Wavelet-Based Solution Of Integral Equations For Acoustic Scattering

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ABSTRACT

In this work, the multi-resolution wavelet analysis is used to solve Helmholtz integral equation for acoustic scattering. The integral equation is solved using moment method with wavelet basis. The unknown field is expressed as a two fold summation of shifted and dilated forms of a properly chosen mother wavelet. The wavelet expansion covers the scatterer surface for distributing the wavelet localized functions. A simpler formulation of a square wavelet operator is proposed and tested in this investigation to obtain the moment matrix. The proposed operator saves some traditional stages of wavelet transform and accordingly a part of the computations required. The square matrix inversion can be implemented easily on different media. The resulting matrix can be made sparse by applying an appropriate threshold. The solution of such sparse matrix saves a large portion of the computational load. The accuracy of the proposed solution is compared to the exact solution of the problem. Computational savings are illustrated for acoustic scattering on a sphere for different wave numbers and wavelet bases order.

SOMMAIRE

Dans ce travail, l'analyse de ondelette est employee pour résoudre l'équation intégrale de Helmholtz pour la dispersion acoustique. L'équation intégrale est résolue en utilisant la méthode de moment avec la base de ondelette. Le champ inconnu est exprimé comme une addition de deux fois des formes décalées et dilatées d'un ondelette correctement choisi de mère. L'expansion de ondelette couvre la surface de diffuseur pour distribuer les functions localisées par ondelette. Un opérateur carré de ondelette est propose dans une formulation plus simple et examiné pour que ce problème obtienne la matrice de moment. L'opérateur proposé sauve quelques étapes traditionnelles de ondelette transforment et en conséquence une partie des calculs priés. L'inversion carrée de matrice peut être mise en application facilement sur différents médias. L'application d'un seuil approprié sur la matrice résultante la rend clairsemée. La solution d'une telle matrice clairsemée sauve une grande partie du volume des calculs. L'exactitude de la solution proposée est examinée par l'intermédiaire de la comparaison avec la solution exacte du problème. L'épargne informatique est illustrée pour la dispersion acoustique sur une sphère pour des nombres de vague et l'ordre différents de bases de ondelette.

1. INTRODUCTION

The surface Helmholtz integral equation is a common approach for the problem of acoustic scattering by obstacles. A general procedure for finding a solution of Helmholtz integral equation that is accurate enough for many practical purposes is the method of moments. Standard moment-method approaches are well suited for the solution of scattering problems as long as the length scale is comparable to the wavelength [1]. The moment method is essentially a discretization scheme whereby a general operator equation is transformed into a matrix equation which can be solved numerically. This transformation is affected by projections on subspaces, which for acoustic scattering bodies are of finite dimensions. The resulting matrix is always dense when the conventional expansion and testing functions are used. Recently, there has been much interest in using wavelet basis to sparsify that dense moment matrix [2]-[4]. In this work, the technique of moment matrix sparsification is used for solving the acoustic scattering problem.

If wavelet basis is used, the moment matrix sparsity depends mainly on two factors. First, the choice of mother wavelet function can help in obtaining the moment matrix which is amenable to be sparsified. The other factor is the sparsification threshold which may yield maximum sparsity while sufficient accuracy of the solution is retained.

In this work, the scattering field is expanded in terms of wavelet basis functions. A mother wavelet basis is chosen to cope with the acoustic problem details. The dilated and shifted version of the chosen basis are distributed along the surface of the scatterer. This distribution ensures localized accurate fitting with different field modes on all surface points. The substitution of such expansion into the integral equation results in a moment matrix. This matrix can be thresholded appropriately to obtain a sparse matrix. The resulting moment matrix is rearranged such that a simple formulation of a square matrix operator is introduced. The proposed operator can be applied to the conventional moment matrix.

This formulation differs from other traditional transformation methods [2], [3] in that only single matrix operator is used. The square operator matrix helps in implementation and speeds up the computations on different media. The squaring is kept by using wrap around filter coefficients along the matrix rows. The filtering is implemented as circular convolutions [5]. The resulting moment matrix is then thresholded to be sparsified. Different thresholds are tested with different Daubechies wavelet basis orders. The mathematical formulation of the problem follows the work of [6] for axissymmetric bodies and we consider similar test cases especially for the rigid sphere which has well known analytical solution [7].

In summary this work addresses the following issues:

a- Introduce a square operator for wavelet-based solution of Helmholtz integral equation which saves a matrix multiplication stage of traditional methods.

b- Test the wavelet-based solution on acoustic scattering on a sphere as an axisymmetric body based on the formulation in [6] at non-characteristic wavenumbers ($ka<\pi$) while the nonuniqueness associated with the characteristic wavenumbers is discussed in [8].

c- The proposed solution reduces the required memory storage and processing time of the acoustic scattering problem as compared to existing direct methods.

d- The memory storage and processing time reduction is a result of the obtained sparse moment matrix after applying wavelet expansion.

e- The results proves the success of the method in saving computational load while the accuracy is retained as compared to exact solution of the problem which is available for simple geometries like a sphere [6], [7], [9] and [10].

The paper is organized as follows. In Section 2, the application of wavelet transform is discussed for solving integral equation of acoustic scattering. Section 3 introduces the multiresolution analysis using wavelet and the proposed operator is presented in Section 4. The numerical implementation is discussed in Section 5. The proposed algorithm is, then, summarized in Section 6. Numerical results are presented in Section, 7 using different wavelet basis orders and sparsifying thresholds. In Section 8, some concluding remarks are given.

2. A Summary Of The Integral Equation Formulation In Acoustics

The boundary integral formulation for acoustic scattering is valid for an acoustic medium B' exterior to a finite body B with surface S on which a unit normal $\boldsymbol{n}$, pointing into B', is defined. The body B is submerged into an infinite linear acoustic medium. When a harmonic acoustic wave ϕ^i impinges upon that body B, the resulting integral equation for smooth boundaries has the following form [9];

$$\chi(P)\,\phi(P) = \int_S \big(\phi(Q)\,G'(P,Q) - G(P,Q)\,\phi'(Q)\big)\,dS_Q + 4\,\pi\phi^i(P) \tag{1}$$

Equation 1 is the standard surface Helmholtz integral equation. where $\phi(P) = \phi(r_P)\,e^{i\omega t}$ at a point P and Q is a point on the body surface.

Where $\phi'(Q) = \dfrac{\partial\phi(Q)}{\partial n}$

and $G'(P,Q) = \dfrac{\partial G(P,Q)}{\partial n}$

$\boldsymbol{n}$ is the unit vector normal to the surface of the scatterer body and into the surrounding space, and n is the distance along the external normal vector $\boldsymbol{n}$.

The free-space Green's function G for Helmholtz wave equation is given by

$G(P,Q) = e^{-ikR}/R$, where R is the distance between the field point P and a source point Q and k is the wave number. The distance is given by

$$R = \left|\mathbf{r_P} - \mathbf{r_Q}\right|$$

The coefficient $\chi(P)$ has the value 0 for P in B, the value 4π for P in B', and the value 2π for P on a smooth S (there is a unique tangent to S at such a point P).

At the surface of a hard scatterer , the normal component $\boldsymbol{u.n}$ of the fluid particle velocity $\mathbf{u}$ is zero; i.e. $\dfrac{\partial\phi}{\partial n} = 0$ while, at the surface of a soft scatterer, the excess pressure is zero; i.e. $\phi = 0$.

Both the body shape and the acoustic variables are independent of the angle of the revolution of the body for a fully axisymmetric scattering case. For scattering, this implies that the direction of the incident wave must coincide with the axis of revolution of the body. The singularity regularization is similar to that used in [6]. This formulation can be summarized as follows.

Consider an axisymmetric body, the integrals in equation (1) can be rewritten using a cylindrical coordinate system (ρ, θ, z) as follows:

For hard scatterer

$$\int_L \phi(Q)\left[\int_0^{2\pi}\frac{\partial}{\partial n}(G(P,Q))d\theta(Q)\right]\rho(Q)dL(Q)$$
$$= \chi(P)\,\phi(P)\; - 4\pi\,\phi^i(P) \qquad \textbf{(1a)}$$

Or for soft scatterer

$$\int_L \frac{\partial\phi(Q)}{\partial n}\left[\int_0^{2\pi}(G(P,Q))d\theta(Q)\right]\rho(Q)dL(Q)$$
$$= \chi(P)\,\phi(P)\; + 4\pi\,\phi^i(P) \qquad \textbf{(1b)}$$

These integrals can be rewritten in the form

$$\int_L \varphi(Q)\;\; K1(P,Q)\;\; \rho(Q)dL(Q)$$
$$= \chi(P)\;\phi(P)\; - 4\pi\,\phi^i(P) \qquad \textbf{(2)}$$

Or

$$\int_L \frac{\partial\varphi(Q)}{\partial n}\;\; K2(P,Q)\;\; \rho(Q)dL(Q) \qquad \textbf{(3)}$$
$$= \chi(P)\phi(P)\; + 4\pi\,\phi^i(P)$$

where the axisymmetric assumption implies that the field $\phi(P)$ and its derivative are independent of θ(*P*) and the differential area element is defined as

$dS(Q) = \rho(Q)\, d\theta(Q)\, dL(Q)$

where $dL(Q)$ is the differential length of the generator *L* of the body at a surface point *Q*, where *Q* now is interpreted as an arbitrary point on L only.

The evaluation of the integrands in Equations (2) and (3) requires the evaluation of the following:

$$K1(P,Q) = \int_0^{2\pi}\frac{\partial}{\partial n}\left(\frac{e^{-ikR(P,Q)}}{R(P,Q)}\right)d\theta(Q) \qquad \textbf{(4)}$$

$$K2(P,Q) = \int_0^{2\pi}\left(\frac{e^{-ikR(P,Q)}}{R(P,Q)}\right)d\theta(Q) \qquad \textbf{(5)}$$

The integrands in Equations (4) and (5) are singular and the singularities can be removed using the scheme developed by Seybert et. al. [6].

3. Wavelet Multiresolution Analysis

The concepts of wavelet expansion and multiresolution analysis will be summarized in this section. A set of subspaces $\{S_j\}$ where $j\varepsilon Z$ is said to be a multiresolution approximation of $L^2(R)$ if the following relations are applied

$$S_j \subset S_{j+1} \qquad \forall j\varepsilon Z$$

$$\bigcup_{j\in Z} S_j = L^2 \qquad \forall j\varepsilon Z$$

$$\bigcap_{j\in Z} S_j = \{0\}$$

$$f(2x) \in S_{j-1} \;\Leftrightarrow\; f(x) \in S_j$$

$$f(x) \in S_j \;\Leftrightarrow\; f(x - 2^{-j}n) \in S_j \qquad \forall j, n\varepsilon Z$$

where Z is the set of integers.

A wavelet family is generated from what is called mother wavelet. All wavelets of = a family share the same properties and their collection constitutes a complete basis. A wavelet ψ_{jk} is defined as follows:

$$\psi_{jk}(x) = 2^{j/2}\;\psi(2^j x - k) \qquad \textbf{(6)}$$

where j and k are indices indicating scale and location of a particular wavelet. Accordingly, the wavelet family is a collection of wavelet functions $\psi_{jk}(x)$ that are translated along the real axis x, then dilated by 2^j times and the new dilated wavelet is translated along the real line again. The wavelet function must have local (or almost local) support in both spatial and frequency domains.

The decomposition of a discrete signal in orthonormal bases functions is called a multiresolution analysis. An approximation of a function $f(x) \in L^2(R)$ at a resolution of 2^{-j}, can be defined as the projection on different wavelet functions

$$f(x) = \sum_j \sum_k a_{jk}\;\psi_{jk}(x) \qquad j,k = 0..M \qquad \textbf{(7)}$$

where a_{jk} is the amplitude of each wavelet at different resolutions (scales) and locations.

A formal approximation of the unknown function at a given resolution, with a finite number of successive length scales, requires both scaling and translation operators across the expansion dimension. From a practical point of view, the approximate solution of an integral equation can be expressed as a summation of an approximate function $C(x)$ and a series of orthogonal wavelets for the finer details.

The approximation function $C(x)$ can represent the d-c value along the solution domain [1]. More general form of equation (7) can be written as follows

$$f(x) = a_0 C(x) + \sum_{m=m1}^{m2} \sum_{n=n1}^{n2} a_{mn}\ \psi_{mn}(x) \tag{8}$$

where a_0 and a_{mn} are yet to be determined coefficients and

$$C(x) = \begin{cases} 1 & x \in \text{ problem boundary} \\ 0 & \text{otherwise} \end{cases}$$

The summation in Equation (8) is over values of m ranging from $m1$, which corresponds to the larger characteristic length scale, to $m2$, which corresponds to the desired resolution in scaling. Recalling that with reference to $\psi_{00}(x)$, the effective width of $\psi_{mn}(x)$ is changed gradually by a factor of 2^{-m} and its center-a point on the wavelet grid- is shifted by the distance $n2^{-m}$. For a given value of m, the number of wavelet functions, $N(m)=n_2-n_1$, is set so that their centers fall within the problem domain and outside. Hence as m increases, more wavelets and more grid points are involved at each resolution level.

The discrimination between two classes of wavelet families should be considered in the selection of wavelet family for multiresolution analysis. Further, these functions should be orthonormal. The following two examples exhibit the localization properties of wavelets.

Shannon family $\psi(x) = \dfrac{\sin(\pi x/2)}{\pi x/2} \cos(\dfrac{3\pi x}{2})$

Haar family $\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x \le 1 \\ 0 & x > 1 \end{cases}$

The above two wavelet family examples are opposite of each other in terms of their localization properties. The Haar wavelet has good space localization but poor space frequency localization. Its spectrum is non zero when the frequency tends to ∞. It does not have compact support in the space frequency domain. In contrast, the Shannon wavelet has non-compact support in space, hence it has poor space localization. In this work, the space localization is essential to cover the scatterer surface [11].

4. Wavelet Expansion Of Integral Equations

Using the method of moments [12] in solving an integral equation of the form in equations (2) or (3) by substituting (8) into (2) or (3) for an acoustic field function $\phi(Q)$ instead of $f(x)$, we get

$$a_0 \Lambda(P) + \sum_{m=m1}^{m2} \sum_{N(m)} a_{mn}\ F_{mn}(P,Q) = g(P) \tag{9}$$

where $N(m) = n_2(m) - n_1(m)$, the number of points required to cover a domain L at a resolution 2^{-m},

$$\Lambda(P) = \int_L C(Q)\ K(P,Q)\ d(Q) \qquad Q \in L$$

and

$$F_{mn}(P) = \int_L \psi_{mn}(Q)\ K(P,Q)\ dQ$$

where $Q \in L$

In multiresolution analysis in space and frequency each spatial domain is analyzed through different scales or resolutions. Accordingly, the wavelets can be renumbered according to their spatial centers and scales.

Rearranging the elements and coefficients of the Equation (9) and casting them into a matrix form, we get

$$\mathbf{K\ W\ Y} = \mathbf{G} \tag{10}$$

where

$$\mathbf{W^T} = \begin{bmatrix} C(q_1) & C(q_2) & \ldots & C(q_N) \\ \psi_{11}(q_1) & \psi_{12}(q_2) & \ldots & \psi_{1N}(q_N) \\ \ldots & \ldots & \ldots & \ldots \\ \psi_{N1}(q_1) & \psi_{N2}(q_1) & \ldots & \psi_{NN}(q_1) \end{bmatrix}$$

where, matrix $\mathbf{W}$ is cast into squared dimension of $N \wedge N$ and it can be called the wavelet operator or filter.

And,

$$\mathbf{K} = \begin{bmatrix} K(p_1,q_1) & K(p_1,q_2) & \ldots & K(p_1,q_N) \\ K(p_2,q_1) & K(p_2,q_2) & \ldots & K(p_2,q_N) \\ \ldots & \ldots & \ldots & \ldots \\ K(p_N,q_1) & K(p_N,q_2) & \ldots & K(p_N,q_N) \end{bmatrix}$$

$G^T = [\ G(p1) \quad G(p2)\ .. \qquad G(pN)]$

and

$Y^T = [A_{n1}\ .\ A_{n1+1}\ \ldots\ldots A_{n2}\ ]$

The unknown coefficients vector **Y** contains subvectors for each scale level. Each subvector represents a distinct length scale. The elements of each subvector are ordered according to their locations. These unknowns are the wavelet amplitudes along the domain *L*.

$A_{ni} = [a_{ni+1}\ .\ a_{ni+2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{ni+N(ni)}\ ]$

The effective support (nonzero elements) in the operator matrix **W** of a wavelet ψ_{mn} as the interval outside of which the wavelet is practically zero.

5. Numerical Formulation

Equation (10) can be written at different node points i_p and assuming the index of surface elements i_q, the following discretized form of equation (10), for N nodes on the surface, can be written as follows

$$\mathbf{A\ Y = G}$$

and $$\boldsymbol{\varphi} = \mathbf{W^T Y} \qquad (11)$$

where **A** is an NxN matrix. $\boldsymbol{\varphi}$ and **G** are N vectors. For the hard scatterer $\frac{\partial \phi}{\partial n} = 0$, and hence, we can write

$$K(i_p,i_q) = \sum_{i_q=1}^{N} K1(P,Q) \quad \rho(i_q) dL(i_q) \qquad \forall\, i_p \neq i_q$$

$$K(i_p,i_p) = \sum_{i_q=1}^{N} K1(P,Q) \quad \rho(i_q) dL(i_q) - 2\pi \qquad \forall \quad i_p = i_q$$

and

$$G(i_p) = -4\pi\phi^i(i_p)\ \forall i_p = 1..N \qquad (12)$$

where $\boldsymbol{\varphi}$ is an N -dimension vector representing the field strength on the scatterer surface and $\boldsymbol{\varphi}^i$ is the incident field.

The numerical example investigated here is the scattering problem of a plane incident wave impinging on a rigid sphere. The incoming unit plane wave travels toward the scatterer along the positive direction of z-axis in the cylindrical coordinates described as e^{-ikz}. The surface field $\boldsymbol{\varphi}$ is computed using the proposed method. The results will be verified via comparison with the analytical solution. The benefits of the method will be validated by comparing the accuracy and sparsity ratio for different wavelet bases support lengths and sparsity thresholds.

On the surface of a hard sphere, the analytical solution of equation (1) for plane incident wave can be expressed as [7]

$$\phi = \frac{-i}{(ka)^2} \sum_{n=0}^{\infty} (-i)^n (2n+1) \frac{P_n(\cos\vartheta)}{h_n^{(2)'}(ka)} \qquad (13)$$

where ϕ is the total field as defined in (3) and ϑ is the co-latitude angle and the incidence angle is taken to be *zero* in this application. P_n is the Legendre polynomial of order n and h_n is the spherical hankel function and a is the radius of the sphere.

Equation (11) is solved using the proposed system of equation (3)-(8) and the proposed discretization scheme considering different N divisions. Using Daubechies wavelet, the resulting operator matrix **W** is square and the maximum number of scaling levels M is defined by $N=2^M$.

The results obtained are then compared to the analytical solution. The normalized error is defined as the ratio between the field (ϕ) error to the analytical solution as follows

$$\text{Normalized Error ratio} = \frac{\| \phi_{wvl} - \phi_{ana} \|}{\| \phi_{ana} \|} \qquad (14)$$

where ϕ_{wvl} is the resulted solution form Equation (11) and ϕ_{ana} is the analytical solution given in (13) and $\|.\|$ is the $L2$ norm.

The second comparison parameter is the percentage sparsity S and it is defined as

$$S = N_o/N^2 \ x\ 100\% \qquad (15)$$

where N_o is the number of zeros in the matrix **A** after thresholding and N is the matrix dimension.

The numerical results are obtained using the direct discretization of the integration in equation (12). These results are compared with that obtained by the proposed method. The comparison discusses the effect of wavelet expansion on saving the computational burden through matrix sparsity and accuracy of the solution. Different thresholds are tested for increasing matrix sparsity and the accuracy computed for each trial.

6. SOLUTION ALGORITHM

For electromagnetic problems, it was reported that almost identical results are obtained using Daubechies and wavelet-like bases [13] and [14]. Daubechies wavelets are strictly localized in space and approximately localized in spatial frequency as discussed in Section 3 since Haar wavelet is a special case of Daubechies family of order 1. Increasing vanishing moment order produces smoother broader basis function with sharper spectral cut-off frequencies [14]. Also, these wavelets can approximate finer resolutions near boundaries and corners of scattering surfaces. In general, classical wavelets seem to be good in computing low frequency scattering and antenna problems [15]. For these reasons and due to similar mathematical formulation of acoustic scattering, Daubechies wavelets are more appropriate for that problem. Many recent works employed Daubechies wavelets in solving scattering problems [2], [3], [4] and [16].

The solution method can be summarized in the following steps which can be used to solve an integral equation of the second kind.

1. Construct a grid of points at the surface of the considered body.
2. Build system of equations for the integral equation kernel at all points of the grid given by step 1.
3. Build the wavelet multiresolution operator as given in equation (11) considering the Daubechies wavelet filters [17].
4. Apply the operator to the kernel matrix (Direct multiplication of square matrices).
5. Scan the resulting matrix elements and get the maximum element value

$$\text{MAXA} = \text{MAX}(\mathbf{A})$$

6. Scan the matrix and compare each element with a threshold as a ratio of the obtained maximum (MAXA) in step 5 and nullify the element which is less than that threshold as follows

$$\textit{For}\ \forall\ i,j$$

$$\textit{If}\ \mathbf{A}(i,j) < (THR * MAXA)\quad \textit{then}\ \ \mathbf{A}(i,j)=0$$

where *THR* is taken with different values as indicated on figures. An average value is around 10^{-3} [5].

7. Solve the resulting sparse matrix considering the given field variation along the solution domain.

7. RESULTS

The problem of acoustic scattering on a hard sphere is solved both numerically using wavelets and analytically. Figures 1 and 3 show comparisons of both solutions graphically for two different Daubechies (2 & 3) wavelet orders for ka=2, N=32 and 10^{-3} threshold. Each figure contains the exact solution of the corresponding problem for comparison.

Figures 2 and 4 show trend graphs between the sparsity and for the cases of Figures 1 and 2. The trend behavior indicates oscillatory behavior of the error at ka=2 when Db2 is used. This behavior can be interpreted based on very small variation in error within the order of 0.015%.

Figure 4 shows monotonic decrease in error with sparsity as expected. These trends also show that the error variation within the used sparsity threshold range is small (in the order of 0.1% at most).

Table 1 A comparison between different solutions using several wavelet bases

Ka	basis	%S	THR	Error	N
1	Db1	33.59	10^{-2}	0.031	16
2	Db2	44.53	10^{-3}	0.044	32
2	Db3	50.97	10^{-3}	0.043	32
2	Db4	61.04	10^{-3}	0.044	32
2.5	Db2	46.29	10^{-3}	0.067	32
2.5	Db3	32.27	10^{-3}	0.067	32

Test results on different wavelet orders, thresholds and noncharacteristic wavenumbers ($ka<\pi$) are summarized in Table1. These results show that the error dose not exceed about 6% and sparsity reaches about 60%. Lower thresholds can give higher sparsity with little deterioration in accuracy as expected from results in Figures 2 and 4.

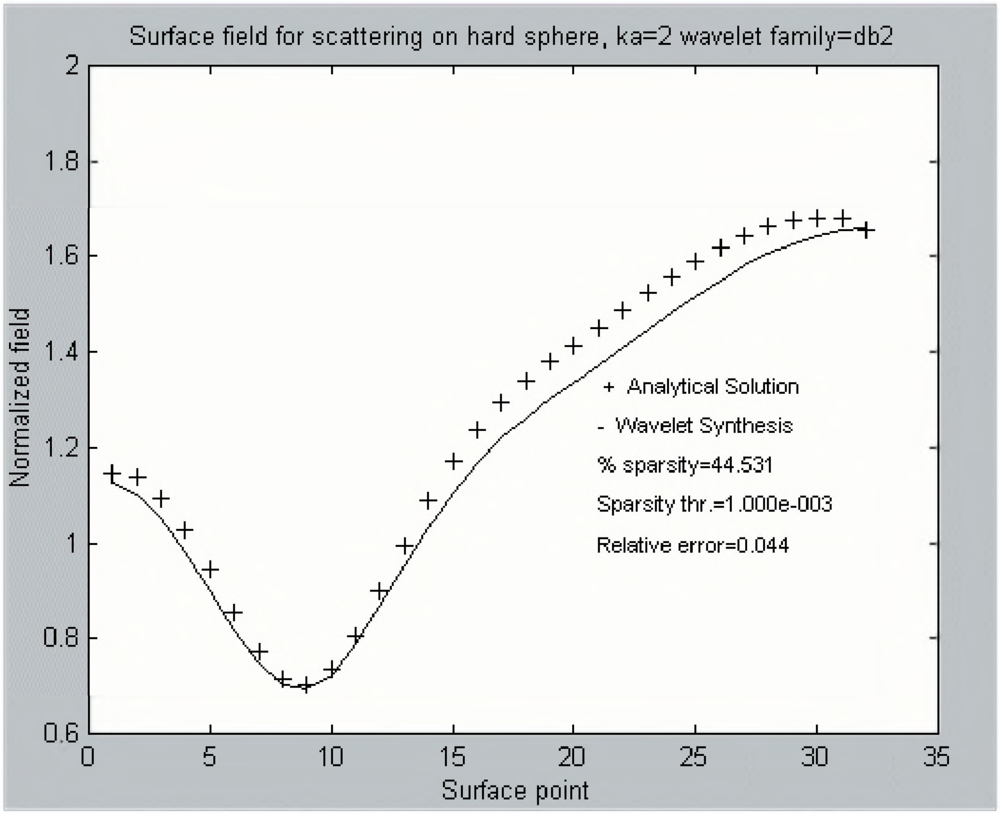

Figure 1. Scattering of plane wave of *ka=2* and db2 wavelet basis

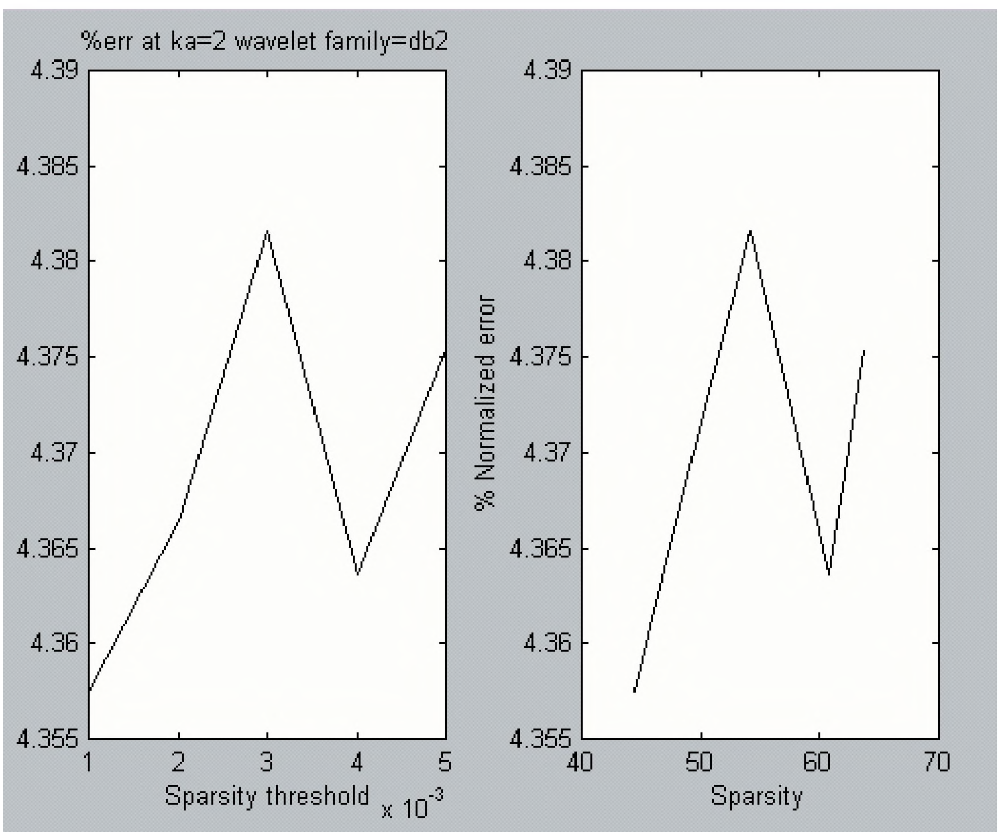

Figure 2. Error trending for db2 basis and *ka*=2

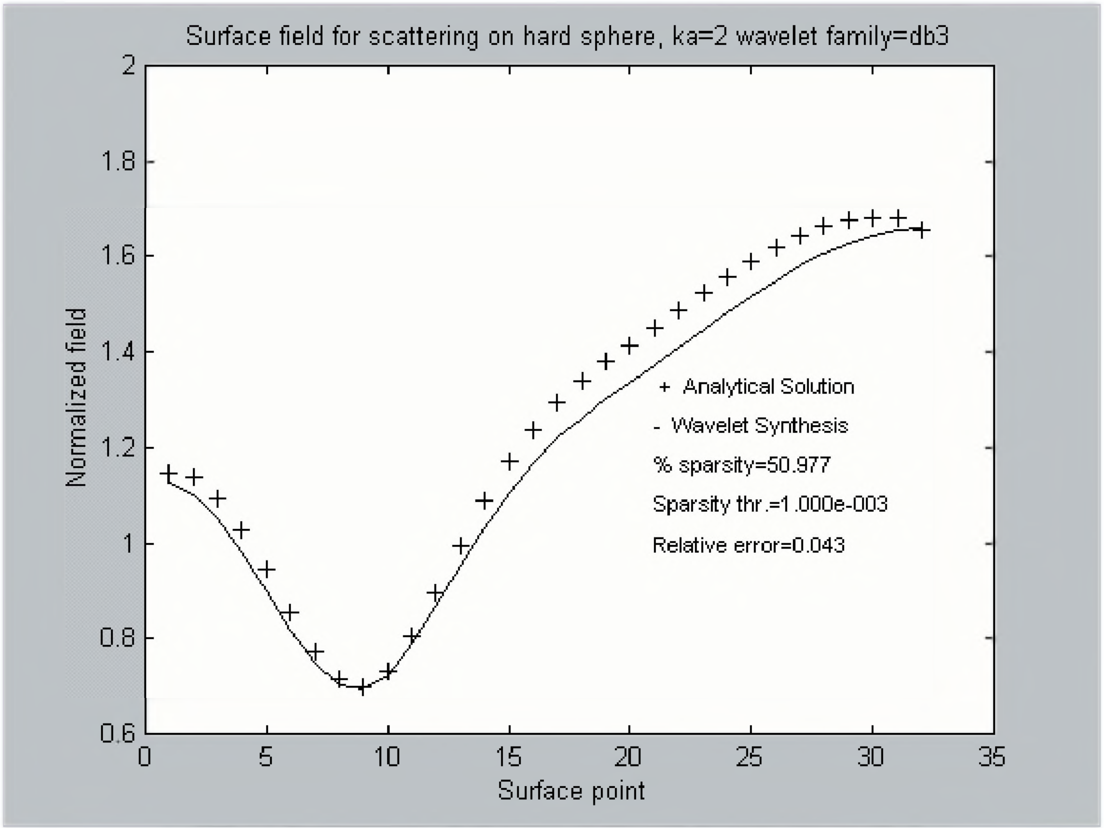

Figure 3. Scattering of plane wave of *ka=2* and db3 wavelet basis

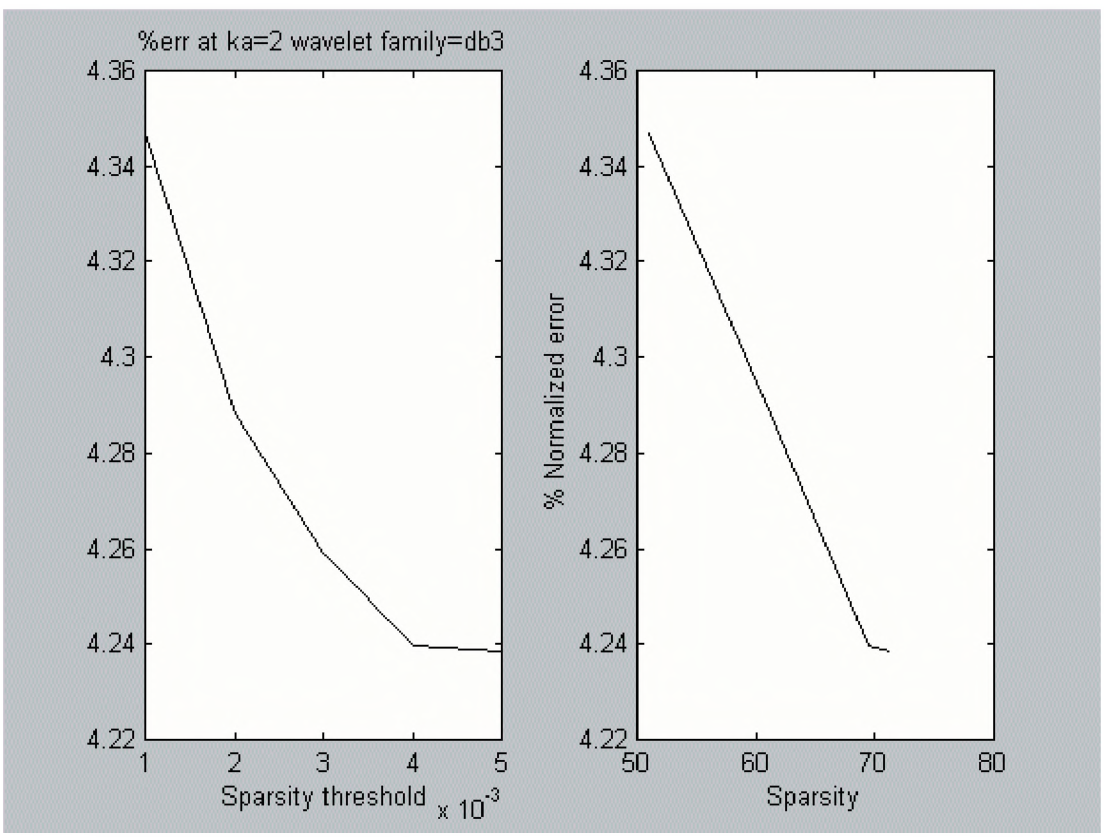

Figure 4. Error trending for db3 basis and *ka*=2

8. Conclusions

A multiresolution wavelet analysis has been applied to solve Helmholtz integral equation for acoustic scattering. The integral equation is solved using moment method with wavelet basis. The wavelet expansion covers the scatterer surface for distributing the wavelet localized functions. Applying an appropriate threshold on the resulting matrix makes it sparse. The sparsity of the resulted matrix can be efficiently utilized to get faster solution of the problem for larger dimensions.

Different comparisons are conducted for different wavelet bases, sparsification thresholds versus solution accuracy. The accuracy of the proposed solution is assessed with respect to exact solution of the problem. The results show that Daubechies wavelet family is successful for the acoustic scattering applications like electromagnetic problems as presented in many references. The results also prove that the proposed square operator can be successfully applied to solve the scattering problem.

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