WAVE RADIATION AND DIFFRACTION BY A SUBMERGED SPHERE IN A CHANNEL

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[Received 17 April 1997. Revises 18 August 1997 and 7 January 1998]

SUMMARY

Wave radiation and diffraction by a submerged sphere is analysed based on the linearized velocity potential theory. The solution is obtained based on the multipole expansion which satisfies the boundary conditions on the free surface, at infinity and on side walls. The coefficients in the expansion are obtained by imposing the body surface boundary condition. Results are obtained for the added mass, the damping coefficient, the exciting force and the drift force. Some of the results have been compared with those obtained from alternative equations and they are all in excellent agreement. The effect of the trapped mode is also discussed. It is found that this effect becomes evident when the sphere is close to the free surface and the channel is very narrow.

1. Introduction

THERE ARE a considerable number of publications concerning the hydrodynamic problem of a body in a channel. Eatock Taylor and Hung (**1**), for example, calculated the drift force on an articulated cylinder in a regular wave. Their work was based on the assumption that the problem associated with a cylinder in the channel can be approximated by an array of cylinders in the open sea, which is formed by treating the side walls as mirrors. They found from the numerical results that the force will converge as the number of cylinders used increases. The method they adopted can deal with not only an array of cylinders formed by the mirror images but also other formations of the multi-cylinder problem. But the technique is not particularly efficient for the channel flow problem. The reason for this is that the flow around each cylinder is different because of the 'end effect'. The unknowns will increase linearly with the number of cylinders.

The end effect will disappear only when an infinite number of cylinders is used. When the cylinder is placed at the centre of the channel the unknowns corresponding to each cylinder become identical and they can be taken out of the summation over the cylinders. These unknowns can then be obtained by imposing the boundary condition on one cylinder. This procedure is adopted by Yeung and Sphaier (**2**). Although for practical computational purposes the number of cylinders has to be truncated at a finite value, the technique allows a large number of cylinders to be used because the number of unknowns does not depend on the number of cylinders. This means that the features associated with the channel can be captured more accurately, such as the spikes of the hydrodynamic forces at resonance.

As pointed out by Linton and Evans (**3**) the above methods are in fact trying to use a group of circular waves to model the waves in a channel. These methods may give satisfactory results for hydrodynamic force on the cylinder and for the flow near the cylinder, but they cannot model the waves in the channel far away from the cylinder. Linton and Evans (**3**) used a different method. They considered a vertical circular cylinder. The solution is written in terms of an infinite series. Each term in the series satisfies the governing equation and all the boundary conditions apart from that on the body surface. In particular, because each term satisfies the boundary condition on the side walls, it produces typical channel waves at infinity rather than circular waves. The technique has been found to be efficient for this type of problem. In particular, it can capture the trapped modes (**4**) effectively. The principle of this technique has been subsequently used in many other publications (**5**, **6**, **7**, **8**).

In this paper, we shall consider wave radiation and diffraction by a submerged sphere in the channel. There is a substantial body of work on the submerged sphere in the open sea. Gray (**9**), Srokosz (**10**) and Wang (**11**) for example considered a sphere in an infinite water depth, while Linton (**12**) and Wu *et al.* (**13**) considered a sphere in finite water depth. Further work on a sphere in the open sea includes the multi-body problem (**14**), a body undergoing large amplitude motion (**15**), wave diffraction and radiation by a sphere at forward speed (**16**, **17**) and a sphere moving in a circular path (**18**). All these publications start by expanding the solution in the spherical system in terms of the Legendre function. Because the free surface boundary condition can be imposed more easily in the Cartesian system while the body surface boundary condition is easier to impose in the spherical system, the method requires some transformation between the two systems. Although this sometimes leads to a quite long analysis, nowadays it has become a standard procedure.

For the channel problem, it becomes less straightforward. In addition to those mentioned above, the condition on the side walls also has to be satisfied. One way to do that is to adopt the technique used by Linton and Evans (**3**). The side wall condition can be satisfied by introducing an additional term in the integral form. Ursell (**19**) has obtained the result for the case of a sphere at the centre of the channel. The equation is in a rather compact and elegant form, but it is problematic from a computational point of view, because the numerical integration has to be performed in a complex plane. Here we adopt a procedure similar to but different from that of Yeung and Sphaier (**2**). We first consider an array of an infinite number of sources formed by the mirror images. The total potential is equal to the sum of that due to each source in the open sea. The infinite series is then transformed into a different one based on the wave component in the channel (**20**). The potential obtained in this way clearly satisfies the boundary condition on the side walls. The multipole expansion (**21**) for the sphere can then be obtained by taking the derivatives with the position of the source. The result has no restriction on the position of the sphere. The calculation can be made by truncating the series at a finite number. But this is different from the truncation in Yeung and Sphaier (**2**). The truncation here specifies the number of wave components in the channel while in Yeung and Sphaier (**2**) it specifies the number of bodies. Therefore the typical wave pattern in the channel flow can be captured accurately here, even in the far field. For this reason, the present work is also similar to that of Linton and Evans (**3**). The difference is that in their case the boundary condition on the side walls is satisfied in an integral form while here the same condition is satisfied in a series form.

The practical relevance of work in this area has been highlighted in many publications (for example, (**3**)). Some important features associated with the channel flow are well understood, such as the effects of the resonance and the trapped mode which corresponds to a free oscillation of finite energy. This work does not seem to offer any new features associated with these effects. What is surprising, however, is that the effect of the trapped mode on the sphere is not as evident as expected. The principal reason for this is due to the fact that the trapped mode is too close to the resonant frequency in many cases. Their difference becomes visible only when the sphere is very close to the free surface and the tank is very narrow. The real significance of this work is that a method similar to, but not the same as, existing ones has been developed to deal with a sphere in a confined fluid domain. The success of the analysis also means that we may be able to consider the sphere in other confined domains. The problem in a circular tank for example is particularly relevant to ship manoeuvring.

Calculations have been made in this paper for various cases. Results are obtained for the added mass and the damping coefficient, the exciting force and the drift force by integrating the pressure over the body surface. It is found that these results satisfy various known mathematical identities and they are in excellent agreement with the results obtained from alternative equations. Some results related to the trapped mode are also provided.

2. Governing equations

We consider the problem of a submerged sphere of radius a in a channel of width $2d$. We define a Cartesian coordinate system $Oxyz$ so that the origin is located on the undisturbed free surface, x points in the longitudinal direction of the channel and z points upwards. We also define a spherical coordinate system (r, θ, β) so the origin is located at the mean position of the centre of the sphere. These two systems are related by the following equations:

$$x = r \sin\theta \cos\beta + x_0, \tag{1a}$$

$$y = r \sin\theta \sin\beta + y_0, \tag{1b}$$

$$z = r \cos\theta + z_0, \tag{1c}$$

where (x_0, y_0, z_0) are the coordinates of the mean position of the centre of the sphere in the Cartesian system.

The total potential Φ in a regular wave of frequency ω may be expanded as

$$\Phi(x, y, z, t) = \mathrm{Re}[\phi(x, y, z)e^{i\omega t}] = \mathrm{Re}[\eta_0(\phi_0 + \phi_7)e^{i\omega t}] + \mathrm{Re}\left[\sum_{j=1}^{6} i\omega\eta_j\phi_j(x, y, z)e^{i\omega t}\right], \tag{2}$$

where ϕ_j ($j = 1, ..., 6$) are radiation potentials corresponding to six degree-of-freedom oscillations of the body; η_j ($j = 1, ..., 6$) are corresponding motion amplitudes; ϕ_0 and ϕ_7 are the potentials due to the incident and diffracted waves respectively; and η_0 is the incoming wave amplitude. Based on the assumptions of the linearized theory, the components of the potential satisfy the following equations:

$$\nabla^2 \phi_j = 0 \tag{3}$$

in the fluid domain;

$$\frac{\partial \phi_j}{\partial_z} - \nu \phi_j = 0 \tag{4}$$

on the free surface S_F or $z = 0$, where $\nu = \omega^2/g$;

$$\frac{\partial_z \phi_j}{\partial n} = n_j, \quad j = 1, ..., 6, \tag{5a}$$

$$\frac{\partial \phi_j}{\partial n} = -\frac{\partial \phi_0}{\partial n}, \quad j = 7, \tag{5b}$$

on the body surface S_0, where

$$(n_1, n_2, n_3) = (n_x, n_y, n_z), \tag{6a}$$

$$(n_4, n_5, n_6) = \mathbf{X} \times \mathbf{n}, \tag{6b}$$

with $\mathbf{X}$ denoting the position vector of a point on S_0 relative to the centre of the body. On the side walls of the channel we have

$$\frac{\partial \phi_j}{\partial y} = 0, \quad y = \pm d. \tag{7}$$

The radiation condition requires the wave to propagate outwards.

3. The multipole expansion

For a single source, the potential can be written as (**15**, **22**)

$$4\pi G = \sum_{s=-\infty}^{\infty} \left[\frac{1}{r_s} - \frac{1}{r'_s} \right] + \frac{2}{d} \sum_{s=0}^{\infty} \varepsilon_s \cos \alpha_s (d - y) \cos \alpha_s (d - y_0) \times \int_L \frac{t e^{t(z+z_0)} \cos[\gamma(x - x_0)]}{(t - \nu)\gamma} dt, \tag{8}$$

where $\varepsilon_s = 1$ when $s = 0$ and $\varepsilon_s = 2$ when $s > 0$, and

$$r_s = [(x - x_0)^2 + (y - 2sd - (-1)^s y_0)^2 + (z - z_0)^2]^{\frac{1}{2}}, \tag{9a}$$

$$r'_s = [(x - x_0)^2 + (y - 2sd - (-1)^s y_0)^2 + (z + z_0)^2]^{\frac{1}{2}}, \tag{9b}$$

$$\gamma = (t^2 - \alpha_s^2)^{\frac{1}{2}}, \tag{9c}$$

$$\alpha_s = s\pi/2d. \tag{9d}$$

The integration route L is from α_s to ∞ and it passes over the singularity at $t = \nu$ when $\nu > \alpha_s$.

Applying the operator

$$(D_\mp)^m \left(\frac{\partial}{\partial z_0}\right)^{n-m} = \left(\frac{\partial}{\partial x_0} \mp i\frac{\partial}{\partial y_0}\right)^m \left(\frac{\partial}{\partial z_0}\right)^{n-m} \tag{10}$$

to equation (8) and using the result provided to the author by Ursell through a private communication:

$$\begin{aligned}\frac{1}{(n-m)!}(D_\mp)^m \left(\frac{\partial}{\partial z_0}\right)^{n-m} \left(\frac{1}{r_s}\right) &= (-1)^m \frac{1}{(n-m)!} \\ &\quad \times \left(\frac{\partial}{\partial x} \mp i(-1)^s \frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z_0}\right)^{n-m} \left(\frac{1}{r_s}\right) \\ &= (-1)^m \exp[\mp(-1)^s im\beta_s] \frac{P_n^m(\cos\theta_s)}{r_s^{n+1}},\end{aligned} \tag{11}$$

where the spherical system (r_s, θ_s, β_s) is similar to (r, θ, β) but with the centre at $(x_0, 2sd + (-1)^s y_0, z_0)$, we have

$$\begin{aligned}(G_\mp)_n^m &= \frac{4\pi(-1)^m}{(n-m)!}(D_\mp)^m \left(\frac{\partial}{\partial z_0}\right)^{n-m} G \\ &= \sum_{s=-\infty}^{\infty} \left\{ \exp[\mp(-1)^s im\beta_s] \frac{P_n^m(\cos\theta_s)}{r_s^{n+1}} \right. \\ &\quad \left. -(-1)^{n-m} \exp[\mp(-1)^s im\beta_s'] \frac{P_n^m(\cos\theta_s')}{r_s'^{n+1}} \right\} \\ &\quad + \frac{2(-1)^m}{d(n-m)!}(D_\mp)^m \sum_{s=0}^{\infty} \varepsilon_s \cos\alpha_s(d-y)\cos\alpha_s(d-y_0) \\ &\quad \times \int_L \frac{t^{n-m+1} e^{t(z+z_0)} \cos[\gamma(x-x_0)]}{(t-\nu)\gamma} dt.\end{aligned} \tag{12}$$

To calculate the derivative in the above equation, we write

$$\begin{aligned}&(D_\mp)^m \cos[\alpha_s(d-y_0)]\cos[\gamma(x-x_0)] \\ &\quad = \tfrac{1}{4}(D_\mp)^m \{\exp[i\alpha_s(d-y_0)] + \exp[-i\alpha_s(d-y_0)]\}\{\exp[i\gamma(x-x_0)] \\ &\qquad + \exp[-i\gamma(x-x_0)]\} \\ &\quad = \tfrac{1}{4}(D_\mp)^m \{\exp[i\alpha_s(d-y_0) + i\gamma(x-x_0)] + \exp[i\alpha_s(d-y_0) - i\gamma(x-x_0)] \\ &\qquad + \exp[-i\alpha_s(d-y_0) + i\gamma(x-x_0)] + \exp[-i\alpha_s(d-y_0) - i\gamma(x-x_0)]\} \\ &\quad = \frac{i^m}{4}\{(-\gamma \pm i\alpha_s)^m [\exp[i\alpha_s(d-y_0) + i\gamma(x-x_0)] \\ &\qquad + (\gamma \pm i\alpha_s)^m \exp[i\alpha_s(d-y_0) - i\gamma(x-x_0)] \\ &\qquad + (-\gamma \mp i\alpha_s)^m \exp[-i\alpha_s(d-y_0) + i\gamma(x-x_0)] \\ &\qquad + (\gamma \mp i\alpha_s)^m \exp[-i\alpha_s(d-y_0) - i\gamma(x-x_0)]\}.\end{aligned} \tag{13}$$

If we define

$$\sin\psi = \alpha_s/t, \quad \cos\psi = \gamma/t, \tag{14}$$

we have

$$(\gamma \pm \alpha_s i)^m = t^m e^{\pm im\psi}. \tag{15}$$

Equation (13) then becomes

$$\begin{aligned}(D_{\mp})^m &\cos[\alpha_s(d-y_0)]\cos[\gamma(x-x_0)]\\ &= \frac{(it)^m}{2}\{(-1)^m e^{i\gamma(x-x_0)}\cos[\alpha_s(d-y_0)\mp m\psi]\\ &\quad + e^{-i\gamma(x-x_0)}\cos[\alpha_s(d-y_0)\pm m\psi]\}.\end{aligned} \tag{16}$$

Substituting this equation into (12), we have

$$\begin{aligned}(G_{\mp})_n^m = \sum_{s=-\infty}^{\infty} &\left\{\exp[\mp(-1)^s im\beta_s]\frac{P_n^m(\cos\theta_s)}{r_s^{n+1}}\right.\\ &\left. -(-1)^{n-m}\exp[\mp(-1)^s im\beta_s']\frac{P_n^m(\cos\theta_s')}{r_s'^{n+1}}\right\}\\ &+\frac{(-i)^m}{d(n-m)!}\sum_{s=0}^{\infty}\varepsilon_s\cos\alpha_s(d-y)\\ &\times\left\{(-1)^m\int_L \frac{t^{n+1}e^{t(z+z_0)+i\gamma(x-x_0)}\cos[\alpha_s(d-y_0)\mp m\psi]}{(t-\nu)\gamma}dt\right.\\ &\left. +\int_L \frac{t^{n+1}e^{t(z+z_0)-i\gamma(x-x_0)}\cos[\alpha_s(d-y_0)\pm m\psi]}{(t-\nu)\gamma}dt\right\}.\end{aligned} \tag{17}$$

If we use (**23**)

$$\exp[\eta r(\cos\theta \mp i\sin\theta\cos(\alpha+\beta))] = \sum_{p=0}^{\infty}\sum_{q=0}^{p}\varepsilon_q(\pm i)^q\frac{(\eta r)^p}{(p+q)!}P_p^q(\cos\theta)\cos q(\alpha+\beta) \tag{18a}$$

and (**24**)

$$\begin{aligned}\exp(\mp im\beta_s)\frac{P_n^m(\cos\theta_s)}{r_s^{n+1}} &= \sum_{p=0}^{\infty}\sum_{q=-p}^{p}(-1)^{p+q}\frac{(n+p-m-q)!}{(n-m)!(p-q)!}\\ &\times\frac{1}{r_{s0}^{n+p+1}}P_{n+p}^{m+q}(\cos\theta_{s0})e^{\mp i(m+q)\beta_{s0}}r^pP_p^{-q}(\cos\theta)e^{\pm iq\beta}, \quad s\neq 0,\end{aligned} \tag{18b}$$

where $(r_{s0}, \theta_{s0}, \beta_{s0})$ are the spherical coordinates of the centre of the sphere in system (r_s, θ_s, β_s), equation (14) becomes

$$
\begin{aligned}
(G_{\mp})_n^m &= \exp(\mp im\beta)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&+ \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty}\sum_{p=0}^{\infty}\sum_{q=-p}^{p}(-1)^{p+q}\frac{(n+p-m-q)!}{(n-m)!(p-q)!}\frac{1}{r_{s0}^{n+p+1}}P_{n+p}^{m+q}(\cos\theta_{s0}) \\
&\times e^{\mp(-1)^s i(m+q)\beta_{s0}}r^p P_p^{-q}(\cos\theta)e^{\pm(-1)^s iq\beta} \\
&- \sum_{s=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{q=-p}^{p}(-1)^{p+q+n+m}\frac{(n+p-m-q)!}{(n-m)!(p-q)!}\frac{1}{r_{s0}'^{n+p+1}}P_{n+p}^{m+q}(\cos\theta_{s0}') \\
&\times e^{\mp(-1)^s i(m+q)\beta_{s0}'}r^p P_p^{-q}(\cos\theta)e^{\pm(-1)^s iq\beta} \\
&+\frac{(-i)^m}{2d(n-m)!}\sum_{s=0}^{\infty}\varepsilon_s\sum_{p=0}^{\infty}\sum_{q=0}^{p}\varepsilon_q\frac{(-i)^q}{(p+q)!} \\
&\times\Bigg\{(-1)^m e^{i\alpha_s(d-y_0)}r^p P_p^q(\cos\theta) \\
&\times\int_L\frac{t^{n+p+1}e^{2tz_0}\cos[\alpha_s(d-y_0)\mp m\psi]\cos q(\psi+\beta)}{(t-\nu)\gamma}dt \\
&+(-1)^m e^{-i\alpha_s(d-y_0)}r^p P_p^q(\cos\theta) \\
&\times\int_L\frac{t^{n+p+1}e^{2tz_0}\cos[\alpha_s(d-y_0)\mp m\psi]\cos q(\psi-\beta)}{(t-\nu)\gamma}dt \\
&+(-1)^q e^{i\alpha_s(d-y_0)}r^p P_p^q(\cos\theta) \\
&\times\int_L\frac{t^{n+p+1}e^{2tz_0}\cos[\alpha_s(d-y_0)\pm m\psi]\cos q(\psi-\beta)}{(t-\nu)\gamma}dt \\
&+(-1)^q e^{-i\alpha_s(d-y_0)}r^p P_p^q(\cos\theta) \\
&\times\int_L\frac{t^{n+p+1}e^{2tz_0}\cos[\alpha_s(d-y_0)\pm m\psi]\cos q(\psi+\beta)}{(t-\nu)\gamma}dt\Bigg\}. \qquad (19)
\end{aligned}
$$

The potential can then be written as

$$
\phi = \sum_{n=1}^{\infty}\sum_{m=0}^{n}a^{n+1}[A_n^m(G_-)_n^m + B_n^m(G_+)_n^m]. \qquad (20)
$$

Since $(G_-)_n^0 = (G_+)_n^0$ in this equation, we may take $B_n^0 = 0$ without loss of generality.

For $y_0 = 0$, Callan (**25**) has also derived the multipole expansion for the sphere in the channel following a different procedure. But no solution is offered in his work.

4. The solution procedure

The coefficients in (20) can be obtained by imposing the boundary condition on the body surface. It is evident that $\phi_j = 0$ when $j = 4, 5, 6$. When $j = 1, 2, 3$, we have on $r = a$

$$\frac{\partial \phi_1}{\partial r} = \sin\theta\cos\beta, \quad \frac{\partial \phi_2}{\partial r} = \sin\theta\sin\beta, \quad \frac{\partial \phi_3}{\partial r} = \cos\theta. \tag{21}$$

Since the incident potential is

$$\begin{aligned}\phi_0 &= \frac{ig}{\omega} e^{\nu z - i\nu x} \\ &= \frac{ig}{\omega} e^{\nu z_0 - i\nu x_0} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \varepsilon_q i^q \frac{(\nu r)^p}{(p+q)!} P_p^q(\cos\theta)\cos q\beta \end{aligned} \tag{22}$$

we have

$$\frac{\partial \phi_7}{\partial r} = -i\omega e^{\nu z_0 - i\nu x_0} \sum_{p=1}^{\infty} \sum_{q=0}^{p} \varepsilon_q i^q \frac{p(\nu a)^{p-1}}{(p+q)!} P_p^q(\cos\theta)\cos q\beta \tag{23}$$

at $r = a$ for the diffraction potential.

Substituting equations (19) and (20) into (21) and (23), we have

$$\begin{aligned}
&-A_n^m \frac{n+1}{a} \\
&+ \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'} A_{n'}^{m'} (-1)^{n+m} H(m_s) \frac{(n'+n-m'+m_s)!\, n a^{n'+n}}{(n'-m')!(n+m)!\, r_{s0}^{n'+n+1}} \\
&\times P_{n'+n}^{m'-m}(\cos\theta_{s0}) e^{-i(m_s'-m)\beta_{s0}} \\
&- \sum_{s=-\infty}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'} A_{n'}^{m'} (-1)^{n+m+n'+m'} H(m_s) \frac{(n'+n-m'+m_s)!\, n a^{n'+n}}{(n'-m')!(n+m)!\, r'^{\,n'+n+1}_{s0}} \\
&\times P_{n'+n}^{m'-m_s}(\cos\theta'_{s0}) e^{-i(m_s'-m)\beta'_{s0}} \\
&+ \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'} B_{n'}^{m'} (-1)^{n+m} J(m_s) \frac{(n'+n-m'-m_s)!\, n a^{n'+n}}{(n'-m')!(n+m)!\, r_{s0}^{n'+n+1}} \\
&\times P_{n'+n}^{m+m_s}(\cos\theta_{s0}) e^{i(m_s'+m)\beta_{s0}} \\
&- \sum_{s=-\infty}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'} B_{n'}^{m'} (-1)^{n+m+n'+m'} J(m_s) \frac{(n'+n-m'-m_s)!\, n a^{n'+n}}{(n'-m')!(n+m)!\, r'^{\,n'+n+1}_{s0}} \\
&\times P_{n'+n}^{m+m_s}(\cos\theta'_{s0}) e^{i(m_s'+m)\beta'_{s0}} \\
&+ \frac{1}{d} \sum_{s=0}^{\infty} \varepsilon_s \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'} A_{n'}^{m'} \frac{(-i)^{m'+m}}{(n'-m')!} \frac{n a^{n+n'}}{(n+m)!} I_1(s, n, n', m, m')
\end{aligned}$$

$$+\frac{1}{d}\sum_{s=0}^{\infty}\varepsilon_s\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}B_{n'}^{m'}\frac{(-i)^{m'+m}}{(n'-m')!}\frac{na^{n+n'}}{(n+m)!}I_2(s,n,n',m,m')=f_j(n,m),\tag{24a}$$

$$\begin{aligned}
&-B_n^m\frac{n+1}{a}\\
&+\sum_{\substack{s=-\infty\\ s\neq 0}}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n+m}J(m_s)\frac{(n'+n-m'-m_s)!}{(n'-m')!(n+m)!}\frac{na^{n'+n}}{r_{s0}^{n'+n+1}}\\
&\quad\times P_{n'+n}^{m'+m_s}(\cos\theta_{s0})e^{-i(m'_s+m)\beta_{s0}}\\
&-\sum_{s=-\infty}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n+m+n'+m'}J(m_s)\frac{(n'+n-m'-m_s)!}{(n'-m')!(n+m)!}\frac{a^{n'+n}}{r_{s0}'^{n'+n+1}}\\
&\quad\times P_{n'+n}^{m'+m_s}(\cos\theta'_{s0})e^{-i(m'_s+m)\beta'_{s0}}\\
&+\sum_{\substack{s=-\infty\\ s\neq 0}}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}B_{n'}^{m'}(-1)^{n+m}H(m_s)\frac{(n'+n-m'+m_s)!}{(n'-m')!(n+m)!}\frac{na^{n'+n}}{r_{s0}^{n'+n+1}}\\
&\quad\times P_{n'+n}^{m'-m_s}(\cos\theta_{s0})e^{i(m'_s-m)\beta_{s0}}\\
&-\sum_{s=-\infty}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}B_{n'}^{m'}(-1)^{n+m+n'+m'}H(m_s)\frac{(n'+n-m'+m_s)!}{(n'-m')!(n+m)!}\frac{a^{n'+n}}{r_{s0}'^{n'+n+1}}\\
&\quad\times P_{n'+n}^{m'-m_s}(\cos\theta'_{s0})e^{i(m'_s-m)\beta'_{s0}}\\
&+\frac{1}{d}\sum_{s=0}^{\infty}\varepsilon_s\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}\frac{i^{m'+m}}{(n'-m')!}\frac{na^{n+n'}}{(n+m)!}I_2(s,n,n',m,m')\\
&+\frac{1}{d}\sum_{s=0}^{\infty}\varepsilon_s\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}B_{n'}^{m'}\frac{i^{m'+m}}{(n'-m')!}\frac{na^{n+n'}}{(n+m)!}I_1(s,n,n',m,m')=g_j(n,m),
\end{aligned}\tag{24b}$$

where

$$m_s=(-1)^s m,\quad m'_s=(-1)m',\tag{25a}$$

$$H(m_s)=\begin{cases}1, & m_s\geqslant 0,\\ (-1)^{m_s}, & m_s<0,\end{cases}\qquad J(m_s)=\begin{cases}1, & m_s\leqslant 0,\\ (-1)^{m_s}, & m_s>0,\end{cases}\tag{25b}$$

$$\begin{aligned}
I_1(s,n,n',m,m')&=\int_L\frac{t^{n'+n+1}e^{2tz_0}}{(t-\nu)\gamma}\\
&\times\{(-1)^{m'}\cos[\alpha_s(d-y_0)-m'\psi]\cos[\alpha_s(d-y_0)-m\psi]\\
&+(-1)^m\cos[\alpha_s(d-y_0)+m'\psi]\cos[\alpha_s(d-y_0)+m\psi]\}dt,
\end{aligned}\tag{25c}$$

$$I_2(s,n,n',m,m')=\int_L\frac{t^{n'+n+1}e^{2tz_0}}{(t-\nu)\gamma}$$

$$
\times\{(-1)^{m'}\cos[\alpha_s(d-y_0)+m'\psi]\cos[\alpha_s(d-y_0)-m\psi]
+(-1)^{m}\cos[\alpha_s(d-y_0)-m'\psi]\cos[\alpha_s(d-y_0)+m\psi]\}dt, \quad (25d)
$$

and

$$
f_1(n,m) = -\tfrac{1}{2}\delta_{1n}\delta_{1m},\quad f_2(n,m)=\frac{1}{2i}\delta_{1n}\delta_{1m},\quad f_3(n,m)=\delta_{1n}\delta_{0m}, \quad (26a)
$$

$$
g_1(n,m) = -\tfrac{1}{2}\delta_{1n}\delta_{1m},\quad g_2(n,m)=-\frac{1}{2i}\delta_{1n}\delta_{1m},\quad g_3(n,m)=0, \quad (26b)
$$

$$
f_7(n,m)=g_7(n,m)=-\omega i^{m+1}e^{\nu z_0-i\nu x_0}\frac{n(\nu a)^{n-1}}{(n+m)!}. \quad (26c)
$$

It should be understood that in these equations $B_n^0 = 0$ and those lines corresponding to $m=0$ in equation (24b) should be deleted.

5. Hydrodynamic forces

Once the solutions of equations (24) have been found, the added mass μ_{lj}, the damping coefficient λ_{lj} and the exciting force F_j can be obtained from the following equations:

$$
\mu_{lj}-i\omega^{-1}\lambda_{lj}=\rho\int_{S_0}\phi_l n_j dS, \quad (27a)
$$

$$
F_j=\rho i\omega\eta_0\int_{S_0}(\phi_7+\phi_0)n_j dS, \quad (27b)
$$

where ρ is the density of the fluid. Following the derivation of Wu (**14**) (noting that A_{nm} and B_{nm} in that paper should be replaced by $A_n^m + B_n^m$ and $-iA_n^m + iB_n^m$ when $m \neq 0$, and replaced by A_n^m and 0 when $m=0$), we have

$$
\mu_{l1}-i\omega^{-1}\lambda_{l1}=\tfrac{4}{3}\rho\pi a^2[3A_1^1(l)+af_l(1,1)+3B_1^1(l)+ag_l(1,1)], \quad (28a)
$$

$$
\mu_{l2}-i\omega^{-1}\lambda_{l2}=\tfrac{4}{3}\rho\pi a^2 i[-3A_1^1(l)-af_l(1,1)+3B_1^1(l)+ag_l(1,1)], \quad (28b)
$$

$$
\mu_{l3}-i\omega^{-1}\lambda_{l3}=-\tfrac{4}{3}\rho\pi a^2[3A_1^0(l)+af_l(1,0)+3B_1^0(l)+ag_l(1,0)], \quad (28c)
$$

$$
F_1=-4\rho i\omega\eta_0\pi a^2[A_1^1(7)+B_1^1(7)], \quad (29a)
$$

$$
F_2=4\rho\omega\eta_0\pi a^2[-A_1^1(7)+B_1^1(7)], \quad (29b)
$$

$$
F_3=4\rho i\omega\eta_0\pi a^2[A_1^0(7)+B_1^0(7)], \quad (29c)
$$

where l in $A_n^m(l)$ and $B_n^m(l)$ indicates that the coefficients correspond to ϕ_l.

It is well known that the exciting force can be obtained from the radiation potential. Following the derivation of Wu (**22**), we have

$$
F_j=-4\rho\omega^2\eta_0\pi a^2 e^{\nu z_0-i\nu x_0}\sum_{n=1}^{\infty}\sum_{m=0}^{n} i^m\frac{(\nu a)^{n-1}}{(n-m)!}[A_n^m(j)+B_n^m(j)]. \quad (30)
$$

The damping coefficients can be obtained from a far-field equation

$$\lambda_{lj} = -\frac{\rho\omega i}{2}\int_{S_\infty}\left(\phi_l\frac{\partial\phi_j^*}{\partial n} - \phi_j^*\frac{\partial\phi_l}{\partial n}\right)dS, \tag{31}$$

where S_∞ are two vertical planes at $x = \pm\infty$ and the asterisk indicates the complex conjugate. The far-field expansion of the potential at $x = \pm\infty$ can be obtained from equation (17). Noting that the integration route L passes over the singularity and using

$$\mathrm{pv}\int_a^\infty \frac{f(k)e^{ikx}}{k-\mu}dk = \pm i\pi e^{i\mu x}f(\mu),\quad \mathrm{pv}\int_a^\infty \frac{f(k)e^{-ikx}}{k-\mu}dk = \mp i\pi e^{-i\mu x}f(\mu),$$
$$x \to \pm\infty,\quad \mu > a, \tag{32}$$

where pv indicates the principal value integration, we have

$$\phi = \frac{2\pi}{d}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(-i)^{m+1}}{(n-m)!}\sum_{s=0}\frac{\varepsilon_s}{\gamma_s}\cos[\alpha_s(d-y)](\nu a)^{n+1}e^{\nu(z+z_0)}$$
$$\times\begin{cases}\{A_n^m\cos[\alpha_s(d-y_0)+m\psi_s] + B_n^m\cos[\alpha_s(d-y_0)-m\psi_s]\}e^{-i\gamma_s(x-x_0)}, & x\to\infty,\\ \{A_n^m\cos[\alpha_s(d-y_0)-m\psi_s] + B_n^m\cos[\alpha_s(d-y_0)+m\psi_s]\}(-1)^m e^{i\gamma_s(x-x_0)}, & x\to-\infty,\end{cases} \tag{33}$$

where

$$\gamma_s = (\nu^2-\alpha_s^2)^{\frac{1}{2}},\qquad \psi_s = \tan^{-1}(\alpha_s/\gamma_s). \tag{34}$$

The upper limit of the summation over s in equation (33) is determined by $\alpha_s < \nu$. Substituting (33) into (31) we have

$$\lambda_{lj} = \rho\omega\nu\frac{4\pi^2a^2e^{2\nu z_0}}{d}\sum_{s=0}\frac{\varepsilon_s}{\gamma_s}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}\frac{(-i)^m i^{m'}(\nu a)^{n+n'}}{(n-m)!(n'-m')!}$$
$$\times\langle\{A_n^m(l)\cos[\alpha_s(d-y_0)+m\psi_s] + B_n^m(l)\cos[\alpha_s(d-y_0)-m\psi_s]\}$$
$$\times\{A_{n'}^{m'*}(j)\cos[\alpha_s(d-y_0)+m'\psi_s] + B_{n'}^{m'*}(j)\cos[\alpha_s(d-y_0)-m'\psi_s]\}$$
$$+(-1)^{m+m'}\{A_n^m(l)\cos[\alpha_s(d-y_0)-m\psi_s] + B_n^m(l)\cos[\alpha_s(d-y_0)+m\psi_s]\}$$
$$\times\{A_{n'}^{m'*}(j)\cos[\alpha_s(d-y_0)-m'\psi_s] + B_{n'}^{m'*}(j)\cos[\alpha_s(d-y_0)+m'\psi_s]\}\rangle. \tag{35}$$

This equation is similar to that derived by Linton and Evans (**4**).

Once the solutions of equations (24) have been found, we can also obtain the

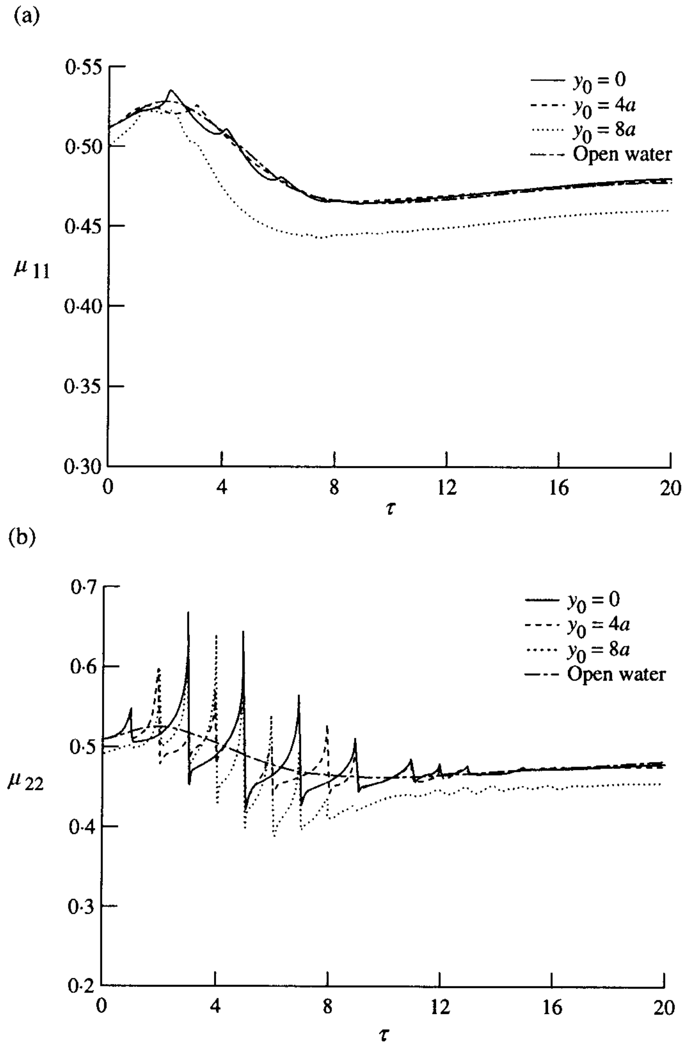


FIG. 1. (a) Surge added mass, (b) sway added mass

(c)

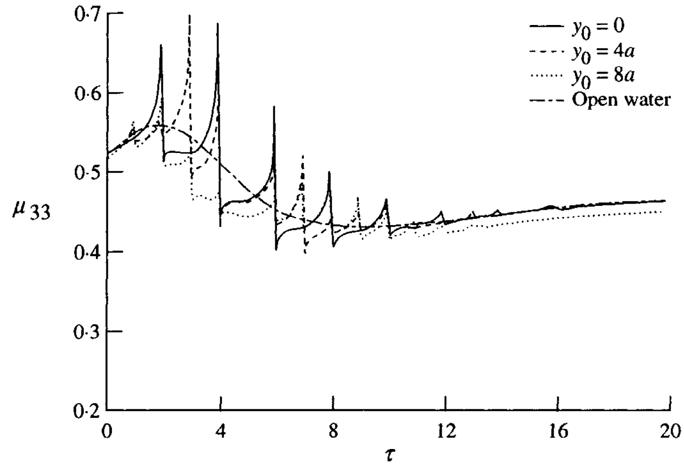


FIG. 1. (c) Heave added mass

steady drift force $\bar{F}_j$. It can be shown (**14**) that

$$\bar{F}_1 = -2\rho\pi \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{1}{\varepsilon_m}\frac{(n+m+2)!}{(n-m)!}\frac{n+2}{n+1}\mathrm{Re}(A_{n+1}^{m+1}A_n^{m*} + B_{n+1}^{m+1}B_n^{m*})$$
$$+2\rho\pi \sum_{n=2}^{\infty}\sum_{m=0}^{n-2} \frac{1}{\varepsilon_m}\frac{(n+m)!}{(n-m-2)!}\frac{n+1}{n}\mathrm{Re}(A_{n-1}^{m+1}A_n^{m*} + B_{n-1}^{m+1}B_n^{m*}), \quad (36a)$$

$$\bar{F}_2 = 2\rho\pi \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{1}{\varepsilon_m}\frac{(n+m+2)!}{(n-m)!}\frac{n+2}{n+1}\mathrm{Re}(-B_{n+1}^{m+1}A_n^{m*} + A_{n+1}^{m+1}B_n^{m*})$$
$$-2\rho\pi \sum_{n=2}^{\infty}\sum_{m=0}^{n-2} \frac{1}{\varepsilon_m}\frac{(n+m)!}{(n-m-2)!}\frac{n+1}{n}\mathrm{Re}(-B_{n-1}^{m+1}A_n^{m*} + A_{n-1}^{m+1}B_n^{m*}), \quad (36b)$$

$$\bar{F}_3 = 4\rho\pi \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{1}{\varepsilon_m}\frac{(n+m+1)!}{(n-m)!}\frac{n+2}{n+1}\mathrm{Re}(A_{n+1}^{m}A_n^{m*} + B_{n+1}^{m}B_n^{m*}), \quad (36c)$$

where

$$A_n^m = \eta_0 A_n^m(7) + i\omega\sum_{j=1}^{3}\eta_j A_n^m(j), \quad B_n^m = \eta_0 B_n^m(7) + i\omega\sum_{j=1}^{3}\eta_j B_n^m(j). \quad (37)$$

The drift force in the x-direction can also be obtained from a far-field equation which can be written as

$$\bar{F}_1 = \frac{\rho}{4}\int_{S_\infty}\left(\phi\frac{\partial^2\phi^*}{\partial n\partial x} - \frac{\partial\phi}{\partial n}\frac{\partial\phi^*}{\partial x}\right)dS, \quad (38)$$

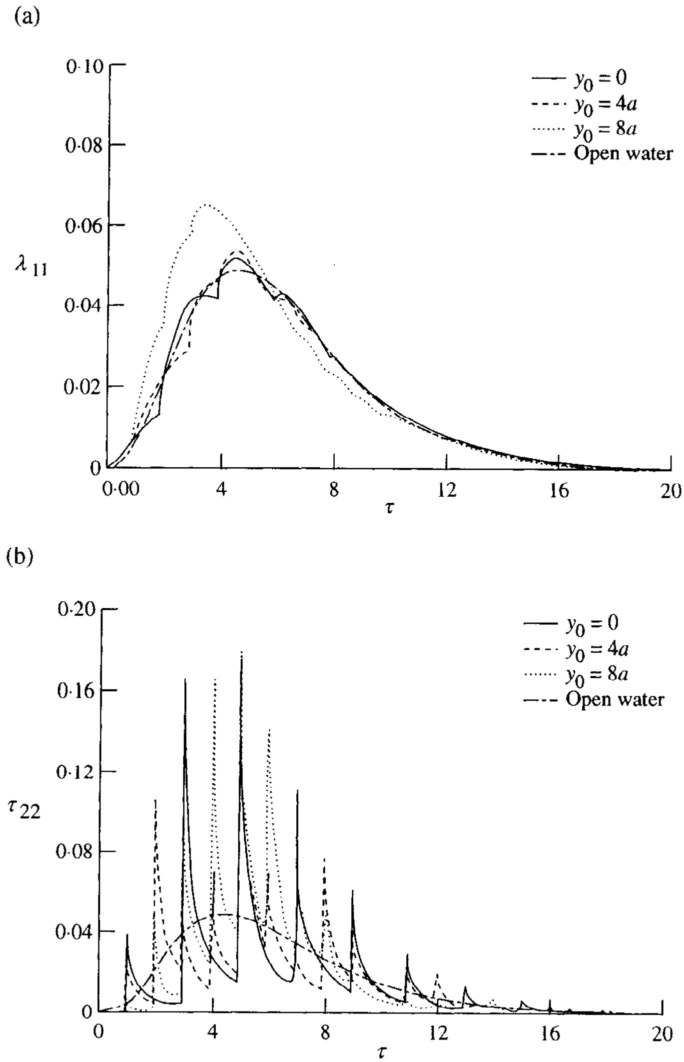


FIG. 2. (a) Surge damping coefficient, (b) sway damping coefficient

(c)

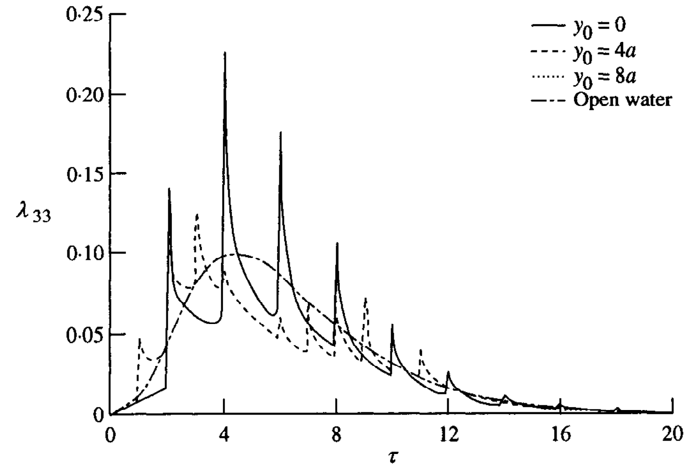


FIG. 2. (c) Heave damping coefficient

where ϕ is given in equation (2). Substituting (22) and (33) into (38), we obtain

$$\bar{F}_1 = 2\rho\omega\pi e^{\nu z_0} a\eta_0 \mathrm{Re}\left(e^{i\nu x_0}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(-i)^m}{(n-m)!}(\nu a)^n(A_n^m + B_n^m)\right)$$

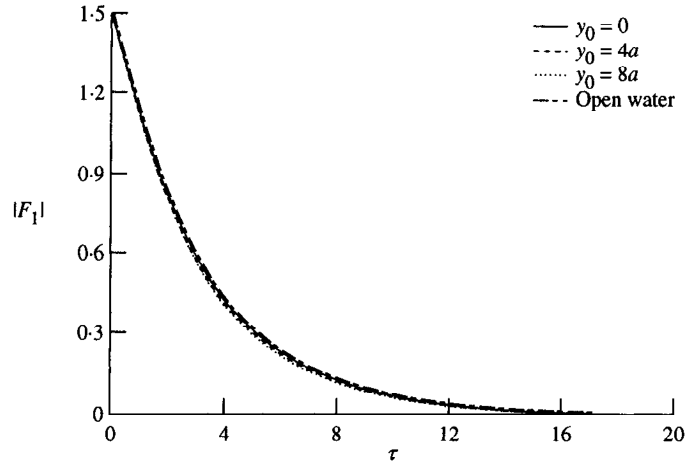
$$-\frac{2\rho\pi^2\nu a^2}{d}e^{2\nu z_0}\sum_{s=0}\varepsilon_s\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}\frac{(-i)^m i^{m'}(\nu a)^{n+n'}}{(n-m)!(n'-m')!}$$
$$\times\langle\{A_n^m\cos[\alpha_s(d-y_0)+m\psi_s]+B_n^m\cos[\alpha_s(d-y_0)-m\psi_s]\}$$
$$\times\{A_{n'}^{m'*}\cos[\alpha_s(d-y_0)+m'\psi_s]+B_{n'}^{m'*}\cos[\alpha_s(d-y_0)-m'\psi_s]\}$$
$$-(-1)^{m+m'}\{A_n^m\cos[\alpha_s(d-y_0)-m\psi_s]+B_n^m\cos[\alpha_s(d-y_0)+m\psi_s]\}$$
$$\times\{A_{n'}^{m'*}\cos[\alpha_s(d-y_0)-m'\psi_s]+B_{n'}^{m'*}\cos[\alpha_s(d-y_0)+m'\psi_s]\}\rangle. \quad (39)$$

where A_n^m and B_n^m are defined in equation (37). Once again it is important to understand that in all these equations (from (28) to (39)), $B_n^0 = 0$. The results below are obtained from equations (28), (29) and (36), while equations (30), (35) and (39) are used for comparison and excellent agreement has been found.

6. Numerical results

We consider a sphere submerged at $z_0 = -2a$ in a channel with $d = 10a$. The calculation has been made by truncating the infinite series in equations (24) at $n = 5$. Further increase of n gives graphically indistinguishable results in the figures given here. All results have been obtained by direct integration of the pressure over the body surface. Alternative equations have been used for comparison and excellent agreement has been found. Figures 1 and 2 give the added mass and the damping

(a)



(b)

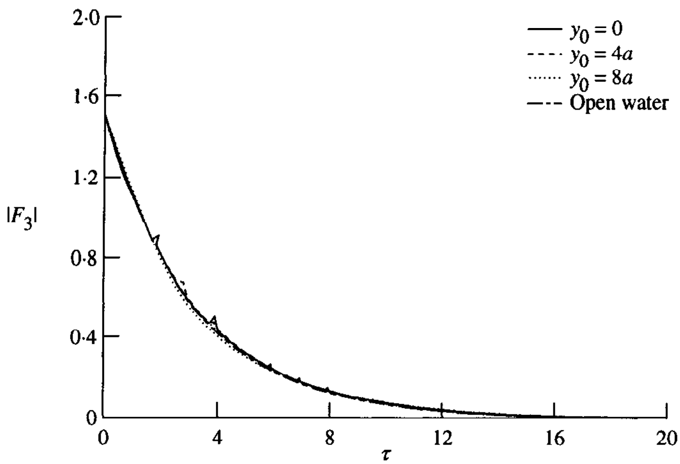


FIG. 3. (a) Surge exciting force, (b) heave exciting force

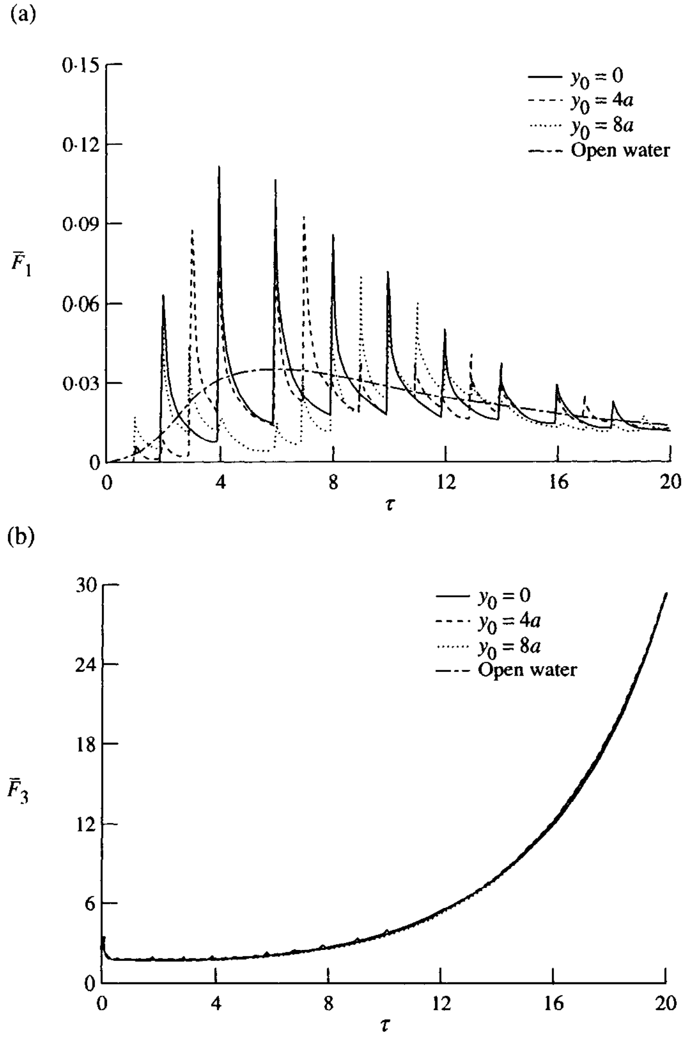


FIG. 4. (a) Surge drift force, (b) heave drift force

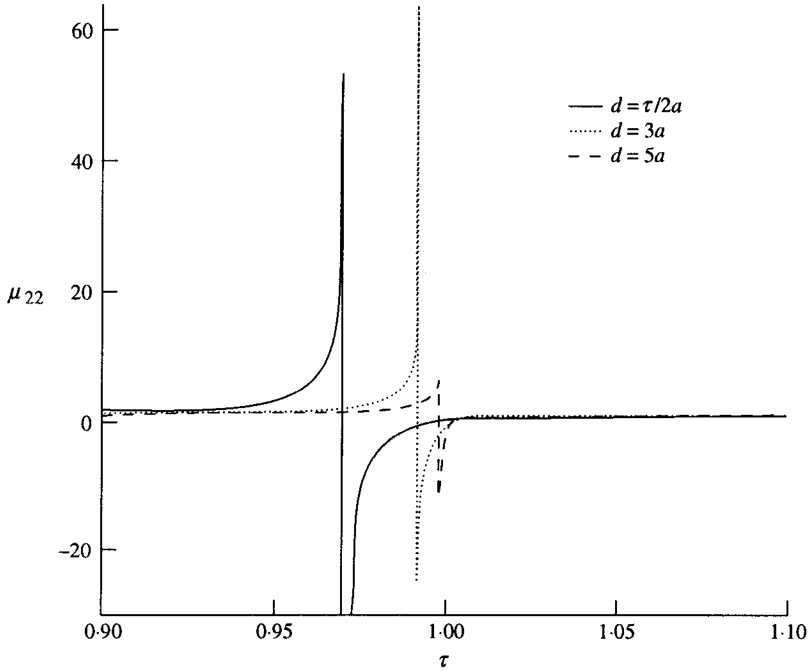


FIG. 5. Sway added mass near the trapped mode

coefficients for the sphere being placed at $y_0 = 0, 4a, 8a$. The added mass has been non-dimensionalized by $\rho 4\pi a^3/3$ and the damping coefficient by $\rho\omega 4\pi a^3/3$. The results have been plotted against τ ($= \nu 2d/\pi$). When τ is an integer n, it corresponds to a natural frequency of the channel. Consequently, the results near these frequencies change sharply. This is similar to the cases mentioned in the Introduction. When $y_0 = 0$, the surge and heave motions become symmetric about $y = 0$ while the sway motion becomes antisymmetric. Correspondingly, the spikes occur only at even modes (τ an even integer) and at odd modes (τ an odd integer). The calculation near the natural frequency is made up to $\tau = n \pm 0{\cdot}05$. The results at $\tau = n - 0{\cdot}05$ and $\tau = n + 0{\cdot}05$ are linked for the purpose of plotting. It does not suggest that the results between $\tau = n - 0{\cdot}05$ and $\tau = n + 0.05$ behave in that way. In fact it should be pointed out that the results at resonant frequencies may not be meaningful, as the linearized theory is not valid at these frequencies.

Figures 3 gives the exciting force non-dimensionalized by $\rho g \nu \eta_0 4\pi a^3/3$. When $y_0 = 0$, the sway exciting force is zero because of symmetry. Although the sway exciting force is not zero when $y_0 = 4a, 8a$, the result has been found to be very small and it is therefore not shown in the figure. Figures 3 shows that the side walls of the channel have little effect on the exciting force. The results in all cases are very close to those obtained in the open sea.

Figures 4 gives the drift force on a stationary sphere. The result has been non-dimensionalized by $\rho g(\nu\eta_0)^2 e^{2\nu z_0} 4\pi a^3/3$. Figure 4a shows that the surge drift force oscillates about the corresponding result for the open sea and changes sharply near the natural frequencies. The side walls, however, have little effect on the heave drift force as shown in Fig. 4b. The sway drift force has been ignored because it is either zero or very small.

One of the remarkable features of the channel problem is that the trapped modes (**26**) may exist. In the above results, however, no visible spikes occur apart from at resonant frequencies of the channel. The missing trapped-mode effect may be due to the fact that the trapped-mode frequency, estimated from the equation derived by McIver (**27**), is too close to the natural frequency of the channel. In fact McIver's equation can be written as

$$\tau = 1/(1 + \lambda^6 e^{-2\lambda f}/4)^{\frac{1}{2}}, \tag{40}$$

where $\lambda = \pi a/d$, $f = -z_0/a$. The equation is derived based on the assumptions that $a \ll |z_0|$, $a \ll d$, and therefore it is an approximation. For the present case, the equation gives $\tau \approx 0{\cdot}9999658$. The effect of the trapped mode may therefore have been overshadowed by the effect due to the natural frequency at $\tau = 1$. We may use equation (40) to estimate the trapped mode in other cases. For a given submergence, when $\lambda = 3/f$ the equation will give the smallest possible trapped mode at

$$\tau = 1/(1 + (3/f)^6 e^{-6}/4)^{\frac{1}{2}}. \tag{41}$$

Thus for the case $f = 2$ considered in the previous figures, equation (41) will give $\tau \approx 0{\cdot}9964892$ which is still very close to the resonant frequency. If we choose $f = 1{\cdot}5$, equation (41) gives $\tau = 0{\cdot}9807509$ and the corresponding channel width becomes $d = \pi a/2$. Figure 5 gives the sway added mass in this case for a sphere on the channel centre-line. It is found that the result changes from $\mu_{22} \approx 53$ at $\tau = 0{\cdot}970$ to $\mu_{22} \approx -458$ at $\tau = 0{\cdot}971$. This suggests that the trapped mode may exist between $\tau = 0{\cdot}970$ and $\tau = 0{\cdot}971$, which is a little different from the approximation $\tau = 0{\cdot}9807509$. In the same figure, we have also given the sway added mass for the same sphere in the channel with $d = 3a$ and $d = 5a$. It is found that the trapped mode may exist between $\tau = 0{\cdot}992$ and $\tau = 0{\cdot}993$ in the first case, and between $\tau = 0{\cdot}998$ and $\tau = 0{\cdot}999$ in the second case. These results show that the effect of the trapped mode becomes evident only for a sphere close to the free surface and in a very narrow channel.

7. Conclusions

The linearized hydrodynamic problem of a submerged sphere in a channel has been solved based on the multipole expansion. The results obtained show the classic behaviour of a body in a channel, as observed in the publications mentioned in the Introduction. The added mass and the damping coefficients have been found to change sharply near the natural frequency. For the case $d = 10a$ presented in the paper, the side walls have little effect on the exciting force. The effect on the vertical drift force is small, but it is significant on the surge drift force. The effect of the trapped mode is not evident in many cases. It becomes significant only when the sphere is close to the free surface and the channel is very narrow.

Acknowledgement

The work has been based on the private notes of Professor F. Ursell, FRS. Without his invaluable input, this paper could not have been written.

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