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WAVELET TRANSFORMS IN EUCLIDEAN SPACES —THEIR RELATION WITH WAVE FRONT SETS AND BESOV, TRIEBEL-LIZORKIN SPACES—

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Abstract. We define a class of wavelet transforms as a continuous and microlocal version of the Littlewood-Paley decompositions. Hörmander's wave front sets as well as the Besov and Triebel-Lizorkin spaces may be characterized in terms of our wavelet transforms.

Introduction. We define a class of wavelet transforms as a continuous and micro-local version of the Littlewood-Paley decompositions. Hörmander's wave front sets as well as the Besov and Triebel-Lizorkin spaces may be characterized in terms of our wavelet transforms. We remark that the components of our decompositions are not linearly independent but can be treated as if they were. This paper consists of two parts. The first part treats the comparison between the wave front sets defined by our wavelet transforms and Hörmander's wave front sets. The second part gives the characterization of the Besov and Triebel-Lizorkin spaces by using our wavelet transforms.

First, we define our wavelet transforms as follows:

DEFINITION 1. Suppose that a function $\psi(x)$ (called a wavelet) has the following properties: $\psi(x)\in\mathcal{S}(\boldsymbol{R}^n)$, $\hat{\psi}(\xi)\in C_0^\infty(\boldsymbol{R}^n)$ and $\hat{\psi}(\xi)\geqq 0$. Let $\Omega=\operatorname{supp}\hat{\psi}(\xi)$ be in a neighbourhood of $(0,\dots,0,1)$. When $n=1$, $\Omega\subset(0,\infty)$, while when $n\geqq 2$, Ω is connected, does not contain the origin 0 and $\psi(x)=\psi(rx)$ for any $r\in SO(n)$ satisfying $r(0,\dots,0,1)=(0,\dots,0,1)$. Let r_ξ be any rotation which sends $\xi/|\xi|$ to $(0,\dots,0,1)$. Then our wavelet transform is defined as follows: for $f(t)\in\mathcal{S}'(\boldsymbol{R}^n)$, $(x,\xi)\in\boldsymbol{R}^{2n}$,

$$W_\psi f(x,\xi)=\begin{cases}\int_{\boldsymbol{R}} f(t)|\xi|^{1/2}\overline{\psi(\xi(t-x))}\,dt\,, & \text{if} \quad n=1\,,\\ \int_{\boldsymbol{R}^n} f(t)|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(t-x))}\,dt\,, & \text{if} \quad n\geqq 2\,.\end{cases}$$

Here $\mathcal{S}(\boldsymbol{R}^n)$ stands for the Schwartz class and $C_0^\infty(\boldsymbol{R}^n)$ consists of functions which are smooth and compactly supported.

REMARK 1. $W_\psi f(x,\xi)$ is rewritten as follows:

$$\int_{\boldsymbol{R}^n}\hat{f}(\tau)\cdot|\xi|^{-n/2}\hat{\psi}(|\xi|^{-1}r_\xi\tau)\cdot e^{i\tau x}d\tau\,.$$

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From this, the meaning of our wavelet transforms is clear.

REMARK 2. Our wavelet transforms in $\boldsymbol{R}^n$ are the reduced versions of those defined by Murenzi [6].

REMARK 3. The domain of a wavelet transformation is usually the L_2-space (see [1]), but can be extended to $\mathscr{S}'(\boldsymbol{R}^n)$, that is, the dual space of $\mathscr{S}(\boldsymbol{R}^n)$.

Now, we define our wave front set $WF_\psi(f)$ $(\subset \boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi)$ of $f \in \mathscr{S}'(\boldsymbol{R}^n)$ as follows:

DEFINITION 2. We say $(x_0, \xi^0) \notin WF_\psi(f)$ if there exists a neighbourhood $U(x_0)$ of x_0 and a conic neighbourhood $\Gamma(\xi^0)$ of ξ^0 such that $W_\psi f(x, \xi) = O(|\xi|^{-N})$ as $|\xi|$ tends to ∞ for any $N \in \boldsymbol{N}$ in $U(x_0) \times \Gamma(\xi^0)$. Here $\boldsymbol{N}$ stands for the set of all positive integers.

Moreover, we define the refinement $WF^{(s)}_\psi(f)$ as follows:

DEFINITION 3.

$$(x_0, \xi^0) \notin WF^{(s)}_\psi(f) \Leftrightarrow \iint_{U(x_0) \times \Gamma(\xi^0)} |W_\psi f(x, \xi)|^2 (1 + |\xi|^2)^s dx d\xi < \infty .$$

It is easy to prove that if $f \in L_2(\boldsymbol{R}^n)$, then $WF_\psi(f)$ is contained in the closure of $\bigcup_{s \geqq 0} WF^{(s)}_\psi(f)$.

We need the following definition to state Theorem 1.

DEFINITION 4. For $\Omega = \operatorname{supp} \psi$, let $\operatorname{cone} \Omega = \{t\xi; \xi \in \Omega, t > 0\}$. Let W be a subset of $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ conical in the ξ variables and denote by $\operatorname{proj}_x W$ the projection of W onto the x-space. We say $(x_0, \xi^0) \notin \bar{W}^\psi$ if $x_0 \notin \operatorname{porj}_x W$ and $\xi^0 \in \boldsymbol{R}^n$, or $x_0 \in \operatorname{proj}_x W$ and $r(\operatorname{cone} \Omega)$ does not intersect $\{\xi \in \boldsymbol{R}^n; (x_0, \xi) \in W\}$ for any $r \in SO(n)$ with $r(\operatorname{cone} \Omega)$ containing ξ^0.

That is to say, the set $\bar{W}^\psi$ is the expanded set of W only in the frequency space.

THEOREM 1. *Let $f \in L_2(\boldsymbol{R}^n)$, and $s \geqq 0$. When $n = 1$, $WF^{(s)}_\psi(f) = WF^{(s)}(f)$. When $n \geqq 2$, $WF^{(s)}_\psi(f) \subseteq \overline{WS^{(s)}(f)}^\psi$ and $WF^{(s)}(f) \subseteq \overline{WF^{(s)}_\psi(f)}^\psi$. We have the same inclusion relations between $WF_\psi(f)$ and $WF(f)$.*

In the second part of this paper we characterize the Besov and Triebel-Lizorkin spaces by using our wavelet transforms. We use continuous decompositions not only for the radial direction but also for the unit sphere of the frequency space. We recall the definitions of those spaces by Peetre [4] and Triebel [5].

DEFINITION 5. Let $\phi(x)$ be a rapidly decreasing function whose Fourier transform is compactly supported in $1/2 \leqq |\xi| \leqq 2$. Moreover, suppose that any half line starting from the origin intersects $\operatorname{supp} \hat{\phi}(\xi)$. Let $\phi_r(x)$ be $r^n \phi(rx)$. Then $\hat{\phi}_r(\xi)$ is equal to $\hat{\phi}(\xi/r)$.

DEFINITION. A function f is said to belong to the Besov space $\dot{B}^s_{p,q}(\boldsymbol{R}^n)$ ($s>0$, $1\leqq p, q\leqq\infty$) if

$$\| \| r^s(\phi_r * f(x)) \|_{L_p(dx)} \|_{L_q(dr/r)} < \infty .$$

DEFINITION. A function f is said to belong to the Triebel-Lizorkin space $\dot{F}^s_{p,q}(\boldsymbol{R}^n)$ ($s>0$, $1\leqq p<\infty$, $1\leqq q\leqq\infty$) if

$$\| \| r^s(\phi_r * f(x)) \|_{L_q(dr/r)} \|_{L_p(dx)} < \infty .$$

THEOREM 2. *A function f belongs to $\dot{B}^s_{p,q}(\boldsymbol{R}^n)$ ($s>0$, $1\leqq p, q\leqq\infty$) if and only if the following condition holds*:

$$\| \| |\xi|^{s+n/2} W_\psi f(x, \xi) \|_{L_p(dx)} \|_{L_q(d\xi/|\xi|^n)} < \infty .$$

THEOREM 3. *A function f belongs to $\dot{F}^s_{p,q}(\boldsymbol{R}^n)$ ($s>0$, $1\leqq p<\infty$, $1\leqq q\leqq\infty$) if and only if the following condition holds*:

$$\| \| |\xi|^{s+n/2} W_\psi f(x, \xi) \|_{L_q(d\xi/|\xi|^n)} \|_{L_p(dx)} < \infty .$$

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1. Proof of Theorem 1. As we have already defined, a wavelet $\psi(x)$ is essentially of two parameters, since it is rotationally invariant around Θ when $n\geqq 2$. For the purpose of proving Theorem 1, we need three propositions. For simplicity, we assume $\Theta=(0, \dots, 1)$.

PROPOSITION 1 (Parseval's formula and the inversion formula). *For $f, g\in L_2(\boldsymbol{R}^n)$, we have*

$$\iint W_\psi f(x, \xi) \overline{W_\psi g(x, \xi)} dx d\xi = C_\psi \int f(t)\overline{g(t)} dt ,$$

where

$$C_\psi = (2\pi)^n \int |\hat{\psi}(\xi)|^2 d\xi / |\xi|^n .$$

From this, we also have

$$f(t) = C_\psi^{-1} \iint W_\psi f(x, \xi) \cdot |\xi|^{n/2} \psi(|\xi| r_\xi(t-x)) dx d\xi ,$$

when $n\geqq 2$. When $n=1$, $|\xi| r_\xi(t-x)$ is replaced by $\xi(t-x)$. For $f\in\mathscr{S}'$, this inversion formula holds in the sense of distribution.

PROPOSITION 2 (the local property). *If x_0 does not belong to* supp *f, then there exists a neighbourhood $U(x_0)$ of x_0 such that $W_\psi f(x, \xi)$ is rapidly decreasing in ξ uniformly in $x \in U(x_0)$.*

PROPOSITION 3 (the global Sobolev property).

$$f \in H^s(\boldsymbol{R}^n) \Leftrightarrow \iint |W_\psi f(x, \xi)|^2 (1+|\xi|^2)^s dx d\xi < \infty .$$

PROOF OF PROPOSITION 1. We use the method which Daubechies [1] employed to prove the statement in the case $n=1$. We have

$$(1) \qquad \iint W_\psi f(x, \xi) \overline{W_\psi g(x, \xi)} dx d\xi$$

$$= \iint \left[\int \hat{f}(\tau) \overline{|\xi|^{-n/2} \hat{\psi}(|\xi|^{-1} r_\xi \tau) e^{-i\tau \cdot x}} d\tau \right] \cdot \overline{W_\psi g(x, \xi)} dx d\xi$$

$$= (2\pi)^n \int d\xi |\xi|^{-n} \int d\tau \hat{f}(\tau) \cdot \overline{\hat{g}(\tau)} \cdot \hat{\psi}(|\xi|^{-1} r_\xi \tau)^2 .$$

We change the variable ξ to $\omega = |\xi|^{-1} r_\xi \tau$. The right hand side of (1) is equal to

$$(2\pi)^n \int d\tau \hat{f}(\tau) \cdot \overline{\hat{g}(\tau)} \int d\omega / |\omega|^n \hat{\psi}(\omega)^2 = C_\psi \int f(x) \cdot \overline{g(x)} dx .$$

q.e.d.

PROOF OF PROPOSITION 2. We take n as 1. The proof is similar to that in the case $n \geqq 2$. Because there exists a neighbourhood $U_1(x_0)$ of x_0 such that $f=0$ in $U_1(x_0)$, it follows that

$$|W_\psi f(x, \xi)| = \left| \int f(t) |\xi|^{1/2} \overline{\psi(\xi(t-x))} dt \right|$$

$$\leqq \|f\|_{L_2} \cdot \left(\int_{U_1(x_0)^c} |\xi| |\psi(\xi(t-x))|^2 dt \right)^{1/2} .$$

Since ψ belongs to $\mathscr{S}(\boldsymbol{R})$, there exists a neighbourhood $U(x_0) \Subset U_1(x_0)$ satisfying this proposition. q.e.d.

PROOF OF PROPOSITION 3. It suffices to prove the statement when $n \geqq 2$. The proof follows from direct application of Parseval's formula. We have

$$\iint |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dxd\xi$$
$$=(2\pi)^n\int d\tau|\hat{f}(\tau)|^2\int d\xi/|\xi|^n(1+|\xi|^2)^s\hat{\psi}(|\xi|^{-1}r_\xi\tau)^2$$
$$=(2\pi)^n\int d\tau|\hat{f}(\tau)|^2\int d\omega/|\omega|^n(1+|\tau|^2/|\omega|^2)^s\hat{\psi}(\omega)^2 .$$

We use the polar coordinate representation of $\omega=(r,\theta)$ and denote $\int\hat{\psi}(r,\theta)^2d\theta$ by $S(r)$. Then the second integral in the last term equals $\int dr/rS(r)(1+|\tau|^2/r^2)^s$. By the assumption on $\hat{\psi}(\omega)$, $\operatorname{supp} S(r)$ is a compact set contained in $(0,\infty)$. Thus $\int dr/rS(r)(1+|\tau|^2/r^2)^s\approx(1+|\tau|^2)^s$. q.e.d.

Now, we are ready to prove Theorem 1.

PROOF OF THEOREM 1. We may assume $n\geqq 2$.

Step 1. Suppose that $(0,\xi^0)$ does not belong to the set $\overline{WF^{(s)}(f)}^\psi$. We suppose $0\in\operatorname{proj}_x WF^{(s)}(f)$. Let $\Gamma(\xi^0)$ be the union of $r(\operatorname{cone}\Omega)$ for all rotations r such that $\xi^0\in r(\operatorname{cone}\Omega)$. Then there exist a function $\phi(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that $\phi=1$ near $x=0$ and a conic neighbourhood $\Gamma'(\xi^0)\subset\Gamma(\xi^0)$ such that $\int_{\Gamma'(\xi^0)}|(\hat{\phi f})(\xi)|^2(1+|\xi|^2)^s d\xi<\infty$. What we want to say is that there exist a conic neighbourhood $\tilde{\Gamma}(\xi^0)$ of ξ^0 and a neighbourhood $U(0)$ of 0, satisfying

$$\iint_{U(0)\times\tilde{\Gamma}(\xi^0)} |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dxd\xi<\infty .$$

Using the inversion formula, we divide $W_\psi f(x,\xi)$ into two parts:

$$W_\psi f(x,\xi)=|\xi|^{n/2}\int(\phi f)(t)\cdot\overline{\psi(|\xi|r_\xi(t-x))}dt+|\xi|^{n/2}\int((1-\varphi)f(t)\cdot\overline{\psi(|\xi|r_\xi(t-x))}dt .$$

If we take a set $U(0)\Subset\{\phi(x)\equiv 1\}$, then by the argument of Proposition 2, the second term is rapidly decreasing in $|\xi|$ uniformly in $x\in U(0)$. Therefore, $(0,\xi_0)\notin WF_\psi^{(s)}((1-\phi)f)$. On the other hand, if we take $\tilde{\Gamma}(\xi^0)$ sufficiently small, then we obtain

$$\iint_{U(0)\times\tilde{\Gamma}(\xi^0)} |W_\psi(\phi f)(x,\xi)|^2(1+|\xi|^2)^s dxd\xi$$
$$\leq\int_{\tilde{\Gamma}(\xi^0)}d\xi\int_{\boldsymbol{R}_x^n}|W_\psi(\phi f)(x,\xi)|^2(1+|\xi|^2)^s dxd\xi$$
$$=(2\pi)^n\int_{\boldsymbol{R}_\tau^n}d\tau|(\hat{\phi f})(\tau)|^2\int_{\tilde{\Gamma}(\xi^0)}d\xi/|\xi|^n(1+|\xi|^2)^s\hat{\psi}(|\xi|^{-1}r_\xi\tau)^2 .$$

We change the variable ξ to $\omega=|\xi|^{-1}r_\xi\tau$ as before. We may assume that $\omega\in\Omega$. Therefore, we can see that τ stays in $\Gamma'(\xi^0)$ because we took $\tilde{\Gamma}(\xi^0)$ sufficiently small. Thus the last term of the above inequality is bounded by

$$(2\pi)^n\int_{\Gamma'(\xi^0)}d\tau|(\widehat{\phi f})(\tau)|^2\int_\Omega d\omega/|\omega|^n(1+|\tau|^2/|\omega|^2)^s\hat{\psi}(\omega)^2$$
$$\leqq C\int_{\Gamma'(\xi^0)}|(\widehat{\phi f})(\tau)|^2(1+|\tau|^2)^s d\tau<\infty\,,$$

where C is a constant. Therefore, we have $(0,\xi^0)\notin WF_\psi^{(s)}(\phi f)$.

Step 2. Suppose that $(0,\xi^0)$ does not belong to the set $\overline{WF_\psi^{(s)}(f)}^\psi$. If we take a conic neighbourhood $\Gamma'(\xi^0)$ of ξ^0 as in Step 1, then there exists a neighbourhood $U(0)$ of $x=0$ such that

$$\iint_{U(0)\times\Gamma'(\xi^0)}|W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dxd\xi<\infty\,.$$

Now using the inversion formula, we divide f into two parts $f=f_{\Gamma'}+f_{\Gamma'^c}$, where

$$f_{\Gamma'}(t)=C_\psi^{-1}\iint_{\Gamma'(\xi^0)\times\boldsymbol{R}_x^n}W_\psi f(x,\xi)\cdot|\xi|^{n/2}\psi(|\xi|r_\xi(t-x))dxd\xi\,,$$

$$f_{\Gamma'^c}(t)=C_\psi^{-1}\iint_{\Gamma'(\xi^0)^c\times\boldsymbol{R}_x^n}W_\psi f(x,\xi)\cdot|\xi|^{n/2}\psi(|\xi|r_\xi(t-x))dxd\xi\,.$$

Then

$$\widehat{f_{\Gamma'^c}}(\tau)=C_\psi^{-1}\int_{\Gamma'(\xi^0)^c}\int_{\boldsymbol{R}_x^n}W_\psi f(x,\xi)\cdot|\xi|^{-n/2}\hat{\psi}(|\xi|^{-1}r_\xi\tau)e^{-i\tau\cdot x}dxd\xi\,.$$

If we take a sufficiently small conic neighbourhood $\tilde{\Gamma}(\xi^0)$ of ξ^0, then we obtain

$$\hat{\psi}(|\xi|^{-1}r_\xi\tau)\equiv 0\qquad\text{for any }\tau\in\tilde{\Gamma}(\xi^0)\quad\text{and}\quad\text{for any }\xi\in\Gamma'(\xi^0)^c\,.$$

Therefore, it follows that $(0,\xi^0)\notin WF^{(s)}(f_{\Gamma'^c})$.

Next, we choose a function $\phi(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that supp $\phi(x)$ is compactly supported in $U(0)$ and that $\phi(x)\equiv 1$ in some neighbourhood $U_1(0)$ of 0. Then, we further divide $f_{\Gamma'}(t)$ into two parts $f_{\Gamma'}=f_{\Gamma',\phi}+f_{\Gamma',1-\phi}$, where

$$f_{\Gamma',\phi}(t)=C_\psi^{-1}\iint_{\Gamma'(\xi^0)\times\boldsymbol{R}_x^n}\phi(x)\cdot W_\psi f(x,\xi)\cdot|\xi|^{n/2}\psi(|\xi|r_\xi(t-x))dxd\xi\,,$$

$$f_{\Gamma',1-\phi}(t)=C_\psi^{-1}\iint_{\Gamma'(\xi^0)\times\boldsymbol{R}_x^n}(1-\phi(x))\cdot W_\psi f(x,\xi)\cdot|\xi|^{n/2}\psi(|\xi|r_\xi(t-x))dxd\xi\,.$$

Let $U_2(0)\Subset\{\phi(x)\equiv 1\}$. Then we can easily see that $f_{\Gamma',1-\phi}(t)\in C^\infty(U_2(0))$ by Proposition 2 and the exchange of order of differentiation and integration. Therefore, it follows that $(0,\xi^0)\notin WF^{(s)}(f_{\Gamma',1-\phi})$. Lastly, we want to show that $(0,\xi^0)\notin WF^{(s)}(f_{\Gamma',\phi})$. This is the heart of the matter in proving Theorem 1. In fact, we can show more strongly that $f_{\Gamma',\phi}$ belongs globally to the Sobolev space $H^s(\boldsymbol{R}^n)$. Its Fourier transform is given by

$$\widehat{f_{\Gamma',\phi}}(\tau)=C_\psi^{-1}\iint_{\Gamma'(\xi^0)\times\boldsymbol{R}_x^n}\phi(x)\cdot W_\psi f(x,\xi)\cdot|\xi|^{-n/2}\hat{\psi}(|\xi|^{-1}r_\xi\tau)e^{-i\tau\cdot x}dxd\xi\,.$$

If we put $g(x,\xi)=\phi(x)\cdot W_\psi f(x,\xi)\cdot(1+|\xi|^2)^{s/2}$, then it follows from the hypothesis and from the fact that $\operatorname{supp}\phi(x)$ is contained in $U(0)$ that

$$\iint_{\Gamma'(\xi^0)\times\boldsymbol{R}_x^n}|g(x,\xi)|^2dxd\xi<\infty\,.$$

If we denote the Fourier transform of $g(x,\xi)$ with respect to x by $\hat{g}(\tau,\xi)$, we have

$$\begin{aligned}&\widehat{f_{\Gamma',\phi}}(\tau)(1+|\tau|^2)^{s/2}\\&\quad=C_\psi^{-1}\iint_{\Gamma'(\xi^0)\times\boldsymbol{R}_x^n}g(x,\xi)e^{-i\tau\cdot x}\cdot|\xi|^{-n/2}\hat{\psi}(|\xi|^{-1}r_\xi\tau)((1+|\tau|^2)/(1+|\xi|^2))^{s/2}dxd\xi\\&\quad=C_\psi^{-1}(2\pi)^{n/2}\int_{\Gamma'(\xi^0)}\hat{g}(\tau,\xi)\cdot K(\tau,\xi)d\xi\,.\end{aligned}$$

Here, $K(\tau,\xi)$ is defined by $|\xi|^{-n/2}\hat{\psi}(|\xi|^{-1}r_\xi\tau)((1+|\tau|^2)/(1+|\xi|^2))^{s/2}$.

Since $\operatorname{supp}\hat{\psi}$ is compact, there exists a constant C such that

$$|K(\tau,\xi)|\leqq C|\xi|^{-n/2}\hat{\psi}(|\xi|^{-1}r_\xi\tau)\,.$$

Therefore, since $0\notin\operatorname{supp}\hat{\psi}$, by the argument given in the proof of Proposition 1, the integral $\int|K(\tau,\xi)|^2d\xi$ is bounded with bound $(2\pi)^{-n}C_\psi C^2$. Consequently, we obtain the following inequality:

$$\begin{aligned}\int|\widehat{f_{\Gamma',\phi}}(\tau)|^2(1+|\tau|^2)^s d\tau&\leqq C_\psi^{-1}C^2\int\left(\int_{\Gamma'(\xi^0)}|\hat{g}(\tau,\xi)|^2d\xi\right)d\tau\\&=C'\int_{\Gamma'(\xi^0)}d\xi\int_{\boldsymbol{R}_\tau^n}|\hat{g}(\tau,\xi)|^2d\tau\end{aligned}$$

$$=C'\iint_{\Gamma'(\xi^0)\times \boldsymbol{R}_x^n} |g(x,\xi)|^2 dx d\xi<\infty .$$

q.e.d.

2. Proof of Theorems 2 and 3.

PROOF OF THEOREM 2. Sufficiency: For simplicity, let $\psi_{|\xi|,r_\xi}*f(x)=|\xi|^{n/2}W_\psi f(x,\xi)$ and $\phi_{|\xi|}=\int \psi_{|\xi|,r_\xi}d\theta_\xi$, where $d\theta_\xi$ is the Haar measure on S^{n-1}. Then, $\hat{\phi}_r(\xi)$ is compactly supported in the domain $C_1 r\leqq|\xi|\leqq C_2 r$ because supp $\hat{\psi}$ is compact and does not contain 0, and any half line starting from the origin intersects supp $\hat{\phi}_r(\xi)$. We have

$$\left(\int\left|\int \psi_{|\xi|,r_\xi}*f(x)d\theta_\xi\right|^p dx\right)^{q/p}\leqq\left(\int\left(\int|\psi_{|\xi|,r_\xi}*f(x)|^p dx\right)^{1/p}d\theta_\xi\right)^q$$
$$\leqq\int\left(\int|\psi_{|\xi|,r_\xi}*f(x)|^p dx\right)^{q/p}d\theta_\xi .$$

The first inequality is due to the continuous version of the Minkowski inequality and the second one is due to the Hölder inequality. Integrating both sides of this inequality with respect to $|\xi|^{sq-1}d|\xi|$, we can see that the usual Besov norm is bounded by the Besov norm via the wavelet transform.

Necessity: Let

$$\hat{\sigma}_r(\tau)^2=(2\pi)^n C_\psi^{-1}\int\hat{\psi}(r^{-1}r_\xi\tau)^2 d\theta_\xi$$

(cf. the proof of Proposition 1). Then, supp $\hat{\sigma}_r(\tau)$ is contained in the domain $C_1 r\leqq|\tau|\leqq C_2 r$ and

$$\int\hat{\sigma}_r(\tau)^2 dr/r=1 ,$$

that is,

$$\int\sigma_r*\sigma_r(x)dr/r=\delta(x) .$$

By using this continuous decomposition of unity, we have

$$\|\psi_{|\xi|,r_\xi}*f(x)\|_{L_p(dx)}\leqq\int\|\psi_{|\xi|,r_\xi}*\sigma_r\|_{L_1(dx)}\cdot\|f*\sigma_r\|_{L_p(dx)}dr/r . \tag{2}$$

Because the Fourier transform of $\psi_{|\xi|,r_\xi}*\sigma_r$ is identically zero unless $C_3|\xi|\leqq r\leqq C_4|\xi|$, and because the L_1-norm of $\psi_{|\xi|,r_\xi}$ and that of σ_r is bounded, the last term in (2) does not exceed

$$(3)\qquad C\int_{C_3|\xi|}^{C_4|\xi|}\|f*\sigma_r\|_{L_p(dx)}dr/r=C\int_{C_3}^{C_4}\|f*\sigma_{t|\xi|}(x)\|_{L_p(dx)}dt/t\,.$$

The right hand side of (3) is independent of the rotation $d\theta_\xi$ and moreover

$$\big\||\xi|^s\|f*\sigma_{t|\xi|}\|_{L_p(dx)}\big\|_{L_q(d\xi/|\xi|^n)}=t^{-s}\big\||\xi|^s\|f*\sigma_{|\xi|}\|_{L_p(dx)}\big\|_{L_q(d\xi/|\xi|^n)}\,.$$

Thus we can conclude that the Besov norm via the wavelet transform is bounded by the usual Besov norm. q.e.d.

PROOF OF THEOREM 3. Sufficiency: As in Theorem 2, let $\phi_{|\xi|}=\int\psi_{|\xi|,r_\xi}d\theta_\xi$. Then, we have

$$\begin{aligned}||\xi|^s(\phi_{|\xi|}*f(x))|^q&\leqq\left|\int\Big(|\psi_{|\xi|,r_\xi}*f(x)|\,|\xi|^s\Big)d\theta_\xi\right|^q\\&\leqq\int|(\psi_{|\xi|,r_\xi}*f(x))|\,\xi|^s|^q d\theta_\xi\,.\end{aligned}$$

Hence, we can easily see that the usual Triebel-Lizorkin norm is bounded by the norm via the wavelet transform.

Necessity: This part needs very deep results which are continuous versions of the work of Fefferman-Stein [2] and Triebel [5].

First, we state the results without proof. The proof is carried out in the same way as in the discrete case. See [2], [5].

Claim 1 (continuous version of [2]). Let $f(x, y)$ be a function of $(x, y)\in \boldsymbol{R}_x^n\times\boldsymbol{R}_y^n$ and $Mf(x, y)$ the Hardy-Littlewood maximal function of $f(x, y)$ with respect to x. Then, we have

$$\left\|\left(\int|Mf(x,y)|^q dy/|y|^n\right)^{1/q}\right\|_{L_p(dx)}\leqq C_{p,q}\left\|\left(\int|f(x,y)|^q dy/|y|^n\right)^{1/q}\right\|_{L_p(dx)},$$

where $1<p<\infty$, $1<q\leqq\infty$.

Claim 2 (continuous version of the maximal inequalities in [5]). Let p, q, r satisfy

$$0<p<\infty\,,\quad 0<q\leqq\infty\quad\text{and}\quad 0<r<\min(p,q)\,.$$

Let $\hat{f}(\xi, y)$ be the Fourier transform of $f(x, y)$ with respect to x and let Ω_y be a set containing $\operatorname{supp}\hat{f}(\cdot, y)$. Let the diameter d_y of Ω_y be a continuous function of y and d_y be positive. Then the following inequality holds:

$$\left\|\left(\int\sup_{z\in\boldsymbol{R}^n}\left(\frac{|f(x-z,y)|}{1+|d_{|y|}z|^{n/r}}\right)^q dy/|y|^n\right)^{1/q}\right\|_{L_p(dx)}\leqq C\left\|\left(\int|f(x,y)|^q dy/|y|^n\right)^{1/q}\right\|_{L_p(dx)}.$$

Claim 3 (continuous version of the multiplier theorem in [5]). Let

$$0<p<\infty, \quad 0<q\leqq\infty \quad \text{and} \quad \kappa>n\left(\frac{1}{2}+\frac{1}{\min(p,q)}\right).$$

Let Ω_y, d_y be as in Claim 2. Let $M(x, y)$ be a function on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_y$. Then the following inequality holds:

$$\left\|\left(\int((M(\cdot,y)*f(\cdot,y))(x))^q dy/|y|^n\right)^{1/q}\right\|_{L_p(dx)}$$
$$\leqq C\sup_{y\in\boldsymbol{R}^n}\|\hat{M}(d_y\cdot,y)\|_{H^\kappa}\cdot\left\|\left(\int|f(x,y)|^q dy/|y|^n\right)^{1/q}\right\|_{L_p(dx)}.$$

Claim 1 is essential in proving Claim 2. To prove Claim 3, we need Claim 2 and the following inequality:

$$|(M(\cdot,y)*f(\cdot,y))(x)|\leqq C\sup_{z\in\boldsymbol{R}^n}\frac{|f(x-z,y)|}{1+|d_{|y|}z|^{n/r}}\cdot\|\hat{M}(d_y\cdot,y)\|_{H^\kappa},$$

where $0<r<\min(p,q)$ and $\kappa>n/2+n/r$.

As in the proof of Theorem 2, we use the continuous decomposition of unity: $\int\sigma_r*\sigma_r(x)dr/r=\delta(x)$. We have

$$(4)\qquad \|\,\||\xi|^s(\psi_{|\xi|,r_\xi}*f(x))\|_{L_q(d\xi/|\xi|^n)}\|_{L_p(dx)}$$
$$=\left\|\left\|\int|\xi|^s(\psi_{|\xi|,r_\xi}*\sigma_r)*(f*\sigma_r)(x)dr/r\right\|_{L_q(d\xi/|\xi|^n)}\right\|_{L_p(dx)}$$
$$\leqq\int_{C_1}^{C_2}dt/t\|\,\||\xi|^s(\psi_{|\xi|,r_\xi}*\sigma_{t|\xi|})*(f*\sigma_{t|\xi|})\|_{L_q(d\xi/|\xi|^n)}\|_{L_p(dx)}.$$

We apply Claim 3 to the integrand on the right hand side of (4) with $d_{|\xi|}=C|\xi|$ and $M(x,\xi)=\psi_{|\xi|,r_\xi}*\sigma_{t|\xi|}(x)$. Since $(\psi_{|\xi|,r_\xi}*\sigma_{t|\xi|})\hat{}(\tau)=\hat{\psi}(|\xi|^{-1}r_\xi\tau)\cdot\hat{\sigma}((t|\xi|)^{-1}\tau)$, the term $\sup_\xi\|\hat{\psi}(|\xi|^{-1}r_\xi C|\xi|\cdot\tau)\cdot\hat{\sigma}((t|\xi|)^{-1}C|\xi|\cdot\tau)\|_{H^\kappa}$ is bounded. Therefore, the Triebel-Lizorkin norm via the wavelet transform is bounded by the usual norm. q.e.d.

REMARK 4. In Theorems 2 and 3, the case where $p+q<2$ and $0<p\leqq 1$ or $0<q\leqq 1$ remains to be dealt with. Such troubles occur because we used the Hölder inequality and the Minkowski inequality in the proof of Theorems 2 and 3.

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