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Wave propagation simulation in a linear viscoacoustic medium

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SUMMARY

A new approach for viscoacoustic wave propagation is developed. The Boltzmann's superposition principle based on the general standard linear solid rheology is implemented in the equation of motion by the introduction of memory variables. This approach replaces the conventional convolutional rheological relation, and thus the complete time history of the material is no longer required, and the equations of motion become a coupled first-order linear system in time. The propagation in time is done by a direct expansion of the evolution operator by a Chebycheff polynomial series. The resulting method is highly accurate and effects such as the numerical dispersion often encountered in time-stepping methods are avoided. The numerical algorithm is tested in the problem of wave propagation in a homogeneous viscoacoustic medium. For this purpose the 1-D and 2-D viscoacoustic analytical solutions were derived using the correspondence principle.

Key words: Attenuation, dispersion, viscoacoustic, wave-propagation simulation

1 INTRODUCTION

Seismic forward modelling has been advancing considerably in recent years (Kosloff & Baysal 1982; Alford *et al.* 1974; Gazdag 1981). With the progress in new methods and new computer technology, it has become possible to solve the governing wave equations with a high degree of precision. Most of the progress, however, has been in increasing the dimensionality and frequency resolution (Alford *et al.* 1974), while using a simple rheological description. An exception has been the incorporation of the elastic rheology to replace the acoustic assumption (Kosloff *et al.* 1984; Virieux 1986; Blake *et al.* 1982). However, an accurate description of wave propagation requires a rheology which accounts for additional factors like anelasticity, anisotropy, scattering from heterogenieties, and propagation through a porous medium (Borcherdt 1982; Payton 1983; Biot 1956a,b).

This paper concentrates on the issue of attenuation of seismic waves. However, the subject of wave attenuation has practical value in many other fields such as, for example, ocean acoustics, non-destructive material testing (Burk & Weiss 1979; Szilard 1982) and polymer physics (Ferry 1970). The object is to introduce a modelling method which includes attenuation, which is correct physically and also fits experimental data. In particular, in seismics it is important that the material rheology gives causal behaviour and approximately a constant Q factor in the seismic frequency band (Liu *et al.* 1976), although the theory must also deal with any type of complex modulus function, no matter the frequency range. In addition to attenuating the events on the time section, wave propagation simulation which includes these physical effects will have dispersion which in turn will affect the shape of the observed arrivals.

Liu *et al.* (1976) showed that a viscoelastic rheology with multiple relaxation mechanisms gives a framework that can explain experimental observation of wave propagation through the earth, and Earth-type materials. In particular they showed that, with a suitable choice of material parameters, constant Q values can be obtained, and a dispersion relation which qualitatively explains differences in seismic-wave velocities in different frequency ranges. A wave propagating in a real material induces a non-instantaneous deformation, but not all of the energy can be recovered, as is the case with a purely elastic solid. The energy that is not dissipated also is delivered in a finite time. This relaxation time may be a consequence of many processes such as interstitial atom relaxation, grain boundary relaxation, thermoelasticity, diffusional motion of dislocations and point defects, etc. The general standard linear solid rheology explains these processes very well. Some of them can be modelled with one mechanism and others using a spectrum of relaxation mechanisms.

Use of rheologies which are simpler to program, such as the attenuation mechanism used for absorbing boundary conditions (Cerjan *et al.* 1985; Kosloff & Kosloff 1986; Kummer & Behle 1984; Levander 1985; Lysmer & Kuhlmeyer 1969), or the Maxwell and Kelvin–Voigt models

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(e.g. Blake *et al.* 1982), result in modelling algorithms which can give attenuation, but fail to produce realistic amplitudes.

However, for direct methods in the time-domain the convolutional kernel contained in the stress–strain constitutive relation (Boltzmann's superposition principle) is difficult to implement in the equations of motion. This problem was addressed previously by Day & Minster (1984), where they used an approach based on Padé approximants to transform the convolution integral into a convergent sequence of constant-coefficient differential operators of increasing order, which is equivalent to the rheology of multiple relaxation mechanisms mentioned above. Then, for each mechanism a first-order differential equation was obtained, which together with the scalar equation of motion, were solved by a finite-difference scheme.

The central problem in the modelling of the viscoelastic behaviour is the correct description of the memory kernel. Difficulties of this kind have been found in the physics and chemistry of dense media where a similar memory kernel arises. A solution to this problem has been proposed by Adelman (1980), who showed that the memory kernel can be successfully approximated by a chain of linearly coupled ghost atoms whose frequencies correspond to moments of the memory kernel spectrum. In this work, a similar approach which uses ghost degrees of freedom is applied to reduce the integro-differential equation to solvable coupled differential equations.

This work is concerned with viscoacoustic wave propagation. Extension of the algorithm to viscoelastic rheology will be carried out in a future work. The first section briefly derives the basic equations of a viscoacoustic medium. The derived equations include the well known convolutional relation between pressure and volumetric strain. In the next section we calculate the complex modulus, quality factor and phase velocity which characterize the viscoacoustic solid. Subsequently, the basic equations are reformulated with an introduction of memory variables which circumvent the convolutional relation. In the following sections, a time-integration technique based on the work of Tal Ezer (1986) is introduced. The numerical algorithm is then tested against the problem of wave propagation in a homogeneous viscoacoustic medium. The 1-D and 2-D viscoacoustic solutions used for the comparisons are obtained by using the correspondence principle.

2 BASIC EQUATION OF THE VISCOACOUSTIC MOTION

The description of wave propagation is based on momentum conservation, combined with the relations which describe the rheology of the medium. The standard linear solid rheology was adopted as a starting point for describing the response function. A detailed description of the standard linear solid response function and its use in explaining the rheology of the Earth is given in Fung (1965), Liu *et al.* (1976) and Hudson (1980).

For the n-dimensional solid the equations of momentum conservation are given by

$$\partial\sigma_{ji}/\partial x_j = \rho\ddot{\mathbf{u}}_i + \mathbf{f}_i \qquad i, j = 1, \ldots, n, \tag{1}$$

where x_j denote a set of Cartesian coordinates, $\mathbf{u}_i(x_k, t)$ is the displacement vector, $\sigma_{ji}(x_k, t)$ is the stress tensor, $\rho(x_k, t)$ is the density, and $\mathbf{f}_i(x_k, t)$ are the body forces. In equation (1), as in the remainder of this work, a dot above a variable denotes time differentiation. In addition, the convention in which the repetition of indices implies summation is used.

In a general acoustic medium the components of the stress tensor can be written as

$$\sigma_{ij} = -p\delta_{ij}, \tag{2}$$

where $p(x_k, t)$ is the pressure field acting on the medium. Equation (1) then becomes

$$-\partial p/\partial x_i = \rho\ddot{\mathbf{u}}_i + \mathbf{f}_i. \tag{3}$$

Dividing both sides of (3) by the density and taking the divergence yields

$$-\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right) = \ddot{e} + s, \tag{4}$$

where

$$e = \partial u_i/\partial x_i = e_{ii} \tag{5}$$

is the trace of the strain tensor, or the dilatation, and

$$s = \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}f_i\right). \tag{6}$$

The stress–strain relation for a generalized standard linear solid with many relaxation mechanisms for the viscoacoustic case is given by (Christensen 1982, p. 14),

$$\sum_{k=0}^{m} c_k \frac{d^k p}{dt^k} = \sum_{k=0}^{m} d_k \frac{d^k e}{dt^k} \qquad m \in N, \tag{7}$$

where d^k/dt^k denote kth order time derivative, and c_k and d_k are coefficients related to the material properties of the medium, subjected to the following constraints on the initial conditions

$$\sum_{r=k}^{m} c_r p^{r-k}(0) = \sum_{r=k}^{m} d_r e^{r-k}(0),$$

with $p^{r-k}(0)$ and $e^{r-k}(0)$ indicating the $(r-k)$th order derivative of the pressure and dilatation evaluated at $t = 0$. Alternatively, by solving equation (7) with Laplace transform methods, the pressure field can be expressed explicitly by

$$p(t) = -M_R \int_{-\infty}^{t} \dot{e}(\tau)\left[1 - \sum_{l=1}^{L}\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\exp\left(-\frac{(t-\tau)}{\tau_{\sigma l}}\right)\right] d\tau, \tag{8}$$

where $\tau_{\sigma l}(x_k)$ and $\tau_{\epsilon l}(x_k)$ denote material relaxation times for the lth mechanism, L is the number of relaxation mechanisms and $M_R(x_k)$ is the relaxed modulus of the medium (Liu *et al.* 1976). Equation (8) is the formulation of the Boltzmann's superposition principle, such that the current pressure is determined by the superposition of the responses at previous times.

Equations (4) and (8) fully describe the deformation of the viscoacoustic medium, and in principle could be a basis for a numerical solution algorithm. However, the convolution integrals in (8) pose difficulties because they require a

knowledge of the full strain history, unlike elastic relations which involve only current values of variables. Moreover, the relation (7) is equally difficult to apply. Consequently, in the next sections the rheological relations are reformulated to yield a more convenient description.

3 COMPLEX MODULUS, DISPERSION RELATION, QUALITY FACTOR AND PHASE VELOCITY FOR THE VISCOACOUSTIC SOLID

Equation (8) can be expressed in terms of a convolution as

$$-p(t) = \dot{e}(t)^* H(t)\psi(t) = e(t)^* \frac{d}{dt}[H(t)\psi(t)], \tag{9}$$

where

$$\psi(t) = \mathrm{M}_R\left[1 - \sum_{l=1}^{L}\left(1 - \frac{\tau_{\epsilon_l}}{\tau_{\sigma_l}}\right)e^{t/\tau_{\sigma_l}}\right] \tag{10}$$

is the relaxation function of the medium, and $H(t)$ is the Heaviside function.

$$-\bar{p}(\omega) = \bar{e}(\omega)\mathbf{F}\left\{\frac{d}{dt}[H(t)\psi(t)]\right\}, \tag{11}$$

where the operator **F** performs the time Fourier transform.

From equation (11) we identify the complex modulus of the medium $\mathrm{M}_C(\omega)$,

$$\mathrm{M}_C(\omega) = \mathbf{F}\left\{\frac{d}{dt}[H(t)\psi(t)]\right\}. \tag{12}$$

Differentiating and using equation (10) we obtain

$$\mathrm{M}_C(\omega) = \mathbf{F}\left\{\delta(t)\psi(t) + H(t)\sum_{l=1}^{L}\phi_l(0)e^{-t/\tau_{\sigma_l}}\right\}, \tag{13}$$

where $\delta(t)$ is the Dirac's function, and $\phi_l(t)$ is defined as the response function corresponding to the l-mechanism:

$$\phi_l(t) = \frac{\mathrm{M}_R}{\tau_{\sigma_l}}\left(1 - \frac{\tau_{\epsilon_l}}{\tau_{\sigma_l}}\right)e^{t/\tau_{\sigma_l}}. \tag{14}$$

Taking the Fourier transform in (13), we find that

$$\mathrm{M}_C(\omega) = \psi(0) + \sum_{l=1}^{L}\phi_l(0)\int_0^\infty e^{-[i\omega + (1/\tau_{\sigma_l})]t}\,dt. \tag{15}$$

After integration, the complex modulus can be expressed in a compact form as

$$\mathrm{M}_C(\omega) = \mathrm{M}_R\left(1 - L + \sum_{l=1}^{L}\frac{1 + i\omega\tau_{\epsilon_l}}{1 + i\omega\tau_{\sigma_l}}\right). \tag{16}$$

As $\omega \to \infty$, $\mathrm{M}_C(\omega) \to \mathrm{M}_u$, the unrelaxed modulus defined by

$$\mathrm{M}_u = \mathrm{M}_R\left[1 - \sum_{l=1}^{L}\left(1 - \frac{\tau_{\epsilon_l}}{\tau_{\sigma_l}}\right)\right], \tag{17}$$

and when $\omega \to 0$, $\mathrm{M}_C(\omega) \to \mathrm{M}_R$, the relaxed modulus.

Real materials behave elastically at both very low and very high frequencies. The relaxation function (10), which is based on the general standard linear solid rheology, describes correctly this behaviour (Liu *et al.* 1976). In this work we choose the acoustic behaviour in the low-frequency limit. For a standard linear solid mechanical model, the acoustic limit is reached when the dashpot is eliminated, implying $\tau_{\epsilon_l} \to 0$ and $\tau_{\sigma_l} \to 0$ (Ben-Menahem & Singh 1981, p. 856). This is equivalent to $\omega \to 0$, as it can be seen from (16), hence the relaxed and acoustic modulus coincide. In practice, however, we do not need to restrict the representation of real materials to mechanical models, so we choose the acoustic or 'non-dispersive' behaviour in the high-frequency limit (see Ben-Menahem & Singh 1981, p. 873). In conclusion, when τ_{σ_l} and $\tau_{\epsilon_l} \to 0$ or $\tau_{\sigma_l} = \tau_{\epsilon_l}$ in (16), the complex bulk modulus equals the relaxed bulk modulus and the acoustic case is obtained.

We define the complex velocity $v(\omega)$ as

$$v(\omega) = \left[\frac{M_C(\omega)}{\rho}\right]^{1/2} = \frac{\omega}{k}, \tag{18}$$

where ω or k can be complex. Relating equation (16) to (18) we obtain the dispersion relation:

$$\frac{\omega^2}{\omega_0^2} = 1 - L + \sum_{l=1}^{L}\frac{1 + i\omega\tau_{\epsilon_l}}{1 + i\omega\tau_{\sigma_l}}, \tag{19}$$

with $\omega_0 = c_a k$ and $c_a = (\mathrm{M}_R/\rho)^{1/2}$, the acoustic velocity.

In order to obtain the temporal quality factor $Q_t(\omega)$ and the phase velocity $c(\omega)$, we solve (19) numerically for the complex frequency $\omega_C = \omega + i\omega_I$. The value of ω_C depends on all the material constants ρ, M_R, τ_{ϵ_l}, τ_{σ_l}, $l = 1, \ldots, L$ as well as on the real wave number k. Then we define (Pilant 1979, p. 327),

$$Q_t(\omega) = \omega/2\omega_I \tag{20}$$

and the phase velocity,

$$c(\omega) = \omega/k. \tag{21}$$

4 INTRODUCTION OF THE MEMORY VARIABLES

In this section we show that by the introduction of memory variables the convolutional integral in (8) can be avoided. Integration of equation (8) by parts yields:

$$p(t) = -e(t)\mathrm{M}_R\left[1 - \sum_{l=1}^{L}\left(1 - \frac{\tau_{\epsilon_l}}{\tau_{\sigma_l}}\right)\right] - \sum_{l=1}^{L}\int_{-\infty}^{t} e(\tau)\phi_l(t-\tau)\,d\tau, \tag{22}$$

where $\phi_l(t)$ is given by equation (14). At this stage it is important to note that equation (22) is also derived for physical processes other than the standard linear solid. In that case the kernel ϕ_l is replaced by a different expression (Adelman 1980). The function ϕ_l obeys the differential equation

$$\dot{\phi}_l(t) = -\phi_l(t)/\tau_{\sigma_l}. \tag{23}$$

We now define L memory variables e_{1_l} by

$$e_{1_l}(t) = \int_{-\infty}^{t} e(\tau)\phi_l(t-\tau)\,d\tau \qquad l = 1, \ldots, L. \tag{24}$$

Because of equation (23) we have

$$\dot{e}_{1_l}(t) = \frac{e_{1_l}(t)}{\tau_{\sigma_l}} + \phi_l(0)e(t). \tag{25}$$

The stress–strain relation (22) can now be expressed as

$$p(t) = -\left(\mathrm{M}_u e(t) + \sum_{l=1}^{L} e_{1_l}\right), \tag{26}$$

where M_u is the unrelaxed modulus (17). Equations (4), (25) and (26) fully describe the response of the solid and will be the basis for a numerical solution algorithm described in the following section.

5 NUMERICAL ALGORITHM

In order to solve the problem numerically the fomulae governing wave propagation need to be recast as a coupled first-order system of ordinary differential equations in time. A combination of (4) and (26) gives

$$\ddot{e} = \mathbf{D}\left(\mathrm{M}_u e + \sum_{l=1}^{L} e_{1_l}\right) - s, \tag{27}$$

where $\mathbf{D}$ is a spatial operator defined by

$$\mathbf{D} = \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial}{\partial x_i}\right). \tag{28}$$

Then, using (25) and (27), the system to be solved can be written in compact notation as:

$$\dot{\mathbf{E}} = \mathbf{M}\mathbf{E} + \mathbf{S}, \tag{29}$$

where $\mathbf{M}$ is a spatial operator matrix of dimension $L+2$, given by

$$\mathbf{M} = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \mathbf{D}\mathrm{M}_u & 0 & \mathbf{D} & \mathbf{D} & \cdots & \mathbf{D} \\ \phi_1(0) & 0 & \dfrac{-1}{\tau_{\sigma_1}} & 0 & \cdots & 0 \\ \phi_2(0) & 0 & 0 & \dfrac{-1}{\tau_{\sigma_2}} & \cdots & 0 \\ & \cdot & \cdot & \cdot & \cdots & \cdot \\ & \cdot & \cdot & \cdot & \cdots & \cdot \\ & \cdot & \cdot & \cdot & \cdots & \cdot \\ \phi_L(0) & 0 & 0 & 0 & \cdots & \dfrac{-1}{\tau_{\sigma_L}} \end{vmatrix}, \tag{30}$$

with

$$\mathbf{E}^T = [e, \dot{e}, e_{1_1}, e_{1_2}, \ldots, e_{1_L}], \tag{31}$$

and the source vector given by

$$\mathbf{S}^T = [0, s, 0, 0, \ldots, 0]. \tag{32}$$

For the spatial derivative terms in equation (29) we use the Fourier method (Kosloff & Baysal 1982), which consists of a spatial discretization and calculation of spatial derivatives using the fast Fourier transform (FFT). Considering a 3-D medium with N_x, N_y and N_z denoting the respective number of grid points in the x, y and z directions, the system (29) becomes a coupled system of $L \cdot N \equiv L \cdot N_x \cdot N_y \cdot N_z$ ordinary differential equations in the unknown variables at the grid points. Equation (29) then can be expressed as

$$\dot{\mathbf{E}}_N = \mathbf{M}_N \mathbf{E}_N + \mathbf{S}_N, \tag{33}$$

subject to the initial conditions

$$\mathbf{E}_N(0) = \mathbf{E}_N^0, \tag{34}$$

where $\mathbf{E}_N$, $\mathbf{M}_N$ and $\mathbf{S}_N$ are the discrete representations of $\mathbf{E}$, $\mathbf{M}$ and $\mathbf{S}$, respectively. The solution to (33) subject to (34) is formally given by

$$\mathbf{E}_N = e^{\mathbf{M}_N t}\mathbf{E}_N^0 + \int_0^t e^{\mathbf{M}_N \tau}\mathbf{S}_N(t-\tau)\, d\tau. \tag{35}$$

We consider a separable source term $\mathbf{S}_N(t) = \mathbf{A}_N h(t)$, where $\mathbf{A}_N$ is the spatial distribution of the source, usually chosen to be a delta or a highly localized function. The function $h(t)$ is the source time history. Considering zero initial condition and replacing the source term, equation (35) becomes

$$\mathbf{E}_N = \left[\int_0^t e^{\mathbf{M}_N \tau} h(t-\tau)\, d\tau\right]\mathbf{A}_N. \tag{36}$$

To solve this equation we use a new time integration technique based on the well-known Chebycheff expansion of the function e^z (Tal-Ezer, 1986),

$$e^z \approx H_K(z) = \sum_{k=0}^{K} C_k J_k(\tau R) Q_k\left[\frac{z}{\tau R}\right], \tag{37}$$

where $|z| \leq tR$ and z lies close to the imaginary axis. $C_0 = 1$ and $C_k = 2$ for $k \geq 1$. J_k is the Bessel function of order k and Q_k are the modified Chebycheff polynomials which satisfy the recurrence relation

$$Q_{k+1}(q) = 2qQ_k(q) + Q_{k-1}(q), \tag{38}$$

with

$$Q_0 = 1 \quad \text{and} \quad Q_1 = q. \tag{39}$$

Replacing $\tau\mathbf{M}_N$ for z in (37), equation (36) becomes

$$\mathbf{E}_N(t) \approx \mathbf{E}_N^K = \sum_{k=0}^{K} C_k b_k(tR) Q_k\left[\frac{\mathbf{M}_N}{R}\right]\mathbf{A}_N, \tag{40}$$

where the coefficients b_k are given by

$$b_k = \int_0^t J_k(\tau R) h(t-\tau)\, d\tau. \tag{41}$$

In Tal-Ezer's (1986) article it was shown that to ensure convergence, the parameter R must be chosen larger than the region spanned by the eigenvalues of $\mathbf{M}_N$, which should be in the vicinity of the imaginary axis. It is known (Abramowitz & Stegun (1972) that $J_k(\theta)$ converges to zero exponentially when k increases beyond θ, so we must take the cut-off K to be greater than tR to find a solution. Actually, the value of K can be determined by finding the range in which the coefficients b_k differ greatly from zero.

As the terms in the expansion (40) decay exponentially for k greater than tR, machine accuracy can be obtained. Therefore stability problems, as with differencing in time, do not exist with this algorithm.

The starting values for the recursion are given by

$$Q_0(\mathbf{M}_N/R)\mathbf{A}_N = \mathbf{A}_N, \quad \text{and} \quad Q_1(\mathbf{M}_N/R)\mathbf{A}_N = (\mathbf{M}_N/R)\mathbf{A}_N, \tag{42}$$

according to equation (39). Additional terms are generated by (38) with $\mathbf{M}_N/R$ replacing q, while $\mathbf{E}_N^K$ is summed by

(40). The final value of $\mathbf{E}_N^K$ is obtained after summing over K terms.

The solution can be propagated in time again by considering $\mathbf{E}_N^K(t_0)$ as an initial condition, by means of the homogeneous solution in equation (35). It yields

$$\mathbf{E}_N(t) \approx \mathbf{E}_N^K = \sum_{k=0}^{K} C_k J_k(tR) Q_k\left[\frac{\mathbf{M}_N}{R}\right]\mathbf{E}_N(t_0), \tag{43}$$

where t_0 is the size of the first time-increment which should be larger than the duration of the source time history. The calculation of time histories at a given point of the material does not require significant additional computational effort since the terms $Q_k(\mathbf{M}_N/R)\mathbf{E}_N$ are calculated in any case; only additional sets of Bessel functions need to be generated. The intermediate results are calculated according to

$$\mathbf{E}_N(t') \approx \mathbf{E}_N^K = \sum_{k=0}^{K} C_k J_k(t'R) Q_k\left[\frac{\mathbf{M}_N}{R}\right]\mathbf{E}_N(t_0), \tag{44}$$

for $t' < t$.

It is shown in Appendix A that the matrix $\mathbf{M}_N$ has eigenvalues corresponding to propagating waves close to the imaginary axis, but also has eigenvalues corresponding to non propagating modes, located away from this axis, near the negative real axis. As mentioned before, in order to use the algorithm all the eigenvalues must be close to the imaginary axis. However, a slight modification of the integration technique can restore convergence. Multiplying equation (35) by $e^{\gamma \mathbf{I} t}$, where γ is constant and $\mathbf{I}$ is the identity matrix, gives

$$e^{\gamma \mathbf{I} t}\mathbf{E}_N = e^{(\mathbf{M}_N+\gamma\mathbf{I})t} e^{\mathbf{M}_N t}\mathbf{E}_N^0 + \int_0^t e^{(\mathbf{M}_N+\gamma\mathbf{I})\tau}\mathbf{S}'_N(t-\tau)\,d\tau, \tag{45}$$

where

$$\mathbf{S}'_N(t) = e^{\gamma \mathbf{I} t}\mathbf{S}_N(t). \tag{46}$$

The solution of the problem is the same as before, but instead of expanding the matrix $\mathbf{M}_N$, we expand $(\mathbf{M}_N + \gamma\mathbf{I})$. Afterwards, the results are multipled by the diagonal matrix $e^{-\gamma \mathbf{I} t}$. This modification causes a shift of value γ in the eigenvalues of the operator.

The parameter R should be chosen so that the region of convergence includes the eigenvalues of the matrix $(\mathbf{M}_N + \gamma\mathbf{I})$. From Appendix A we have that the largest real eigenvalue of the evolution operator is approximately $\Lambda = -1/\min(\tau_{\sigma l})$, $l = 1, \ldots, L$. We have found that the algorithm converges reasonably well by choosing

$$R = \tfrac{3}{2}(\Lambda^2 + R_I^2)^{1/2}, \tag{47}$$

and

$$\gamma = \Lambda/2, \tag{48}$$

where R_I is the imaginary part of the largest eigenvalue corresponding to the propagating modes ($R_I = 630.15$/s in Appendix A). The numerical results presented in the next section show that these choices yield the required spectral convergence.

6 COMPARISON WITH ANALYTIC RESULTS

In this section we present numerical results to illustrate the resolution properties of the time integration method and to compare viscoacoustic and acoustic wavefields.

6.1 1-D wave propagation

In order to illustrate the accuracy of the time integration technique we consider the initial value problem in a 1-D viscoacoustic medium. To obtain the viscoacoustic analytical solution we first solve the acoustic problem and then apply the correspondence principle. The acoustic wave equation with constant density is

$$\ddot{e} - c_a^2 \frac{\partial^2 e}{\partial x^2} = 0, \tag{49}$$

where c_a is the acoustic velocity. The solution of equation (49) under the initial condition $e(x, 0) = g(x)$ is (Pilant 1979, p. 40),

$$e(x, t) = \tfrac{1}{2}[g(x + c_a t) + g(x - c_a t)]. \tag{50}$$

Assuming

$$g(x) = e^{-\eta K_0^2 x^2}\cos(\epsilon \pi K_0 x), \tag{51}$$

with K_0 the cut-off wave number, and ϵ and η constants, the time Fourier transform of equation (50) is

$$\tilde{e}(x, \omega, c_a) = \pi\left(\frac{\pi}{\eta}\right)^{1/2}\frac{1}{k_0 c_a}\cos\left(\omega\frac{x}{c_a}\right) \times\left[\exp\left[-\frac{\pi^2}{\eta}\left(\frac{\epsilon}{2} - \frac{\omega}{k_0 c_a}\right)^2\right] + \exp\left[-\frac{\pi^2}{\eta}\left(\frac{\epsilon}{2} + \frac{\omega}{k_0 c_a}\right)^2\right]\right], \tag{52}$$

where $k_0 = 2\pi K_0$.

Quoting Bland (1960, p. 96), the correspondence principle states: 'The solution for a dynamic problem for a viscoelastic material can be obtained from the solution for the corresponding problem for an elastic solid by applying the one-sided Fourier transform to the elastic solution, replacing the elastic constants by the corresponding viscoelastic complex moduli and finally inverting the transform.' In this case we replace the acoustic velocity in equation (52) by the complex velocity defined by equation (18). The viscoacoustic solution is then

$$\tilde{e}_v[x, \omega] = \tilde{e}[x, \omega, v(\omega)]. \tag{53}$$

The solution in the time domain $e_v(x, t)$ is obtained by performing an inverse Fast Fourier transform on equation (53).

The relaxed modulus of the medium is chosen $\mathbf{M}_R = 8$ GPa, and the density, $\rho = 2000\ \mathrm{kg\,m^{-3}}$, which give an acoustic velocity, $c_a = 2000\ \mathrm{m\,s^{-1}}$. The analytical solution is computed at a distance $x_0 = 400$ m, and at the time level $t_0 = 0.2$s, that in the acoustic case corresponds to the main peak of the signal with amplitude $2e(x_0, t_0) = g(x_0 - c_a t_0) = 1$. The anelasticity is defined by the five sets of relaxation times given in Table 1. Fig. 1(a) shows the temporal quality factor Q_t, and Fig. 1(b) the phase velocity dispersion, both

Table 1. Relaxation times (seconds).

l	$\tau_{\epsilon l}$	$\tau_{\sigma l}$
1	0.3196389	0.3169863
2	0.0850242	0.0842641
3	0.0226019	0.0224143
4	0.0060121	0.0059584
5	0.0016009	0.0015823

Table 2. Resolution properties.

Analytical	K	Numerical	Error
0.7528533138	275	0.7832041480	$<10^{-1}$
	280	0.7589254561	$<10^{-2}$
	285	0.7533112375	$<10^{-3}$
	290	0.7527847257	$<10^{-4}$
	300	0.7528512004	$<10^{-6}$
	320	0.7528533138	$<10^{-10}$

as a function of the frequency. The parameters of $g(x)$ are $K_0 = 1/40\,\mathrm{m}^{-1}$, $\eta = 0.5$ and $\epsilon = 1$. The exact solution is obtained by using double precision arithmetic. It yields

$$2e_v(x_0, t_0) = 0.7528533138.$$

The numerical solution is computed using a number of grid points, $N = 198$ and a grid spacing $DX = 10\,\mathrm{m}$. This sufficient sampling ensures that the error comes solely from the time-integration method. According to the theory, K should be greater than $t_0 R$ to obtain enough spectral accuracy. From equation (47) and Appendix A, where we compute the eigenvalues for the smallest relaxation times, we have that $t_0 R = 266.96$ for this example.

Table 2 illustrates the resolution properties of the scheme. The result for $K = 325$ shows that, while the minimum value required to have an acceptable resolution (two digits) is $K_{\min} = 280$, increasing K by only 14 per cent is enough to match the analytical solution by up to 10 digits.

6.2 2-D wave propagation

The problem involves wave propagation in a 2-D homogeneous viscoacoustic medium. The calculations use a 132×132 grid with a grid spacing $DX = DZ = 20\,\mathrm{m}$. The motion is initiated by a point force which is applied at the centre of the grid. The time history of the source is given by equation (B9) with $\eta = 0.5$, $\epsilon = 1$, $t_0 = 0.06\,\mathrm{s}$ and $f_0 = 50\,\mathrm{Hz}$. The relaxed modulus, denisty, and relaxation times (Table 1) of the 1-D problem are also used here. They give an almost constant temporal quality factor $Q_t \approx 100$ (see Fig. 1a), a typical value in the exploration seismic band.

The analytical solution of the problem is obtained in Appendix B, where we calculate the 2-D viscoacoustic Green's function by using the correspondence principle. Fig. 2(a–c) compares numerical and analytical time histories at stations located at distances of 200, 500 and 800 m, respectively. As the figure shows, the comparison between numerical and analytical solutions is excellent.

Figure 3(a–c) compares time histories at the same stations, between viscoacoustic and purely acoustic forward-modelling algorithms. As expected, at large distances the discrepancy between the solutions becomes pronounced. At the distance of 800 m (Fig. 3c), it can be appreciated clearly that the viscoacoustic pulse arrives earlier than the acoustic one. This is because the phase velocity in the viscoacoustic solid is greater than the acoustic or relaxed velocity for the whole frequency range (see Fig. 1b).

In virtue of the complete accuracy both in time and space for band-limited functions, it can be concluded that numerical dispersion is not present in this numerical algorithm. This is very important in anelastic wave propagation where the numerical dispersion can be taken as physical dispersion.

7 CONCLUSIONS

The paper has dealt with viscoacoustic wave propagation. The model used to simulate this propagation is based on the

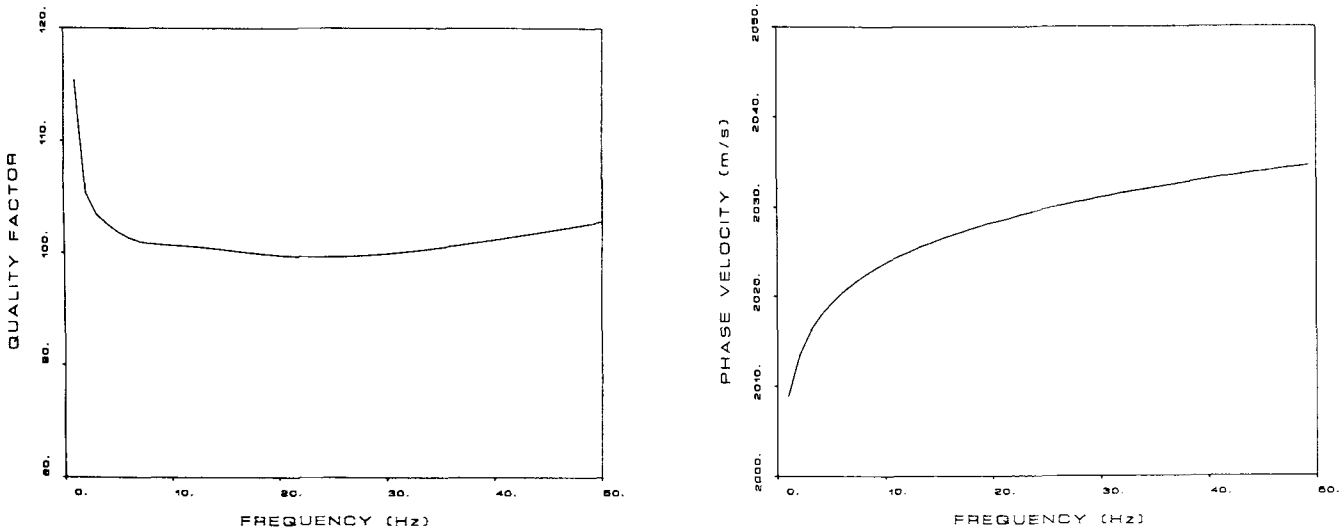

Figure 1. (a) Temporal quality factor versus frequency. (b) Phase velocity versus frequency. The medium is defined by a relaxed modulus $M_R = 8\,\mathrm{GPa}$, a density $\rho = 2000\,\mathrm{kg\,m^{-3}}$, and the five sets of relaxation times given in Table 1.

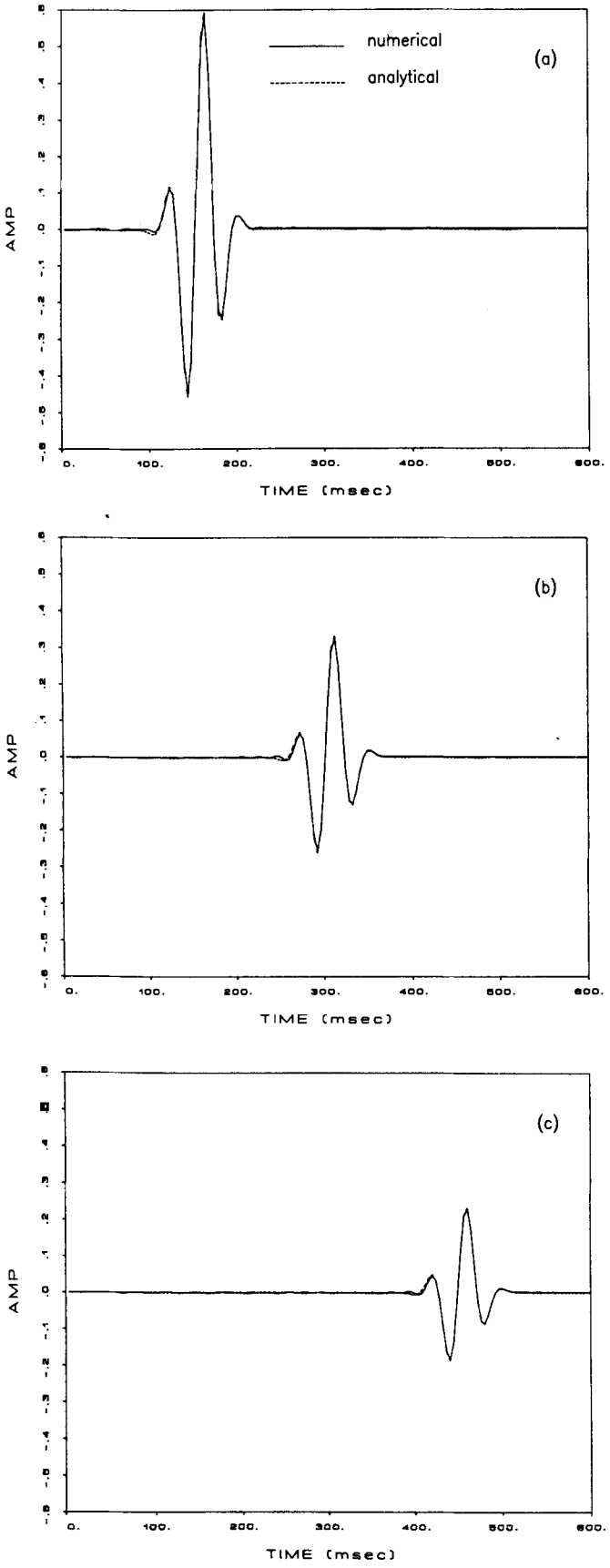

Figure 2. Theoretical and numerical time histories at three stations located at distances of (a) 200 m, (b) 500 m and (c) 800 m, respectively, from the source position.

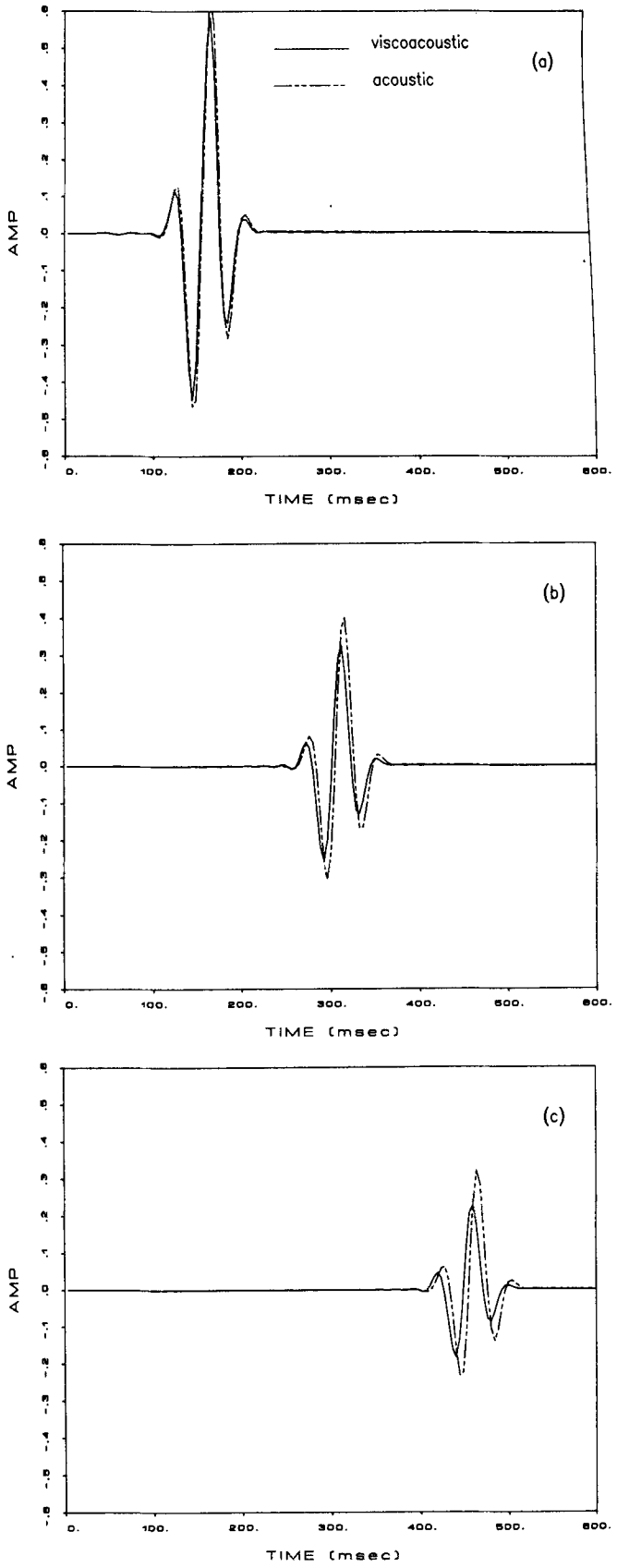

Figure 3. Time history comparison between the viscoacoustic and acoustic forward-modelling algorithms at three stations located at distances of (a) 200 m, (b) 500 m and (c) 800 m, respectively, from the source position.

linear solid material rheology. This rheology is the most general within the class of linear constitutive relations. In the case of wave propagation the model is able to fit experimental data, in particular the constant Q and linear dispersion relation which has often been observed in the laboratory (Wuenschel 1965), and field experiments (Liu *et al.* 1976). Moreover non-constant Q values can also fit the theory.

For the study of viscoacoustic wave propagation a new formulation of the material rheology was introduced. This formulation is based on the introduction of memory variables, and is equivalent to the convolutional or differential formulations which are known in the literature. However, the memory variable approach seems more suitable for numerical calculations because it does require knowledge of the complete history of the material or the solution of high-order differential equations.

The numerical implementation in this study was based on the approach introduced by Tal-Ezer (1986), in which the evolution operator is expanded in a Chebycheff polynomial series. Because this approach gives very accurate time integration the problem of numerical dispersion is therefore avoided. This is very important in viscoacoustic propagation where numerical dispersion could be confused with the real physical dispersion. The accuracy of the numerical simulations was verified in the comparisons with theoretical solutions obtained with the correspondence principle. Moreover, comparison between the viscoacoustic and acoustic wavefields illustrates the differences in a typical Earth material. The theory in this work can be extended to viscoelastic wave propagation. Possible applications include realistic simulations of wave propagation through the Earth and also in the field of material science.

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APPENDIX A

Eigenvalues of the propagation matrix M

To simplify the procedure, one mechanism with relaxation times τ_σ and τ_ϵ is considered. A plane wave solution to equation (29) is assumed of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \tag{A1}$$

where ω is the complex angular frequency, $\mathbf{k}$ is the real wave-number vector and $\mathbf{x}$ is the position vector. Considering constant material properties and a zero source

term, a substitution of (A1) in (29) yields

$$i\omega\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{M_u k^2}{\rho} & 0 & -\dfrac{k^2}{\rho} \\ \phi(0) & 0 & \dfrac{-1}{\tau_\sigma} \end{bmatrix} \mathbf{E}, \tag{A2}$$

where

$$\phi(0) = \frac{M_R}{\tau_\sigma}\left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right). \tag{A3}$$

Equation (A2) is an eigenvalue equation for the eigenvalues $\lambda = i\omega$.

The characteristic equation of (A2) is given by

$$\lambda^3 + \frac{\lambda^2}{\tau_\sigma} + \lambda\frac{\tau_\epsilon}{\tau_\sigma}\omega_0^2 + \frac{\omega_0^2}{\tau_\sigma} = 0, \tag{A4}$$

with $\omega_0 = c_a k$, and c_a the relaxed wave velocity defined by

$$c_a = \left[\frac{M_R}{\rho}\right]^{1/2}.$$

For instance, for $\tau_\epsilon = 0.0016009\,\text{s}$, $\tau_\sigma = 0.0015823\,\text{s}$, $c_a = 2000\,\text{m s}^{-1}$ and $k = \pi/DX$, $DX = 10\,\text{m}$, the Nyquist (maximum) wave number, the eigenvalues are $\lambda_1 = (-1.85 + 630.15i)/\text{s}$, $\lambda_2 = \lambda_1^*$, and $\lambda_3 = -628.30i/\text{s}$. The first two eigenvalues correspond to propagating waves, and the third corresponds to a static mode that attenuates with time. When τ_ϵ is close in value to $\tau_\sigma (Q_t \gg 1)$, it holds that $\lambda_3 = -1/\tau_\sigma$ approximately.

APPENDIX B

Calculation of the 2-D Green's function using the correspondence principle

To find the Green's function $G(x, z, x_0, z_0, t)$ for a 2-D acoustic medium, we solve the inhomogeneous scalar wave equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial z^2} - \frac{1}{c_a^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(x - x_0)\delta(z - z_0)\delta(t), \tag{B1}$$

where x, z are the observer coordinates, x_0, z_0 are the source coordinates, t is the time and c_a is the acoustic-wave velocity of the medium. The solution to equation (B1) is given by (Morse & Feshbach 1953),

$$G(x, z, x_0, z_0, t) = 2H\left(t - \frac{r}{c_a}\right)\left(t^2 - \frac{r^2}{c_a^2}\right)^{-1/2}, \tag{B2}$$

with

$$r = [(x - x_0)^2 + (z - z_0)]^{1/2}, \tag{B3}$$

and H is the Heaviside function. Taking Fourier transform with respect to time in equation (B2) gives

$$\tilde{G}(x, z, x_0, z_0, \omega) = 2\int_{r/c_a}^{\infty}\left(t^2 - \frac{r^2}{c_a^2}\right)^{-1/2} e^{-i\omega t}\,dt, \tag{B4}$$

with ω real. By making a change of variable $\tau = c_a(t/r)$, equation (B4) reads

$$\tilde{G}(x, z, x_0, z_0, \omega) = 2\int_1^{\infty}(t/c_a)\tau\,d\tau. \tag{B5}$$

This expression is merely the integral representation of the zero-order Hankel function of the second kind (Morse & Feshbach 1953):

$$\tilde{G}(x, z, x_0, z_0, \omega) = -i\pi H_0^{(2)}\left(\frac{\omega}{c_a}r\right). \tag{B6}$$

Using the correspondence principle, we replace the real wave number in equation (B6) by the complex wave number:

$$\frac{\omega}{c_a} \to \frac{\omega}{v(\omega)}, \tag{B7}$$

where $v(\omega)$ is the complex velocity of the medium given by equation (18).

With the wave number defined by equation (B7) we express the 2-D viscoacoustic Green's function in ω-space as

$$\tilde{G}_v(x, z, x_0, z_0, \omega) = -i\pi H_0^{(2)}\left(\frac{\omega}{v(\omega)}r\right), \tag{B8a}$$

for $\omega \geqslant 0$ and

$$\tilde{G}_v(x, z, x_0, z_0, \omega) = \tilde{G}_v^*(x, z, x_0, z_0, -\omega). \tag{B8b}$$

Equation (B8) ensures that the Fourier transform of the viscoacoustic Green's function is real.

Because the Hankel function in equation (B8) has a singularity at $\omega = 0$, we multiply $\tilde{G}_v(x, z, x_0, z_0, \omega)$ by the Fourier transform of a shifted zero-phase Ricker wavelet defined by

$$F(t) = e^{-\eta f_0^2(t - t_0)^2}\cos\epsilon\pi f_0(t - t_0), \tag{B9}$$

where $f_0 = 2\pi\Omega_0$ is the cut-off frequency, t_0 is the time shift and η and ϵ are constants. The Fourier transform of $F(t)$ is

$$\tilde{F}(\omega) = \pi\left(\frac{\pi}{\eta}\right)^{1/2}\frac{1}{\Omega_0}e^{i\omega t_0} \times\left[\exp -\frac{\pi^2}{\eta}\left(\frac{\epsilon}{2} - \frac{\omega}{\Omega_0}\right)^2 + \exp -\frac{\pi^2}{\eta}\left(\frac{\epsilon}{2} + \frac{\omega}{\Omega_0}\right)^2\right]. \tag{B10}$$

Multiplying the transformed Green's function (B8) by $\tilde{F}(\omega)$ we obtain

$$\tilde{\Phi}(r, \omega) = \begin{cases} \tilde{G}_v(x, z, x_0, z_0, \omega)\tilde{F}(\omega) & \omega \neq 0, \\ 0 & \omega = 0, \end{cases} \tag{B11}$$

avoiding with this definition, the singularity (actually this is an approximation because strictly is $\tilde{F}(0) = 0$). Because the inverse Fourier transform of $\tilde{\Phi}(r, \omega)$ has no exact analytical expression, we invert it numerically by using the discrete fast Fourier transform, obtaining the time function $\Phi(r, t)$.