Wave Propagation in Shallow Water

Robert A. Dalrymple, F. ASCE

Center for Applied Coastal Research
University of Delaware
Newark, DE 19716

Contents

1 Introduction

The propagation of waves from offshore to onshore is a difficult problem due to the mathematical complexities of the governing equations and the degree of uncertainty of the bathymetry over which the waves must travel.

In this lecture, the various methods of transforming offshore wave trains to shallow water are discussed. First, single wave trains will be covered and then the shoaling of spectra will be introduced. Various wave transformation methods, such as ray tracing and parabolic modelling, are covered.

2 Propagation of Wave Trains

The simplest model of a wave train is due to Airy (1845). The displacement of the water surface from its mean location (at $z = 0$) for a wave train propagating in the x direction is

$$\eta(x,t) = a\cos(kx - \sigma t) \tag{2.1}$$

where the amplitude of the wave motion is a, k is the wave number, defined as $k = 2\pi/L$, where L is the wave length, and σ is the angular frequency, defined as $\sigma = 2\pi/T$, where T is the wave period. If the wave is propagating in an arbitrary direction, then the wave form is most easily described as

$$\begin{aligned} \eta(x,y,t) &= a\cos(k\cos\theta\, x + k\sin\theta\, y - \sigma t) && (2.2)\\ &= a\cos S(x,y,t) && (2.3) \end{aligned}$$

where θ is the angle that the wave train makes with the x axis. The argument of the cosine function is the phase function, $S(x,y,t)$, where $S = k\cos\theta\, x + k\sin\theta\, y - \sigma t$. Defining the wave number vector with components in the (x,y) direction, $\vec{k} = (k\cos\theta, k\sin\theta)$ and $\vec{x} = (x,y)$, we have $S = \vec{k}\cdot\vec{x} - \sigma t$. The direction of the wave number vector is the wave direction. Waves crests are located where the phase function has values of $2n\pi$, n= 0, 1, 2, ... The locations of the crests moves in space according to S = constant. For example, following the crest associated with $S = 0$ leads to

$$\frac{\vec{k}}{|k|}\cdot\vec{x} = \frac{\sigma}{|k|}t \tag{2.4}$$

where the left hand side is the distance in the wave direction travelled in time t and the ratio $\sigma/|k|$ is the wave phase speed, C.

Wave trains shoal and refract as they propagate from one water depth to another because the wave length (and therefore the speed of the wave) changes with depth. The local wave length from Airy wave theory (see, e.g., Dean and Dalrymple, 1984) is related to the local water depth, h, and wave period as given by the dispersion relationship, which can be written in several ways:

$$L = L_o \tanh kh \tag{2.5}$$

where

$$L_o = gT^2/2\pi, \text{ the deep water wave length} \tag{2.6}$$

or, after introducing the definitions for k and σ,

$$\sigma^2 = gk\tanh kh \tag{2.7}$$

The dispersion relationship (2.5 or 2.7) indicates that the wave length in shallow water is always shorter than that in deep water, L_o, which is solely dependent on the period, T, of the waves.

The transcendental nature of the dispersion equation makes it difficult to solve. A Newton-Raphson iterative method is often used. This technique requires a starting estimate of the solution.

Recently Fenton and McKee (1990) provided an approximate equation which gives solutions within 1.6% of the exact dispersion relationship:

$$L = \frac{g}{2\pi}T^2 \frac{1}{\left(\coth\left((2\pi\sqrt{h/g}/T)^{3/2}\right)\right)^{2/3}}$$

This approximation can provide the initial starting point for the Newton-Raphson technique or it be used to provide the final solution if its accuracy is sufficient. In the following two figures, the exact dispersion relationship is solved for wave number and wave length, given the water depth and wave period, with the Fenton/McKee approximation as a starting value in both FORTRAN and Mathematica, which is a higher level computer language–as seen by the fewer number of lines required. The FORTRAN program converges to a relative error of 0.000001 in 3 or less iterations.

The wave phase speed (or celerity) in Eq. (2.4) is also given by $C = L/T$. From (2.5), the celerity can be written as

$$C = C_o \tanh kh.$$

where C_o is the deep water value of the celerity ($C_o = gT/2\pi$). The wave celerity decreases monotonically with depth.

The wavelength change is also reflected in the rate at which the energy is transported by the waves, or the group velocity, C_g, which is defined as

$$C_g = nC = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)C$$

The factor n is $1/2$ in deep water and is unity in shallow water, where the group velocity becomes small, since the depth is small.

Conservation of wave energy implies that for normal wave incidence,

$$EnC_g = (EnC_g)_o$$

leading to

$$H = H_o\sqrt{\frac{C_{g,o}}{C_g}} = H_o K_s$$

where K_s is the shoaling factor (which is tabulated and graphed in the **Shore Protection Manual**).

2.1 Refraction

Refraction occurs when an obliquely incident wave encounters a change of depth. In this case, a portion of the wave crest is in shallower water than the rest of the wave crest. This part of the wave travels slower than the other and hence the wave crest changes direction.

For straight and parallel contours, it is easy to show that Snell's law (from optics) holds for wave direction. Given the wave direction, θ_1, in deeper water, Snell's law gives the direction in shallower water, θ_2,

$$\frac{\sin\theta_1}{C_1} = \frac{\sin\theta_2}{C_2} \tag{2.8}$$

This simple relationship provides a means to compute the wave direction in shallow water given the deep water direction by simply computing the deep water and shallow water celerities. The intervening contours play no role in the ultimate wave direction.

If the bottom contours are not straight and parallel, then *ray tracing* is possible, and in fact was the first practical means to compute wave refraction. By using Snell's law at each contour crossing, assuming straight and parallel contours locally, a wave ray can be drawn from offshore to onshore or in reverse (e.g., U.S. Army **Shore Protection Manual**, 1984).

```
c*****************************************************************
c    determine the wavelength, wavenumber from linear theory
c
c*****************************************************************

      common /const/g, pi
      g=9.81
      pi=3.1415927
      write(6,*) ' Input the water depth (m) and wave period'
      read(5,*) h,T
      call wvnum(h,T,xk)
      xl=2.*pi/xk
      write(6,*) ' The wave length is ',xl,' m'
      write(6,*) ' The wave number is', xk
      stop
      end
C*****************************************************************
       subroutine wvnum(dpt,per,xk)

c      this calculates linear wave number by Newton's method

       common /const/ g, pi
       xkh0=((2.*pi/per)**2.)*dpt/g
       coth=1./tanh( xkh0**(3./4.))
       xkh=xkh0*(coth)**(2./3.)
       do 4 i=1,10
          th=tanh(xkh)

          ch=cosh(xkh)

          f=xkh0-xkh*th

          fprime=-xkh/ch**2-th

          dxkh=-f/fprime

          if(abs(dxkh/xkh).lt. 0.000001) go to 9
    4     xkh=xkh+dxkh
       write(*,*) ' ten iterations failed for kh'
    9  xk=xkh/dpt
       return

       end
```

Figure 1: Newton-Raphson Solution of the Dispersion Relationship in FORTRAN.

```
(*  Dispersion Relationship in Mathematica

          Robert A. Dalrymple, December 1991
*****************************************************)

WaveNumber[h_,T_]:=
  Block[{kh,kh0,k,y},
    kh0=(2 Pi/T)^2 * h/9.81;
    kh=N[kh0 (Coth[(kh0)^(0.75)])^(2/3)];
    kh=y /. FindRoot[kh0==y Tanh[y],{y,kh},AccuracyGoal->5,
          MaxIterations ->20];
    k=kh/h
    ]

****************************************************

h=Input["Water depth (m) = "]
T=Input["Wave period = "]
k=WaveNumber[h,T]
L=N[2 Pi/k]
Print[" The wave length is ",L]
Print[" The wave number is ",k]
```

Figure 2: Newton-Raphson Solution of the Dispersion Relationship in Mathematica.

For irregular bathymetry, where the assumption of locally straight and parallel contours could lead to erroneous results, the irrotationality of the wave number has been used to generalize the Snell's law result. This equation follows from the definitions (2.2):

$$\vec{k} = \nabla S(x,y,t) \tag{2.9}$$

$$\sigma = -\frac{\partial S}{\partial t} \tag{2.10}$$

Identically, $\nabla x \vec{k} = 0$. This can be expanded to

$$\frac{\partial k \sin\theta}{\partial x} - \frac{\partial k \cos\theta}{\partial y} = 0 \tag{2.11}$$

For straight and parallel contours (x onshore, y alongshore), the derivatives in y are zero and this equation reduces to Snell's Law, given above. For realistic bathymetry, this equation can be solved for θ in a number of ways, as shown below.

2.1.1 Ray Tracing

Historically, Eq. (2.11) has been converted to an equation along a wave ray (e.g., Dean and Dalrymple, 1984):

$$\frac{\partial\theta}{\partial s} = -\frac{1}{C}\frac{\partial C}{\partial n},$$

where s, n are coordinates along and normal to the ray. This equation is Fermat's Principle, which follows from the statement that light always follows the shortest transmission path between two points.

Associated with the direction change is a wave height change due to convergence/divergence of the rays (Munk and Arthur, 1952). If b_o is the original spacing between two adjacent wave rays and b is the local spacing of the rays, and defining β as b/b_o, Munk and Arthur derived the following second order differential equation for β:

$$\frac{\partial^2\beta}{\partial s^2} + p\frac{\partial\beta}{\partial s} + q\beta = 0 \tag{2.12}$$

where

$$p = -\frac{\cos\theta}{C}\frac{\partial C}{\partial x} - \frac{\sin\theta}{C}\frac{\partial C}{\partial y};\ q = \frac{\sin^2\theta}{C}\frac{\partial^2 C}{\partial x^2} - \frac{\sin 2\theta}{C}\frac{\partial^2 C}{\partial x \partial y} + \frac{\cos^2}{C}\frac{\partial^2 C}{\partial y^2} \tag{2.13}$$

The local wave height is found from

$$H = H_o\sqrt{\frac{C_o}{2C_g}}\sqrt{\frac{1}{\beta}} \tag{2.14}$$

where C_g is the local group velocity.

Noda (1974) solved the refraction/shoaling problem by simultaneously solving the set of four first order differential equations with a fourth-order Runge-Kutta scheme:

$$\begin{aligned} \frac{\partial x}{\partial s} &= \cos\theta;\ \frac{\partial y}{\partial s} = \sin\theta \\ \frac{\partial\theta}{\partial s} &= \frac{1}{C}[\sin\theta\frac{\partial C}{\partial x} - \cos\theta\frac{\partial C}{\partial y}] \\ \frac{\partial\beta}{\partial s} &= r;\ \frac{\partial r}{\partial s} = -pr - q\beta \end{aligned}$$

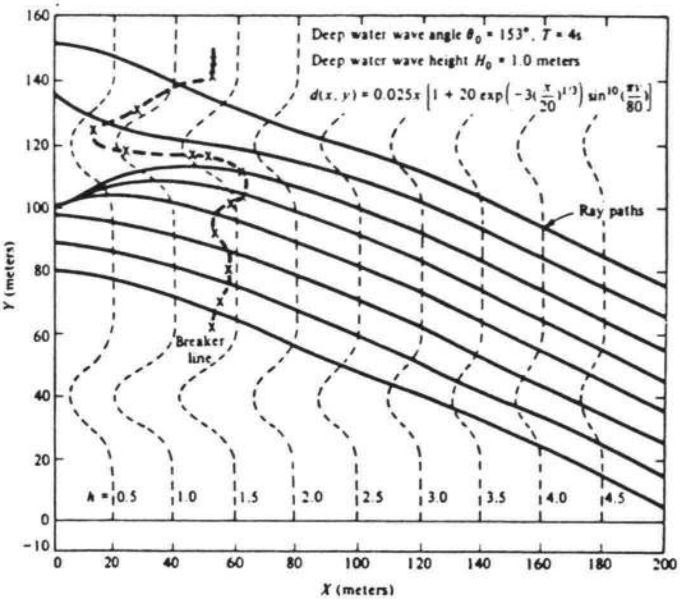

Figure 3: Rays for Oblique Incidence Over a Rip Channel, Noda (1974)

where the last two equations are two first order equations obtained from splitting the second order equation (2.12). An example of Noda's results are given in Figure 3.

Jonsson and Christoffersen (1984) have developed ray tracing procedures for waves on currents. The conservation of wave action (which replaces the wave energy conservation for waves on currents; see below) is used.

Ray tracing has drawbacks. The computation of rays does not guarantee that the area of interest is densely covered with rays, providing information for the design wave height and direction (although ray tracing can be used backwards from the site to offshore). Also, the crossing of wave rays leads to trouble in interpretation and most ray tracing methods neglect diffractive effects.

2.1.2 Grid Models

More recently, refraction calculations have been carried out by solving the irrotationality condition (2.11) on a rectangular grid (Perlin and Dean, 1983; Dalrymple, 1988, with Lax-Wendroff modification, 1991). The gridded results can then be used for input to other models of interest–wave-induced circulation, for example. These models entail dividing the offshore region into a grid, say, $x = m\Delta x$, $m = 1, 2, \ldots, M$ and $y = n\Delta y$, $n = 1, 2, \ldots, N$. See figure 4.

Now, Eqn. (2.11) must be solved in finite difference form. Dalrymple (1991) rewrites this equation as

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = 0 \tag{2.15}$$

where $A = k\cos\theta$ and $B = k\sin\theta = \sqrt{k^2 - A^2}$. (Knowing k at every grid location from the dispersion relationship (2.7), and given A, B is calculable.) The two-step Lax-Wendroff method first involves taking a half step in both the x and y directions.

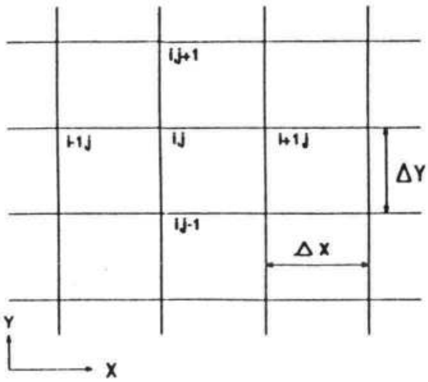

Figure 4: Schematic Diagram for Grid Models

Step 1.

$$\frac{A_{m+1/2,n+1/2} - (A_{m,n+1} + A_{m,n})/2}{\Delta x/2} - \left(\frac{B_{m,n+1} - B_{m,n}}{\Delta y}\right) = 0 \tag{2.16}$$

which is solved for $A_{m+1/2,n+1/2}$. $B_{m+1/2,n+1/2}$ is then determined from its definition.

Step 2.

$$\frac{A_{m+1,n} - A_{m,n}}{\Delta x} - \frac{B_{m+1/2,n+1/2} - B_{m+1/2,n-1/2}}{\Delta y} = 0 \tag{2.17}$$

These two equations permit second order accurate differencing to be carried out. Additionally this method does not require any iteration (for the linear dispersion relationship). The results of the two-step Lax-Wendroff method are the wave direction at all locations of the grid.

$$\theta_{m,n} = \sin^{-1}\left(\frac{A_{m,n}}{k_{m,n}}\right)$$

The wave height follows from the wave energy equation or the wave action equation (for waves on currents). The steady-state wave energy equation is

$$\nabla \cdot \left(E\vec{C_g}\right) = \frac{\partial EC_g \cos\theta}{\partial x} + \frac{\partial EC_g \sin\theta}{\partial y} = -\epsilon_d \tag{2.18}$$

where ϵ_d represents energy losses due to such things as bottom friction or percolation. If ϵ_d is zero, then this equation can be rewritten as

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = 0 \tag{2.19}$$

with A and B redefined accordingly. This equation also can be solved by the two-step Lax-Wendroff method.

For waves on currents, the wave energy is replaced by the wave action, developed by Bretherton and Garrett (1968), which is the wave energy divided by the relative frequency, σ, E/σ, where σ is changed due to the presence of the current:

$$\sigma = \omega - \vec{k} \cdot \vec{U} \tag{2.20}$$

where σ is the wave frequency with respect to coordinate system moving with the current, $\vec{U} = (U, V)$ and ω is the absolute frequency $(2\pi/T)$. The equation for the relative frequency depends on the wave number implicitly since the formula for σ is given by (2.7).

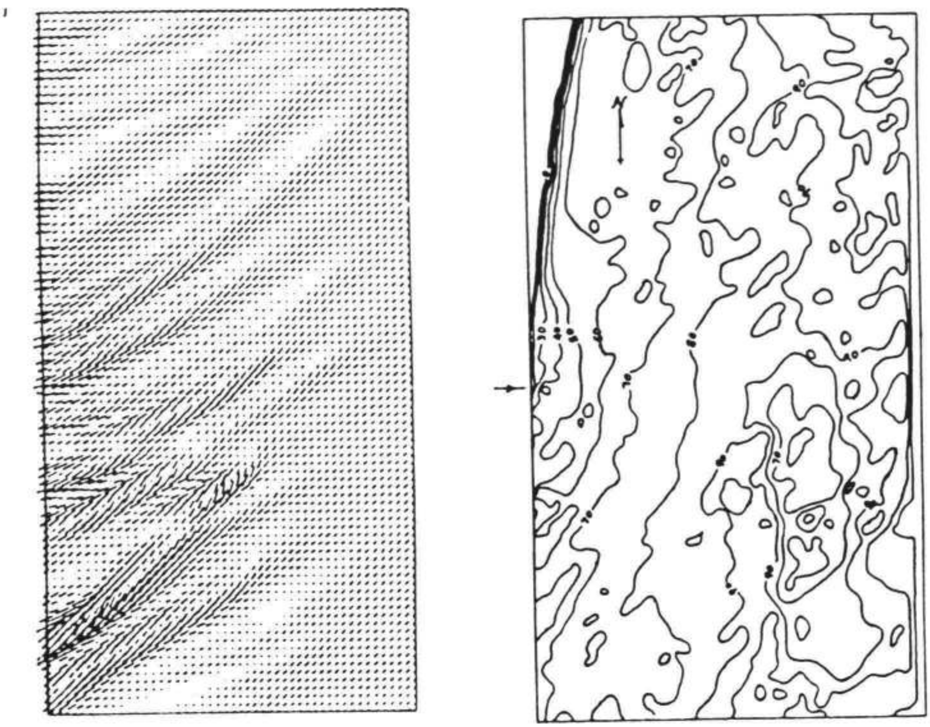

Figure 5: Bathymetry and Wave Vectors Given By REFRACT, Dalrymple (1988). Soundings are in feet and Area Depicted is 24 by 44 n. miles. Arrow Lengths are Proportional to Wave Height. The Wave Period is 12 s. and Incident Wave Height is 2 m.

The conservation of wave action (in the absence of dissipation) is

$$\frac{\partial (E/\sigma)}{\partial t} + \nabla \cdot \left(\frac{E}{\sigma}(\vec{U} + \vec{C_g}) \right) = 0 \tag{2.21}$$

For the case of no currents, the conservation of wave action reduces to the conservation of energy equation. Dalrymple (1988, 1991) used the steady-state wave action equation (2.21) and the dispersion relationship (2.20) for waves on currents. Figure (5) shows the bathymetry and resulting wave direction and height vectors (the length of the vector is proportional to the wave height).

2.2 Diffraction

The presence of surface piercing obstacles provides a good example of diffraction. The mathematical theory for waves passing a semi-infinite breakwater is given by Sommerfeld (1849) and where A is now a spatially varying amplitude. The governing equation for the potential is the Laplace equation, with associated bottom and free surface boundary conditions:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2.23}$$

Substituting our assumed form for ϕ, yields the following Helmholz equation for A,

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + k^2 A = 0 \tag{2.24}$$

This elliptic equation must be solved subject to certain boundary conditions. In the case of harbor oscillations, Lee (1971) introduced the use of boundary integral methods in coastal engineering by solving (2.24) for arbitrarily shaped harbors with vertical sides and constant depth.

2.3 Combined Refraction and Diffraction

The parabolic approximation provides a convenient method to predict the waves when refraction, shoaling, and diffraction occur simultaneously. Further, it allows dropping the requirement for a downwave boundary condition. Examining the assumed form for ϕ for the case of diffraction, we notice that the amplitude A must vary rapidly with x to account for the wavelike behavior of the waves in the x direction. For waves propagating in the x direction, this behavior is of the form e^{ikx}. For waves propagating nearly in the x direction, this function will provide for most of the wave oscillation. Therefore, we will assume that the local velocity potential is described by

$$\phi(x,y,z,t) = A(x,y)\frac{\cosh k(h+z)}{\cosh kh} e^{ikx} e^{-i\sigma t}$$

In this case, we expect that $A(x,y)$ will vary slowly in x. Substituting into the Laplace equation (2.23) and treating a constant depth problem, we have

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + 2ik\frac{\partial A}{\partial x} = 0 \tag{2.25}$$

The first term can be shown to be small compared to the others for small wave angles[1]; therefore, we arrive at the simple parabolic equation for constant depth,

$$\frac{\partial A}{\partial x} = \frac{i}{2k}\frac{\partial^2 A}{\partial y^2} \tag{2.27}$$

For plane waves, we find that A is

$$A(x,y) = ae^{-\frac{ik}{2}\sin^2\theta\, x + ik\sin\theta\, y} \tag{2.28}$$

If we compare the approximate solution, $A(x,y)\, e^{ikx}$, to the plane wave solution for the elliptic equation (2.24), we have

$$ae^{ik\cos\theta\, x + ik\sin\theta\, y} \text{ compared to } ae^{ik(1-\frac{1}{2}\sin^2\theta)x + ik\sin\theta\, y} \tag{2.29}$$

[1] If $A = ae^{i(k(\cos\theta - 1)\, x + k\sin\theta\, y)}$, which is the plane wave solution (2.2) without the $e^{ikx}\, e^{-i\sigma t}$, then

$$\frac{\partial^2 A}{\partial x^2} = -k^2(\cos\theta - 1)^2 A \tag{2.26}$$

For small angles, $\cos\theta \to 1 - \theta^2/2\ldots$; therefore, this term is much smaller, $O(\theta^4)$, than the other terms, $O(\theta^2)$, in Eq. (2.25) for small θ.

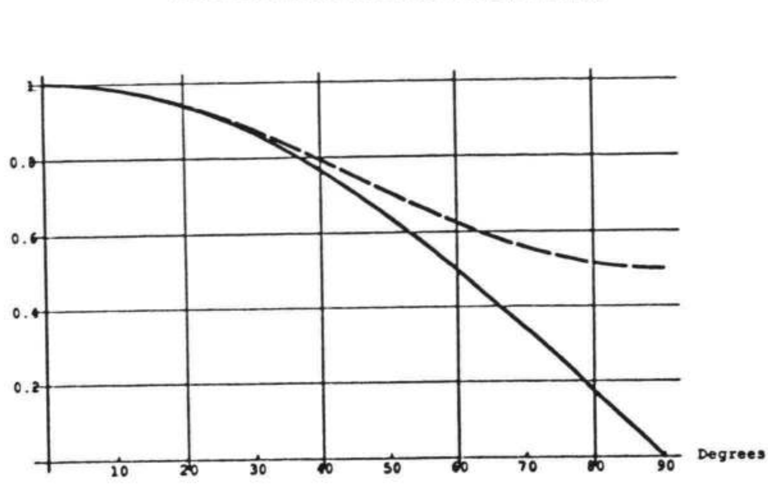

Figure 6: Errors Between Simple Parabolic Model (Dashed Line) and the Exact Solution (Solid Line)

This shows that the simple parabolic approximation for plane waves represents $\cos\theta = \sqrt{1-\sin^2\theta}$ as the first two terms in a binomial expansion, $1-\sin^2\theta/2$. For small θ, there is very little error; however, for θ greater than 45°, there can be significant errors, thus limiting the effectiveness of the simple parabolic equation method. This is shown in Figure 6. Kirby (1986) provides a means to extend the parabolic method to wider angles of wave incidence.

2.4 Mild Slope Equation

Berkhoff (1972) introduced the mild-slope equation for the calculation of waves over mildly sloping bathymetry. If the total potential is

$$\phi = \tilde{\phi}\,\frac{\cosh k(h+z)}{\cosh kh}\,e^{i\sigma t},$$

then by integrating over depth, using the hyperbolic function as an integrating factor, this three-dimensional elliptic equation reduces to an approximate two-dimensional equation of the following form:

$$\frac{\partial}{\partial x}\left(CC_g\frac{\partial\tilde{\phi}}{\partial x}\right)+\frac{\partial}{\partial y}\left(CC_g\frac{\partial\tilde{\phi}}{\partial y}\right)+k^2CC_g\tilde{\phi}=0 \tag{2.30}$$

If we substitute $\tilde{\phi} = A(x,y)\,e^{iS(x,y)}$, where the amplitude, A, and the phase, S, are real, then we can separate the mild slope equation into two equations:

$$k^2-|\nabla S|^2+\frac{\nabla\cdot CC_g\nabla A}{CC_gA} = 0 \tag{2.31}$$

$$\nabla\cdot\left(C_gA^2\sigma\frac{\nabla S}{|\nabla S|}\right) = 0 \tag{2.32}$$

The first equation is the *eikonal equation*, which provides the real wavenumber, ∇S, in terms of the wavenumber k given from the dispersion relationship (2.7) and a correction term due to diffraction, resulting from local gradients in amplitude A and wave speed. The second equation is a conservation equation for wave action (or wave energy in the absence of currents).

Ebersole (1985) solved these last two equations and the irrotationality condition (2.11) in finite difference form, resulting in a model called RCPWAVE, which is used by the U.S. Army Corps of Engineers. The disadvantage of RCPWAVE (and all models which use the irrotationality condition, such as REFRACT) is the use of the irrotationality of the wave number condition, which restricts the model to situations where the wave phase is single-valued, precluding amphidromic points in the phase and intersecting wave trains[2].

The mild slope equation (2.30) has been solved a variety of ways. Finite element methods were used first (e.g., Berkhoff, 1972; Bettess and Zienkiewicz, 1977; Houston, 1981). For large areas, these methods can lead to very large grids and matrices which must be inverted.

Radder (1979) introduced a parabolic representation of the mild slope equation, which had the advantage of removing the downwave boundary condition, which is often unknown, while still retaining the diffraction capabilities of the model. Kirby and Dalrymple (1983) introduced a nonlinear parabolic representation, which included the third order Stokes correction to the wave speed, leading to the development of the numerical code, REF/DIF. This model, which neglects backscattered waves, only requires the efficient inverting of tri-diagonal matrices of the size of the width of the model grid. Kirby and Dalrymple (1984) showed excellent agreement between laboratory data of Berkhoff, Booij, and Radder (1982) and their model, particularly when the nonlinear corrections were used. Other numerical codes exist, such as CREDIZ, from the Rijkswaterstaat in the Netherlands and the model of Tsay and Liu (1989).

The mild slope equation has also been solved by separating it into three time-dependent equations, a technique first used by Ito and Tanimoto (1972). Copeland (1985) and Madsen and Larsen (1987) present examples of this method. The advantages are that the elliptic equation is replaced by equations similar to long wave equations and large matrices do not have to be inverted, but, as Kirby and Rasmussen (1991) have pointed out, this methodology is valid only for strictly periodic wave trains.

3 Energy Dissipation

Waves lose energy through a variety of processes, such as breaking (treated separately here), interaction with the bottom and by reflection.

The steady-state conservation of wave energy flux is given by

$$\nabla \cdot (E\vec{C_g}) = -\epsilon_d \tag{3.1}$$

The ϵ_d is the energy dissipation rate.

Bottom friction creates a loss of wave energy as the waves must work against the bottom shear stress, τ_b. Here, $\epsilon_d = \overline{\vec{\tau_b} \cdot \vec{u_b}}$, where u_b is the wave-induced velocity at the bottom. For waves in the x direction and a turbulent shear stress given by $\tau_b = \rho f u_b|u_b|/8$, where f is the Darcy-Weisbach friction factor, see, e.g., Kamphuis (1975), Putnam and Johnson (1949) found that

$$\epsilon_d = \frac{\rho f u_b^3}{6\pi} \tag{3.2}$$

[2]The problem arises in expressing the wave form in terms of an amplitude and a phase. If the wave form is zero somewhere, then A is zero and S is undefined and non-differential. Therefore the phase is no longer single valued.

The loss of wave energy is also given by k_i, which is the damping rate with distance of the wave height, as given in this form

$$a(x) = a_0 e^{-k_i x}$$

Using this equation in the conservation equation (3.1) gives

$$k_i = -\frac{\epsilon_d}{2C_g E}$$

Liu and Dalrymple (1984) examined the damping of waves over a sandy bed and found that the wave number is complex, $k = (k_r + ik_i)$, due to the influence of the energy loss due to the induced flow in the porous medium. Neglecting flow accelerations, the complex dispersion relationship for this case is

$$\sigma^2 = gk \tanh kh - i\left(\frac{\sigma K}{\nu}\right)(gk - \sigma^2 \tanh kh) \tag{3.3}$$

Here, K is the soil permeability and ν is the kinematic viscosity of the fluid. For small permeabilites, the real part of the wave number is unchanged from that given by the impermeable bed case (2.7), while the imaginary part of the wave number is approximately

$$k_i = \frac{2(\sigma K/\nu)k_r}{2k_r h + \sinh 2k_r h} \tag{3.4}$$

See also Dalrymple and Dean (1984) for this case. Liu and Dalrymple also resolved the contradictions between previous models for this problem.

Dalrymple, Kirby, and Hwang (1984) examine the inclusion of damping into the mild slope equation (2.30), following the work of Booij (1981).

$$\frac{\partial}{\partial x}\left(CC_g \frac{\partial \tilde{\phi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(CC_g \frac{\partial \tilde{\phi}}{\partial y}\right) + (k^2 CC_g + i\sigma w\tilde{\phi}) = 0 \tag{3.5}$$

where w is the dissipation term, related to ϵ_d, by $wE = \epsilon_d$ and k_i for small k_i, as $w = 2C_g k_i$. They further show various forms of k_i for viscous muds, porous bottoms, surface films, laminar bottom boundary layers, and seaweed. Areas of extreme damping can cause the waves to diffract around such regions of damping. Figure 7 shows the wave field created by a rectangular region representing an area of kelp, which are damping the waves.

4 Shallow Water Wave Equations

The above sections have referred to the solutions of the Laplace or mild slope equation, which are best used for intermediate water depths. In shallow water, which is defined as when the ratio of water depth to wave length is small, $h/L << 1/20$, then other wave theories become more efficient in describing the wave forms. This is not to say that solutions to Laplace equations are incorrect in shallow water; indeed, the Stream Function wave theory of Dean (1965) is valid in shallow water, but requires very high order wave theory.

The most common equation used in shallow water are the Boussinesq equations. For variable depth, these equations are (Peregrine, 1967):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial x} + \frac{h}{2}\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u}{\partial t}\right) - \frac{h^2}{6}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) \tag{4.1}$$

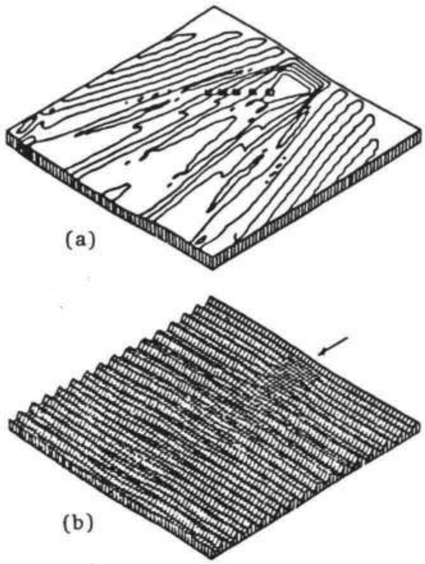

Figure 7: Wave Field in and around Strong Damping Region. a) Contours of Transmission Coefficient and b) Instantaneous Water Surface Elevation, from Dalrymple, Kirby and Hwang (1984).

$$\frac{\partial \eta}{\partial t} + \frac{\partial (h+\eta)u}{\partial x} = 0 \tag{4.2}$$

For constant depth, the solitary wave and the cnoidal waves are solutions. For variable depth and a two dimensional problem, numerical solutions by a number of techniques are available. Some of the more well-known include that of Abbott, Peterson and Skovgaard (1978)–the Jupiter 21 model.

If the initial wave field is expanded in terms of slowly varying (in x) Fourier modes, Boussinesq equations yield a set of coupled evolution equations that predict the amplitude and phase of the Fourier modes with distance. Field applications of the spectral Boussinesq theory show that the model predictions agree very well with normally incident ocean waves (Freilich and Guza, 1984). Elgar and Guza (1986) show that the model is also able to predict the skewness of the shoaling wave field, which is important for sediment transport considerations. Liu, Yoon and Kirby (1985) developed a parabolic approach to the Boussinesq equations to permit modelling of directional seas.

Within the surf zone, wave breaking creates a radically different wave field. The nonlinear shallow water equations, which predict waves which steepen and break in shallow water have been used by Hibberd and Peregrine (1979) to predict bores in the surf zone. The methodology involves numerical integrations with the Lax-Wendrof technique, which preserves 'shock' fronts across the surf zone. Packwood (1983) added friction and permeability to this model, while Ryrie (1983) allowed for oblique incidences. Engineering models of this method for regular and irregular waves, including time dependent swash oscillations and set-up, have been developed by Kobayashi, Otta, and Roy (1987) and Kobayashi, Cox, and Wurjanto (1990). Their models are IBREAK and RBREAK.

5 Wave Breaking

Waves become unstable in shallow water and will often break. The historical criterion for wave breaking is that the wave will break when the wave height is some fraction of the water depth, $H_b = \kappa h_b$, where the subscripts denote breaking values. The breaking index, κ, is a function of the bottom slope and wave steepness and ranges in value from 0.78 for horizontal bottoms to over 1.5 on steeper slopes. The **Shore Protection Manual** provides curves for the breaking index versus slope and offshore wave steepness.

Within the surf zone, the energy flux equation (3.1) still holds; what is required is the appropriate form for the energy loss (ϵ_d) due to breaking.

The Dally, Dean, and Dalrymple (1985) model assumes that there is a stable wave height after breaking equal to some fraction of the water depth and that the rate of energy dissipation in the surf zone is proportional to the difference between the actual wave energy flux and the stable wave energy flux, $(EC_G)_s$. The model has the following form,

$$\frac{dEC_G}{dx} = -\frac{K}{h}\left(EC_G - (EC_G)_s\right) \tag{5.1}$$

The stable wave height is given by $H_s = \gamma h$, where γ is of order 0.4. Figure 8 shows a comparison this model with data, using $K = 0.15$. This model has been used in a variety of wave models, such as REF/DIF.

For spectral wave breaking, the distribution of breaking waves has to be considered. Battjes and Janssen (1978) truncated the Rayleigh wave height distribution at the breaking wave height and utilized a turbulent bore model to dissipate wave energy. A turbulent bore (hydraulic jump) dissipates energy as

$$\epsilon_d = \frac{1}{4}\rho g \frac{(h_2 - h_1)^3}{h_1 h_2} q \tag{5.2}$$

where h_1 and h_2 are the depth before and after the bore, and q is the discharge per unit area. Relating the water level difference to the wave height and introducing $q = Ch/L$ for a periodic bore, this dissipation is modified to

$$\epsilon_d = \frac{1}{8\pi}\rho g \sigma \frac{(BH)^3}{h} \tag{5.3}$$

B is a breaker coefficient of O(1), resulting from $(h_2 - h_1) = BH$. The last step is the introduction of the probability distribution function for the breaking wave height, $p_b(H)$, (Thornton and Guza, 1983) and integrating over all wave heights.

$$\epsilon_d = \frac{1}{8\pi}\rho g B^3 \int_0^\infty H^3 p_b(H)\, dH \tag{5.4}$$

Utilizing field data, Thornton and Guza devised two different forms of the breaking wave probability distribution. The first was a simple model that permits analytic solutions for wave height and the second provided a better fit to the data. They are

$$\text{Simple Model: } \epsilon_d = \frac{3}{32\sqrt{\pi}}\rho g \frac{B^3\sigma}{\gamma^4 h^5} H_{rms}^7 \tag{5.5}$$

$$\text{Complete Model: } \epsilon_d = \frac{3}{32\sqrt{\pi}}\rho g \frac{B^3\sigma}{\gamma^2 h^3} H_{rms}^5 \left[1 - \frac{1}{(1 + (H_{rms}/\gamma h)^2)^{5/2}}\right] \tag{5.6}$$

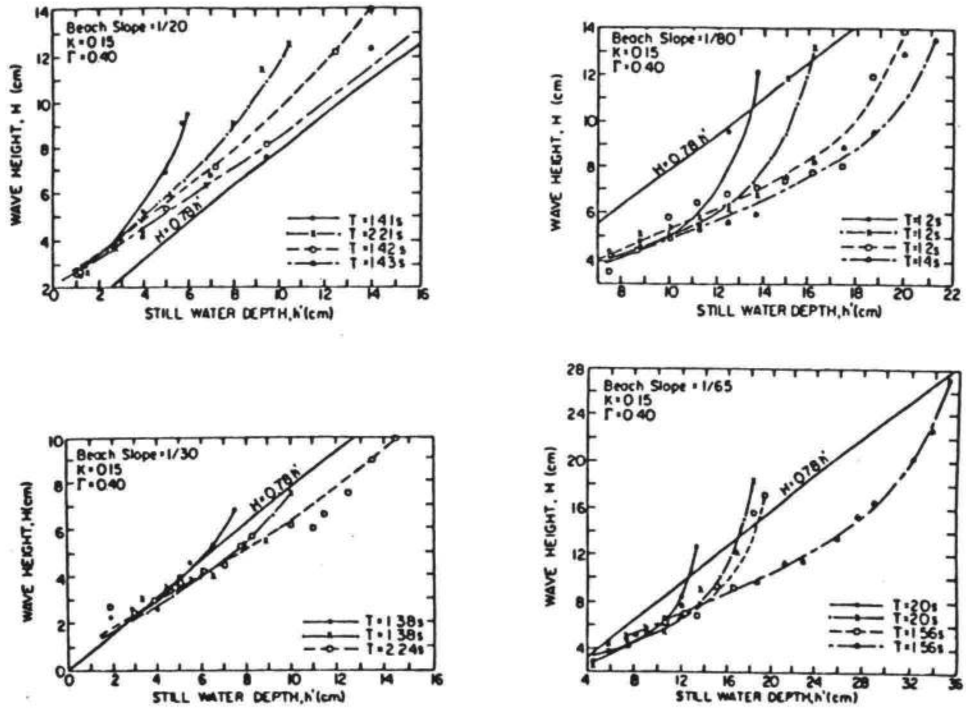

Figure 8: Comparison of Dally *et al.*, (1985) to Wave Data of Horikawa and Kuo (1966). Figure Also Shows the Spilling Breaker Approximation as Solid Line.

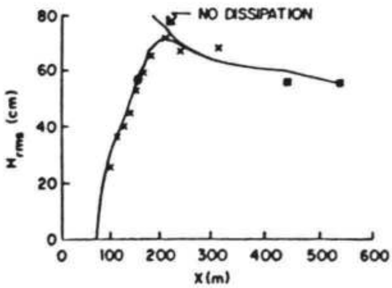

Figure 9: H_{rms} versus Distance Across the Surf Zone, from Thornton and Guza (1983)

The parameter γ is the empirical relationship between the rms wave height and the depth, $H_{rms} = \gamma h$, and γ is about 0.42.

For this model, the energy loss equation (3.1) is written in terms of the root-mean-square wave height,

$$\frac{\partial EC_g}{\partial x} = \frac{\partial \left(\frac{1}{8}\rho g H_{rms}^2 C_g\right)}{\partial x} = -\epsilon_d \tag{5.7}$$

Figure 9 shows the shoaling and then the wave height decrease due to breaking across a real beach using a model based on the complete breaking model. This shows that the rms wave height is predicted very well by the model.

6 Spectral Models for Shoaling and Refraction

In the previous portion of this chapter, a single wave train was discussed. However, in a realistic sea state, the water surface can be decomposed into a large number of wave trains with different frequencies and directions. The sea state is then described by a spectrum. Studies of wave fields (independent of direction) have lead to a variety of frequency spectra, $S(\sigma)$; for example, the Pierson-Moscowitz, Bretschneider, JONSWAP, and Mitsuyasu spectra. As direction resolving capabilies have improved, we now use directional spectra, which have the form, $S(\sigma, \theta)$. Most often, $S(\sigma, \theta)$ is separated as $S(\sigma)D(\sigma, \theta)$, where $D(\sigma, \theta)$ is the directional distribution of the waves.

For shoaling of spectra from offshore to the shoreline, two different methods have been used. The first is to consider the shoaling of the spectra directly using the wave energy or the wave action equation and the second has been to use the Boussinesq equations.

For simplified bathymetry, shoaling and refraction of spectra is relatively straightforward. LeMéhaute and Wang (1982) show that the spectrum is shoaled over straight and parallel contours by

$$S(\sigma, \theta) = \frac{kC_{g0}}{k_0 C_g} S_0 \left(\sigma, \sin^{-1}\left[\frac{k}{k_0}\sin\theta\right]\right) \tag{6.1}$$

Freilich, Elgar and Guza (1990) show that this simple model does reasonable well (within 30%) for shoaling between 10 m and 4 m; however, wave-wave interactions due to wave nonlinearity which are neglected in this model can be important. In Figure 10, the measured offshore and inshore directional spectra are shown as contour plots. In Figure 11, the results given by the linear transformation model (6.1) are shown.

Karlson (1969) was the first to develop a finite difference model for the refraction of spectra

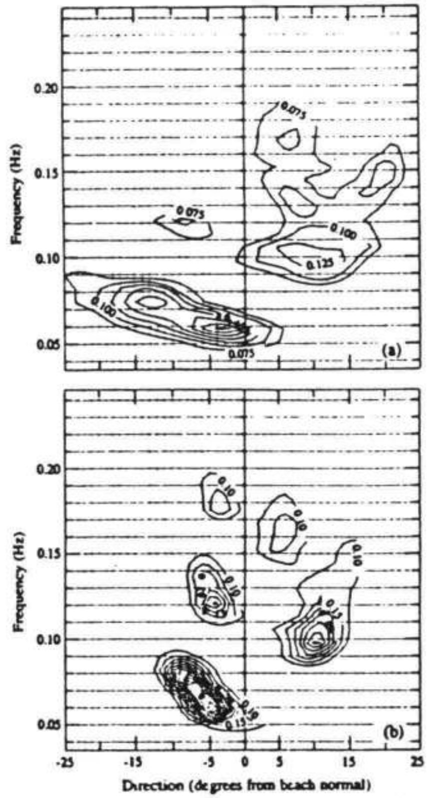

Figure 10: Measured Mean Frequency-directional Spectra at a) 10 m and b) at 4 m depth, from Freilich, Guza, and Elgar (1990)

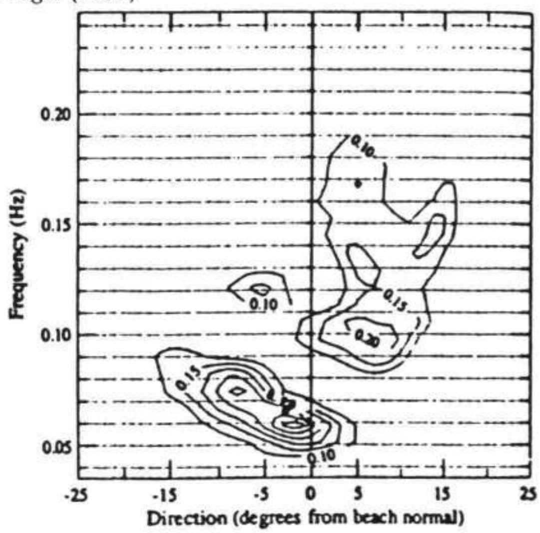

Figure 11: Predicted Mean Frequency-directional Spectra at 4 m depth, Using the Linear Transformation Model, from Freilich, Guza, and Elgar (1990)

from offshore to onshore. Due to the prevalence of currents, most models now include the refraction of the spectrum by currents as well as the bathymetry. Sakai, Koseki and Iwagaki (1983) and Hirosue and Sakai (1986) solve the wave action equation using finite differences. The total derivative of the wave action, defined here as $A(x, y, \theta, t)$ is

$$\frac{\partial A}{\partial t} + \frac{\partial A v_x}{\partial x} + \frac{\partial A v_y}{\partial y} + \frac{\partial A v_\theta}{\partial \theta} = 0 \tag{6.2}$$

where the velocity components v_x, v_y are composed of the mean current components plus the group velocity of the waves:

$$v_x = U + C_g \cos\theta \tag{6.3}$$

$$v_y = V + C_g \sin\theta \tag{6.4}$$

A new term in this expression is derivative with respect to θ, which accounts for changes in wave direction. Assuming steady state conditions, v_θ is found by carrying out the derivatives in the irrotationality condition (2.11) to find $\partial\theta/\partial t$ (Brink-Kjaer, Christoffersen, and Jonsson (1984).

$$v_\theta = -\frac{1}{k}\frac{\partial \sigma}{\partial h}\frac{\partial h}{\partial n} - \frac{\vec{k}}{k}\frac{\partial \vec{U}}{\partial n} \tag{6.5}$$

where the normal (to the wave direction) derivative operator is

$$\frac{\partial}{\partial n} = (\sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}) \tag{6.6}$$

For computational purposes, Hirosue and Sakai (1986) used the wave action at the offshore boundary in terms of 18 discrete directions and 19 frequency bins and all wave action was assumed to propagate in the onshore direction. The values of the wave action on the next onshore grid line is computed by solving finite difference approximations to (6.2) for each direction and each frequency range.

Booij, Holthuijsen, and Herbers (1985) and Holthuijsen and Booij (1990) use the wave action equation method, but in a simplified form by integration of the steady state wave action equation (6.2) over frequency. Two wave action moments are defined:

$$m_n(\theta) = \int_0^\infty \sigma^n A(\sigma, \theta) d\sigma \text{ for } n = 0, 1 \tag{6.7}$$

(Note that a mean frequency is defined by $\sigma_0 = m_1/m_0$.) Two differential equations are obtained by integrating the wave action equation for with weights of $\sigma^n, n = 0, 1$. Using finite difference representations, these equations, including additional terms for wind generation, bottom friction and wave blocking, which is the stopping of waves by opposing currents, are solved. This is the HISWA model.

Collins (1972) and Abernethy and Gilbert (1975), for the case of no currents, and Mathiesen (1984), Brink-Kjaer (1984), and Yamaguchi and Hatada (1990), with currents, use the fact that the wave action is conserved along wave rays to develop backward ray tracing models, which involve computing rays for numerous pairs of frequency and direction from a given point to the offshore boundary, where boundary values of the wave spectrum are given. By summing over all pairs of frequencies and directions, the local wave spectrum is determined at the point of interest. This is very labor intensive; for a calculation of the refraction due to a current eddy, Mathiesen (1987) used 1450 rays per point!

Parabolic spectral shoaling models, involving the superposition of linear solutions to the mild slope equation of many different frequencies, have been carried out by Panchang, Wei, Pearce and Briggs (1990) for comparison to laboratory wave data. A linear parabolic model was run for many directions and frequencies. The amplitude of a wave within a frequency/direction bin was given by $\sqrt{2E(\sigma)D(\theta)\Delta\sigma\Delta\theta}$. The results of the many model runs were summed at a given location for the significant wave height,

$$H_s^2 = \sum_{i=1}^{n} H_i^2 \tag{6.8}$$

where n is the number of model runs used (up to 615). They found good agreement with the laboratory data. Another method was used by O'Reilly and Guza (1991), who computed the response at different locations to unit amplitude waves of different directions for a given frequency. This provides the impulse response function for that direction/frequency. Once the impulse response function is known, then any given offshore directional spectrum can be converted to a shallow water directional spectrum, including the effects of diffraction (if the nonlinearities in the wave field may be neglected). More work needs to be carried out in the use of parabolic models for spectral calculations.

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8 Symbols

A	=	wave number component; amplitude of potential
B	=	wave number component: breaking coefficient
C	=	wave celerity (speed)
C_g	=	group velocity
H	=	wave height
K	=	soil permeability
K_s	=	shoaling factor
L	=	wave length
S	=	wave phase function; spectrum
T	=	wave period
U	=	mean current component in x direction
V	=	mean current component in y direction
a	=	wave amplitude
f	=	friction factor
g	=	acceleration of gravity
h	=	water depth
k	=	wave number
n	=	C_g/C
p	=	probability
s	=	ray distance
u_b	=	bottom velocity
w	=	dissipation term (Booij)
x	=	horizontal coordinate direction
y	=	horizontal coordinate direction
z	=	vertical coordinate direction
β	=	ray spacing
ϵ_d	=	energy dissipation
η	=	water surface displacement
γ	=	breaking index
ν	=	fluid viscosity
ω	=	absolute wave frequency (fixed observer)
ϕ	=	velocity potential
ρ	=	fluid density
σ	=	angular frequency
θ	=	wave direction