alpha_2 (\nabla \bar{v}_2)^T - \\ & - \frac{2}{3} \alpha_2 \nabla \cdot \bar{v}_2 I + J_{2i}), \quad i = s, 1, f, \end{aligned} \quad (32)$$

$$\begin{aligned} \alpha_f \bar{\tau}_f = & K_2(\alpha_f \nabla \cdot \bar{u}_f + \Delta\alpha_f - M)I + \mu_2(\alpha_f \nabla \bar{v}_f + \alpha_f (\nabla \bar{v}_f)^T - \\ & - \frac{2}{3} \alpha_f \nabla \cdot \bar{v}_f I + J_{fi}), \quad i = s, 1, 2. \end{aligned} \quad (33)$$

Under the small deformations assumption, the interfaces of the phases are not allowed to experience large deformations. If we assume  $\partial u_i / \partial t \gg \nu_i \cdot \nabla u_i$ , then  $J_{ij} = \partial K_{ij} / \partial t$ .  $K_{ij}$  and  $J_{ij}$  couple the shear deformation of the phases. In almost all studies associated with the deformation of the solid matrix, these coupling terms are neglected assuming that all shear resistance is provided by the matrix only.

The microscopic boundary condition at the fluid-fluid interface (Equation (17)) shows that there is a jump in the stresses of the immiscible fluids because of the presence of interfacial tension and curvature of the interface. Assuming smooth pressure variations within the averaging volume, we can write

$$\bar{P}_1^* - \bar{P}_2^* = P_{\text{cap}}(S_1), \quad (34)$$

where  $\bar{P}_1^*$  and  $\bar{P}_2^*$  are the intrinsic averaged pressures.  $P_{\text{cap}}$  also known as capillary pressure is assumed to be a function of  $S_1$  (saturation of the non-wetting phase) only. We recall that phases 1 and 2 are the nonwetting and wetting fluid phases, respectively.  $S_1$  is related to the volume fractions by

$$S_i = \frac{\alpha_i}{1 - \alpha_s - \alpha_f}, \quad i = 1, 2. \quad (35)$$

Then  $S_1 + S_2 = 1$ . Note that the fluid pressures we work with are the incremental pressures. Then, as a first-order approximation we can write

$$\bar{P}_1^* - \bar{P}_2^* = \frac{dP_{\text{cap}}}{dS_1} \Delta S_1 = P'_{\text{cap}} \Delta S_1 \quad (36)$$

provided that change in saturation  $\Delta S_1$  is small.

Deformation of a porous medium can be investigated by independently considering the volume change behavior (nonzero trace) and shear deformation behavior (zero trace). In the following, we first consider the part of constitutive relations associated with the volume changes. After examining the shear deformations, we combine these two to finalize the macroscopic constitutive relations.

To explore the constitutive relations associated with the volume changes, we start by introducing  $\bar{P}_j$  as the average trace of the volume averaged stress tensor of phase  $j$

$$-\alpha_j \bar{P}_j = \frac{1}{3} \text{tr}(\alpha_j \bar{\tau}_j) = K_j(\alpha_j \nabla \cdot \bar{u}_j + \Delta \alpha_j), \quad j = s, 1, \quad (37)$$

$$-\alpha_2 \bar{P}_2 = \frac{1}{3} \text{tr}(\alpha_2 \bar{\tau}_2) = K_2(\alpha_2 \nabla \cdot \bar{u}_2 + \Delta \alpha_2 - M), \quad (38)$$

$$-\alpha_f \bar{P}_f = \frac{1}{3} \text{tr}(\alpha_f \bar{\tau}_f) = K_2(\alpha_f \nabla \cdot \bar{u}_f + \Delta \alpha_f + M). \quad (39)$$

Equations (37)–(39) do not contain any rotational deformations.

In the following, we derive an expression for the dilatation  $\nabla \cdot \bar{u}_s$  for saturated fractured porous media by superposing three cases (Tuncay and Corapcioglu, 1995). We analyze three different stress state conditions individually. In each of these cases, we obtain an expression for the dilatation of the solid matrix  $\nabla \cdot \bar{u}_s$  by introducing macroscopic material coefficients when necessary. Then we will superpose these expressions to obtain a relation for  $\nabla \cdot \bar{u}_s$  when  $\bar{P}_s$ ,  $\bar{P}_f$  and  $\bar{P}_p$  are simultaneously present.  $\bar{P}_f$  and  $\bar{P}_p$  are intrinsic averaged pressures in the fractures and in the pores, respectively. Superposition is justified by the linearity of the system.

In the first case, we consider a drained porous medium, i.e.,  $\bar{P}_f = \bar{P}_p = 0$ . Introducing the drained bulk modulus of the fractured porous medium  $K_{\text{fr}}$ , we write

$$-\alpha_s \bar{P}_s = K_{\text{fr}} \nabla \cdot \bar{u}_s \quad (40)$$

By substituting Equation (40) in (37), we obtain

$$\Delta \alpha_s = - \left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{\text{fr}}} \right) \bar{P}_s. \quad (41)$$

$K_{\text{fr}}$  can be evaluated experimentally by testing a drained fractured porous sample.

In the second case, we consider a stress state where  $\bar{P}_s = \bar{P}_f = \bar{P}_p$ . This case corresponds to a fractured porous medium immersed in a fluid subjected to external pressure. Because of the homogeneity and isotropy of the medium, all volume fractions remain constant and Equation (37) yields

$$-\bar{P}_s = K_s \nabla \cdot \bar{u}_s. \quad (42)$$

In the third case which is a thought experiment, we assume that the volume fraction change of the fractures is zero and, furthermore,  $\bar{P}_p = 0$ . As in the second case, the fractured porous sample is immersed in a fluid. The pressure in the fractures  $\bar{P}_f$  is equal to the applied pressure. Then  $\bar{P}_s$  is

$$\bar{P}_f = \alpha_s \bar{P}_s + \alpha_f \bar{P}_f \quad \text{or} \quad \bar{P}_s = \frac{1 - \alpha_f}{\alpha_s} \bar{P}_f. \quad (43)$$

Introducing  $K_{fr}^m$  as the drained bulk modulus of the nonfractured porous matrix, we write

$$-\alpha_s \bar{P}_s = K_{fr}^m \nabla \cdot \bar{u}_s. \quad (44)$$

Experimentally  $K_{fr}^m$  can be determined by extracting a nonfractured sample from the fractured porous medium. The third case will be discussed in more details after the superposition of three cases. The volume fraction change of the solid phase can be solved from Equations (37) and (44) as

$$\Delta \alpha_s = - \left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{fr}^m} \right) \bar{P}_s. \quad (45)$$

The stress states can be summarized as

$$\begin{aligned} \text{Case 1 : } & \bar{P}_s = P_{c1}, \quad \bar{P}_p = 0, \quad \bar{P}_f = 0, \\ \text{Case 2 : } & \bar{P}_s = P_{c2}, \quad \bar{P}_p = P_{c2}, \quad \bar{P}_f = P_{c2}, \\ \text{Case 3 : } & \bar{P}_s = \frac{1 - \alpha_f}{\alpha_s} P_{c3}, \quad \bar{P}_p = 0, \quad \bar{P}_f = P_{c3}, \end{aligned} \quad (46)$$

where subscripts 1, 2 and 3 refer to cases 1, 2 and 3, respectively, Since we seek expressions when  $\bar{P}_s$ ,  $\bar{P}_p$  and  $\bar{P}_f$  are simultaneously present in the system,  $P_{c1}$ ,  $P_{c2}$  and  $P_{c3}$  must satisfy

$$P_{c1} + P_{c2} + \frac{1 - \alpha_f}{\alpha_s} P_{c3} = \bar{P}_s, \quad P_{c2} = \bar{P}_p, \quad P_{c3} + P_{c2} = \bar{P}_f. \quad (47)$$

Solution of Equations (47) for  $P_{c1}$ ,  $P_{c2}$  and  $P_{c3}$  yields

$$\begin{aligned} P_{c1} + \bar{P}_s - \bar{P}_p - \frac{1 - \alpha_f}{\alpha_s} (\bar{P}_f - \bar{P}_p), \quad P_{c2} &= \bar{P}_p, \\ P_{c3} &= P_f - \bar{P}_p. \end{aligned} \quad (48)$$

In the third case, we observe that  $P_{c3} = \bar{P}_f - \bar{P}_p$ . Hence, case 3 corresponds to the dilatation of the fractured solid matrix due to the pressure difference between

the pores and fractures. In other words, there are three components of the matrix dilatation associated with three bulk moduli,  $K_s, K_{fr}, K_{fr}^m$ .

The dilatation of the fractured solid matrix is obtained by superposing Equations (40), (42) and (44) and substituting the expressions for  $\bar{P}_s$  (Equations (46)) as

$$\nabla \cdot \bar{u}_s = -\frac{\alpha_s P_{c1}}{K_{fr}} - \frac{P_{c2}}{K_s} - \frac{\alpha_s (1 - \alpha_f)}{K_{fr}^m} \frac{1}{\alpha_s} P_{c3}. \quad (49)$$

Substitution of Equation (48) in (49) yields

$$\begin{aligned} \nabla \cdot \bar{u}_s = & -\frac{\alpha_s}{K_{fr}} \left( \bar{P}_s - \bar{P}_p - \frac{1 - \alpha_f}{\alpha_s} (\bar{P}_f - \bar{P}_p) \right) - \\ & -\frac{\bar{P}_p}{K_s} - \frac{1 - \alpha_f}{K_{fr}^m} (\bar{P}_f - \bar{P}_p). \end{aligned} \quad (50)$$

Similarly,  $\Delta\alpha_s$  is obtained from Equations (41) and (45) as

$$\begin{aligned} \Delta\alpha_s = & -\left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{fr}} \right) \left( \bar{P}_s - \bar{P}_p - \frac{1 - \alpha_f}{\alpha_s} (\bar{P}_f - \bar{P}_p) \right) - \\ & -\left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{fr}} \right) \frac{1 - \alpha_f}{\alpha_s} (\bar{P}_f - \bar{P}_p). \end{aligned} \quad (51)$$

We assume that in case of two fluids in the primary pores  $\bar{P}_p$  is given by

$$\bar{P}_p = S_1 \bar{P}_1 + (1 - S_1) \bar{P}_2. \quad (52)$$

We note that, among the cases mentioned above, the volume fraction of fractures changes only in case 1. Since the sum of changes in volume fractions is zero, i.e.,  $\Delta\alpha_s + \Delta\alpha_1 + \Delta\alpha_2 + \Delta\alpha_f = 0$ , from Equations (51) and (52), we can write the following equation for case 1

$$\begin{aligned} \Delta\alpha_f + \Delta\alpha_1 + \Delta\alpha_2 & \\ = -\Delta\alpha_s = & \left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{fr}} \right) \left( \bar{P}_s - S_1 \bar{P}_1 - (1 - S_1) \bar{P}_2 - \right. \\ & \left. - \frac{1 - \alpha_f}{\alpha_s} (\bar{P}_f - S_1 \bar{P}_1 - (1 - S_1) \bar{P}_2) \right). \end{aligned} \quad (53)$$

We propose the following expression for the change in volume fraction of fractures

$$\begin{aligned} \Delta\alpha_f = & F \left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{fr}} \right) \left( \bar{P}_s - S_1 \bar{P}_1 - (1 - S_1) \bar{P}_2 - \right. \\ & \left. - \frac{1 - \alpha_f}{\alpha_s} (\bar{P}_f - S_1 \bar{P}_1 - (1 - S_1) \bar{P}_2) \right), \end{aligned} \quad (54)$$

where  $F$  is a material property of the fractured porous medium. When a drained fractured porous medium is subjected to external loads, both the volume fraction of fractures and primary pores change. Since the sum of changes in volume fractions is zero, a fraction of  $\Delta\alpha_s$  should be equal and opposite to sign to  $\Delta\alpha_f$ . If the bulk modulus of the porous blocks  $K_{fr}^m$  is much greater than the bulk modulus of the fractured medium  $K_{fr}$ , i.e.,  $K_{fr}^m \gg K_{fr}$ , the volume of porous blocks does not change and  $F$  approaches unity. We can express the changes in volume fraction of phases 1 and 2 as

$$\begin{aligned} \Delta\alpha_1 &= (1 - \alpha_s^0 - \alpha_f^0)\Delta S_1 - S_1^0(\Delta\alpha_s + \Delta\alpha_f) - \\ &\quad - \Delta S_1(\Delta\alpha_s + \Delta\alpha_f) \\ &\approx (1 - \alpha_s - \alpha_f)\Delta S_1 - S_1(\Delta\alpha_s + \Delta\alpha_f), \end{aligned} \tag{55}$$

$$\begin{aligned} \Delta\alpha_2 &= -(1 - \alpha_s^0 - \alpha_f^0)\Delta S_1 - (1 - S_1^0)(\Delta\alpha_s + \Delta\alpha_f) + \\ &\quad + \Delta S_1(\Delta\alpha_s + \Delta\alpha_f) \\ &\approx -(1 - \alpha_s - \alpha_f)\Delta S_1 - (1 - S_1)(\Delta\alpha_s + \Delta\alpha_f). \end{aligned} \tag{56}$$

$\Delta S_1$ ,  $\Delta\alpha_s$ , and  $\Delta\alpha_f$  can be eliminated from Equations (37)–(39) by using Equations (36), (50) and (54) to obtain

$$\begin{aligned} -\alpha_s \bar{P}_s &= a_{11} \nabla \cdot \bar{u}_s + a_{12} \nabla \cdot \bar{u}_1 + a_{13} \nabla \cdot \bar{u}_2 + a_{14} \nabla \cdot \bar{u}_f + \\ &\quad + \left( \frac{a_{14}}{\alpha_f} - \frac{a_{13}}{\alpha_2} \right) M, \end{aligned} \tag{57}$$

$$\begin{aligned} -\alpha_1 \bar{P}_1 &= a_{21} \nabla \cdot \bar{u}_s + a_{22} \nabla \cdot \bar{u}_1 + a_{23} \nabla \cdot \bar{u}_2 + a_{24} \nabla \cdot \bar{u}_f + \\ &\quad + \left( \frac{a_{24}}{\alpha_f} - \frac{a_{23}}{\alpha_2} \right) M, \end{aligned} \tag{58}$$

$$\begin{aligned} -\alpha_2 \bar{P}_2 &= a_{31} \nabla \cdot \bar{u}_s + a_{32} \nabla \cdot \bar{u}_1 + a_{33} \nabla \cdot \bar{u}_2 + a_{34} \nabla \cdot \bar{u}_f + \\ &\quad + \left( \frac{a_{34}}{\alpha_f} - \frac{a_{33}}{\alpha_2} \right) M, \end{aligned} \tag{59}$$

$$\begin{aligned} -\alpha_f \bar{P}_f &= a_{41} \nabla \cdot \bar{u}_s + a_{42} \nabla \cdot \bar{u}_1 + a_{43} \nabla \cdot \bar{u}_2 + a_{44} \nabla \cdot \bar{u}_f + \\ &\quad + \left( \frac{a_{44}}{\alpha_f} - \frac{a_{43}}{\alpha_2} \right) M. \end{aligned} \tag{60}$$

The constants are given by

$$\begin{aligned}
 a_{11}A_3 = & [\alpha_s\alpha_f(\alpha_p + E_1K_{fr}) - \alpha_pE_3K_2(1 + E_2K_{fr}) - \\
 & - E_1E_3K_{fr}K_2(1 - \alpha_f)](A_1\alpha_sK_{fr}K_s) - \\
 & - \{\alpha_f\alpha_p\alpha_sA_2 + \alpha_pE_3[-(1 - \alpha_f)K_2^2S_1 - \\
 & - P'_{cap}S_1(1 - S_1)(K_2(1 - \alpha_fS_1) + \alpha_fS_1K_1) - \\
 & - K_1K_2(1 - S_1(1 - \alpha_f))]\}(K_{fr}K_s), \quad (61)
 \end{aligned}$$

$$\begin{aligned}
 a_{12}A_3 = & [\alpha_s\alpha_f(\alpha_p + E_1K_{fr}) - \alpha_pE_3K_2(1 + E_2K_{fr}) - \\
 & - E_1E_3K_{fr}K_2(1 - \alpha_f)][K_2 + P'_{cap}S_1(1 - S_1)](\alpha_1K_sK_1), \quad (62)
 \end{aligned}$$

$$\begin{aligned}
 a_{13}A_3 = & [\alpha_s\alpha_f(\alpha_p + E_1K_{fr}) - \alpha_pE_3K_2(1 + E_2K_{fr}) - \\
 & - E_1E_3K_{fr}K_2(1 - \alpha_f)][K_2 + P'_{cap}S_1(1 - S_1)](\alpha_2K_sK_2), \quad (63)
 \end{aligned}$$

$$\begin{aligned}
 a_{14}A_3 = & -\{[-\alpha_s\alpha_p(\alpha_f + E_2K_{fr})A_2 + [\alpha_p + K_{fr}(E_1(1 - \alpha_f) + \\
 & + \alpha_pE_2)]][K_3A_1]\}(\alpha_fK_sK_2), \quad (64)
 \end{aligned}$$

$$\begin{aligned}
 a_{21}A_3 = & [-\alpha_f\alpha_s + E_3K_2(1 + E_2K_{fr})](\alpha_1\alpha_sK_sK_1) \times \\
 & \times [K_2 + P'_{cap}S_1(1 - S_1)] + \\
 & + (\alpha_f\alpha_s - \alpha_fE_3K_s - E_3K_2 + \alpha_fE_3K_2)(\alpha_1K_{fr}K_1) \times \\
 & \times [K_2 + P'_{cap}S_1(1 - S_1)], \quad (65)
 \end{aligned}$$

$$\begin{aligned}
 a_{22}A_3 = & -K_sS_1(\alpha_f\alpha_p\alpha_s - \alpha_pE_3K_2(1 + E_2K_{fr})) \times \\
 & \times [K_2 + P'_{cap}(1 - S_1)]\alpha_1K_1 - K_2P'_{cap}S_1(1 - S_1)^2 \times \\
 & \times [-\alpha_f\alpha_s(\alpha_p + E_1K_{fr}) + \alpha_pE_3K_2(1 + E_2K_{fr}) + \\
 & + E_1E_3K_{fr}(K_2(1 - \alpha_f) + K_s\alpha_f)](\alpha_1K_1), \quad (66)
 \end{aligned}$$

$$\begin{aligned}
a_{23}A_3 = & \{[\alpha_p E_3 K_2(1 + C_2 K_{fr}) - \alpha_f \alpha_s (\alpha_p + E_1 K_{fr}) + \\
& + E_1 E_3 K_{fr}(K_2(1 - \alpha_f) + \alpha_f K_s)][P'_{cap} S_1(1 - S_1)] + \\
& + K_s[-\alpha_p \alpha_f \alpha_s + \alpha_p E_3 K_2(1 + E_2 K_{fr})]\} \times \\
& \times (\alpha_p K_1 K_2(1 - S_1) S_1)
\end{aligned} \tag{67}$$

$$\begin{aligned}
a_{24}A_3 = & (-\alpha_f \alpha_s - \alpha_s E_2 K_{fr} + E_3 K_s + E_2 E_3 K_{fr} K_s) \times \\
& \times [K_2 + P'_{cap} S_1(1 - S_1)](\alpha_f \alpha_1 K_1 K_2),
\end{aligned} \tag{68}$$

$$\begin{aligned}
a_{31}A_3 = & (-\alpha_s \alpha_f + E_3 K_2(1 + E_2 K_{fr}))[K_1 + P'_{cap} S_1(1 - S_1)] \times \\
& \times (\alpha_s \alpha_2 K_s K_2) + (\alpha_f \alpha_s - \alpha_f E_3 K_s - E_3 K_2 + \alpha_f E_3 K_2) \times \\
& \times [K_1 + P'_{cap} S_1(1 - S_1)](\alpha_2 K_{fr} K_2),
\end{aligned} \tag{69}$$

$$\begin{aligned}
a_{32}A_3 = & \{[\alpha_s E_3 K_2(1 + C_2 K_{fr}) - \alpha_f \alpha_s (\alpha_p + E_1 K_{fr}) + \\
& + E_1 E_3 K_{fr}(K_2(1 - \alpha_f) + \alpha_f K_s)][P'_{cap} S_1(1 - S_1)] + \\
& + K_s[-\alpha_p \alpha_f \alpha_s + \alpha_p E_3 K_2(1 + E_2 K_{fr})]\} \times \\
& \times (\alpha_p K_1 K_2(1 - S_1) S_1),
\end{aligned} \tag{70}$$

$$\begin{aligned}
a_{33}A_3 = & -K_s(1 - S_1)(\alpha_f \alpha_p \alpha_s - \alpha_p E_3 K_2(1 + E_2 K_{fr})) \times \\
& \times (K_1 + P'_{cap} S_1) \alpha_1 K_1 - K_1 P'_{cap} S_1^2(1 - S_1) \times \\
& \times [-\alpha_f \alpha_s (\alpha_p + E_1 K_{fr}) + \alpha_p E_3 K_2](1 + E_2 K_{fr}) + \\
& + E_1 E_3 K_{fr}(K_2(1 - \alpha_f) + K_s \alpha_f)](\alpha_s K_2),
\end{aligned} \tag{71}$$

$$\begin{aligned}
a_{34}A_3 = & (-\alpha_s \alpha_f - \alpha_s E_2 K_{fr} + E_3 K_s + E_2 E_3 K_{fr} K_s) \times \\
& \times [K_1 + P'_{cap} S_1(1 - S_1)](\alpha_2 \alpha_f K_2^2),
\end{aligned} \tag{72}$$

$$a_{41}A_3 = -(E_1E_3A_1\alpha_s\alpha_fK_{fr}K_sK_2) - (-E_3K_sA_2 + E_3A_1) \times \\ \times (\alpha_f\alpha_pK_{fr}K_2), \quad (73)$$

$$a_{42}A_3 = -E_1E_3\alpha_f\alpha_1K_{fr}K_sK_1K_2[K_2 + P'_{cap}S_1(1 - S_1)], \quad (74)$$

$$a_{43}A_3 = -E_1E_3\alpha_f\alpha_2K_{fr}K_sK_2^2[K_1 + P'_{cap}S_1(1 - S_1)], \quad (75)$$

$$a_{44}A_3 = -[(E_1E_3K_{fr}K_s - \alpha_p\alpha_s - \alpha_sE_1K_{fr})\alpha_fA_1 + \\ + \alpha_f\alpha_p\alpha_sK_sA_2](\alpha_fK_2), \quad (76)$$

where

$$\alpha_p = 1 - \alpha_s - \alpha_f, \quad (77)$$

$$A_1 = K_1K_2 + K_1P'_{cap}S_1^2(1 - S_1) + K_2P'_{cap}S_1(1 - S_1)^2, \quad (78)$$

$$A_2 = K_1(1 - S_1) + K_2S_1 + P'_{cap}S_1(1 - S_1), \quad (79)$$

$$A_3 = [\alpha_f\alpha_s(\alpha_p + E_1K_{fr}) - \alpha_pE_3K_2(1 + E_2K_{fr}) - \\ - K_{fr}E_1E_3(K_2(1 - \alpha_f) + \alpha_fK_s)]A_1 + \\ + [-\alpha_p\alpha_f\alpha_s + \alpha_pE_3K_2(1 + E_2K_{fr})]K_sA_2 \quad (80)$$

$$E_1 = \frac{1}{K_s} - \frac{1 - \alpha_f}{K_{fr}^m}, \quad (81)$$

$$E_2 = \frac{1 - \alpha_f}{K_{fr}^m} - \frac{1}{K_{fr}}, \quad (82)$$

$$E_3 = F \left( \frac{\alpha_s}{K_s} - \frac{\alpha_s^2}{K_{fr}} \right). \quad (83)$$

As noted earlier, we assume that all shear resistance of the porous medium is provided by the solid matrix only. Then, we can write

$$\alpha_s\bar{\tau}_s^D + \alpha_1\bar{\tau}_1^D + \alpha_1\bar{\tau}_2^D + \alpha_f\bar{\tau}_f^D \approx G_{fr} \left( \nabla\bar{u}_s + (\nabla\bar{u}_s)^T - \frac{2}{3}\nabla \cdot \bar{u}_s I \right), \quad (84)$$



where  $\bar{\tau}_j^D$  is the deviatoric stress of phase  $j$  and  $G_{fr}$  is the shear modulus of the solid matrix. In other words, the fluids are viscous but the mechanical shear response of the solid medium is provided by the solid matrix only. Fluid viscosities will be taken into consideration later when we discuss the momentum transfers between the phases. We can rewrite the complete constitutive relations by combining Equations (84) and (57)–(60) as

$$\begin{aligned} \alpha_s \bar{\tau}_s = & \left( a_{11} \nabla \cdot \bar{u}_s + a_{12} \nabla \cdot \bar{u}_1 + a_{13} \nabla \cdot \bar{u}_2 + a_{14} \nabla \cdot \bar{u}_f + \right. \\ & \left. + \left( \frac{a_{14}}{\alpha_f} - \frac{a_{13}}{\alpha_2} \right) M \right) I + G_{fr} (\nabla \bar{u}_s + (\nabla \bar{u}_s)^T - \frac{2}{3} \nabla \cdot \bar{u}_s I), \end{aligned} \quad (85)$$

$$\begin{aligned} \alpha_1 \bar{\tau}_1 = & \left( a_{21} \nabla \cdot \bar{u}_s + a_{22} \nabla \cdot \bar{u}_1 + a_{23} \nabla \cdot \bar{u}_2 + a_{24} \nabla \cdot \bar{u}_f + \right. \\ & \left. + \left( \frac{a_{24}}{\alpha_f} - \frac{a_{23}}{\alpha_2} \right) M \right) I, \end{aligned} \quad (86)$$

$$\begin{aligned} \alpha_2 \bar{\tau}_2 = & \left( a_{31} \nabla \cdot \bar{u}_s + a_{32} \nabla \cdot \bar{u}_1 + a_{33} \nabla \cdot \bar{u}_2 + a_{34} \nabla \cdot \bar{u}_f + \right. \\ & \left. + \left( \frac{a_{34}}{\alpha_f} - \frac{a_{33}}{\alpha_2} \right) M \right) I, \end{aligned} \quad (87)$$

$$\begin{aligned} \alpha_f \bar{\tau}_f = & \left( a_{41} \nabla \cdot \bar{u}_s + a_{42} \nabla \cdot \bar{u}_1 + a_{43} \nabla \cdot \bar{u}_2 + a_{44} \nabla \cdot \bar{u}_f + \right. \\ & \left. + \left( \frac{a_{44}}{\alpha_f} - \frac{a_{43}}{\alpha_2} \right) M \right) I. \end{aligned} \quad (88)$$

When the volume fraction of the fractures vanishes, i.e.,  $\alpha_f = M = E_2 = 0$  and  $K_{fr}^m = K_{fr}$ , these constitutive relations reduce to the ones obtained by Tuncay and Corapcioglu (1996) for elastic porous media saturated by two fluids. Furthermore, by setting  $S_1 = 0$ , we obtain the constitutive relations identical to that of Biot (1956a) for saturated porous media with the same definitions given by Biot and Willis (1957).

## 6. Macroscopic Momentum Balance Equations

Employing the averaging theorems (Equations (4)–(6)), the volume average of Equation (15) is obtained as

$$\left\langle \rho_j \frac{\partial v_j}{\partial t} \right\rangle = \nabla \cdot \langle \tau_j \rangle + \frac{1}{V} \int_{S_{ji}} \tau_j \cdot n_j \, dA, \quad j \neq i, \quad i = s, 1, 2, f. \quad (89)$$

Following Pride *et al.* (1992), for low frequencies

$$\left\langle \rho_j \frac{\partial v_j}{\partial t} \right\rangle = \bar{\rho}_j \frac{\partial \langle v_j \rangle}{\partial t}. \quad (90)$$

Average velocities and displacements for all phases are related by

$$\begin{aligned} \langle v_j \rangle &= \left\langle \frac{\partial u_j}{\partial t} \right\rangle = \frac{\partial \langle u_j \rangle}{\partial t} - \frac{1}{V} \int_{S_{ji}} u_j v_j \cdot n_j \, dA, \\ j &\neq i, \quad i = s, 1, 2, f. \end{aligned} \quad (91)$$

Since we are interested in the low frequency wave propagation, i.e., characteristic length of the microscopic scale is smaller than the wavelength, the displacements appearing in the integrand in Equation (91) can be assumed to be constant. Then, Equation (91) can be rewritten by employing Equations (3) and (8) as

$$\begin{aligned} \langle v_j \rangle &\approx \alpha_j \frac{\partial \bar{u}_j}{\partial t} + \bar{u}_j \frac{\partial \alpha_j}{\partial t} - \frac{\bar{u}_j}{V} \int_{S_{ji}} v_j \cdot n_j \, dA = \alpha_j \frac{\partial \bar{u}_j}{\partial t}, \\ j &\neq i, \quad i = s, 1, 2, f. \end{aligned} \quad (92)$$

Substitution of Equation (92) in (89) yields

$$\begin{aligned} \langle \rho_j \rangle \frac{\partial^2 \bar{u}_j}{\partial t^2} &= \nabla \cdot \langle \tau_j \rangle + \frac{1}{V} \int_{S_{ji}} \tau_j \cdot n_j \, dA, \\ j &\neq i, \quad i = s, 1, 2, f. \end{aligned} \quad (93)$$

## 7. Momentum Transfer (Interaction Terms)

One of the challenges in mechanics of porous media is the momentum transfer terms which appear in volume averaged momentum balance equations. Since the integral is over a representative volume of the microstructure, it requires the characterization and solution of the pore-scale equations. This is usually done by assuming a simple periodic microstructure. After solving the pore-scale equations, the solutions are related to the macroscopic variables (Biot, 1956a). An alternative approach is the use of empirical relations. In this study, due to the complexity of the pore-scale problem,

we approximate the interaction terms by assuming the validity of Darcy’s law. Since the theory is formulated for low frequency we propagation, the assumption of laminar flow is a reasonable one. We assume that the momentum transfer between the fluid phases can be neglected. These terms result in the cross permeabilities known as Yuster effect in the literature (Yuster, 1953; Scott and Rose, 1953). The Yuster effect can be neglected for practical purposes (Bear, 1972). Then assuming the validity of relative Darcy’s law in the primary pores, we can write (Bear and Corapcioglu, 1981)

$$\frac{1}{V} \int_{S_{s1}} \tau_s \cdot n_s \, dA = \frac{(1 - \alpha_s - \alpha_f)^2 S_1^2 \mu_1}{K_p k_{r1}} (\bar{v}_1 - \bar{v}_s), \tag{94}$$

$$\frac{1}{V} \int_{S_{s2}} \tau_s \cdot n_s \, dA = \frac{(1 - \alpha_s - \alpha_f)^2 (1 - S_1)^2 \mu_2}{K_p k_{r2}} (\bar{v}_2 - \bar{v}_s), \tag{95}$$

where  $K_p$  is the intrinsic permeability of the nonfractured porous medium and  $k_{ri}$  is the relative permeability of phase  $i$ . Similarly (Bear and Berkowitz, 1987; Beskos, 1989; Wilson and Aifantis, 1982)

$$\frac{1}{V} \int_{S_{sf}} \tau_s \cdot n_s \, dA = \frac{\alpha_f^2 \mu_2}{K_f} (\bar{v}_f - \bar{v}_s), \tag{96}$$

where  $K_f$  is the intrinsic permeability of the fractures.

### 8. Final Set of Equations

Substitution of the constitutive relations (Equations (85)–(88)) and the interaction terms (Equations (94)–(96)) in the averaged momentum balance equations yield

$$\begin{aligned} \langle \rho_s \rangle \frac{\partial^2 \bar{u}_s}{\partial t^2} = & \nabla \cdot \left( \left( a_{11} + \frac{G_{fr}}{3} \right) \nabla \cdot \bar{u}_s + a_{12} \nabla \cdot \bar{u}_1 + a_{13} \nabla \cdot \bar{u}_2 + \right. \\ & \left. + a_{14} \nabla \cdot \bar{u}_f + \left( \frac{a_{14}}{\alpha_f} - \frac{a_{13}}{\alpha_2} \right) M \right) + \nabla \cdot (G_{fr} \nabla \bar{u}_s) + \\ & + C_1 (\bar{v}_1 - \bar{v}_s) + C_2 (\bar{v}_2 - \bar{v}_s) + C_3 (\bar{v}_f - \bar{v}_s) \end{aligned} \tag{97}$$

$$\begin{aligned} \langle \rho_1 \rangle \frac{\partial^2 \bar{u}_1}{\partial t^2} = & \nabla \cdot \left( a_{21} \nabla \cdot \bar{u}_s + a_{22} \nabla \cdot \bar{u}_1 + a_{23} \nabla \cdot \bar{u}_2 + \right. \\ & \left. + a_{24} \nabla \cdot \bar{u}_f + \left( \frac{a_{24}}{\alpha_f} - \frac{a_{23}}{\alpha_2} \right) M \right) - C_1 (\bar{v}_1 - \bar{v}_s), \end{aligned} \tag{98}$$

$$\begin{aligned} \langle \rho_2 \rangle \frac{\partial^2 \bar{u}_2}{\partial t^2} = & \nabla \left( a_{31} \nabla \cdot \bar{u}_s + a_{32} \nabla \cdot \bar{u}_1 + a_{33} \nabla \cdot \bar{u}_2 + \right. \\ & \left. + a_{34} \nabla \cdot \bar{u}_f + \left( \frac{a_{34}}{\alpha_f} - \frac{a_{33}}{\alpha_2} \right) M \right) - C_2 (\bar{v}_2 - \bar{v}_s), \end{aligned} \quad (99)$$

$$\begin{aligned} \langle \rho_f \rangle \frac{\partial^2 \bar{u}_f}{\partial t^2} = & \nabla \left( a_{41} \nabla \cdot \bar{u}_s + a_{42} \nabla \cdot \bar{u}_1 + a_{43} \nabla \cdot \bar{u}_2 + \right. \\ & \left. + a_{44} \nabla \cdot \bar{u}_f + \left( \frac{a_{44}}{\alpha_f} - \frac{a_{43}}{\alpha_2} \right) M \right) - C_3 (\bar{v}_f - \bar{v}_s), \end{aligned} \quad (100)$$

where

$$C_1 = \frac{(1 - \alpha_s - \alpha_f)^2 S_1^2 \mu_1}{K_p k_{r1}}, \quad (101)$$

$$C_2 = \frac{(1 - \alpha_s - \alpha_f)^2 (1 - S_1)^2 \mu_2}{K_p k_{r2}}, \quad (102)$$

$$C_3 = \frac{\alpha_f^2 \mu_2}{K_f}. \quad (103)$$

Equations (97)–(100) (12 equations) with Equations (59)–(60) and (31) are 15 equations for low frequency wave propagation in a fractured porous medium saturated by two fluids with fifteen unknowns  $\bar{u}_s, \bar{u}_1, \bar{u}_2, \bar{u}_f, \bar{P}_2, \bar{P}_f$  and  $M$ . Equations simplify considerably when  $M = 0$ , i.e., no mass exchange between the fractures and porous blocks. When the deformations are small, this can be a rational assumption.

## 9. Conclusions

A theory of wave propagation in fractured porous media saturated by two immiscible fluids is presented. The macroscopic equations are obtained by volume averaging the micro-scale mass and momentum balance equations, and constitutive relations. The main limitations of the theory are the low frequency and small deformations assumptions. The macroscopic constitutive relations contain the bulk modulus of the fractured porous medium, bulk modulus of the nonfractured medium, bulk modulus of the solid grains and shear modulus of the solid matrix. The

capillary pressure effects are taken into account by assuming the validity of the relationship between capillary pressure and saturation. The momentum transfer terms are expressed in terms of intrinsic and relative permeabilities as a first order approximation. The final set of equations has an hyperbolic behavior with dissipation due to interaction terms.

A further study of the subject by the authors (Tuncay and Corapcioglu, 1996a) shows that an analysis of governing equations (Equations (97)–(100)) reveals the existence of two compressional waves in addition to the two compressional waves analogous to the fast and slow compressional waves in Biot's theory. One of these additional waves arises because of fractures whereas the other is associated with the pressure difference between the fluid phases in the porous blocks. So far there are a very limited number of studies available on wave propagation in fractured porous media. However, we hope that theoretical studies in this area will stimulate interest for future experimental research. It should be noted that the second compressional wave in Biot's theory was confirmed experimentally by Plona (1980) a quarter of a century after the theoretical prediction.

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