WAVE SOLUTIONS TRAVELLING ALONG QUADRATIC PATHS FOR THE EQUATION $(\partial u/\partial t) - (k(u)u_x)_x = 0$*

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1. Introduction. In this paper we discuss some aspects of the asymptotic behavior of the following initial-boundary value problem:

$$\begin{aligned} &(\partial u/\partial t) = (k(u)u_x)_x = (\beta(u))_{xx}, \quad t > 0, \quad 0 \le x < \infty \\ &u(0, x) = f(x), \quad 0 \le x < \infty, \quad u(t, 0) = a > 0, \quad u(t, \infty) = 0, \quad t > 0 \end{aligned} \tag{1}$$

where we assume that $k(s) \ge 0$, $k(0) = 0$ (with other assumptions later as needed), and

$$\beta(u) = \int_0^u k(s)\, ds. \tag{2}$$

Problem (1) can describe the flow of a fluid through a porous medium with the density (essentially u) being kept constant at one edge of the medium. The non-negative function $k(u)$ is often called a diffusion coefficient. Further details can be found in the literature referenced. Apart from this problem of filtration, there are doubtless other physical and biological settings for the problem.

The initial value problem for the *DE* in (1) was discussed by Oleinik [11]. In general there is a unique weak solution $u(t, x) \ge 0$ for suitably prescribed $f(x) \ge 0$, $-\infty < x < \infty$, which is a "classical" solution at any point where $u(t, x) > 0$. The transition from a region where $u > 0$ to a region where $u = 0$ need not be smooth since with $k(0) = 0$ the *DE* in (1) changes type. For the case $\beta(u) = u^m$, $m \ge 1$, Aronson [1] discusses the global continuity and differentiability properties of the solution. Again when $m \ge 1$, L. A. Peletier [13] treats problem (1) by conventional "weak" methods, obtaining a complete result in this case for the asymptotic behavior (see also the work of Shampine [15]). Finally Craven and Peletier [6] and Atkinson and Peletier [2], in papers that directly stimulated this present work, assume that the solution of problem (1) has the form $u(t,x) = F(\eta)$ where $\eta = x(t+1)^{-1/2}$. Then it is found that $F(\eta)$ must satisfy the problem

$$(k(F)F')' + \tfrac{1}{2}\eta F' = 0, \quad 0 < \eta < \infty, \quad F(0) = a > 0, \quad F(\infty) = 0. \tag{3}$$

Under the conditions

$$k(0) = 0, \quad k(s) > 0, \quad s > 0, \quad \text{and} \quad \int_0^1 \frac{k(s)}{s}\, ds < \infty,$$

problem (3) is shown to have a unique weak solution.

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Further references, particularly to Soviet literature, may be found in the various papers listed at the end of this paper.

The author undertook this work partly because it is a vehicle for use of the nonlinear semigroup methods of Brezis, Crandall, Dorroh, Kato, and Pazy [3, 4, 5, 7, 8, 12]. Application of these methods seems to result in more conciseness than was previously obtainable with the nonlinear initial-boundary value problem, and it may be fruitful for generalizations.

2. Main results. As $t \to \infty$, the solution of problem (1) evidently does not approach a steady state. To produce a problem with a steady state we introduce moving coordinates as follows:

$$= \frac{x}{(1+t)^{1/2}}, \quad \tau = \log(1+t), \quad v(\tau, \xi) \equiv v\left(\log(1+t), \frac{x}{(1+t)^{1/2}}\right) \equiv u(t, x) \tag{4}$$

whence we arrive at a new problem:

$$\begin{aligned} &\partial v/\partial\tau = (k(v)v_\xi)_\xi + \tfrac{1}{2}\xi v_\xi = (\beta(v))_{\xi\xi} + \tfrac{1}{2}\xi v_\xi, \qquad \tau > 0, \qquad 0 \le \xi \le R \\ &v(0, \xi) = f(\xi), \qquad 0 \le \xi \le R, \qquad v(\tau, 0) = a > 0, \qquad v(\tau, R) = 0, \qquad \tau > 0. \end{aligned} \tag{5}$$

The choice of the finite interval $0 \le \xi \le R$ for problem (5) is deliberate to simplify the analysis. The corresponding region in problem (1) would be $0 \le x \le R(1+t)^{1/2}$, $t \ge 0$.

One sees immediately that the operator on the right in the DE of problem (5) i the same as the operator in the DE of problem (3). If we solve problem (3), obtaining the unique weak solution of Atkinson and Peletier [2] (which is positive on an interval $0 \le \eta < \alpha$, and vanishes on the interval $\alpha \le \eta < \infty$ for some $\alpha > 0$), then this solution is the steady-state solution of problem (5). This supposes, however, that we take $R \ge \alpha$.

Accordingly, we take $R \ge \alpha$ where α is the first vanishing point of the weak solution of (3) [2, pp. 370, 379].

The idea in solving problem (5) is to show that the solution of problem (1) tends as $t \to \infty$ asymptotically to a wave travelling in the τ-direction along the quadratic curves $x/(1+t)^{1/2} = \xi_i =$ const of Fig. 1.

Thus our attention is focused on problem (5). We make the following definitions:

$$A_1 v = -(\beta(v))_{\xi\xi}, \qquad v(0) = a, \qquad v(R) = 0 \tag{6}$$

$$A_2 v = -\tfrac{1}{2}\xi v_\xi, \qquad v(R) = 0 \tag{7}$$

and we consider A_1 to be multivalued. We propose to show that the non-linear operator A_1 is the subdifferential [4, p. 21, ex. 2.1.4] of a convex lower semicontinuous functional in a suitable Hilbert space, and is thus maximal monotone. Next we show that the linear operator A_2 is maximal monotone on the same space, and is also strongly monotone [4, p. 88]. The correct Hilbert space turns out to be H^{-1}; this is the distributional dual of the Sobolev space $H_0^1 \subset H^1 = W_2^1(0, R)_1$ [3, pp. 123, 146; 9, pp. 59, 79].

Using these facts and the theory of nonlinear semigroups essentially as given by H. Brezis [3, 4], we prove the following results. The restrictions on $k(s)$ in the following statement are due partly to the requirements of Atkinson and Peletier in the steady state.

Theorem: Suppose $k(s) \ge 0$ is real and continuous, that $k(0) = 0$ and that $k(s) > 0$ for $s > 0$; that

$$\int_{-\infty}^{0} k(s)\, ds = \infty, \qquad \int_{1}^{\infty} k(s)\, ds = \infty, \quad \text{but that} \quad \int_{0}^{1} \frac{k(s)}{s}\, ds < \infty.$$

Suppose that $f(s) \in H^1$ is such that $f(0) = a > 0$, $f(R) = 0$, and that $\beta(f(s)) \in H^1$. Suppose that $R \geq \alpha$, where $\alpha > 0$ is the first vanishing point in the solution of (3). Then initial-boundary value problem (5) has a unique generalized solution $v(\tau, \xi) \in C([0, \infty); H^1)$ (in the sense of Brezis [3, p. 54]) which tends exponentially to the unique solution of problem (3) as $\tau \to \infty$, in terms of the norm of H^{-1} (and thus pointwise a.e. since the solution is in $D(A_1) \cap D(A_2)$).

COROLLARY: The solution of problem (1) approaches a quadratically travelling wave.

3. Proof of the theorem. The proof takes place in several parts. We write problem (5) simply as follows:

$$\partial u/\partial \tau + (A_1 + A_2)\, u \ni 0, \qquad u(0) = f \tag{8}$$

where $A_1 + A_2$ is multivalued.

A. *The nonlinear operator A_1 is the subdifferential of a convex lower semicontinuous (l.s.c.) function on H^{-1}.* Reviewing definitions, let H_0^1 be the completion of the $C_0^\infty(0, R)$ functions under the H^1 norm, where H^1 is the Sobolev space $W_2^1(0, R)$. Then H^{-1} is the Hilbert space of distributions using H_0^1 as the test space; i.e., H^{-1} is the distributional dual of H_0^1 under the duality relationship $\langle u, v\rangle = \int_0^R u(\xi)v(\xi)\, d\xi$, $u \in H_0^1$, $v \in H^{-1}$.

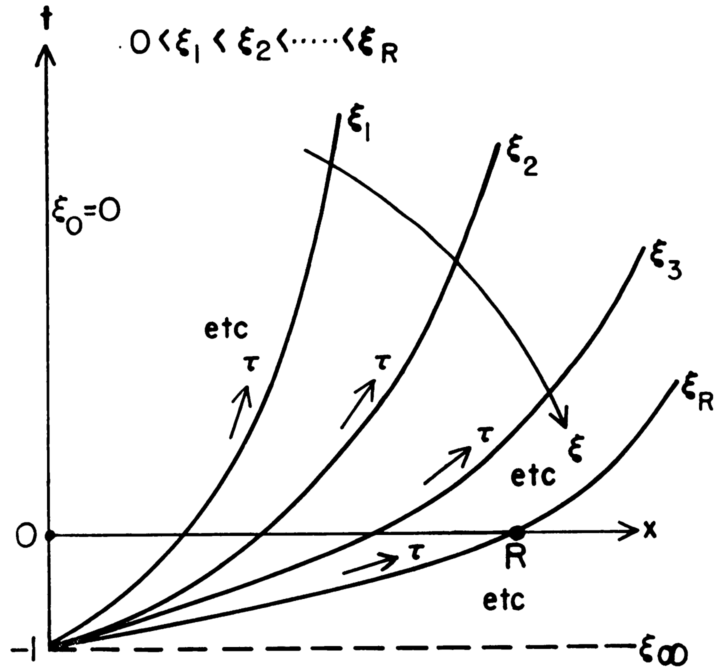

FIG. 1. Quadratic propagation.

The space H^{-1} has inner product $(f, g)_{H^{-1}} = \langle \Lambda^{-1}f, g\rangle$, where $\Lambda = -\Delta = -(\cdot)_{\xi\xi}$ is an isomorphism of H_0^1 onto H^{-1} [9, Th. 9.1].

We employ the following convex l.s.c. functional on H^{-1} [3, p. 123]:

$$\phi_1(u) = \int_0^R j(u)\,d\xi, \qquad u \in L^1(0, R) \cap H^{-1} \ni j(u) \in L^1(0, R) \cap H^{-1}$$
$$= +\infty \quad \text{otherwise.}$$

Here $j(r)$ is a convex l.s.c. functional from R^1 into $(-\infty, \infty]$, and has as subdifferential the operator $\partial j = \beta$, where $\beta(r)$ is given by (2). From the conditions on $k(s)$ given in the theorem, $\mathcal{R}(\beta) = R^1$, and therefore

$$\lim_{|r|\to\beta} \frac{j(r)}{|r|} = +\infty.$$

Suppose that $f \in A_1u$; i.e., suppose $\exists\ u \in L^1 \cap H^{-1}$ and $f \in H^{-1}$ such that

$$-f \in (\beta(u))_{\xi\xi}, \qquad \beta(u(0)) = \beta(a), \qquad \beta(u(R)) = 0 \tag{9}$$

where we transform the boundary conditions, using $\beta(r) \in \nearrow$. Noting that $w(\xi) = w_0(\xi) + g(\xi)$ solves the linear problem

$$-w_{\xi\xi} = f, \qquad w(0) = \beta(a), \qquad w(R) = 0,$$

where $w_0 = \Lambda^{-1}f \in H_0^1$ and $g = (\beta(a)/R)(R - \xi) \in H^1$, we see that (9) states that $w \in \beta(u)$. Therefore let us define the domain:

$$D(A_1) = [u \mid u \in L^1 \cap H^{-1},\ \exists\ w_0 \in H_0^1 \ni w(\xi) = w_0(\xi) + g(\xi) \in \beta(u(\xi)) \text{ a.e.}]. \tag{10}$$

Now $w(\xi) \in \beta(u(\xi))$ a.e. implies that

$$j(v(\xi)) - j(u(\xi)) \geq w(\xi)\cdot(v(\xi) - u(\xi)) \quad \text{a.e. in } (0, R), \tag{11}$$

where $u \in D(A_1)$ and $v \in L^1 \cap H^{-1} \ni j(v) \in L^1 \cap H^{-1}$. Since $j(r)$ is bounded below by an affine function: $j(r) \geq -C_1|r| - C_2$, $C_1 > 0$, $C_2 > 0$ (see [3, p. 125]), and since $w(\xi) \in H^1$ is continuous [9, p. 51], $j(u) \in L^1 \cap H^{-1}$ by (11), so that

$$\int_0^R j(v)\,d\xi - \int_0^R j(u)\,d\xi \geq \int_0^R w(\xi)\cdot(v(\xi) - u(\xi))\,d\xi = \int_0^R (\Lambda^{-1}f + g)(v(\xi) - u(\xi))\,d\xi.$$

This means that $f \in \partial\phi$, where we define the functional

$$\phi(u) = \int_0^R (j(u(\xi)) - g(\xi)u(\xi))\,d\xi, \quad u \in L^1(0, R) \cap H^{-1} \ni j(u) \in L^1(0, R) \cap H^{-1}$$
$$= +\infty \quad \text{otherwise.} \tag{12}$$

The functional ϕ is proper, convex l.s.c. on H^{-1}. Thus $A_1 \subset \partial\phi$.

We show now that $A_1 \equiv \partial\phi$ by proving that A_1 is maximal monotone. Monotonicity of A_1 in H^{-1} follows from the monotonicity of $\beta(r)$. To show maximal monotonicity, one solves the following problem in $L^1 \cap H^{-1}$:

$$-(\beta(u))_{\xi\xi} + u \ni f, \qquad \beta(u(0)) = \beta(a), \qquad \beta(u(R)) = 0,$$

where $f \in H^{-1}$ is arbitrary. This problem is convertible to the form

$$w + \Lambda^{-1}u = \Lambda^{-1}f + g, \qquad w(\xi) \in \beta(u(\xi)) \quad \text{a.e.} \tag{13}$$

Let $\gamma = \beta^{-1}$, so that $D(\gamma) = R^1$. Then (13) can be written as a Hammerstein equation in H_0^1:

$$w_0 + \Lambda^{-1}\gamma(w_0 + g) \ni \Lambda^{-1}f, \qquad u(\xi) \in \gamma(w_0(\xi) + g) \quad \text{a.e.}$$

If $\beta(u)$ in (2) is single-valued, this is the case of Hammerstein's equation for which there is an existence theorem [17, p. 208]. Otherwise one uses the Yosida approximation in the manner of Brezis [3, pp. 125–127]. This completes the proof of proposition A.

B. *The operator A_2 is a maximal monotone operator, which is a strongly monotone operator.* The operator $A_2u = -\frac{1}{2}\xi u_\xi$ with $u(R) = 0$ is linear, and $D(A_2)$ certainly contains the absolutely continuous functions defined on $(0, R)$ which vanish at $\xi = R$. The latter functions being dense in $L^1(0, R)$, $D(A_2)$ is dense in H^{-1}. So as to include the generalized derivatives we write

$$D(A_2) = [u \mid u \in H^{-1} \cap H^1, u(R) = 0] \tag{14}$$

Let us solve explicitly the following DE problem:

$$u_t - \tfrac{1}{2}\xi u_\xi = 0, \qquad u(t, R) = 0 \tag{15}$$

with $u(0, \xi) = f(\xi) \in D(A_2)$. The characteristics satisfy the DE $d\xi/dt = -\frac{1}{2}\xi$. Thus $\xi(t) = \xi_0 \exp(-t/2)$, for various values of $\xi_0 \geq 0$, represents the characteristics in the t, ξ plane. Also $u(t, \xi) = u(0, \xi_0) = \text{const}$ along the curve $\xi(t) = \xi_0 \exp(-t/2)$ for given ξ_0. Through any point (t, ξ) in $(0, R) \times (0, R)$ we can pass a characteristic leading to the initial or boundary data, and thus find $u(t, \xi)$ (see Fig. 2). Beginning with ξ_R, all characteristics bear the value $u = 0$.

The explicit solution has the semigroup property. If we let $u(t, \xi; \hat{t}, \psi(\xi))$ represent

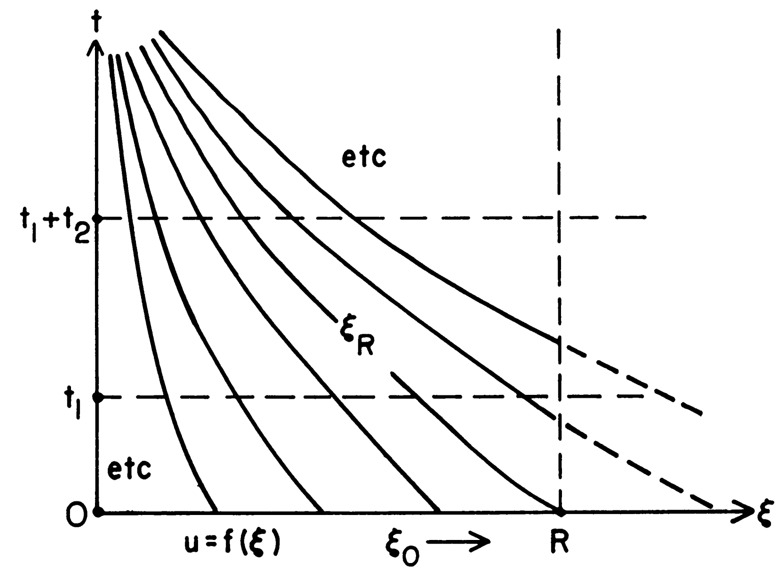

FIG. 2. Characteristic curves.

the solution based on initial conditions $u = \Psi(\xi)$ at some time $\hat{t}$, then the semigroup property satisfied is as follows:

$$u(t_1 + t_2, \xi; 0, f(\xi)) = u(t_2, \xi; t_1, u(t_1, \xi; 0, f(\xi))) \tag{16}$$

The same semigroup solution is produced on $(0, R)$ by solving the DE problem $u_t - \frac{1}{2}\xi u_\xi = 0$, $u(t, \infty) = 0$; i.e., by using the entire infinite interval. We simply restate the initial data as follows:

$$\begin{aligned} f^0(\xi) = u(0, \xi) &= f(\xi), \qquad 0 \le \xi \le R \\ &= 0 \quad , \qquad R < \xi < \infty . \end{aligned} \tag{17}$$

The characteristic curves are the same as in Fig. 2, but they are extended to the ξ-axis where the extended trivial initial data is defined.

Thus we have a semigroup, and we expect that the operator $-A_2$ is its generator in H^{-1}.

The norm of an element $f_1 \in H^{-1}$ is defined as follows:

$$\begin{aligned} \|f_1\|_{H^{-1}}^2 = (f_1, f_1)_{H^{-1}} &= \int_0^R (\Lambda^{-1} f_1) f_1 \, d\xi \\ &= \int_0^R \int_0^R G(\xi, \eta) f_1(\eta) f_1(\xi) \, d\eta \, d\xi \end{aligned} \tag{18}$$

where

$$\begin{aligned} G(\xi; \eta) &= \frac{1}{R} \xi(R - \eta), \qquad 0 \le \xi \le \eta \\ &= \frac{1}{R} \eta(R - \xi), \qquad \eta \le \xi \le R \end{aligned} \tag{19}$$

is the kernel of the linear integral operator Λ^{-1}.

The semigroup solution of problem (15) is strongly continuous in H^{-1}. We have for $f \in D(A_2)$, where f is the initial function,

$$\begin{aligned} \|u(t, \cdot) - f(\cdot)\|_{H^{-1}}^2 &= \left| \int_0^R \int_0^R G(\xi, \eta)\{u(t, \eta) - f(\eta)\}\{u(t, \xi) - f(\xi)\} \, d\eta \, d\xi \right| \\ &\le M \left[\int_0^R |u(t, \xi) - f(\xi)| \, d\xi \right]^2 \end{aligned}$$

where

$$M = \max_{\substack{0 \le \xi \le R \\ 0 \le \eta \le R}} |G(\xi, \eta)| = \frac{R}{4}.$$

Now, however, putting $\xi = \xi_0 \exp(-t/2)$, $d\xi = d\xi_0 \cdot \exp(-t/2)$, we get

$$\begin{aligned} \int_0^R |u(t, \xi) - f(\xi)| \, d\xi &= \int_0^{Re^{t/2}} |u(t, \xi_0 e^{-t/2}) - f(\xi_0 e^{-t/2})| \, d\xi_0 \cdot \exp(-t/2) \\ &= \int_0^{Re^{t/2}} |u(0, \xi_0) - f(\xi_0 e^{-t/2})| \, d\xi_0 \cdot \exp(-t/2) \\ &= \exp(-t/2) \int_0^{Re^{t/2}} |f^0(\xi_0) - f^0(\xi_0 e^{-t/2})| \, d\xi_0 \to 0 \end{aligned}$$

as $t \to 0$ by continuity and a known result [16, p. 396, ex. 18]. Here we have used extended initial data (17) and the constancy of the solution along characteristics. Thus, our semigroup has strong continuity.

The semigroup happens to be contracting in $H^0 = L_2(0, R)$. In H^{-1}, however, it seems only to be uniformly bounded [12, p. 349]. We have, making the substitutions $\xi = \xi_0 \exp(-t/2)$, $\eta = \eta_0 \exp(-t/2)$ in the integration:

$$||u(t, \cdot)||_{H^{-1}}^2 = \int_0^R \int_0^R G(\xi, \eta)u(t, \eta)u(t, \xi)\, d\eta\, d\xi$$

$$= \exp(-t) \int_0^R \int_0^R G(\xi_0 \exp(-t/2), \eta_0 \exp(-t/2))f(\eta_0)f(\xi_0)\, d\eta_0\, d\xi_0 ,$$

which quantity vanishes at $t \to \infty$. Here we have again used extended initial data (17) and the constancy of u on characteristics.

Another semigroup, related to (16), is formed of the family of functions given by $u_1(t, \xi; 0, f) = \exp(t/2)u(t, \xi; 0, f)$, as can be seen by using the linearity of problem (15) and the semigroup property possessed by the function $\exp(t/2)$. This latter semigroup is likewise strongly continuous, and

$$||u_1(t, \cdot)||_{H^{-1}}^2 = \int_0^R \int_0^R G(\xi_0 \exp(-t/2), \eta_0 \exp(-t/2))f(\eta_0)f(\xi_0)\, d\eta_0\, d\xi_0 \qquad (20)$$

which of course vanishes as $t \to \infty$. Let us denote this semigroup as follows: $u_1 = S_1(t)f$ (with the property $S_1(t_1 + t_2)f = S_1(t_1)S_1(t_2)f$).

Both of these semigroups are uniformly bounded. By introducing the equivalent norm $||f||_{H^{-1}}^{(1)} = \sup_{t \geq 0} ||S_1(t)f||_{H^{-1}}$, where $u_1 = S_1(t)f$, the semigroup S_1 is made to be contracting on the Banach space consisting of the elements of H^{-1} normed by $||\cdot||_{H^{-1}}^{(1)}$.

Since the norms $||\cdot||_{H^{-1}}$ and $||\cdot||_{H^{-1}}^{(1)}$ are equivalent:

$$||f||_{H^{-1}} \leq ||f||_{H^{-1}}^{(1)} \leq M_1\, ||f||_{H^{-1}}, \qquad f \in H^{-1}, \qquad M_1 > 1,$$

S_1 is strongly continuous as well as contracting in terms of the new norm. However, we pay for these benefits in that H^{-1} is no longer a Hilbert space. We still have A_2 densely defined, and a result of Phillips [10, p. 686] is applicable: "A necessary and sufficient condition that a linear operator (in our case $A_2 - \frac{1}{2}I$ on Banach space H^{-1}) with dense domain generate a strongly continuous semi-group of contraction operators is that it be maximal accretive." The parallel theorem of Hille-Yosida (see [14, p. 203], setting $\omega = 0$) equates this maximal accretiveness of $A_2 - \frac{1}{2}I$ with the condition:

$$M_1^{-1}\, ||[I - \lambda(A_2 - \tfrac{1}{2}I)]^{-1}||_{H^{-1}} \leq ||[I - \lambda(A_2 - \tfrac{1}{2}I)]^{-1}||_{H^{-1}}^{(1)} \leq 1 , \qquad \lambda < 0.$$

This boundedness of the resolvent for $\mathcal{R}e\lambda < 0$ means, however, that the numerical range of $A_2 - \frac{1}{2}I$ in the real Hilbert space H^{-1} is positive, i.e. $((A_2 - \frac{1}{2}I)f, f)_{H^{-1}} \geq 0$, $f \in H^{-1}$.

Thus the operator $(A_2 - \frac{1}{2}I)u = -\frac{1}{2}u\xi u_\xi - \frac{1}{2}u$, with $u(R) = 0$, is maximal monotone along with the operator A_2 itself (see, for example, Pazy [12, p. 353] with $\omega = -\frac{1}{2}$). Also we have

$$((A_2 - \tfrac{1}{2}I)u, u)_{H^{-1}} = ((-\tfrac{1}{2}\xi u_\xi - \tfrac{1}{2}u), u)_{H^{-1}} \geq 0$$

which implies strong monotonicity for A_2 :

$$(-\tfrac{1}{2}\xi u_\xi , u)_{H^{-1}} \geq \tfrac{1}{2}(u, u)_{H^{-1}} . \qquad (21)$$

This concludes the proof of proposition B.

C. *The operator $A_1 + A_2$ is a maximal monotone operator which is strongly monotone.* The operator $A_1u = -(\beta(u))_{\xi\xi}$, $\beta(u(0)) = \beta(a)$, $\beta(u(R)) = 0$, is the subdifferential of the convex proper l.s.c. functional ϕ on H^{-1} given by (12), and therefore is maximal monotone [4, p. 25, ex. 2.3.4]. The set $D(A_1)$ is given by (10) and the set $D(A_2)$ is given by (14). It is easy to see that with $\beta(u)$ as given in (2), $D(A_1) \cap D(A_2)$ is not void.

We assume that $j(r) > 0$ and $\phi(u) > 0$; otherwise, in view of the property that $j(r)/|r| \to \infty$ as $r \to \infty$, one could merely add a constant (see section A of this proof).

The idea of the proof is to show that $\phi(J_{2\lambda}u) \leq \phi(u) + c\lambda$, $c \geq 0$, $\forall\, u \in H^{-1}$ and $\forall\, \lambda > 0$, where $J_{2\lambda} = (I + \lambda A_2)^{-1}$ is the (linear) resolvent of A_2. Then recourse can be had to known results [3, p. 108, Th. 9; 4, p. 48, Prop. 2.17] to show that $A_1 + A_2 = \partial\phi + A_2$ is maximal monotone.

We let $j_\alpha(r)$ be the Yosida approximation, with $\alpha > 0$, of the convex l.s.c. function $j(r)$ mapping R^1 into $(-\infty, \infty]$ given in the first section of this proof [3, p. 104, Th. 4; 4, p. 39, Prop. 2.11]. We have for $u \in H^{-1} \cap L^1(0, R)$, with notation from (12)

$$\begin{aligned}\hat{j}_\alpha(u(\xi)) - \hat{j}_\alpha((J_{2\lambda}u)(\xi)) &= j_\alpha(u(\xi)) - gu(\xi) - j_\alpha((J_{2\lambda}u)(\xi)) + g(J_{2\lambda}u)(\xi)\\ &\geq [\partial j_\alpha((J_{2\lambda}u)(\xi)) - g][u(\xi) - (J_{2\lambda}u)(\xi)] \qquad \text{a.e. in} \quad (0, R)\\ &= [\partial j_\alpha((J_{2\lambda}u)(\xi)) - g][-\tfrac{1}{2}\lambda\xi((J_{2\lambda}u)(\xi))_\xi]\\ &= -\tfrac{1}{2}\lambda\xi[j_\alpha((J_{2\lambda}u)(\xi)) - g(J_{2\lambda}u)(\xi)]_\xi = -\tfrac{1}{2}\lambda\xi[\hat{j}_\alpha((J_{2\lambda}u)(\xi))]_\xi \qquad (22)\end{aligned}$$

where ∂j_α is the Frechet derivative of j_α and we have used the property $A_{2\lambda} = A_2J_{2\lambda}$ valid if A_2 is linear, where $A_{2\lambda} = (1/\lambda)[I - J_{2\lambda}]$.

Next with regard for the function ϕ defined in (12), we write for the Yosida approximation ϕ_α, $\alpha \geq 0$, with $u \in D(\phi)$,

$$\begin{aligned}\phi_\alpha(J_{2\lambda}u) &= \int_0^R \hat{j}_\alpha((J_{2\lambda}u)(\xi))\, d\xi \leq \int_0^R \hat{j}_\alpha(u(\xi))\, d\xi + \frac{\lambda}{2}\int_0^R \xi[\hat{j}_\alpha((J_{2\lambda}u)(\xi))]_\xi\, d\xi\\ &= \int_0^R \hat{j}_\alpha(u(\xi))\, d\xi + \frac{\lambda}{2}\left\{\xi\hat{j}_\alpha((J_{2\lambda}u)(\xi))\Big|_0^R - \int_0^R \hat{j}_\alpha((J_{2\lambda}u)(\xi)\, d\xi\right\}\\ &= \phi_\alpha(u) + \frac{\lambda}{2}R\hat{j}_\alpha((J_{2\lambda}u)(R)) - \frac{\lambda}{2}\phi_\alpha(J_{2\lambda}u).\end{aligned}$$

where use has been made of the inequality in (22). Then

$$\left(1 + \frac{\lambda}{2}\right)\phi_\alpha(J_{2\lambda}u) \leq \phi_\alpha(u) + \frac{\lambda}{2}Rj_\alpha(0) \leq \phi(u) + \frac{\lambda}{2}Rj(0) \qquad (23)$$

since $j_\alpha \leq j$ and $\phi_\alpha \leq \phi$ [3, p. 104, Th. 4; 4, p. 39, Prop. 2.11], and since $v = J_{2\lambda}u$ is the solution of the DE problem

$$-\frac{\lambda}{2}\xi v_\xi + v = u, \qquad v(R) = 0.$$

Finally noting the (23) holds for all $\alpha > 0$, we have

$$\phi(J_{2\lambda}u) < \phi(u) + \frac{\lambda}{2}Rj(0) \qquad \forall\, \lambda > 0, \qquad \forall\, u \in D(\phi)$$

as was desired; i.e., $c = \tfrac{1}{2}Rj(0) \geq 0$.

The strong monotonicity of $A_1 + A_2$ is immediate from the monotonicity of A_1 and A_2, and inequality (21).

This ends the proof of Statement C.

D. *There exists a unique strongly continuous semigroup of contractions which provides the solution of problem* (5) *provided* $f \in D(A_1) \cap D(A_2)$. *The solution remains in* $D(A_1) \cap D(A_2) \subset H^{-1}$, *and converges exponentially in norm to the unique solution of problem* (3). We again appeal to a theorem given by Brezis [4, p. 54, Th. 3.1]. By $(A_1 + A_2)^0$ below we mean the principal section consisting of the graph of elements in $H^{-1} \times H^{-1}$ of least norm of the sets $(A_1 + A_2)u$, $u \in D(A_1) \cap D(A_2)$. Accordingly we can say on the basis of the foregoing propositions applied to problem (8) that to the problem

$$v_\tau - (\beta(v))_{\xi\xi} - \tfrac{1}{2}\xi v_\xi \ni 0, \qquad \tau > 0, \qquad 0 \le \xi \le R,$$
$$\beta(v(\tau, 0)) = \beta(a), \qquad \beta(v(\tau, R)) = 0, \qquad \tau > 0,$$
$$v(0, \xi) = f(\xi) \in D(A_1) \cap D(A_2) \subset H^{-1} \tag{24}$$

there corresponds a unique function $v(\tau)$ on $[0, \infty]$ into H^{-1}, such that $v(\tau) \in D(A_1) \cap D(A_2)$, $(dv/d\tau) + (A_1 + A_2)v \ni 0$ a.e. on $(0, +\infty)$, $v(0) = f$, $(d^+v/d\tau) + (A_1 + A_2)^0 v = 0$, $\forall\ \tau \in [0, +\infty)$. The mapping $\tau \to (A_1 + A_2)^0 v(\tau)$ is continuous on the right, and the mapping $\tau \to \|(A_1 + A_2)^0 v(\tau)\|_{H^{-1}}$ is decreasing. Also

$$\|v_1(\tau) - v_2(\tau)\|_{H^{-1}} \le \|v_1(0) - v_2(0)\|_{H^{-1}} = \|f_1 - f_2\|_{H^{-1}}$$

$\forall\ \tau \in [0, +\infty)$, where $v_1(\tau) = S(\tau)f_1$, $v_2(\tau) = S(\tau)f_2$, and where $S(\tau)$ is the semigroup generated by $A_1 + A_2$.

We present this function as the solution of initial value problem (5) in the strong sense of Brezis [4, pp. 54, 64].

By two results of Atkinson and Peletier [3, pp. 375, 378], under conditions on $k(s)$ set forth in the main theorem of this paper, for any $a > 0$ there exists a unique weak solution $v_\infty(\xi)$, of steady state problem (3), positive on an interval $[0, \alpha)$ and vanishing on the interval $[\alpha, \infty)$. This is a classical solution except at $\xi = \alpha$; at the latter point it remains continuous, but may have a discontinuity in the (classical) derivative, depending on further assumptions one can make about $k(s)$ at $s = 0$. In fact, if $\lim_{v\to 0+} k'(v) = k'(0) > 0$, then $\lim_{\xi\to\alpha-} v_\infty'(\xi) = -\frac{1}{2}(\alpha/k'(0))$. If $\lim_{k\to 0+} k'(v) = 0$, then $\lim_{\xi\to\alpha-} v_\infty'(\xi) = -\infty$ (see [6, pp. 80, 81]). We note also that $v_\infty(\xi)$ is in $D(A_1) \cap D(A_2) \subset H^{-1} \cap L^1(0, R)$ if $R \ge \alpha$ as we suppose.

By the strong monotonicity of $A_1 + A_2$, and by another theroem of Brezis on asymptotic behavior [4, p. 88, Th. 3.9], since the function $v_\infty(\xi) \in H^{-1}$ satisfies the steady state problem a.e. uniquely, we have

$$\|v(\tau, \xi) - v_\infty(\xi)\|_{H^{-1}} \le \exp(-\tau/2)\, \|v(0, \xi) - v_\infty(\xi)\|_{H^{-1}}. \tag{25}$$

Hence the solution $v(\tau, \xi)$ of problem (5) approaches the steady-state solution of problem (3) in norm, and exponentially in τ as $\tau \to +\infty$. This implies convergence pointwise a.e. as can be shown by contradiction using expression (18) and the fact that $v(\tau, \xi)$ stays in $D(A_1) \cap D(A_2)$ and is thus in H^1. This proves proposition D and the theorem.

4. Remarks. By the theorem, and because of the exponential decay of the transient part of the solution of problem (5), after a short interval in the τ-variable the solution settles down essentially to the solution $v_\infty(\xi)$ of steady state problem (3). For $0 \le \xi < \alpha$, it tends pointwise a.e. to the positive part of $v_\infty(\xi)$; for $\alpha \le \xi \le R$ it tends to zero pointwise a.e.

The solution $v(\tau, \xi)$ of problem (5) remains in $D(A_1) \cap D(A_2)$. Thus it is continuous in ξ, has first generalized derivatives in ξ, and $v(\tau, 0) = a$, $v(\tau, R) = 0$. In terms of the variable τ it is continuous and right-differentiable, with decreasing H^{-1} norm. The full derivative exists almost everywhere.

When we use transformation (4) to go back to x, t coordinates, we see that arbitrary initial data in $D(A_1) \cap D(A_2)$ reuslts through evolution in a wave-like solution, travelling along the quadratic curves of Fig. 1. The speed of propagation is thus finite and decreasing, and the asymptotic wave form is that of Atkinson and Peletier [2].

If, instead of the conditions of our theorem in Sec. 2, we set $k(v) \equiv 1$ in problem (5), there results the familiar linear heat flow problem transformed via Eqs. (4). Problem (5) has a steady state solution in this case, namely

$$v(\xi) = a\left[1 - \frac{1}{\sqrt{\pi}} \int_0^{\xi} \exp\left(-\hat{\xi}^2/4\right) d\hat{\xi}\right] \tag{26}$$

which is unique. This is approached by the transient solution as $\tau \to \infty$ for more or less arbitrary initial data in H^{-1}. One might claim a wave-like phenomenon here also. As an asymptotic solution, (26) is so smooth that no travelling disturbance is perceptible. It is when $k(v) \geq 0$ vanishes somewhere, and the DE changes type, that there is an abrupt travelling disturbance. We conjecture that if $k(v)$ merely becomes "small," this situation is approached.

If one transforms the asymptotic estimate (25) back to t, x coordinates through Eqs. (4), the result is seen to be comparable to that of Peletier [13, p. 546, Th. 1] in 1971. Of course the present work uses a less rigid hypotheses about the function $k(s)$.

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