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WAVELET TRANSFORMS VERSUS FOURIER TRANSFORMS

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ABSTRACT. This note is a very basic introduction to wavelets. It starts with an orthogonal basis of piecewise constant functions, constructed by dilation and translation. The "wavelet transform" maps each $f(x)$ to its coefficients with respect to this basis. The mathematics is simple and the transform is fast (faster than the Fast Fourier Transform, which we briefly explain), but approximation by piecewise constants is poor. To improve this first wavelet, we are led to dilation equations and their unusual solutions. Higher-order wavelets are constructed, and it is surprisingly quick to compute with them — always indirectly and recursively.

We comment informally on the contest between these transforms in signal processing, especially for video and image compression (including high-definition television). So far the Fourier Transform — or its 8 by 8 windowed version, the Discrete Cosine Transform — is often chosen. But wavelets are already competitive, and they are ahead for fingerprints. We present a sample of this developing theory.

1. THE HAAR WAVELET

To explain wavelets we start with an example. It has every property we hope for, except one. If that one defect is accepted, the construction is simple and the computations are fast. By trying to remove the defect, we are led to dilation equations and recursively defined functions and a small world of fascinating new problems — many still unsolved. A sensible person would stop after the first wavelet, but fortunately mathematics goes on.

The basic example is easier to draw than to describe:

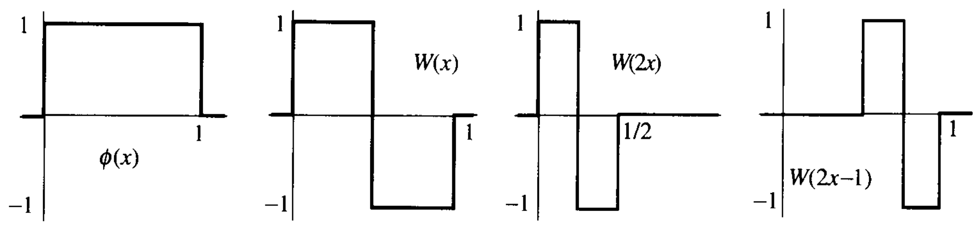

FIGURE 1. Scaling function $\phi(x)$, wavelet $W(x)$, and the next level of detail.

Already you see the two essential operations: *translation* and *dilation*. The step from $W(2x)$ to $W(2x-1)$ is translation. The step from $W(x)$ to $W(2x)$ is dilation. Starting from a single function, the graphs are shifted and compressed. The next level contains $W(4x)$, $W(4x-1)$, $W(4x-2)$, $W(4x-3)$.

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Each is supported on an interval of length $\frac{1}{4}$. In the end we have Haar's infinite family of functions:

$$W_{jk}(x) = W(2^j x - k) \quad \text{(together with } \phi(x)\text{)}.$$

When the range of indices is $j \geq 0$ and $0 \leq k < 2^j$, these functions form a remarkable basis for $L^2[0, 1]$. We extend it below to a basis for $L^2(\mathbf{R})$.

The four functions in Figure 1 are piecewise constant. Every function that is constant on each quarter-interval is a combination of these four. Moreover, the inner product $\int \phi(x) W(x)\,dx$ is zero — and so are the other inner products. This property extends to all j and k: *The translations and dilations of W are mutually orthogonal.* We accept this as the definition of a wavelet, although variations are definitely useful in practice. The goal looks easy enough, but the example is deceptively simple.

This orthogonal Haar basis is not a recent invention [1]. It is reminiscent of the Walsh basis in [2] — but the difference is important.† For Walsh and Hadamard, the last two basis functions are changed to $W(2x) \pm W(2x-1)$. All of their "binary sinusoids" are supported on the whole interval $0 \leq x \leq 1$. This global support is the one drawback to sines and cosines; otherwise, Fourier is virtually unbeatable. To represent a local function, vanishing outside a short interval of space or time, a global basis requires extreme cancellation. Reasonable accuracy needs many terms of the Fourier series. *Wavelets give a local basis.*

You see the consequences. If the signal $f(x)$ disappears after $x = \frac{1}{4}$, only a quarter of the later basis functions are involved. The wavelet expansion directly reflects the properties of f in physical space, while the Fourier expansion is perfect in frequency space. Earlier attempts at a "windowed Fourier transform" were ad hoc — wavelets are a systematic construction of a local basis.

The great value of orthogonality is to make expansion coefficients easy to compute. Suppose the values of $f(x)$, constant on four quarter-intervals, are $9, 1, 2, 0$. Its Haar wavelet expansion expresses this vector y as a combination of the basis functions:

$$\begin{bmatrix} 9 \\ 1 \\ 2 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

The wavelet coefficients b_{jk} are $3, 2, 4, 1$; they form the wavelet transform of f. The connection between the vectors y and b is the matrix W_4, in whose orthogonal columns you recognize the graphs of Figure 1:

$$y = W_4 b \quad \text{is} \quad \begin{bmatrix} 9 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

This is exactly comparable to the *Discrete Fourier Transform*, in which $f(x) = \sum a_k e^{ikx}$ stops after four terms. Now the vector y contains the values of f

† Rademacher was first to propose an orthogonal family of ± 1 functions; it was not complete. After Walsh constructed a complete set, Rademacher's Part II was regrettably unpublished and seems to be lost (but Schur saw it).

at four points:

$$y = F_4 a \quad \text{is} \quad \begin{bmatrix} f(0\pi/2) \\ f(1\pi/2) \\ f(2\pi/2) \\ f(3\pi/2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

This Fourier matrix also has orthogonal columns. The n by n matrix F_n follows the same pattern, with $\omega = e^{2\pi i/n}$ in place of $i = e^{2\pi i/4}$. Multiplied by $1/\sqrt{n}$ to give orthonormal columns, it is the most important of all unitary matrices. The wavelet matrix sometimes offers modest competition.

To invert a real orthogonal matrix we transpose it. To invert a unitary matrix, transpose its complex conjugate. After accounting for the factors that enter when columns are not unit vectors, the inverse matrices are

$$W_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix} \quad \text{and} \quad F_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (-i) & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix}.$$

The essential point is that the inverse matrices have the same form as the originals. If we can transform quickly, we can invert quickly — between coefficients and function values. The Fourier coefficients come from values at n points. The Haar coefficients come from values on n subintervals.

2. Fast Fourier Transform and Fast Wavelet Transform

The Fourier matrix is full — it has no zero entries. Multiplication of F_n times a vector a, done directly, requires n^2 separate multiplications. We are evaluating an n-term Fourier series at n points. The series is $\sum_0^{n-1} a_k e^{ikx}$, and the points are $x = 2\pi j/n$.

The wavelet matrix is sparse — many of its entries are zero. Taken together, the third and fourth columns of W fill a single column; the fifth, sixth, seventh, and eighth columns would fill one more column. With $n = 2^\ell$, we fill only $\ell + 1$ columns. The total number of nonzero entries in W_n is $n(\ell + 1)$. This already shows the effect of a more local basis. Multiplication of W_n times a vector b, done directly, requires only $n(\log_2 n + 1)$ separate multiplications.

Both of these matrix multiplications can be made faster. For $F_n a$, this is achieved by the Fast Fourier Transform — the most valuable numerical algorithm in our lifetime. It changes n^2 to $\frac{1}{2} n \log_2 n$ by a reorganization of the steps — which is simply a factorization of the Fourier matrix. A typical calculation with $n = 2^{10}$ changes $(1024)(1024)$ multiplications to $(5)(1024)$. This saving by a factor greater than 200 is genuine. The result is that the FFT has revolutionized signal processing. Whole industries are changed from slow to fast by this one idea — which is pure mathematics.

The wavelet matrix W_n also allows a neat factorization into very sparse matrices. The operation count drops from $O(n \log n)$ all the way to $O(n)$. For our piecewise constant wavelet the only operations are add and subtract; in fact, W_2 is the same as F_2. Both fast transforms have $\ell = \log_2 n$ steps, in the passage from n down to 1. For the FFT, each step requires $\frac{1}{2} n$ multiplications (as shown below). For the Fast Wavelet Transform, the cost of each successive step is cut in half. It is a beautiful "pyramid scheme" created by Burt and

Adelson and Mallat and others. The total cost has a factor $1+\frac{1}{2}+\frac{1}{4}+\cdots$ that stays below 2. This is why the final outcome for the FWT is $O(n)$ without the logarithm ℓ.

The matrix factorizations are so simple, especially for $n=4$, that it seems worthwhile to display them. The FFT has two copies of the half-size transform F_2 in the middle:

$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \tag{1}$$

The permutation on the right puts the even a's (a_0 and a_2) ahead of the odd a's (a_1 and a_3). Then come *separate half-size transforms* on the evens and odds. The matrix at the left combines these two half-size outputs in a way that produces the correct full-size answer. By multiplying those three matrices we recover F_4.

The factorization of W_4 is a little different:

$$W_4 = \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{2}$$

At the next level of detail (for W_8), the same 2 by 2 matrix appears four times in the left factor. The permutation matrix puts columns $0, 2, 4, 6$ of that factor ahead of $1, 3, 5, 7$. The third factor has W_4 in one corner and I_4 in the other corner (just as W_4 above ends with W_2 and I_2 — this factorization is the matrix form of the pyramid algorithm). It is the identity matrices I_4 and I_2 that save multiplications. Altogether W_2 appears 4 times at the left of W_8, then 2 times at the left of W_4, and then once at the right. The multiplication count from these $n-1$ small matrices is $O(n)$ — the Holy Grail of complexity theory.

Walsh would have another copy of the 2 by 2 matrix in the last corner, instead of I_2. Now the product has orthogonal columns with all entries ± 1 — the Walsh basis. Allowing W_2 or I_2, W_4 or I_4, W_8 or I_8, $\ldots$ in the third factors, the matrix products exhibit a whole family of orthogonal bases. This is a *wavelet packet*, with great flexibility. Then a "best basis" algorithm aims for a choice that concentrates most of f into a few basis vectors. That is the goal — to compress information.

The same principle of factorization applies for any power of 2, say $n = 1024$. For Fourier, the entries of F are powers of $\omega = e^{2\pi i/1024}$. The row and column indices go from 0 to 1023 instead of 1 to 1024. The zeroth row and column are filled with $\omega^0 = 1$. *The entry in row j, column k of F is ω^{jk}.* This is the term e^{ikx} evaluated at $x = 2\pi j/1024$. The multiplication $F_{1024}a$ computes the series $\sum a_k\,\omega^{jk}$ for $j=0$ to 1023.

The key to the matrix factorization is just this. Squaring the 1024th root of unity gives the 512th root: $(\omega^2)^{512} = 1$. This was the reason behind the middle factor in (1), where i is the fourth root and i^2 is the square root. It is the essential link between F_{1024} and F_{512}. The first stage of the FFT is the great factorization rediscovered by Cooley and Tukey (and described in 1805 by Gauss):

$$F_{1024} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{shuffle} \end{bmatrix}. \tag{3}$$

I_{512} is the identity matrix. D_{512} is the diagonal matrix with entries $(1, \omega, \dots, \omega^{511})$, requiring about 512 multiplications. The two copies of F_{512} in the middle give a matrix only *half full* compared to F_{1024} — here is the crucial saving. The shuffle separates the incoming vector a into $(a_0, a_2, \dots, a_{1022})$ with even indices and the odd part $(a_1, a_3, \dots, a_{1023})$.

Equation (3) is an imitation of equation (1), eight levels higher. Both are easily verified. Computational linear algebra has become a world of matrix factorizations, and this one is outstanding.

You have anticipated what comes next. Each F_{512} is reduced in the same way to two half-size transforms $F = F_{256}$. The work is cut in half again, except for an additional 512 multiplications from the diagonal matrices $D = D_{256}$:

$$\begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{even-odd gives} \\ \text{0 and 2 mod 4} \\ \text{even-odd gives} \\ \text{1 and 3 mod 4} \end{bmatrix}. \tag{4}$$

For $n = 1024$ there are $\ell = 10$ levels, and each level has $\frac{1}{2}n = 512$ multiplications from the first factor — to reassemble the half-size outputs from the level below. Those D's yield the final count $\frac{1}{2}n\ell$.

In practice, $\ell = \log_2 n$ is controlled by splitting the signal into smaller blocks. With $n = 8$, the scale length of the transform is closer to the scale length of most images. This is the *short time* Fourier transform, which is the transform of a "windowed" function wf. The multiplier w is the characteristic function of the window. (Smoothing is necessary! Otherwise this blocking of the image can be visually unacceptable. The ridges of fingerprints are broken up very badly, and windowing was unsuccessful in tests by the FBI.) In other applications the implementation may favor the FFT — theoretical complexity is rarely the whole story.

A more gradual exposition of the Fourier matrix and the FFT is in the monographs [3, 4] and the textbooks [5, 6] — and in many other sources [see 7]. (In the lower level text [8], it is intended more for reference than for teaching. On the other hand, this is just a matrix-vector multiplication!) FFT codes are freely available on *netlib*, and generally each machine has its own special software.

For higher-order wavelets, the FWT still involves many copies of a single small matrix. The entries of this matrix are coefficients c_k from the "dilation equation". We move from fast algorithms to a quite different part of mathematics — with the goal of constructing new orthogonal bases. The basis functions are unusual, for a good reason.

3. Wavelets by multiresolution analysis

The defect in piecewise constant wavelets is that they are very poor at approximation. Representing a smooth function requires many pieces. For wavelets this means many levels — the number 2^j must be large for an acceptable accuracy. It is similar to the rectangle rule for integration, or Euler's method for a differential equation, or forward differences $\Delta y/\Delta x$ as estimates of dy/dx.

Each is a simple and natural first approach, but inadequate in the end. Through all of scientific computing runs this common theme: Increase the accuracy at least to second order. What this means is: *Get the linear term right.*

For integration, we move to the trapezoidal rule and midpoint rule. For derivatives, second-order accuracy comes with centered differences. The whole point of Newton's method for solving $f(x) = 0$ is to follow the tangent line. All these are exact when f is linear. For wavelets to be accurate, $W(x)$ and $\phi(x)$ need the same improvement. Every $ax + b$ must be a linear combination of translates.

Piecewise polynomials (splines and finite elements) are often based on the "hat" function — the integral of Haar's $W(x)$. But this piecewise linear function does not produce orthogonal wavelets with a local basis. The requirement of *orthogonality to dilations* conflicts strongly with the demand for compact support — so much so that it was originally doubted whether one function could satisfy both requirements and still produce $ax + b$. It was the achievement of Ingrid Daubechies [9] to construct such a function.

We now outline the construction of wavelets. The reader will understand that we only touch on parts of the theory and on selected applications. An excellent account of the history is in [10]. Meyer and Lemarié describe the earliest wavelets (including Gabor's). Then comes the beautiful pattern of multiresolution analysis uncovered by Mallat — which is hidden by the simplicity of the Haar basis. Mallat's analysis found expression in the Daubechies wavelets.

Begin on the interval $[0, 1]$. The space V_0 spanned by $\phi(x)$ is orthogonal to the space W_0 spanned by $W(x)$. Their sum $V_1 = V_0 \oplus W_0$ consists of all piecewise constant functions on half-intervals. A different basis for V_1 is $\phi(2x) = \frac{1}{2}(\phi(x) + W(x))$ and $\phi(2x-1) = \frac{1}{2}(\phi(x) - W(x))$. Notice especially that $V_0 \subset V_1$. *The function $\phi(x)$ is a combination of $\phi(2x)$ and $\phi(2x-1)$.* This is the dilation equation, for Haar's example.

Now extend that pattern to the spaces V_j and W_j of dimension 2^j:

$$V_j = \text{span of the translates } \phi(2^j x - k) \text{ for fixed } j,$$
$$W_j = \text{span of the wavelets } W(2^j x - k) \text{ for fixed } j.$$

The next space V_2 is spanned by $\phi(4x)$, $\phi(4x-1)$, $\phi(4x-2)$, $\phi(4x-3)$. It contains all piecewise constant functions on quarter-intervals. That space was also spanned by the four functions $\phi(x)$, $W(x)$, $W(2x)$, $W(2x-1)$ at the start of this paper. Therefore, V_2 decomposes into V_1 and W_1 just as V_1 decomposes into V_0 and W_0:

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1. \tag{5}$$

At every level, *the wavelet space W_j is the "difference" between V_{j+1} and V_j*:

$$V_{j+1} = V_j \oplus W_j = V_0 \oplus W_0 \oplus \cdots \oplus W_j. \tag{6}$$

The translates of wavelets on the right are also translates of scaling functions on the left. For the construction of wavelets, this offers a totally different approach. Instead of creating $W(x)$ and the spaces W_j, we can create $\phi(x)$ and the spaces V_j. It is a choice between the terms W_j of an infinite series or their partial sums V_j. Historically the constructions began with $W(x)$. Today the constructions begin with $\phi(x)$. It has proved easier to work with sums than differences.

A first step is to change from $[0, 1]$ to the whole line $\mathbf{R}$. The translation index k is unrestricted. The subspaces V_j and W_j are infinite-dimensional (L^2 closures of translates). One basis for $L^2(\mathbf{R})$ consists of $\phi(x-k)$ and $W_{jk}(x) = W(2^j x - k)$ with $j \geq 0$, $k \in \mathbf{Z}$. Another basis contains all W_{jk} with $j, k \in \mathbf{Z}$. Then the dilation index j is also unrestricted — for $j = -1$ the functions $\phi(2^{-1}x - k)$ are constant on intervals of length 2. The decomposition into $V_j \oplus W_j$ continues to hold! The sequence of closed subspaces V_j has the following basic properties for $-\infty < j < \infty$:

$$V_j \subset V_{j+1} \text{ and } \bigcap V_j = \{0\} \text{ and } \bigcup V_j \text{ is dense in } L^2(\mathbf{R});$$

$$f(x) \text{ is in } V_j \text{ if and only if } f(2x) \text{ is in } V_{j+1};$$

$$V_0 \text{ has an orthogonal basis of translates } \phi(x-k),\ k \in \mathbf{Z}.$$

These properties yield a "*multiresolution analysis*" — the pattern that other wavelets will follow. V_j will be spanned by $\phi(2^j x - k)$. W_j will be its orthogonal complement in V_{j+1}. Mallat proved, under mild hypotheses, that W_j is also spanned by translates [11]; these are the wavelets.

Dilation is built into multiresolution analysis by the property that $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$. This applies in particular to $\phi(x)$. It must be a combination of translates of $\phi(2x)$. That is the hidden pattern, which has become central to this subject. We have reached the dilation equation.

4. The dilation equation

In the words of [10], "*la perspective est complètement changée.*" The construction of wavelets now begins with the scaling function ϕ. The dilation equation (or refinement equation or two-scale difference equation) connects $\phi(x)$ to translates of $\phi(2x)$:

$$\phi(x) = \sum_{k=0}^{N} c_k\, \phi(2x-k). \tag{7}$$

The coefficients for Haar are $c_0 = c_1 = 1$. The box function ϕ is the sum of two half-width boxes. That is equation (7). Then W is a combination of the same translates (because $W_0 \subset V_1$). The coefficients for $W = \phi(2x) - \phi(2x-1)$ are 1 and -1. It is absolutely remarkable that W uses the same coefficients as ϕ, but in reverse order and with alternating signs:

$$W(x) = \sum_{1-N}^{1} (-1)^k\, c_{1-k}\, \phi(2x-k). \tag{8}$$

This construction makes W orthogonal to ϕ and its translates. (For those translates to be orthogonal to each other, see below.) The key is that every vector c_0, c_1, c_2, c_3 is automatically orthogonal to $c_3, -c_2, c_1, -c_0$ and all even translates like $0, 0, c_3, -c_2$.

When N is odd, c_{1-k} can be replaced in (8) by c_{N-k}. This shift by $N-1$ is even. Then the sum goes from 0 to N and $W(x)$ looks especially attractive.

Everything hinges on the c's. They dominate all that follows. They determine (and are determined by) ϕ, they determine W, and they go into the matrix factorization (2). In the applications, convolution with ϕ is an averaging operator — it produces smooth functions (and a blurred picture). Convolution with W is a differencing operator, which picks out details.

The convolution of the box with itself is the piecewise linear hat function — equal to 1 at $x = 1$ and supported on the interval $[0, 2]$. It satisfies the dilation equation with $c_0 = \frac{1}{2}$, $c_1 = 1$, $c_2 = \frac{1}{2}$. But there is a condition on the c's in order that the wavelet basis $W(2^j x - k)$ shall be orthogonal. The three coefficients $\frac{1}{2}, 1, \frac{1}{2}$ do not satisfy that condition. Daubechies found the unique c_0, c_1, c_2, c_3 (four coefficients are necessary) to give orthogonality plus second-order approximation. Then the question becomes: *How to solve the dilation equation*?

Note added in proof. A new construction has just appeared that uses *two* scaling functions ϕ_i and wavelets W_i. Their translates are still orthogonal [38]. The combination $\phi_1(x)+\phi_1(x-1)+\phi_2(x)$ is the hat function, so second-order accuracy is achieved. The remarkable property is that these are "*short functions*": ϕ_1 is supported on $[0, 1]$ and ϕ_2 on $[0, 2]$. They satisfy a matrix dilation equation.

These short wavelets open new possibilities for application, since the greatest difficulties are always at boundaries. The success of the finite element method is largely based on the very local character of its basis functions. Splines have longer support (and more smoothness), wavelets have even longer support (and orthogonality). The translates of a long basis function overrun the boundary.

There are two principal methods to solve dilation equations. One is by Fourier transform, the other is by matrix products. Both give ϕ as a limit, not as an explicit function. We never discover the exact value $\phi(\sqrt{2})$. It is amazing to compute with a function we do not know — but the applications only require the c's. When complicated functions come from a simple rule, we know from increasing experience what to do: Stay with the simple rule.

Solution of the dilation equation by Fourier transform. Without the "2" we would have an ordinary difference equation — entirely familiar. The presence of two scales, x and $2x$, is the problem. A warning comes from Weierstrass and de Rham and Takagi — their nowhere differentiable functions are all built on multiple scales like $\sum a^n \cos(b^n x)$. The Fourier transform easily handles translation by k in equation (7), but $2x$ in physical space becomes $\xi/2$ in frequency space:

$$\hat{\phi}(\xi) = \frac{1}{2}\sum c_k\, e^{ik\xi/2}\,\hat{\phi}\left(\frac{\xi}{2}\right) = P\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right). \tag{9}$$

The "symbol" is $P(\xi) = \frac{1}{2}\sum c_k\, e^{ik\xi}$. With $\xi = 0$ in (9) we find $P(0) = 1$ or $\sum c_k = 2$ — the first requirement on the c's. This allows us to look for a solution normalized by $\hat{\phi}(0) = \int \phi(x)\,dx = 1$. It does not ensure that we find a ϕ that is continuous or even in L^1. What we do find is an infinite product, by recursion from $\xi/2$ to $\xi/4$ and onward:

$$\hat{\phi}(\xi) = P\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right) = P\left(\frac{\xi}{2}\right)P\left(\frac{\xi}{4}\right)\hat{\phi}\left(\frac{\xi}{4}\right) = \cdots = \prod_{j=1}^{\infty} P\left(\frac{\xi}{2^j}\right).$$

This solution ϕ may be only a distribution. Its smoothness becomes clearer by matrix methods.

Solution by matrix products [12, 13]. When ϕ is known at the integers, the dilation equation gives ϕ at half-integers such as $x = \frac{3}{2}$. Since $2x - k$ is an integer, we just evaluate $\sum c_k \phi(2x-k)$. Then the equation gives ϕ at quarter-integers as combinations of ϕ at half-integers. The combinations are built into the entries of two matrices A and B, and the recursion is taking their products.

To start we need ϕ at the integers. With $N = 3$, for example, set $x = 1$ and $x = 2$ in the dilation equation:

$$\begin{aligned} \phi(1) &= c_1 \phi(1) + c_0 \phi(2), \\ \phi(2) &= c_3 \phi(1) + c_2 \phi(2). \end{aligned} \tag{10}$$

Impose the conditions $c_1 + c_3 = 1$ and $c_0 + c_2 = 1$. Then the 2 by 2 matrix in (10), formed from these c's, has $\lambda = 1$ as an eigenvalue. The eigenvector is $(\phi(1), \phi(2))$. It follows from (7) that ϕ will vanish outside $0 \le x \le N$.

To see the step from integers to half-integers in matrix form, convert the scalar dilation equation to a first-order equation for the vector $v(x)$:

$$v(x) = \begin{bmatrix} \phi(x) \\ \phi(x+1) \\ \phi(x+2) \end{bmatrix}, \qquad A = \begin{bmatrix} c_0 & 0 & 0 \\ c_2 & c_1 & c_0 \\ 0 & c_3 & c_2 \end{bmatrix}, \qquad B = \begin{bmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 \end{bmatrix}.$$

The equation turns out to be $v(x) = Av(2x)$ for $0 \le x \le \frac{1}{2}$ and $v(x) = Bv(2x-1)$ for $\frac{1}{2} \le x \le 1$. By recursion this yields v at any dyadic point — whose binary expansion is finite. Each 0 or 1 in the expansion decides between A and B. For example

$$v(.01001) = (ABAAB)v(0). \tag{11}$$

Important: *The matrix B has entries c_{2i-j}*. So does A, when the indexing starts with $i = j = 0$. The dilation equation itself is $\phi = C\phi$, with an operator C of this new kind. Without the 2 it would be a Toeplitz operator, constant along each diagonal, but now every other row is removed. Engineers call it "convolution followed by decimation". (The word *downsampling* is also used — possibly a euphemism for decimation.) Note that the derivative of the dilation equation is $\phi' = 2C\phi'$. Successive derivatives introduce powers of 2. The eigenvalues of these operators C are $1, \frac{1}{2}, \frac{1}{4}, \ldots$, until $\phi^{(n)}$ is not defined in the space at hand. The sum condition $\sum c_{\text{even}} = \sum c_{\text{odd}} = 1$ is always imposed — it assures in Condition A_1 below that we have first-order approximation at least.

When x is not a dyadic point $p/2^n$, the recursion in (11) does not terminate. The binary expansion $x = .0100101\ldots$ corresponds to an infinite product $ABAABAB\ldots$. The convergence of such a product is by no means assured. It is a major problem to find a direct test on the c's that is equivalent to convergence — for matrix products *in every order.* We briefly describe what is known for arbitrary A and B.

For a single matrix A, the growth of the powers A^n is governed by the spectral radius $\rho(A) = \max |\lambda_i|$. Any norm of A^n is roughly the nth power of this largest eigenvalue. Taking nth roots makes this precise:

$$\lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A).$$

The powers approach zero if and only if $\rho(A) < 1$.

For two or more matrices, the same process produces the *joint spectral radius* [14]. The powers A^n are replaced by products Π_n of n A's and B's. The maximum of $\|\Pi_n\|$, allowing products in all orders, is still submultiplicative. The limit of nth roots (also the infimum) is the joint spectral radius:

$$\lim_{n\to\infty} (\max \|\Pi_n\|)^{1/n} = \rho(A, B). \tag{12}$$

The difficulty is not to define $\rho(A, B)$ but to compute it. For symmetric or normal or commuting or upper triangular matrices it is the larger of $\rho(A)$ and $\rho(B)$. Otherwise eigenvalues of products are not controlled by products of eigenvalues. An example with zero eigenvalues, $\rho(A) = 0 = \rho(B)$, is

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \qquad AB = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case $\rho(A, B) = \|AB\|^{1/2} = 2$. The product $ABABAB\ldots$ diverges. In general ρ is a function of the matrix entries, bounded above by norms and below by eigenvalues. Since one possible infinite product is a repetition of any particular Π_n (in the example it was AB), the spectral radius of that single matrix gives a lower bound on the joint radius:

$$(\rho(\Pi_n))^{1/n} \leq \rho(A, B).$$

A beautiful theorem of Berger and Wang [15] asserts that these eigenvalues of products yield the same limit (now a supremum) that was approached by norms:

$$\limsup_{n\to\infty} (\max \rho(\Pi_n))^{1/n} = \rho(A, B). \tag{13}$$

It is conjectured by Lagarias and Wang that equality is reached at a finite product Π_n. Heil and the author noticed a corollary of the Berger-Wang theorem: ρ is a continuous function of A and B. It is upper-semicontinuous from (12) and lower-semicontinuous from (13).

Returning to the dilation equation, the matrices A and B share the left eigenvector $(1, 1, 1)$. On the complementary subspace, they reduce to

$$A' = \begin{bmatrix} c_0 & 0 \\ -c_3 & 1 - c_0 - c_3 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 - c_0 - c_3 & -c_0 \\ 0 & c_3 \end{bmatrix}.$$

It is $\rho(A', B')$ that decides the size of $\phi(x) - \phi(y)$. Continuity follows from $\rho < 1$ [16]. Then ϕ and W belong to C^α for all α less than $-\log_2 \rho$. (When $\alpha > 1$, derivatives of integer order $[\alpha]$ have Hölder exponent $\alpha - [\alpha]$.) In Sobolev spaces H^s, Eirola and Villemoes [17, 18] showed how an ordinary spectral radius — *computable* — gives the exact regularity s.

5. Accuracy and orthogonality

For the Daubechies coefficients, the dilation equation does produce a continuous $\phi(x)$ with Hölder exponent 0.55 (it is differentiable almost everywhere). Then (8) constructs the wavelet. Figure 2 shows ϕ and W with c_0, c_1, c_2, $c_3 = \frac{1}{4}(1+\sqrt{3})$, $\frac{1}{4}(3+\sqrt{3})$, $\frac{1}{4}(3-\sqrt{3})$, $\frac{1}{4}(1-\sqrt{3})$.

What is special about the four Daubechies coefficients? They satisfy the requirement A_2 for second-order accuracy and the separate requirement O for orthogonality. We can state Condition A_2 in several forms. In terms of W, the moments $\int W(x)\,dx$ and $\int x\,W(x)\,dx$ are zero. Then the Fourier

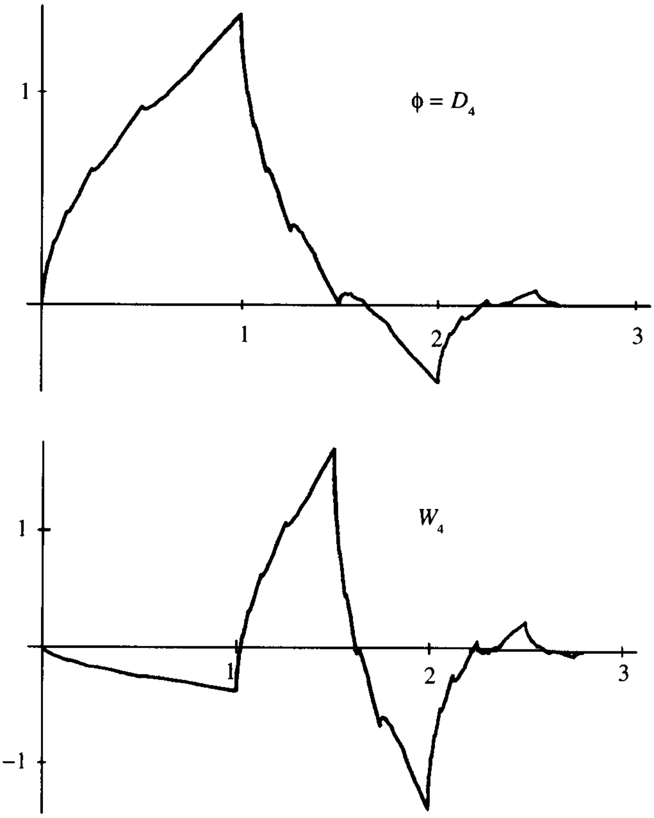

FIGURE 2. The family $W_4(2^j x - k)$ is orthogonal. Translates of D_4 can reproduce any $ax+b$. Daubechies also found D_{2p} with orthogonality and pth order accuracy.

transform of (8) yields $P(\pi) = P'(\pi) = 0$. In terms of the c's (or the symbol $P(\xi) = \frac{1}{2}\sum c_k e^{ik\xi}$), the condition for accuracy of order p is A_p:

$$(14) \qquad \sum(-1)^k k^m c_k = 0 \text{ for } m < p \quad \text{or equivalently} \quad P(\xi + \pi) = O(|\xi|^p).$$

This assures that translates of ϕ reproduce (locally) the powers $1, x, \ldots, x^{p-1}$ [19]. The zero moments are the orthogonality of these powers to W. Then the Taylor series of $f(x)$ can be matched to degree p at each meshpoint. The error in wavelet approximation is of order h^p, where $h = 2^{-j}$ is the mesh width or translation step of the local functions $W(2^j x)$. The price for each extra order of accuracy is two extra coefficients c_k — which spreads the support of ϕ and W by two intervals. A reasonable compromise is $p = 3$. The new short wavelets may offer an alternative.

Condition A_p also produces zeros in the infinite product $\hat{\phi}(\xi) = \Pi P(\xi/2^j)$. Every nonzero integer has the form $n = 2^{j-1}m$, m odd. Then $\hat{\phi}(2\pi n)$ has the factor $P(2\pi n/2^j) = P(m\pi) = P(\pi)$. Therefore, the pth order zero at $\xi = \pi$ in Condition A_p ensures a pth order zero of $\hat{\phi}$ at each $\xi = 2\pi n$. This is the test for the translates of ϕ to reproduce $1, x, \ldots, x^{p-1}$. That step closes the circle and means approximation to order p. Please forgive this brief recapitulation of an older theory — the novelty of wavelets is their orthogonality. This is tested by Condition O:

$$(15)\qquad \sum c_k\, c_{k-2m} = 2\,\delta_{0m} \quad \text{or equivalently} \quad |P(\xi)|^2 + |P(\xi+\pi)|^2 \equiv 1.$$

The first condition follows directly from $(\phi(x), \phi(x-m)) = \delta_{0m}$. The dilation equation converts this to $(\sum c_k\, \phi(2x-k), \sum c_\ell\, \phi(2x-2m-\ell)) = \delta_{0m}$. It is the "perfect reconstruction condition" of digital signal processing [20–22]. It assures that the L^2 norm is preserved, when the signal $f(x)$ is separated by a low-pass filter L and a high-pass filter H. The two parts have $\| Lf \|^2 + \| Hf \|^2 = \| f \|^2$. *A filter is just a convolution.* In frequency space that makes it a multiplication. *Low-pass* means that constants and low frequencies survive — we multiply by a symbol $P(\xi)$ that is near 1 for small $|\xi|$. *High-pass* means the opposite, and for wavelets the multiplier is essentially $P(\xi+\pi)$. The two convolutions are "mirror filters".

In the discrete case, the filters L and H (with downsampling to remove every second row) fit into an orthogonal matrix:

$$(16)\qquad \begin{bmatrix} L \\ \\ H \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & & \\ & & c_0 & c_1 & c_2 & c_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_3 & -c_2 & c_1 & -c_0 & & \\ & & c_3 & -c_2 & c_1 & -c_0 \end{bmatrix}.$$

This matrix enters each step of the wavelet transform, from vector y to wavelet coefficients b. The pyramid algorithm executes that transform by *recursion with rescaling.* We display two steps for a general wavelet and then specifically for Haar on $[0, 1]$:

$$(17)\qquad \begin{bmatrix} \begin{smallmatrix} L \\ H \end{smallmatrix} & \\ & I \end{bmatrix} \begin{bmatrix} L \\ H \end{bmatrix} \text{ is } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & \sqrt{2} & \\ & & & \sqrt{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & \\ & & 1 & 1 \\ 1 & -1 & & \\ & & 1 & -1 \end{bmatrix}.$$

This product is still an orthogonal matrix. When the columns of W_4 in §1 are normalized to be unit vectors, this is its inverse (and its transpose). The recursion decomposes a function into wavelets, and the reverse algorithm reconstructs it. The 2 by 2 matrix has low-pass coefficients $1, 1$ from ϕ and high-pass coefficients $1, -1$ from W. Normalized by $\frac{1}{2}$, they satisfy Condition O (note $e^{i\pi} = -1$), and they preserve the ℓ^2 norm:

$$\left| \frac{1+e^{i\xi}}{2} \right|^2 + \left| \frac{1+e^{i(\xi+\pi)}}{2} \right|^2 \equiv 1.$$

Figure 3 shows how those terms $|P(\xi)|^2$ and $|P(\xi+\pi)|^2$ are mirror functions that add to 1. It also shows how four coefficients give a flatter response — with higher accuracy at $\xi = 0$. Then $|P|^2$ has a fourth-order zero at $\xi = \pi$.

The design of filters (the choice of convolution) is a central problem of signal processing — a field of enormous size and importance. The natural formulation is in frequency space. Its application here is to multirate filters and "subband coding", with a sequence of scales $2^j x$.

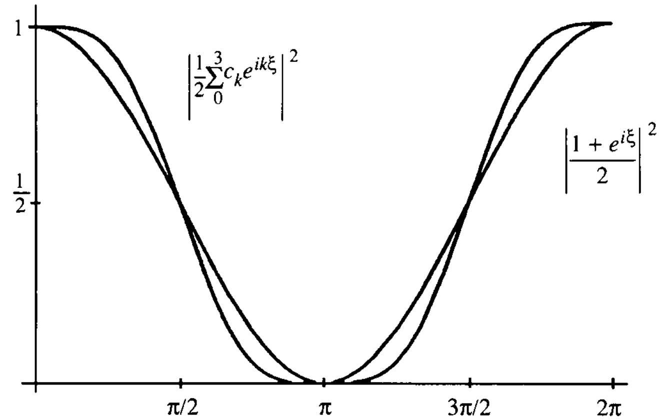

FIGURE 3. Condition O for Haar ($p = 1$) and Daubechies ($p = 2$).

Note. Orthogonality of the family $\phi(x-k)$ leads by the Poisson summation formula to $\sum|\hat{\phi}(\xi+2\pi n)|^2 = 1$. Applying the dilation equation (7) and separating even n from odd n shows how the second form of Condition O is connected to orthogonality:

$$\begin{aligned}
&\sum|\hat{\phi}(\xi+2\pi n)|^2 \\
&\quad = \sum\left|P\left(\frac{\xi}{2}+\pi n\right)\right|^2\left|\hat{\phi}\left(\frac{\xi}{2}+\pi n\right)\right|^2 \\
&\quad = \left|P\left(\frac{\xi}{2}\right)\right|^2\sum\left|\hat{\phi}\left(\frac{\xi}{2}+\pi 2m\right)\right|^2+\left|P\left(\frac{\xi}{2}+\pi\right)\right|^2\sum\left|\hat{\phi}\left(\frac{\xi}{2}+\pi(2m+1)\right)\right|^2 \\
&\quad = \left|P\left(\frac{\xi}{2}\right)\right|^2+\left|P\left(\frac{\xi}{2}+\pi\right)\right|^2 \quad (= 1 \text{ by Condition O}).
\end{aligned}$$

The same ideas apply to W. For dilation by 3^j or M^j instead of 2^j, Heller has constructed [23] the two wavelets or $M-1$ wavelets that yield approximation of order p. The orthogonality condition becomes $\sum_0^{M-1} |P(\xi+2\pi j/M)|^2 = 1$.

We note a technical hypothesis that must be added to Condition O. It was found by Cohen and in a new form by Lawton (see [24, pp. 177–194]). Without it, $c_0 = c_3 = 1$ passes test O. Those coefficients give a stretched box function $\phi = \frac{1}{3}\chi_{[0,3]}$ that is not orthogonal to $\phi(x-1)$. The matrix with L and H above will be only an isometry — it has columns of zeros. The filters satisfy $LL^* = HH^* = I$ and $LH^* = HL^* = 0$ but not $L^*L + H^*H = I$. The extra hypothesis is applied to this matrix A, or after Fourier transform to the operator $\mathscr{A}$:

$$A_{ij} = \sum_0^N c_k c_{j-2i+k} \quad \text{or} \quad \mathscr{A}f(\xi) = \left|P\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|P\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right).$$

The matrix A with $|i| < N$ and $|j| < N$ has two eigenvectors for $\lambda = 1$. Their components are $v_m = \delta_{0m}$ and $w_m = (\phi(x), \phi(x-m))$. Those must be the same! Then the extra condition, added to O, is that $\lambda = 1$ shall be a simple eigenvalue.

In summary, Daubechies used the minimum number $2p$ of coefficients c_k to satisfy the accuracy condition A_p together with orthogonality. These wavelets furnish *unconditional bases* for the key spaces of harmonic analysis (L^p , Hölder, Besov, Hardy space H^1 , BMO , ...). The Haar-Walsh construction fits functions with no extra smoothness [25]. Higher-order wavelets fit Sobolev spaces, where functions have derivatives in L^p (see [11, pp. 24–27]). With marginal exponent $p = 1$ or even $p < 1$, the wavelet transform still maps onto the right discrete spaces.

6. The contest: Fourier vs. wavelets

This brief report is included to give some idea of the decisions now being reached about standards for video compression. The reader will understand that the practical and financial consequences are very great. Starting from an image in which each color at each small square (pixel) is assigned a numerical shading between 0 and 255, the goal is to compress all that data to reduce the transmission cost. Since $256 = 2^8$, we have 8 bits for each of red-green-blue. The bit-rate of transmission is set by the channel capacity, the compression rule is decided by the filters and quantizers, and the picture quality is subjective. Standard images are so familiar that experts know what to look for — like tasting wine or tea.

Think of the problem mathematically. We are given $f(x, y, t)$, with x-y axes on the TV screen and the image f changing with time t. For digital signals all variables are discrete, but a continuous function is close — or piecewise continuous when the image has edges. Probably f changes gradually as the camera moves. We could treat f as a sequence of still images to compress independently, which seems inefficient. But the direction of movement is unpredictable, and too much effort spent on extrapolation is also inefficient. A compromise is to encode every fifth or tenth image and, between those, to work with the time differences Δf — which have less information and can be compressed further.

Fourier methods generally use real transforms (cosines). The picture is broken into blocks, often 8 by 8. This improvement in the scale length is more important than the control of $\log n$ in the FFT cost. (It may well be more important than the choice of Fourier.) After twenty years of refinement, the algorithms are still being fought over and improved. Wavelets are a recent entry, not yet among the heavyweights. The accuracy test A_p is often set aside in the goal of constructing "brick wall filters" — whose symbols $P(\xi)$ are near to characteristic functions. An exact zero-one function in Figure 3 is of course impossible — the designers are frustrated by a small theorem in mathematics. (Compact support of f and $\hat{f}$ occurs only for $f \equiv 0$.) In any case the Fourier transform of a step function has oscillations that can murder a pleasing signal — so a compromise is reached.

Orthogonality is not set aside. It is the key constraint. There may be eight or more bands (8 times 8 in two dimensions) instead of two. Condition O has at least eight terms $|P(\xi + k\pi/8)|^2$. After applying the convolutions, the energy or entropy in the high frequencies is usually small and the compression of that part of the signal is increased — to avoid wasting bits. The actual encoding or "quantization" is a separate and very subtle problem, mapping the real numbers to $\{1, \dots, N\}$. A vector quantizer is a map from $\mathbf{R}^d$, and the best are not just tensor products [28]. Its construction is probably more important to a successful compression than refining the filter.

Audio signals have fewer dimensions and more bands — as many as 512. One goal of compression is a smaller CD disk. Auditory information seems to come in octaves of roughly equal energy — the energy density decays like $1/\xi$. Also physically, the cochlea has several critical bands per octave. (An active problem in audio compression is to use psychoacoustic information about the ear.) Since $\int d\xi/\xi$ is the same from 1 to 2 and 2 to 4 and 4 to 8 (by a theorem we teach freshmen!), subband coding stands a good chance.

That is a barely adequate description of a fascinating contest. It is applied analysis (and maybe free enterprise) at its best. For video compression, the Motion Picture Experts Group held a competition in Japan late in 1991. About thirty companies entered algorithms. Most were based on cosine transforms, a few on wavelets. The best were all windowed Fourier. Wavelets were down the list but not unhappy. Consolation was freely offered and accepted. The choice for HDTV, with high definition, may be different from this MPEG standard to send a rougher picture at a lower bit-rate.

I must emphasize: The real contest is far from over. There are promising wavelets (Wilson bases and coiflets) that were too recent to enter. Hardware is only beginning to come—the first wavelet chips are available. MPEG did not see the best that all transforms can do.

In principle, wavelets are better for images, and Fourier is the right choice for music. Images have sharp edges; music is sinusoidal. The jth Fourier coefficient of a step function is of order $1/j$. The wavelet coefficients (mostly zero) are multiples of $2^{-j/2}$. The L^2 error drops exponentially, not polynomially, when N terms are kept. To confirm this comparison, Donoho took digitized photos of his statistics class. He discarded 95% of the wavelet and the Fourier coefficients, kept the largest 5%, and reconstructed two pictures. (The wavelets were "coiflets" [24], with greater smoothness and symmetry but longer support. Fourier blocks were not tried.) Every student preferred the picture from wavelets.

The underlying rule for basis functions seems to be this: choose scale lengths that match the image and allow for spatial variability. Smoothness is visually important, and D_4 is being superseded. Wavelets are not the only possible construction, but they have opened the door to new bases. In the mathematical contest (perhaps eventually in the business contest) unconditional bases are the winners.

We close by mentioning fingerprints. The FBI has more than 30 million in filing cabinets, counting only criminals. Comparing one to thousands of others is a daunting task. Every improvement leads to new matches and the solution of old crimes. The images need to be digitized.

The definitive information for matching fingerprints is in the "minutiae" of ridge endings and bifurcations [29]. At 500 pixels per inch, with 256 levels of gray, each card has 10^7 bytes of data. Compression is essential and $20:1$ is the goal. The standard from the Joint Photographic Experts Group (JPEG) is Fourier-based, with 8 by 8 blocks, and the ridges are broken. The competition is now between wavelet algorithms associated with Los Alamos and Yale [30–33] — fixed basis versus "best basis", $\ell < 100$ subbands or $\ell > 1000$, vector or scalar quantization. There is also a choice of coding for wavelet coefficients (mostly near zero when the basis is good). The best wavelets may be *biorthogonal* — coming from two wavelets W_1 and W_2. This allows a left-right symmetry [24], which is absent in Figure 2. The fingerprint decision is a true contest in applying pure mathematics.

Acknowledgment

I thank Peter Heller for a long conversation about the MPEG contest and its rules.

Additional note. After completing this paper I learned, with pleasure and amazement, that a thesis which I had promised to supervise ("formally", in the most informal sense of that word) was to contain the filter design for MIT's entry in the HDTV competition. The Ph.D. candidate is Peter Monta. The competition is still ahead (in 1992). Whether won or lost, I am sure the degree will be granted! These paragraphs briefly indicate how the standards for High Definition Television aim to yield a very sharp picture.

The key is high resolution, which requires a higher bit-rate of transmission. For the MPEG contest in Japan — to compress videos onto CD's and computers — the rate was 1 megabit/second. For the HDTV contest that number is closer to 24. Both compression ratios are about 100 to 1. (The better picture has more pixels.) The audio signal gets $\frac{1}{2}$ megabit/sec for its four stereo channels; closed captions use less. In contrast, conventional television has no compression at all — in principle, you see everything. The color standard was set in 1953, and the black and white standard about 1941.

The FCC will judge between an AT&T/Zenith entry, two MIT/General Instruments entries, and a partly European entry from Philips and others. These finalists are all digital, an advance which surprised the *New York Times*. Monta proposed a filter that uses seven coefficients or "taps" for low-pass and four for high-pass. Thus the filters are not mirror images as in wavelets, or brick walls either. Two-dimensional images come from tensor products of one-dimensional filters. Their exact coefficients will not be set until the last minute, possibly for secrecy — and cosine transforms may still be chosen in the end.

The red-green-blue components are converted by a 3 by 3 orthogonal matrix to better coordinates. Linear algebra enters, literally the spectral theorem. The luminance axis from the leading eigenvector gives the brightness.

A critical step is motion estimation, to give a quick and close prediction of successive images. A motion vector is estimated for each region in the image [34]. The system transmits only the difference between predicted and actual images — the "motion compensated residual". When that has too much energy, the motion estimator is disabled and the most recent image is sent. This will be the case when there is a scene change. Note that coding decisions are based on the *energy* in different bands (the size of Fourier coefficients). The L^1 norm is probably better. Other features may be used in 2001.

It is very impressive to see an HDTV image. The final verdict has just been promised for the spring of 1993. Wavelets will not be in that standard, but they have no shortage of potential applications [24, 35–37]. A recent one is the LANDSAT 8 satellite, which will locate a grid on the earth with pixel width of 2 yards. The compression algorithm that does win will use good mathematics.

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