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WAVELETS ON MANIFOLDS: AN OPTIMIZED CONSTRUCTION

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Dedicated to Peter Deuflhard on the occasion of his 60th birthday

Abstract. A key ingredient of the construction of biorthogonal wavelet bases for Sobolev spaces on manifolds, which is based on topological isomorphisms is the Hestenes extension operator. Here we firstly investigate whether this particular extension operator can be replaced by another extension operator. Our main theoretical result states that an important class of extension operators based on interpolating boundary values cannot be used in the construction setting required by Dahmen and Schneider. In the second part of this paper, we investigate and optimize the Hestenes extension operator. The results of the optimization process allow us to implement the construction of biorthogonal wavelets from Dahmen and Schneider. As an example, we illustrate a wavelet basis on the 2-sphere.

1. Introduction

This paper is motivated by the results presented in [DS]. There compactly supported biorthogonal wavelet bases for Sobolev spaces $H^m(\Gamma)$ on manifolds Γ were constructed which may be used, for instance, to numerically solve boundary integral equations by wavelet schemes; see, e.g., [Ha]. The starting point is a quadrangulation for Γ, i.e., $\Gamma = \bigcup_{i=1}^N \overline{\Gamma_i}$ for $\Gamma_i := \kappa_i(\square)$ being the parametric image of $\square = (0,1)^d$ under κ_i, and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. The construction in [DS] is based on a topological isomorphism

$$P : \prod_{i=1}^N H^m(\Gamma_i)^\uparrow \to H^m(\Gamma) \tag{1.1}$$

developed previously in [CF] that yields a decomposition of the Sobolev space $H^m(\Gamma)$ into particular closed subspaces $H^m(\Gamma_i)^\uparrow$ of $H^m(\Gamma)$. Here any m is allowed as long as it does not exceed the smoothness of the manifold. Using this decomposition, norm equivalences for $H^m(\Gamma)$ in terms of weighted sequence norms of coefficients of wavelet expansions were given in [DS] for the whole range of admissible m. Such norm equivalences are fundamental for applications of wavelet bases. Other constructions based on domain decomposition and gluing [CTU, DS1] only

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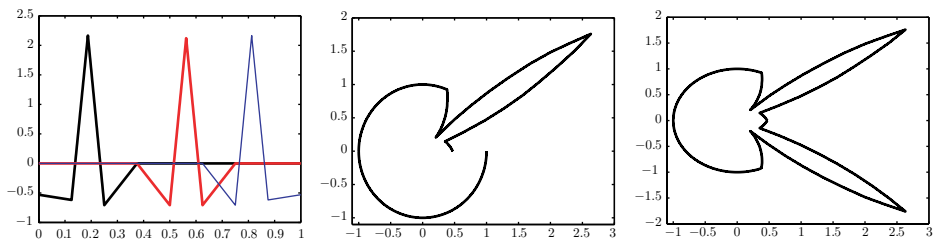

FIGURE 1. Left: Boundary adapted and interior wavelets on the interval from [DKU]. Middle: Right boundary adapted wavelet from the left figure pushed forward to the upper part of the circle S^1 and extended by zero onto the lower part. Right: Same wavelet pushed extended continuously.

achieve norm equivalences $-\frac{1}{2} < m < \frac{3}{2}$ for arbitrary manifolds, or $|m| \leq 1$ for the special case of piecewise affine surfaces with triangular facets [DSt].

Due to the structure of (1.1), wavelet bases Ψ of $H^m(\square)$ will serve as the first basic ingredient. Such bases were constructed in [DKU] (see the left plot in Figure 1). These wavelets are pushed forward to the patches Γ_i first to obtain wavelet bases Ψ_{Γ_i} of $H^m(\Gamma_i)^\uparrow$. Defining $\Psi'_{\Gamma_i} := \Psi \circ \kappa_i^{-1}$ and extending Ψ'_{Γ_i} by zero outside of Γ_i, we see, however, that the collection

$$\Psi' := \Psi'_{\Gamma_1} \cup \cdots \cup \Psi'_{\Gamma_N}$$

is not a candidate for a basis of $H^m(\Gamma)$ for $m > \frac{1}{2}$: for some $\psi \in \Psi$ not vanishing on parts of the boundary $\partial\square$ of $\square$, $\psi \circ \kappa_i^{-1}$ is not continuous in Γ and, hence, $\psi \circ \kappa_i^{-1} \notin H^m(\Gamma)$. In the middle figure in Figure 1 the right boundary wavelet from the left plot is pushed forward to the circle $\Gamma := S^1$ and quadrangulated into the upper and lower half of the circle. It is obviously not continuous in S^1. Extending it continuously onto the lower part of S^1 yields the figure on the right of Figure 1. Hence, the main task of the operator P in (1.1) is to extend the single components $\Psi_i \in H^m(\Gamma_i)^\uparrow$ onto the whole of Γ in some appropriate way. This is done by certain *extension operators* which play the role of the second main ingredient for constructing wavelet bases on manifolds. Recall that for some open subset $\Omega \subset \Gamma$, $E : H^m(\Omega) \to H^m(\Gamma)$ is called an *extension operator* if $Ef|_\Omega = f$ is satisfied for all $f \in H^m(\Omega)$, where $|_\Omega$ denotes the restriction to Ω. Given such extension operators

$$E_i : H^m(\Gamma_i) \to H^m(\Gamma),$$

the collection $\Psi_\Gamma := \Psi_{\Gamma_1} \cup \cdots \cup \Psi_{\Gamma_N}$ with $\Psi_{\Gamma_i} := E_i(\Psi \circ \kappa_i^{-1})$ is a reasonable candidate for a basis of $H^m(\Gamma)$. For the example discussed before, the right figure in Figure 1 displays a possible extension to the whole of S^1.

To establish the equivalence of norms for the isomorphism (1.1), the extension operators E_i as well as their adjoints E_i^* relative to the L^2 inner product have to satisfy quite severe conditions, denoted as property (E):

(CE) *Continuity:* $E_i : H^m(\Gamma_i) \to H^m(\Gamma)$ is continuous.

(CE*) *Continuity of the L^2 adjoint relative to H^m*: The L^2 adjoint $E_i^* : L^2(\Gamma) \to L^2(\Gamma_i)$ of E_i is well defined and continuous as an operator $E_i^* : H^m(\Gamma) \to H^m(\Gamma_i)$; i.e., $\|E_i^* f\|_{H^m(\Gamma_i)} \lesssim \|f\|_{H^m(\Gamma)}$ for all $f \in H^m(\Gamma)$.

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Here and in the rest of the paper $a \lesssim b$ means $a \leq c\,b$ with some constant c which is independent of all parameters on which a and b may depend. An operator that meets these requirements is the Hestenes extension operator [H] used in [DS]. However, a brief look reveals that this operator is initially ill conditioned—it produces strongly oscillating extensions; see Section 6. This fact leads to large constants in the norm equivalence of the isomorphism (1.1).

The main objective of this paper is the construction of well-conditioned extension operators satisfying (E), which makes the implementation of wavelets on manifolds practically feasible. Firstly, this motivates the search for alternative extension operators in Sections 3 and 4. We propose two different such operators, both by interpolating trace values at the boundary: the first of such *trace dependent operators* is based on Taylor expansions, the second employs norm minimizing harmonic extensions, and both satisfy (CE). However, as stated in the main theoretical result of this paper, Theorem 4.7, no extension operator based on the interpolation of boundary values can be further employed as their adjoints do not satisfy (CE*). Nevertheless, these results lead us to finding an extension operator that does not depend on the values of the trace directly. A variant of the Hestenes operator employed in [DS] turns out to be our method of choice which is derived in Section 5 where we also prove that the Hestenes extension emerges as a limit of an interpolation process with polynomials. In particular, we realize H^m, instead of C^m, extensions. In Section 6 the Hestenes extension is examined thoroughly: we perform a number of modifications which substantially reduce the oscillations of the extension and optimize its ingredients in Section 7. The construction of wavelets from [DS] is summarized in Section 8. An implementation of the construction employing the optimized Hestenes extension is based on the software package IGPMlib [IGPM]; the resulting C++ library is documented in [S]. This library is finally used in Section 9 to construct and visualize nontrivial application biorthogonal wavelet bases for $H^1(S^2)$ on the sphere $S^2 \subset \mathbb{R}^3$ based on an appropriate quadrangulation of S^2.

2. Some basic notions

In the rest of this paper, we will only consider extensions of functions in Sobolev spaces of natural order $m \in \mathbb{N}$; see, e.g., [A] for a general reference. For an open interval $I \subset \mathbb{R}$ and $m \in \mathbb{N}$, $1 \leq p \leq \infty$, $f \in L^p(I)$ has weak derivatives of order m in $L^p(I)$ if for all $k \in \mathbb{N}$ with $k \leq m$ there exist functions $f^{(k)} \in L^p(I)$ satisfying

$$\int_I f(x)\partial_x^k \varphi(x)dx = (-1)^k \int_I f^{(k)}(x)\varphi(x)dx$$

for all $\varphi \in C_0^\infty(I)$. Here $C_0^\infty(I)$ denotes the space of arbitrarily smooth functions with compact support in I. The spaces

$$W^{m,p}(I) := \{f \in L^p(I) : f \text{ has weak derivatives } f^{(k)} \in L^p(I) \text{ of order } m\}$$

are the Sobolev spaces of order m, equipped with the norm

$$\|f\|_{W^{m,p}(I)} := \sum_{k=0}^{m} \|f^{(k)}\|_{L^p(I)},$$

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and the norm in $H^m(I) := W^{m,2}(I)$ is induced by the inner product

$$\langle f, g\rangle_{H^m(I)} := \sum_{k=0}^{m} \langle f^{(k)}, g^{(k)}\rangle_{L^2(I)}.$$

Moreover, zero boundary values are incorporated into the vector space $H_0^m(I)$ by defining $H_0^m(I)$ as the closure of $C_0^\infty(I)$ with respect to the norm in $H^m(I)$. In order to be able to assign boundary values to $f \in H^m(I)$ we recall

Theorem 2.1. *For $m \geq 1$ the embedding operator $H^m(I) \hookrightarrow C^{m-1}(\overline{I})$ is well defined and continuous. In particular, $f \in H^m(I)$ can be assigned boundary values $f^{(k)}|_{\partial I}$ for all $k = 0, \ldots, m-1$, called the* trace values *of f.*

As we will see below, the extension operators will be defined by gluing together functions defined on adjacent intervals.

Remark 2.2. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals satisfying $\overline{I_1} \cap \overline{I_2} = \{b\}$. Then one has $f_1 \in H^m(I_1)$, $f_2 \in H^m(I_2)$, $f_1^{(k)}(b) = f_2^{(k)}(b)$ if and only if $\chi_{I_1} f_1 + \chi_{I_2} f_2 \in H^m(I_1 \cup \{b\} \cup I_2)$. Here for any interval I, χ_I denotes the characteristic function on I. By χ_I we sometimes also denote the extension by zero to some set $J \supset I$ which will be clear from the context. Recall also that for $I \subset \mathbb{R}$ an open interval, one has $f \in H_0^m(I)$ if and only if $f \in H^m(I)$ and $f^{(k)}|_{\partial I} = 0$ for all $k = 0, \ldots, m-1$.

3. Univariate extension operators

The multivariate extension operators needed later in Subsection 8.5 can be defined as tensor products of univariate extension operators A. We use the following notation. For $a < b < c < d$, let intervals $I, \tilde{I} \subset \mathbb{R}$ be defined by $I := (b, c), \tilde{I} := (a, d)$. The boundaries of $I, \tilde{I}$ will be denoted by $\partial I = \{b, c\}$, $\partial \tilde{I} = \{a, d\}$, respectively. An operator $A : H^m(I) \to H^m(\tilde{I})$ is called a *univariate extension operator* if A satisfies $(Af)|_I = f$ for all $f \in H^m(I)$. We first boil down the formulation of the crucial property (E) to the univariate case. We say that a univariate extension operator A satisfies property (A) if

- (L) *Locality*: For $f \in H^m(I)$ one has $Af \in H_0^m(\tilde{I})$.
- (CA) *Continuity*: $A : H^m(I) \to H_0^m(\tilde{I})$ is continuous.
- (CA*) *Continuity of* $A^* : H^m(\tilde{I}) \to H^m(I)^*$, defined by $\langle A^* g, f\rangle := \langle g, Af\rangle_{L^2(\tilde{I})}$ for all $f \in H^m(I)$, $g \in H^m(\tilde{I})$, is continuous as an operator $A^* : H^m(\tilde{I}) \to H^m(I)$ with the adjoint property

$$\langle A^* g, f\rangle = \langle A^* g, f\rangle_{L^2(I)} = \langle g, Af\rangle_{L^2(\tilde{I})}.$$

Of course, (CA*) is a stronger requirement than A^* to be continuous as an operator $H^m(\tilde{I}) \to H^m(I)^*$. Here and below, for a vector space V, we denote its dual by V^*, the space of continuous linear functionals $V \to \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing, which is given by $\langle F, f\rangle := F(f)$ for all $F \in V^*, f \in V$.

The detailed discussion of univariate extension operators will be of vital importance for the multivariate case in Subsection 8.5 below and requires the effort of Sections 4–7. Next we discuss two classes of univariate extension operators.

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4. Extension operators based on traces

Remark 2.2 enables us to extend functions by interpolating the first $m-1$ derivatives of $f \in H^m(I)$ explicitly at the boundary ∂I. As we are interested in extensions $Af \in H_0^m(\tilde{I})$, we also have to achieve zero boundary values in the H^m-sense at the boundary $\partial\tilde{I}$. Writing $I_\ell := (a,b)$, $I_r := (c,d)$, we have to find functions $u_\ell \in H^m(I_\ell)$, $u_r \in H^m(I_r)$ that satisfy the boundary conditions

$$u_\ell{}^{(k)}(a) = 0,\ u_\ell{}^{(k)}(b) = f^{(k)}(b) \qquad \text{for } k = 0, \dots, m-1, \tag{4.1}$$

$$u_r{}^{(k)}(d) = 0,\ u_r{}^{(k)}(c) = f^{(k)}(c) \qquad \text{for } k = 0, \dots, m-1. \tag{4.2}$$

Definition 4.1. Let $A : H^m(I) \to H^m(\tilde{I})$ be a continuous extension operator. Then A is called *trace dependent* if for all $f \in H^m(I)$ the definition of Af depends only on the trace values $f^{(k)}(b), f^{(k)}(c), k = 0, \dots, m-1$, of f at the boundary ∂I.

Since trace dependent operators by their very definition require the availability of the trace values of the derivatives of f which are not always accessible, they are of limited applicability. However, these extensions are of theoretical interest and some of their properties will be exploited further on. We now discuss two different solutions to this interpolation problem which only employ values of f at ∂I. In contrast, we will later consider approaches where, in addition, values of f in the interior of I come into play. Note that, due to Theorem 2.1, $f^{(m-1)} \in H^1(I)$ is still continuous, but not necessarily $f^{(m)}$. Thus, interpolation of the first m derivatives can in general not be carried out for H^m-functions. At first we look at an operator based on a Taylor expansion.

4.1. Extension based on a Taylor expansion.

Let $\eta \in C_0^\infty(\tilde{I})$ be a cutoff function displayed in Figure 2 which is defined by

$$\eta(x) = \begin{cases} 0, & x \in (a,d) \setminus [\frac{a+b}{2}, \frac{c+d}{2}], \\ 1, & x \in [\frac{a+3b}{4}, \frac{3c+d}{4}]. \end{cases}$$

Lemma 4.2. *The operator*

$$(E_T f)(x) := \chi_{I_\ell}(x)\eta(x) \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \chi_I(x) f(x) + \chi_{I_r}(x)\eta(x) \sum_{k=0}^{m-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is a continuous linear extension operator $E_T : H^m(I) \to H_0^m(\tilde{I})$.

The proof is straightforward and can be found in [KS].

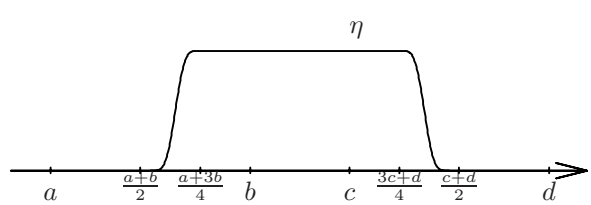

Figure 2. The cutoff function η.

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4.2. **A norm minimizing extension.** Another possible extension is based on finding functions $u_\ell \in H^m(I_\ell),\ u_r \in H^m(I_r)$ which minimize the functionals

$$\tilde{u} \mapsto \|\tilde{u}\|_{H^m(I_\ell)}, \quad \tilde{u} \mapsto \|\tilde{u}\|_{H^m(I_r)}$$

among all functions $\tilde{u}$ satisfying the boundary conditions (4.1), (4.2), respectively. Clearly, defining

(4.3)
$$u_\ell := \operatorname*{argmin}_{\tilde{u}\in H^m(I_\ell),\ \tilde{u} \text{ satisfies (4.1)}} \|\tilde{u}\|^2_{H^m(I_\ell)}, \quad u_r := \operatorname*{argmin}_{\tilde{u}\in H^m(I_r),\ \tilde{u} \text{ satisfies (4.2)}} \|\tilde{u}\|^2_{H^m(I_r)},$$

the operator E defined by

(4.4)
$$(Ef)(x) := \chi_{I_\ell}(x)u_\ell(x) + \chi_I(x)f(x) + \chi_{I_r}(x)u_r(x), \qquad E : H^m(I) \to H^m(\tilde{I}),$$

is the operator with the smallest operator norm among all possible extension operators. We restrict the our discussion to satisfying (4.1). As for (4.2) the situation is analogous, and the following results were established in [KS], where also details of the explicit construction can be found.

Lemma 4.3. *The extension operator $E : H^m(I) \to H^m(\tilde{I})$ defined in* (4.4) *satisfies $\|Ef\|_{H^m(\tilde{I})} \lesssim \|f\|_{H^m(I)}$ for all $f \in H^m(I)$.*

As an application of the extension operator E defined in (4.4) we will later need norm minimizing cutoff functions. By this we mean for $a < b \le c < d$ a function $\eta \in H^m(\mathbb{R})$ satisfying $|\eta(x)| \le 1$ and

$$\eta(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \tilde{I}, \\ 1, & x \in I, \end{cases} \tag{4.5}$$

with minimal H^m-norm. Using E we extend the function $f : I \to \mathbb{R},\ f \equiv 1$ on I, to a function $\eta : \tilde{I} \to \mathbb{R}$ by finding $\eta_\ell \in H^m(I_\ell),\ \eta_r \in H^m(I_r)$ with the boundary conditions (4.1), (4.2) given by

$$\eta_\ell^{(k)}(a) = 0 \text{ for all } 0 \le k \le m-1, \quad \eta_\ell^{(k)}(b) = 0 \text{ for all } 1 \le k \le m-1,\ \eta_\ell(b) = 1,$$
$$\eta_r^{(k)}(d) = 0 \text{ for all } 0 \le k \le m-1, \quad \eta_r^{(k)}(c) = 0 \text{ for all } 1 \le k \le m-1,\ \eta_r(c) = 1.$$

By Remark 2.2 the function $\eta := Ef = \chi_{I_\ell}\eta_\ell + \chi_I f + \chi_{I_r}\eta_r$ extended by 0 outside $\tilde{I}$ is then in $H^m(\mathbb{R})$ and satisfies (4.5). Due to the norm minimizing property of the extension operator E, the new function η has minimal H^m-norm as well.

Lemma 4.4. *For $a = -1, b = c = 0, d = 1,\ m \in \mathbb{N}$ arbitrary, we have constructed a cutoff function $\eta_m \in H^m(\mathbb{R})$ with $\operatorname{supp}\eta_m = [-1, 1]$ satisfying $\eta_m(0) = 1$ and $\eta_m^{(k)}(0) = 0$ for all $1 \le k \le m-1$. These cutoff functions are displayed in Figure* 3 *for $m = 1, 2, 3$.*

4.3. **Trace dependent operators and property (A).** Next we discuss that trace dependent operators cannot be designed to satisfy (A). We recall that property (A) requires the existence of a continuous operator $A^* : H^m(\tilde{I}) \to H^m(I)$, i.e.

$$\|A^*f\|_{H^m(I)} \lesssim \|f\|_{H^m(\tilde{I})} \qquad \text{for all } f \in H^m(\tilde{I}), \tag{4.6}$$

satisfying

$$\langle Af, g\rangle_{L^2(\tilde{I})} = \langle f, A^*g\rangle_{L^2(I)} \qquad \text{for all } f \in H^m(I), g \in H^m(\tilde{I}). \tag{4.7}$$

We first state a simple consequence of the properties (4.6) and (4.7).

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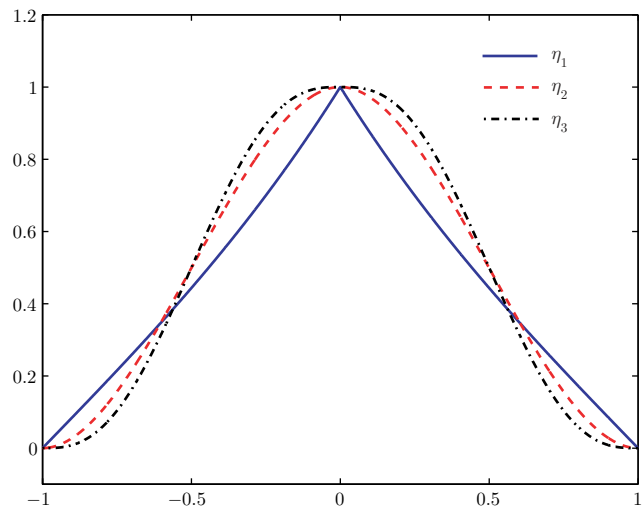

FIGURE 3. Cutoff functions $\eta_m \in H^m((-1,1))$ with minimal norm.

Lemma 4.5. *Let* (4.6) *and* (4.7) *hold. Then for all* $g \in H^m(\tilde{I})$ *the functional*

$$A_g := \langle A\cdot, g\rangle_{L^2(\tilde{I})} : H^m(I) \to \mathbb{R}$$

is bounded in the L^2*-norm, that is,*

$$|A_g(f)| \lesssim \|f\|_{L^2(I)} \qquad \text{for all } f \in H^m(I). \tag{4.8}$$

By density of $H^m(I)$ *in* $L^2(I)$ *this means in particular that* A_g *can be extended uniquely to a functional*

$$A_g : L^2(I) \to \mathbb{R}, \text{ and } A_g \in L^2(I)^*.$$

Proof. By properties (4.6), (4.7), and the Cauchy–Schwartz inequality we have

$$\begin{aligned} |A_g(f)| = \left|\langle Af, g\rangle_{L^2(\tilde{I})}\right| &= \left|\langle f, A^*g\rangle_{L^2(I)}\right| \le \|f\|_{L^2(I)} \|A^*g\|_{L^2(I)} \\ &\le \|f\|_{L^2(I)} \|A^*g\|_{H^m(I)} \lesssim \|f\|_{L^2(I)} \|g\|_{H^m(\tilde{I})}. \end{aligned}$$

Thus, A_g can be extended uniquely to $A_g \in L^2(I)^*$, and (4.8) follows. □

The next result gives a first hint that (4.8) is not valid for trace dependent operators.

Lemma 4.6. *Let* $A : H^m(I) \to H^m(\tilde{I})$ *be a trace dependent operator. Then* A *cannot be extended to a continuous operator* $A : L^2(I) \to L^2(\tilde{I})$*. In particular, for every* $m \in \mathbb{N}$ *there exists a sequence* $f_k \in H^m(I)$ *with constant trace values such that* $f_k \to 0$ *in* $L^2(I)$ *and*

$$\frac{\|Af_k\|_{L^2(\tilde{I})}}{\|f_k\|_{L^2(I)}} \to \infty \qquad \text{for } k \to \infty.$$

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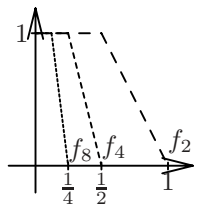

FIGURE 4. The functions f_k defined in (4.9).

Proof. For convenience let $I = (0,1)$, $\tilde{I} = (-1,2)$, $I_\ell = (-1,0)$, $I_r = (1,2)$. For $m = 1$ and $k \in \mathbb{N}$ consider the sequence displayed in Figure 4,

$$f_k(x) = \begin{cases} 1, & x \in (0, \frac{1}{k}], \\ 2 - kx, & x \in (\frac{1}{k}, \frac{2}{k}), \\ 0, & x \in [\frac{2}{k}, 1). \end{cases} \tag{4.9}$$

Due to Remark 2.2, $f_k \in H^1(I)$ for all $k \in \mathbb{N}$ and $f_k \to 0$ in $L^2(I)$, as

$$\int_0^1 |f_k(x)|^2 \, dx = \frac{1}{k} + \int_{\frac{1}{k}}^{\frac{2}{k}} k^2 \left(\frac{2}{k} - x\right)^2 dx = \frac{1}{k} - \frac{k^2}{3}\left(\frac{2}{k} - x\right)^3 \Big|_{\frac{1}{k}}^{\frac{2}{k}} = \frac{1}{k} + \frac{1}{3k} \to 0$$

for $k \to \infty$. As A is trace dependent and $f_k(0) = 1$ for all k, we have $Af_k|_{I_\ell} = Af_\ell|_{I_\ell}$ for all $k, \ell \in \mathbb{N}$, and as $Af \in H^1(\tilde{I})$, it is continuous. Thus $Af_k|_{I_\ell} \not\equiv 0$ for all k, as $f_k(0) = 1$ for all k. Thus, we have $\|Af_k\|_{L^2(I_\ell)} = \|Af_\ell\|_{L^2(I_\ell)} =: c > 0$ for all $k, \ell \in \mathbb{N}$. Since $f_k \to 0$ in $L^2(I)$, we have

$$\frac{\|Af_k\|_{L^2(\tilde{I})}}{\|f_k\|_{L^2(I)}} \geq \frac{\|Af_k\|_{L^2(I_\ell)}}{\|f_k\|_{L^2(I)}} = \frac{c}{\|f_k\|_{L^2(I)}} \to \infty \qquad \text{for } k \to \infty.$$

For larger m we just have to make the sequence f_k smoother in such a way that the boundary values remain unchanged, for instance by computing the $(m-1)$-fold convolution with $\chi_{[0,\frac{1}{2km}]}$, as this will result in convolutions in the desired space $H^m(I)$ with boundary values unchanged. □

The necessary condition (4.8) from Lemma 4.5 now gives us a tool to prove the following main result.

Theorem 4.7. *No trace dependent operator can fulfill* (A).

Proof. Again, for convenience we choose $I = (0,1)$, $\tilde{I} = (-1,2)$, $I_\ell = (-1,0)$, $I_r = (1,2)$. We construct a function $g_\varepsilon \in H^m(\tilde{I})$ such that A_{g_ε} is not continuous as a functional from L^2 to $\mathbb{R}$. This is a contradiction to (4.8) and thus a contradiction to (A). To this end, let $g_\varepsilon \in C^\infty(\tilde{I})$, $g_\varepsilon \geq 0$ and

$$g_\varepsilon(x) := \begin{cases} 0, & x \in (-1, -\varepsilon], \\ 1, & x \in [-\frac{\varepsilon}{2}, 2]. \end{cases}$$

Now we find $\varepsilon > 0$ such that $A_{g_\varepsilon} = \langle A\cdot, g_\varepsilon \rangle_{L^2(\tilde{I})} \notin L^2(I)^*$. For $m \in \mathbb{N}$ let $\{f_k\}_k \subset H^m(I)$ be the sequence introduced in the proof of Lemma 4.6. As stated there, we have $Af_k|_{I_\ell} = Af_\ell|_{I_\ell}$ for all $k, \ell \in \mathbb{N}$ and $Af_k(0) = 1$. Since $Af_k \in H^m(\tilde{I}), Af_k$ is continuous. Thus, there exists $\delta > 0$ not depending on k such that $Af_k(x) > \frac{1}{2}$ for

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all k and all $x \in [-\delta, 0]$. Choose $\varepsilon := \delta$ and g_ε as defined above. As $f_k \geq 0$, $g_\varepsilon \geq 0$ we then have

$$\frac{\left|\langle Af_k, g_\varepsilon\rangle_{L^2(\tilde{I})}\right|}{\|f_k\|_{L^2(I)}} \geq \frac{\langle Af_k, g_\varepsilon\rangle_{L^2(I_\ell)}}{\|f_k\|_{L^2(I)}} \geq \frac{\frac{1}{2}\cdot\frac{\delta}{2}}{\|f_k\|_{L^2(I)}} = \frac{\delta}{4\,\|f_k\|_{L^2(I)}} \to \infty,$$

as $f_k \to 0$ in $L^2(I)$. This means $A_{g_\varepsilon} \notin L^2(I)^*$, which is the desired contradiction. □

5. Working around the trace

Although extension operators which interpolate trace values do not explicitly satisfy (A), they will lead us to deduce an extension operator in Definition 5.3 which turns out to be similar to the Hestenes extension used in [DS]. To this end, we concentrate on extending functions $f \in H^m(0,\infty)$ to $Af \in H^m(\mathbb{R})$. Two-sided extensions are then a simple modification of such an operator. The crucial point of the extension process is the satisfaction of an interpolation property like (4.1) of Af at $x = 0$ on the one hand and continuity of A and A^* on the other hand. First we focus on the interpolation property. What we are looking for is an operator A which satisfies the interpolation property

$$(Af)^{(k)}(x)|_{x=0-} = f^{(k)}(0), \qquad \text{for } k = 0, \dots, m-1, \tag{5.1}$$

together with $Af|_{(-\infty,0)} \in H^m((-\infty,0))$. Consider the Taylor polynomial T_{x_0} of f at $x_0 = 0$ from Subsection 4.1. Obviously,

$$(T_{x_0}f)(x) = \sum_{j=0}^{m-1} \tfrac{f^{(j)}(x_0)}{j!}(x-x_0)^j = \sum_{j=0}^{m-1} \tfrac{f^{(j)}(0)}{j!}x^j \tag{5.2}$$

satisfies the interpolation property (5.1). Recall that the operator E_T in Subsection 4.1 was built in terms of T_{x_0}. We now aim at deriving an approximation of the Taylor polynomial which is not trace dependent, employing divided differences; see, e.g., [Bo] for details.

Definition 5.1. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $\beta_j \in I,\ j = 1, \dots, k$, be pairwise distinct. Then the *divided differences* $[\beta_1, \dots, \beta_k]_f$ of f are defined as

$$\begin{aligned} [\beta_1]_f &:= f(\beta_1), \\ [\beta_1, \dots, \beta_k]_f &:= \tfrac{[\beta_2,\dots,\beta_k]_f - [\beta_1,\dots,\beta_{k-1}]_f}{\beta_k - \beta_1}. \end{aligned} \tag{5.3}$$

For I open and $f \in C^k(I)$, the divided differences satisfy

$$[\beta_1, \dots, \beta_k]_f \to \tfrac{f^{(k-1)}(x_0)}{(k-1)!}, \qquad \text{if } \beta_j \to x_0 \text{ for } j = 1, \dots, k. \tag{5.4}$$

Hence for $f \in C^k(I)$, $[\beta_1, \dots, \beta_k]_f$ can be extended to a continuous function $[\beta_1, \dots, \beta_k]_f : I \times \cdots \times I \to \mathbb{R}$. Due to (5.4), T_{x_0} from (5.2) can also be written as

$$(T_{x_0}f)(x) = (Nf)(x) := \sum_{j=1}^{m} [\beta_1, \dots, \beta_j]_f \prod_{i=1}^{j}(x-\beta_i) = \sum_{j=1}^{m} [\beta_1, \dots, \beta_j]_f x^{j-1}.$$

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Note that we cannot compute the incorporated divided differences explicitly as long as some β_j coincide. This leads us to the following approach. For $\beta_j > 0$ pairwise distinct, find a polynomial $P_\beta f$ solving the interpolation problem

$$(P_\beta f)(\beta_j) = f(\beta_j), \qquad j = 1, \dots, m. \tag{5.5}$$

The solution polynomial can be written in Newton form as

$$(P_\beta f)(x) = \sum_{j=1}^{m} [\beta_1, \dots, \beta_j]_f \prod_{i=1}^{j} (x - \beta_i).$$

Because of (5.4) we have pointwise convergence,

$$(P_\beta f)(x) \to (Nf)(x), \qquad \text{for all } x, \text{ if } \beta_j \to 0 \text{ for all } j = 1, \dots, m. \tag{5.6}$$

Hence for small β_j, the polynomial P_β can be regarded as a good approximation of the Taylor polynomial in (5.2). Of course, it still does not satisfy the interpolation property (5.1). We now aim at building the convergence of the β_j into the interpolating polynomial P_β. Clearly we have

$$\beta_j x \to 0 \qquad \text{for all } j = 1, \dots, m \text{ if } x \to 0.$$

So by (5.6) the convergence $(P_{\beta x} f)(y) \to (Nf)(y)$ for $x \to 0$ holds for all y. As we are just interested in an approximation of the Taylor polynomial $(T_0 f)(x)$ for x close to 0, we expect $(P_{\beta x} f)(x)$ to satisfy the interpolation property (5.1). Before we establish this property in Proposition 5.2, we first find a simple representation for $(P_{\beta x} f)(x)$. We have

$$\begin{aligned}(P_{\beta x} f)(x) &= \sum_{j=1}^{m} [\beta_1 x, \dots, \beta_j x]_f \prod_{i=1}^{j} (x - \beta_i x) \\ &= \sum_{j=1}^{m} f(\beta_j x) \prod_{i=1, i\neq j}^{m} \tfrac{x - \beta_j x}{\beta_j x - \beta_i x} = \sum_{j=1}^{m} f(\beta_j x) \prod_{i=1, i\neq j}^{m} \tfrac{1-\beta_i}{\beta_j - \beta_i}.\end{aligned}$$

Here we have switched in the second identity to the Lagrange representation of the interpolation polynomial $P_{\beta x} f$. Now we are ready to define an operator which satisfies the interpolation property (5.1) which is not an extension operator yet, as it neither extends f to the negative range nor leaves f unchanged on its range.

Proposition 5.2. *For $\beta_j > 0$ for $j = 1, \dots, m$ and $x > 0$, define $I_H : H^m(\mathbb{R}^+) \to H^m(\mathbb{R}^+)$ by*

$$(I_H f)(x) := (P_{\beta x} f)(x) = \sum_{j=1}^{m} f(\beta_j x) \prod_{i=1, i\neq j}^{m} \tfrac{1-\beta_i}{\beta_j - \beta_i}.$$

Then I_H is an interpolation operator *satisfying the interpolation property* (5.1).

Proof. Obviously, $I_H f \in H^m(\mathbb{R}^+)$ if $f \in H^m(\mathbb{R}^+)$. We want to prove

$$(I_H f)^{(k)}(x)|_{x=0} = f^{(k)}(0) \qquad \text{for } k = 0, \dots, m-1.$$

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We have for all $k \le m-1$

$$\begin{aligned}(I_H f)^{(k)}(x)|_{x=0} &= \sum_{j=1}^{m} f^{(k)}(\beta_j x)\beta_j^k \prod_{i\neq j}^{m} \tfrac{1-\beta_i}{\beta_j-\beta_i}\Bigg|_{x=0} \\ &= f^{(k)}(0)\sum_{j=1}^{m}\beta_j^k \prod_{i\neq j}^{m} \tfrac{1-\beta_i}{\beta_j-\beta_i} = f^{(k)}(0)\end{aligned}$$

because of the identity

$$\sum_{j=1}^{m}\beta_j^k \prod_{i\neq j}^{m} \tfrac{1-\beta_i}{\beta_j-\beta_i} = 1$$

in the last equation. To see this, we claim that

$$\sum_{j=1}^{m}\beta_j^k \prod_{i\neq j}^{m} \tfrac{x-\beta_i}{\beta_j-\beta_i} = x^k$$

holds for all $x \in \mathbb{R}$ and $0 \le k \le m-1$. Indeed, both sides of the equation are polynomials of degree $\le m-1$ which coincide on the m sampling points β_j. Hence they are equal. Applying this to $x = 1$ yields the desired identity. □

By a modification we obtain an extension operator.

Proposition 5.3. *For $f \in H^m(\mathbb{R}^+)$, $\beta_j < 0$, $j = 1, \dots, m$, define $E_H : H^m(\mathbb{R}^+) \to H^m(\mathbb{R})$ by*

$$(E_H f)(x) := \chi_{\mathbb{R}^+}(x) f(x) + \chi_{(-\infty,0)} \sum_{j=1}^{m} f(\beta_j x) \prod_{i=1, i\neq j}^{m} \frac{1-\beta_i}{\beta_j-\beta_i}. \tag{5.7}$$

Then E_H is an extension operator *satisfying the interpolation property* (5.1).

Proof. It is clear from the proof of Proposition 5.2 that $E_H f$ satisfies the interpolation property (5.1). To prove $E_H f \in H^m(\mathbb{R})$ we remark that $E_H f|_{(-\infty,0)} \in H^m((-\infty, 0))$. Together with the interpolation property (5.1) and the gluing behavior stated in Remark 2.2, we have $E_H f \in H^m(\mathbb{R})$. □

Note that in (5.7) the β_j are all negative now, ensuring that f is evaluated in its parameter domain $\mathbb{R}^+$ in the second term. The definition in Proposition 5.3 can be generalized to some extent. Revisiting the crucial step in the proof of smoothness in Proposition 5.2, we recall that smoothness was obtained since for $\ell = m$ we had

$$\sum_{j=0}^{\ell}\beta_j^k \prod_{i\neq j}^{\ell} \tfrac{1-\beta_i}{\beta_j-\beta_i} =: \sum_{j=0}^{\ell}\beta_j^k \alpha_j = 1, \qquad \text{for } k = 0, \dots, m-1. \tag{5.8}$$

This is a Vandermonde system for the α_j. Thus, it is ill conditioned for the β_j being close to each other which results in a solution vector α with a large Euclidean norm $\|\alpha\|_2$. If we now allow ℓ to become greater than m and still require the ℓ sampling points β_j to satisfy (5.8), we observe smaller solutions α measured in $\|\cdot\|_2$, solving the system (5.8) by the least squares method. Although in the course of the optimization process we will realize that the operator norm of E_H does not decrease when the number ℓ of sampling points is increased, we restate the definition in Proposition 5.3 for this setting.

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Proposition 5.4. *Given the smoothness level m, let $\ell \geq m$. Let the numbers*

$$\beta_1, \dots, \beta_\ell < 0 \text{ pairwise distinct}, \qquad \alpha_1, \dots, \alpha_\ell,$$

solve the linear Vandermonde system $\sum_{j=1}^{\ell} \alpha_j \beta_j^k = 1$, $k = 0, \dots, m-1$. Then the operator $\tilde{E}_H$, defined by $(\tilde{E}_H f)(x) := \chi_{\mathbb{R}^+}(x) f(x) + \chi_{(-\infty,0)} \sum_{j=1}^{\ell} \alpha_j f(\beta_j x)$, is an extension operator $E : H^m(\mathbb{R}^+) \to H^m(\mathbb{R})$ satisfying the interpolation property (5.1).

The proof is the same as in Proposition 5.3. Although $\tilde{E}_H$ is very similar to the Hestenes extension introduced in [DS], it still suffers from the following deficiencies. First, $\tilde{E}_H$ extends functions in $H^m(\mathbb{R}^+)$ to functions in $H^m(\mathbb{R})$ rather than functions in $H^m(I)$ to functions in $H^m(\tilde{I})$. Second, in (A) we require zero boundary values. Third, $\tilde{E}_H$ is an extension operator on the primal side, but we do not yet have any grip on the adjoint $\tilde{E}_H^*$ relative to the L_2 inner product. Further we need to modify $\tilde{E}_H$ to finally obtain an operator satisfying (A).

6. The Hestenes extension

Given the smoothness level m, let $\ell \geq 2m$. Let real numbers

$$\beta_1, \dots, \beta_\ell < 0 \text{ pairwise distinct}, \qquad \alpha_1, \dots, \alpha_\ell, \tag{6.1}$$

solve the linear Vandermonde system

$$\sum_{j=1}^{\ell} \alpha_j \beta_j^k = 1, \qquad k = -m, \dots, m-1. \tag{6.2}$$

Defining

$$V_m := \begin{pmatrix} \beta_1^{-m} & \cdots & \beta_\ell^{-m} \\ \vdots & \ddots & \vdots \\ \beta_1^{m-1} & \cdots & \beta_\ell^{m-1} \end{pmatrix}, \qquad b := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{6.3}$$

this can be written as the linear system of equations

$$V_m \alpha = b \tag{6.4}$$

for $\alpha := (\alpha_1, \dots, \alpha_\ell)^T \in \mathbb{R}^\ell$. We can assume without loss of generality that the β_j are sorted such that

$$\min_j \beta_j = \beta_1 < \cdots < \beta_\ell = \max_j \beta_j < 0. \tag{6.5}$$

Remark 6.1. 1. Obviously, $\ell = 2m$ suffices to obtain a unique solution of (6.4). For greater ℓ smaller solutions α, with respect to the Euclidean $\|\cdot\|_2$ norm, may be obtained by the least squares approach

$$\alpha = V_m^T (V_m V_m^T)^{-1} b, \tag{6.6}$$

which is the smallest solution of (6.4) with respect to $\|\cdot\|_2$. We will see in Subsection 7.3 that $\ell = 2m$ is actually the best choice.

2. In [DS] it was assumed that $\ell \geq 2m+2$ and that the β_j, α_j satisfy (6.2) in the larger range $k = -m-1, \dots, m$. This is necessary for smooth extensions in the C^m-sense. However, as we are indeed interested in smooth extensions in the H^m-sense, by Remark 2.2 $2m$ conditions suffice. In fact, this observation will mean a great improvement to the condition of the Hestenes extension.

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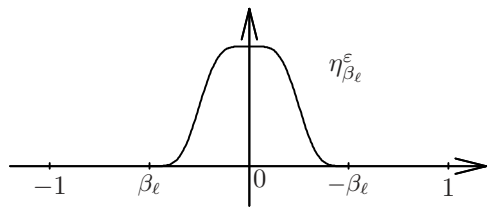

FIGURE 5. The function $\eta^{\varepsilon}_{\beta_\ell}$ with $\varepsilon = 1$.

Additionally, we need a cutoff function $\eta^{\varepsilon}_{\beta_\ell} \in H^m(\mathbb{R})$, where $\varepsilon \in (0,1]$ and β_ℓ is the largest sampling point in (6.5). Let

$$\eta^{\varepsilon}_{\beta_\ell}(x) := \eta_m\left(\tfrac{x}{-\beta_\ell \varepsilon}\right), \tag{6.7}$$

where η_m is the cutoff function defined in Lemma 4.4. We see in Figure 5 that as long as $\beta_\ell > -1$, $\eta^{\varepsilon}_{\beta_\ell}$ satisfies the support property

$$\operatorname{supp} \eta^{\varepsilon}_{\beta_\ell} \subset ([\varepsilon\beta_\ell, -\varepsilon\beta_\ell] \cap [-\varepsilon, \varepsilon]) \tag{6.8}$$

together with the growth property at $x = 0$,

$$\eta^{\varepsilon}_{\beta_\ell}(0) = 1, \quad {\eta^{\varepsilon}_{\beta_\ell}}^{(k)}(0) = 0 \quad \text{for } k = 1, \dots, m-1. \tag{6.9}$$

In [DS] β_1 was bounded by -2 from below, which is indeed not necessary after specifying the cutoff function by (6.7). These prerequisites at hand, we can now state

Definition 6.2. Let $I = (0,1)$, $\tilde{I} = (-1,2)$, $I_\ell = (-1,0)$, $I_r = (1,2)$. The *Hestenes extension* $A : H^m(I) \to H^m(\tilde{I})$ of f is defined by

$$\begin{aligned}(Af)(x) := \chi_I(x) f(x) &+ \chi_{I_\ell}(x) \sum_{j=1}^{\ell} \alpha_j (\eta^{\varepsilon}_{\beta_\ell} f)(\beta_j x) \\ &+ \chi_{I_r}(x) \sum_{j=1}^{\ell} \alpha_j\, \eta^{\varepsilon}_{\beta_\ell}(\beta_j(x-1)) f(1 + \beta_j(x-1)).\end{aligned} \tag{6.10}$$

Its adjoint $A^* : H^m(\tilde{I}) \to H^m(I)$ of A relative to the L^2 inner product can be computed as

$$\begin{aligned}A^* g \quad &= \chi_I(x) g(x) - \chi_{I_\ell}(x-1) \sum_{j=1}^{\ell} \beta_j^{-1} \alpha_j \eta^{\varepsilon}_{\beta_\ell}(x) g(\beta_j^{-1} x) \\ &\quad - \chi_{I_r}(1+x) \sum_{j=1}^{\ell} \beta_j^{-1} \alpha_j \eta^{\varepsilon}_{\beta_\ell}(x-1) g(1 + \beta_j^{-1}(x-1)).\end{aligned} \tag{6.11}$$

Lemma 6.3. *The Hestenes extension A defined in* (6.2) *satisfies* (A). *Furthermore, we have* $\operatorname{supp} Af \subset [-\varepsilon, 1+\varepsilon]$.

Proof. We have to show (L), (CA), (CA*) and that A is well defined; i.e., $Af \in H^m(\tilde{I})$ for all $f \in H^m(I)$. (CA) is proved in Lemma 6.4 below. The proof of (CA*) is analogous in view of (6.11). We start by proving that A maps H^m-functions to H^m-functions. Due to the product rule for H^m-functions we certainly have $Af|_{I_\ell} \in H^m(I_\ell)$, $Af|_I = f \in H^m(I)$, $Af|_{I_r} \in H^m(I_r)$. In view of Remark 2.2 we

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have to check whether the first $m-1$ trace values of $Af|_{I_\ell}$ and f coincide at $x=0$, and of $Af|_{I_r}$ and f at $x=1$, respectively. As $f, \eta^\varepsilon_{\beta_\ell} \in C^{m-1}(\overline{I})$ by Theorem 2.1, we can differentiate (6.10) from the left at $x=0$ using the Leibniz rule and compute for $k=0,\dots,m-1$

$$\begin{aligned}(Af)^{(k)}(0) &= \sum_{j=1}^{\ell} \alpha_j \beta_j^k (\eta^\varepsilon_{\beta_\ell} f)^{(k)}(0) = \sum_{j=1}^{\ell} \alpha_j \beta_j^k \sum_{i=0}^{k} \binom{k}{i} {\eta^\varepsilon_{\beta_\ell}}^{(k-i)}(0) f^{(i)}(0) \\ &= \sum_{j=1}^{\ell} \alpha_j \beta_j^k \eta^\varepsilon_{\beta_\ell}(0) f^{(k)}(0) = f^{(k)}(0)\end{aligned}$$

due to (6.2) and the behaviour of $\eta^\varepsilon_{\beta_\ell}$ at $x=0$ described in (6.9). Differentiating (6.10) from the right at $x=1$ analogously yields

$$(Af)^{(k)}(1) = \sum_{j=1}^{\ell} \alpha_j \beta_j^k \eta(0) f^{(k)}(1) = f^{(k)}(1), \quad k=0,\dots,m.$$

Similarly, we get $A^* : H^m(\tilde{I}) \to H_0^m(I)$, now using (6.2) for $k=-m,\dots,-1$. To see that A^* from (6.11) really coincides with the L^2-adjoint of A, we compute for $f \in L^2(I)$, $g \in L^2(\tilde{I})$

$$\begin{aligned}\int_{\tilde{I}} Afg\,dx &= \int_{[0,1]} f(x)g(x)dx + \int_{[-1,0]} \sum_{j=1}^{\ell} \alpha_j \left(\eta^\varepsilon_{\beta_\ell} f\right)(\beta_j x) g(x)\,dx \\ &\quad + \sum_{j=1}^{\ell} \int_{[1,2]} \alpha_j \eta^\varepsilon_{\beta_\ell}(\beta_j(x-1)) f(1+\beta_j(x-1)) g(x)\,dx \\ &= \int_{[0,1]} f(x)g(x)dx - \sum_{j=1}^{\ell} \int_{[0,-\beta_j]} \beta_j^{-1}\alpha_j \left(\eta^\varepsilon_{\beta_\ell} f\right)(x) g(\beta_j^{-1}x)\,dx \\ &\quad - \sum_{j=1}^{\ell} \int_{[1+\beta_j,1]} \beta_j^{-1}\alpha_j \eta^\varepsilon_{\beta_\ell}(x-1) f(x) g(\beta_j^{-1}(x-1)+1)\,dx \\ &= \int_{[0,1]} f(x)g(x)dx - \int_{[0,1]} \sum_{j=1}^{\ell} \beta_j^{-1}\alpha_j \left(\eta^\varepsilon_{\beta_\ell} f\right)(x) g(\beta_j^{-1}x)\,dx \\ &\quad - \int_{[0,1]} \sum_{j=1}^{\ell} \beta_j^{-1}\alpha_j \eta^\varepsilon_{\beta_\ell}(x-1) f(x) g(\beta_j^{-1}(x-1)+1)\,dx.\end{aligned}$$

Here we simply substituted $x \mapsto \beta_j^{-1}x$ in the second integral and $x \mapsto \beta_j^{-1}(x-1)+1$ in the third integral. We were allowed to extend the integral domains as we have $\eta^\varepsilon_{\beta_\ell}(x)=0$ for $x > -\beta_j$ and $\eta^\varepsilon_{\beta_\ell}(x-1)=0$ for $x < 1+\beta_j$ due to (6.8), bearing in mind that $\beta_j \le \beta_\ell$ for all j.

We are left with showing (L); i.e., we have to prove zero boundary values for the extension Af. We will prove that $\operatorname{supp} Af \subset [-\varepsilon, 1+\varepsilon]$ and that Af has zero boundary values in its support. For the support, note that obviously

$$\operatorname{supp} Af \subset \operatorname{supp} \sum_{j=1}^{\ell} \eta^\varepsilon_{\beta_\ell}(\beta_j x) \subset \bigcup_{j=1}^{\ell} \operatorname{supp} \eta^\varepsilon_{\beta_\ell}(\beta_j x) \subset \operatorname{supp} \eta^\varepsilon_{\beta_\ell}(\beta_\ell x)$$

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because $\beta_j \leq \beta_\ell$ and

$$\operatorname{supp} \eta^\varepsilon_{\beta_\ell}(\beta_\ell x) = \operatorname{supp} \eta_m\left(\tfrac{x}{\varepsilon}\right) = [-\varepsilon, 1+\varepsilon]$$

by the definition of $\eta^\varepsilon_{\beta_\ell}$ in (6.7) and the cutoff function η_m from Lemma 4.4. As $f(\beta_j x)$ is bounded and $\eta^\varepsilon_{\beta_\ell}(\beta_j x)$ has zero boundary values by Lemma 4.4 in its support $\operatorname{supp} \eta^\varepsilon_{\beta_\ell}(\beta_j x) = [\beta_j \varepsilon, -\beta_j \varepsilon]$ (see (6.8)), it is clear that Af also has zero boundary values in its support $[-\varepsilon, 1+\varepsilon]$, as Af is just a sum of the $\eta^\varepsilon_{\beta_\ell}$ weighted with bounded functions. Finally note that although f and g might be evaluated outside their domains I, $\tilde{I}$, respectively, in the definitions of A and A^* the operators are still well defined as in those cases the support of the cutoff function $\eta^\varepsilon_{\beta_\ell}$ ensures that the corresponding term is annihilated. □

We will use an estimate of the operator norm of A to further optimize the parameters involved.

Lemma 6.4. *Let $m \geq 1$ and η_m be as given in Lemma 4.4. In view of (6.5), we have the estimate*

$$\|Af\|_{H^m(\tilde{I})} \leq \left(1 + 2\sqrt{\ell}\frac{(-\beta_1)^{m-\frac{1}{2}}}{(-\beta_\ell)^m} \|\alpha\|_2 \|\eta_m\|_{W^{m,\infty}(\mathbb{R})}\right) \|f\|_{H^m(I)}. \tag{6.12}$$

Proof. From (6.10) we obtain by the triangle inequality

$$\begin{aligned}\|Af\|_{H^m(\tilde{I})} \leq \|f\|_{H^m(I)} &+ \|\sum_{j=1}^{\ell} \alpha_j (\eta^\varepsilon_{\beta_\ell} f)(\beta_j x)\|_{H^m(I_\ell)} \\ &+ \|\sum_{j=1}^{\ell} \alpha_j \eta^\varepsilon_{\beta_\ell}(\beta_j(x-1)) f(1+\beta_j(x-1))\|_{H^m(I_r)}.\end{aligned}$$

We just prove the inequality for the second term; for the third it is analogous:

$$\begin{aligned}\|\sum_{j=1}^{\ell} \alpha_j (\eta^\varepsilon_{\beta_\ell} f)(\beta_j x)\|_{H^m(I_\ell)} &\leq \sum_{j=1}^{\ell} |\alpha_j| \left\|(\eta^\varepsilon_{\beta_\ell} f)(\beta_j x)\right\|_{H^m(I_\ell)} \\ &\leq \left(\sum_{j=1}^{\ell} |\alpha_j| \left|\frac{\beta_j}{\beta_\ell}\right|^{m-\frac{1}{2}}\right) \|\eta_m(x) f(\beta_\ell x)\|_{H^m(I)} \\ &\lesssim \left(\sum_{j=1}^{\ell} |\alpha_j| \, |\beta_j|^{m-\frac{1}{2}}\right) \frac{\|\eta_m\|_{W^{m,\infty}(\mathbb{R})}}{|\beta_\ell|^m} \|f\|_{H^m(I)}.\end{aligned}$$

Here we have used twice that for any $\varphi \in H^m$ and any $\lambda \in \mathbb{R}$ we have $\|\varphi(\lambda x)\|_{H^m} \leq |\lambda|^{-\frac{1}{2}} \cdot \max\{1, |\lambda|^m\} \|\varphi\|_{H^m}$. We can now apply various finite dimensional Hölder

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inequalities to the sum above. Using $\|(a_j b_j)_j\|_1 \le \|(a_j)_j\|_2 \|(b_j)_j\|_2$, we can estimate

$$\begin{aligned}\tfrac{\|\eta_m\|_{W^{m,\infty}(\mathbb{R})}}{|\beta_\ell|^m} \|f\|_{H^m(I)} \sum_{j=1}^{\ell} |\alpha_j|\,|\beta_j|^{m-\frac{1}{2}} \\ &\le \|\alpha\|_2 \left\| \left(|\beta_j|^{m-\frac{1}{2}}\right)_j \right\|_2 \tfrac{\|\eta_m\|_{W^{m,\infty}(\mathbb{R})}}{|\beta_\ell|^m} \|f\|_{H^m(I)} \\ &\le \sqrt{\ell}\tfrac{(-\beta_1)^{m-\frac{1}{2}}}{(-\beta_\ell)^m} \|\alpha\|_2 \|\eta_m\|_{W^{m,\infty}(\mathbb{R})}. \qquad \square\end{aligned}$$

7. Optimization of the Hestenes extension

We now use the estimate (6.12) for optimizing the operator norm of A; i.e., we minimize the functional

$$F(\beta_1,\dots,\beta_\ell) := \ell\frac{(-\beta_1)^{2m-1}}{(-\beta_\ell)^{2m}} \|\alpha\|_2^2 = \ell\frac{(-\beta_1)^{2m-1}}{(-\beta_\ell)^{2m}} \sum_{j=1}^{\ell} \alpha_j(\beta_1,\dots,\beta_\ell)^2. \tag{7.1}$$

We will later apply a gradient method to F. For this to be practicable, we first deduce for $\ell = 2m$ an explicit representation for the α_j in terms of the β_j.

7.1. Optimization for $\ell = 2m$.

Lemma 7.1. *The solution α of the Vandermonde system* (6.2) *for the special case $\ell = 2m$ is given by*

$$\alpha_j = \beta_j^m \prod_{i=1, i\neq j}^{\ell} \left(\tfrac{1-\beta_i}{\beta_j-\beta_i}\right) \tag{7.2}$$

with partial derivatives

$$\partial_{\beta_k}\alpha_j = \begin{cases} \alpha_j \frac{1-\beta_j}{(1-\beta_k)(\beta_j-\beta_k)}, & \text{if } k \neq j, \\ \alpha_j \left(\frac{m}{\beta_j} - \sum_{i=1, i\neq j}^{2m} (\beta_j-\beta_i)^{-1}\right), & \text{if } k = j. \end{cases} \tag{7.3}$$

Proof. Applying Cramer's rule to the matrix system (6.4) we find for the solutions α_j

$$\alpha_j = \frac{\begin{vmatrix} \beta_1^{-m} & \cdots & \beta_{j-1}^{-m} & 1 & \beta_{j+1}^{-m} & \cdots & \beta_\ell^{-m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_1^{m-1} & \cdots & \beta_{j-1}^{m-1} & 1 & \beta_{j+1}^{m-1} & \cdots & \beta_\ell^{m-1} \end{vmatrix}}{|V_m|}, \tag{7.4}$$

where $|M| := \det M$. For the Vandermonde determinant we have

$$|V_m| = \prod_{i=1}^{\ell} \beta_i^{-m} \prod_{i<j}^{\ell} (\beta_j - \beta_i). \tag{7.5}$$

It remains to deduce a formula for the determinant in the numerator in (7.4). Note that the matrix in the numerator has been derived by simply filling the j-th column with 1's, or, equivalently, replacing the real number β_j by 1. So we obtain a modified

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Vandermonde system with the determinant given by (7.5) with β_j replaced by 1. Consequently, combining (7.4), (7.5), and this last observation, we get

$$\alpha_j = \beta_j^m \prod_{i=1, i\neq j}^{\ell} \left(\tfrac{1-\beta_i}{\beta_j-\beta_i}\right).$$

Straightforward computations then yield (7.3). □

Note that we could have also followed the lines of the proof after Proposition 5.2 to prove the explicit representation (7.2). We now use the explicit representation of the α_j (7.2) to minimize the functional F in (7.1) by a gradient projection method described in [B]; i.e., F is being minimized successively by descending along the gradient of F projected onto the linear admissible range given by (6.1).

7.2. **Numerical results of the optimization process for $\ell = 2m$.** Comparing our optimized Hestenes extension with the operator found in [DS], two relaxations are made here. First, $\ell \geq 2m$ of the β_j's suffice, whereas in [DS], $\ell \geq 2m+2$ of the β_j's were used to satisfy (6.2) in the larger range $k = -m-1, \ldots, m$; see Remark 6.1.2. Second, by adapting the support of the cutoff function $\eta^{\varepsilon}_{\beta_\ell}$ in (6.8), it turns out that the restriction of the β_j from above and below (i.e., $-2 \leq \beta_1 < \cdots < \beta_\ell \leq -\frac{1}{2}$) is actually not necessary.

Apart from these changes, the main ingredient of the above optimization process is to find the optimal distribution of the β_j. We consider an equidistant distribution for comparison purposes. As mentioned above in Remark 6.1.1 the operator norm decreases in the equidistant setting when ℓ is increased, solving (6.2) by the least squares method. In numerical tests, it has turned out that choosing $\ell = 1000$ equidistant β_j for the unoptimized comparison operator gives among other choices of ℓ slightly better results with respect to the term (7.1). The left table in Table 1 shows the estimates of the operator norms of this unoptimized operator for $m = 1, 2, 3$. The operator norms of the Hestenes extension stemming from the optimization process in Subsection 7.1 are listed in the right table in Table 1. The values and the distribution of the optimized $\beta_j, j = 1, \ldots, \ell$, are provided in Figure 6 as well as the corresponding $\alpha_j, j = 1, \ldots, \ell$, computed by (7.2).

Remark 7.2. We have not yet explicitly addressed the case $m = 0$. Obviously, extending by zero yields an operator $Z : L^2(I) \to L^2(\tilde{I})$, with its adjoint given by the restriction $Z^*(g) = g|_I$ for all $g \in L^2(\tilde{I})$. For $m = 0$, one can also use the Hestenes operator for the case $m = 1$ with β_2 chosen very small. Then the small support of $\eta^{\varepsilon}_{\beta_\ell}$ ensures that the extensions do not differ from the extensions by zero very much (see the central and right column in Figure 8), and additionally, continuous functions will have continuous extensions.

TABLE 1. Values for F defined in (7.1), not optimized, equidistant distribution of the β_j's for $\ell = 1000$ (left) and optimized for $\ell = 2m$ (right).

m	$\sqrt{\ell}\frac{(-\beta_1)^{m-\frac{1}{2}}}{(-\beta_\ell)^m}\,\lVert\alpha\rVert_2$
1	$5.72e+2$
2	$1.70e+5$
3	$5.33e+7$

m	$\sqrt{\ell}\frac{(-\beta_1)^{m-\frac{1}{2}}}{(-\beta_\ell)^m}\,\lVert\alpha\rVert_2$
1	$8.69e+0$
2	$1.14e+3$
3	$1.39e+5$

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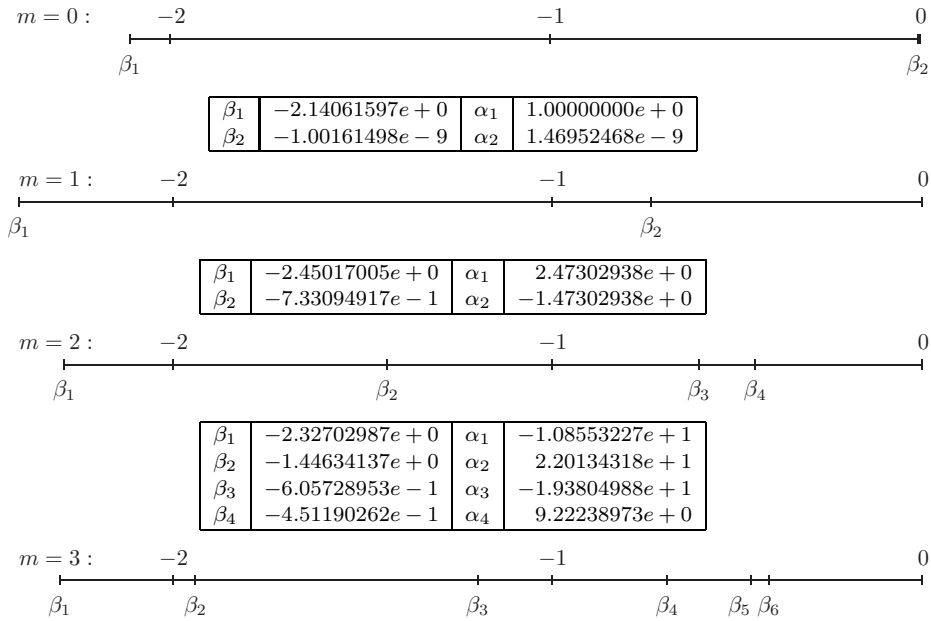

β_1	$-2.14061597e+0$	α_1	$1.00000000e+0$
β_2	$-1.00161498e-9$	α_2	$1.46952468e-9$

β_1	$-2.45017005e+0$	α_1	$2.47302938e+0$
β_2	$-7.33094917e-1$	α_2	$-1.47302938e+0$

β_1	$-2.32702987e+0$	α_1	$-1.08553227e+1$
β_2	$-1.44634137e+0$	α_2	$2.20134318e+1$
β_3	$-6.05728953e-1$	α_3	$-1.93804988e+1$
β_4	$-4.51190262e-1$	α_4	$9.22238973e+0$

β_1	$-2.33906919e+0$	α_1	$1.17173325e+2$
β_2	$-1.97087654e+0$	α_2	$-2.30781180e+2$
β_3	$-1.20324850e+0$	α_3	$2.29596664e+2$
β_4	$-6.88357382e-1$	α_4	$-2.17924281e+2$
β_5	$-4.64512531e-1$	α_5	$2.09953416e+2$
β_6	$-4.11270732e-1$	α_6	$-1.07017944e+2$

FIGURE 6. Optimized β_j, their distribution, and corresponding α_j solving (6.2) for $m = 0, \ldots, 3$.

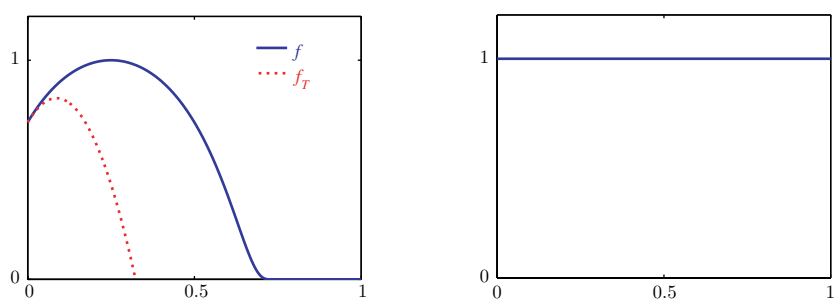

FIGURE 7. The function f defined in (7.6), its Taylor expansion f_T (left), and $g = \chi_{[0,1]}$ (right).

Numerical Examples. We briefly pause to illustrate the optimized extension and to consider extensions for two smooth functions and of the characteristic function.

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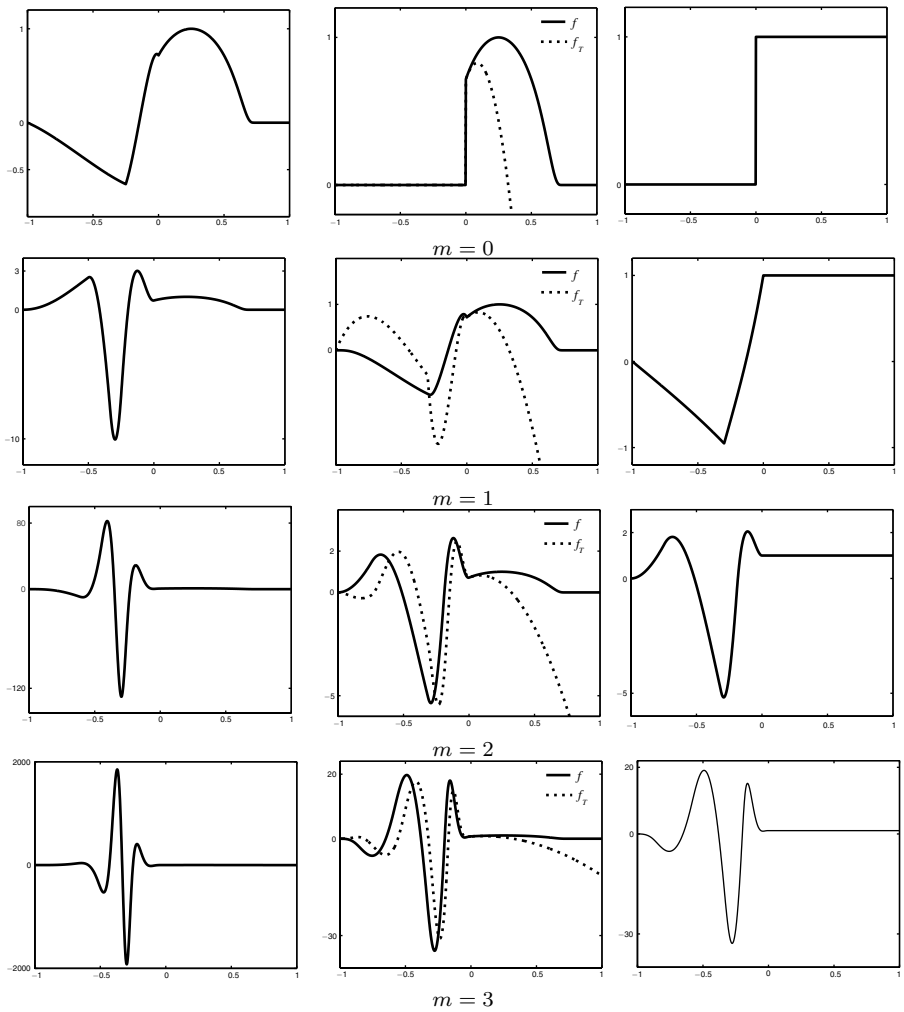

FIGURE 8. Left column: extensions of f not optimized, equidistant distribution of the β_j's for $\ell = 1000$; Center column: extensions of f and f_T optimized; Right column: extensions of $\chi_{[0,1]}$ optimized, $m = 0, \ldots, 3$.

Define

$$f(x) := \begin{cases} \exp\left(1 - \frac{1}{1-\left(2x-\frac{1}{2}\right)^2}\right), & x \in \left(0, \frac{3}{4}\right), \\ 0, & x \in \left[\frac{3}{4}, 1\right), \end{cases} \tag{7.6}$$

$$f_T(x) := e^{-\frac{1}{3}}\left(1 + \frac{32}{9}x - \frac{13 \cdot 128}{81}x^2\right), \qquad g := \chi_{[0,1]};$$

see Figure 7. f_T is constructed to be the Taylor expansion of f up to order 2 at $x = 0$. Figure 8 shows extensions of f, f_T and g. The center column displays extensions of f and its Taylor expansion f_T by the optimized Hestenes extension. In contrast to the trace dependent operators discussed in Section 4, the extensions

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of f and f_T differ in spite of coinciding trace values. For comparison, extensions of f by the unoptimized Hestenes extension are displayed in the left column. The right column show extensions of $g = \chi_{[0,1]}$.

7.3. **Optimization for $\ell > 2m$.** As mentioned in Remark 6.1.1, smaller solutions of α with respect to the Euclidean norm can be obtained in the equidistant case by increasing ℓ. This suggests applying a gradient method to the functional (7.1) for the general case $\ell > 2m$ to obtain further improvements. Although in this setting the representation (7.2) still solves (6.2), it does not coincide with the least squares solution $\alpha = V_m^T(V_m V_m^T)^{-1}b$ introduced in (6.6) (with V_m from (6.3)); hence a minimization of the operator norm cannot be carried out analogously. We use an implicit gradient method instead, using the expression proved in [KS, Remark 10.1]. We apply this general formula to the situation of the least squares solution (6.6) in order to explicitly compute the gradient of (7.1) for the case $\ell > 2m =: \nu$. Defining $V := V_m$ as in (6.3), we want to compute the gradient of the function

$$\beta \mapsto \alpha(\beta) := V^T(VV^T)^{-1}b, \tag{7.7}$$

where here

$$A := VV^T = \left(\sum_{q=1}^{\ell} \beta_q^{-\nu+i+j-2}\right)_{ij=1}^{\nu}. \tag{7.8}$$

The proof of the following technical lemma can be found in [KS].

Lemma 7.3. *The partial derivatives of the function defined in* (7.7) *are given by* (7.9)

$$\partial_{\beta_k}\alpha_j = \begin{cases} \displaystyle\sum_{s=1}^{\nu}(\mu+s)\beta_j^{\mu+s-1}\sum_{t=1}^{\nu}a^{st}b_t \\ \qquad \displaystyle+\sum_{s=1}^{\nu}\beta_j^{\mu+s}\sum_{t=1}^{\nu}b_t\sum_{q,r=1}^{\nu}-a^{sq}a^{rt}(2\mu+q+r)\beta_k^{2\mu+q+r-1}, & k=j \\ \displaystyle\sum_{s=1}^{\nu}\beta_j^{\mu+s}\sum_{t=1}^{\nu}s\sum_{q,r=1}^{\nu}-a^{sq}a^{rt}(2\mu+q+r)\beta_k^{2\mu+q+r-1}, & k\neq j. \end{cases} \tag{7.9}$$

With these partial derivatives at hand we can compute the gradient of the functional (7.1) in order to perform a gradient descent as before for the case $\ell = 2m$. Various results of this optimization procedure for different choices of ℓ show that this relaxation yields larger operator norms of the Hestenes extension; see Table 2.

Now after we have constructed an optimized extension operator satisfying (A), we can proceed with the construction of wavelets.

TABLE 2. Optimized operator norms $\sqrt{\ell}\frac{(-\beta_1)^{m-\frac{1}{2}}}{(-\beta_\ell)^m}\|\alpha\|_2$ for $m = 1, 2, 3$, $\ell \geq 2m$.

$m\backslash\ell$	2	3	4	5	6	7	8
1	8.69	8.97	$9.29e+0$	$1.00e+1$	$1.05e+1$	$9.82e+1$	$1.03e+1$
2			$1.14e+3$	$1.36e+3$	$1.51e+4$	$1.78e+3$	$2.01e+3$
3					$1.39e+5$	$1.47e+5$	$1.68e+5$

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8. Construction of the wavelets

In this section we summarize the construction of wavelets on manifolds in [DS]. We start by subdividing the manifold into quadrangular patches. In a second step, wavelets will be pushed forward to these patches from the parameter domain in Subsection 8.4. Finally these wavelets are extended appropriately to the manifold in a third stage in Subsection 8.5.

8.1. **Deconstruction of the manifold.** We consider manifolds Γ that are the disjoint union of parametric images of the open unit cube $\square = (0,1)^n$ where n is the dimension of the manifold; i.e., we assume that we have sufficiently smooth mappings $\kappa_i : \square \to \Gamma$, $\Gamma_i := \kappa_i(\square)$ and

$$\Gamma = \bigcup_{i=1}^{N} \overline{\Gamma_i}, \Gamma_i \cap \Gamma_j = \emptyset \quad \text{for } i \neq j.$$

Remark 8.1. For technical reasons we require the following property. If $x \in \Gamma$ is a vertex of some patch Γ_i, then it is also a vertex of any other patch Γ_j with $x \in \overline{\Gamma_j}$.

Γ has a boundary; we denote the part where Dirichlet boundary conditions shall be applied by ∂_D. Likewise, ∂_N denotes the part of the boundary of Γ where Neumann boundary conditions will be posed.

Remark 8.2. The definition of parametric mappings imply that for all $i = 1, \ldots, N$ we have a square $\Diamond \Supset \square$ and a bijective mapping $\hat{\kappa}_i$ with the same smoothness as the κ_i such that $\hat{\kappa}_i|_\square = \kappa_i$. $\square \Subset \Diamond$ means that $\square$ is compactly contained in $\Diamond$ and hence there exist $\varepsilon_i^a, \varepsilon_i^b > 0$ such that $\square \subset (-\varepsilon_i^a, 1 + \varepsilon_i^b)^n \subset \Diamond$. Hence we can assume without loss of generality that

$$\Diamond = \times_{i=1}^{n} (-\varepsilon_i^a, 1 + \varepsilon_i^b) =: \square_N^{(i)}. \tag{8.1}$$

This is more specific than in [DS] and will play an important role later.

The manifold Γ is equipped with the scalar product

$$\langle u, v \rangle_\Gamma = \int_\Gamma uv d\mu, \tag{8.2}$$

derived by integrals over the patches Γ_i,

$$\int_{\Gamma_i} u d\mu := \int_\square u \kappa_i \sqrt{\left|\det \nabla \kappa_i^T \nabla \kappa_i\right|} dL^n.$$

We abbreviate

$$\partial \kappa_i := \sqrt{\left|\det \nabla \kappa_i^T \nabla \kappa_i\right|}.$$

Recall the special numbering of the patches given in [DS]. The numbering has two phases, an initialization phase and an induction phase. In the initialization phase we pick an arbitrary patch and call it Γ_1. We set $\mathcal{G}_1 := \{\Gamma_1\}$. In the induction phase, let $\mathcal{G}_i$ be constructed. We then set

$$\mathcal{G}_{i+1} := \mathcal{G}_i \cup \{\text{all patches that share an edge with patches in } \mathcal{G}_i\}$$

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and number the new patches in $\mathcal{G}_{i+1}$ (i.e. the patches in $\mathcal{G}_{i+1} \setminus \mathcal{G}_i$) as follows: take any patch that has not been numbered yet and assign it the next free number. Repeat this process until all patches in $\mathcal{G}_{i+1}$ are numbered. In the rest of the paper the numbering $(\Gamma_i)_{i=1}^N$ will always be assumed to stem from this ordering. An example can be seen in Figure 12 below.

8.2. **Neighbours, inflow and outflow.** To any patch Γ_i we associate a set of *inflow* and *outflow* neighbours. We divide the boundary of Γ_i into an *inflow* and an *outflow boundary*. The outflow boundary $\partial^\uparrow \Gamma_i$ is defined by

$$\partial^\uparrow \Gamma_i := \bigcup \{\text{edges } \nu \text{ of } \Gamma_i \,:\, (\nu \text{ is also an edge of } \Gamma_j \text{ with } j > i) \text{ or } \nu \subset \partial_N\}. \tag{8.3}$$

Analogously we define the inflow boundary $\partial^\downarrow \Gamma_i$ to be

$$\partial^\downarrow \Gamma_i := \bigcup \{\text{edges } \nu \text{ of } \Gamma_i \,:\, (\nu \text{ is also an edge of } \Gamma_j \text{ with } j < i) \text{ or } \nu \subset \partial_D\}. \tag{8.4}$$

We will later define primal wavelets on Γ which have zero boundary values on the inflow boundary of Γ_i on the primal side. For this reason edges on the Dirichlet boundary are assigned to the inflow boundary. Outflow neighbours $\mathcal{N}_i^\uparrow$ and inflow neighbours $\mathcal{N}_i^\downarrow$ are now defined by

$$\mathcal{N}_i^\uparrow := \{\Gamma_j \subset \Gamma \,:\, j > i, \overline{\Gamma_j} \cap (\text{rel int } \partial^\uparrow \Gamma_i) \neq \emptyset\} \tag{8.5}$$

and

$$\mathcal{N}_i^\downarrow := \{\Gamma_j \subset \Gamma \,:\, j < i, \overline{\Gamma_j} \cap (\text{rel int } \partial^\downarrow \Gamma_i) \neq \emptyset\}. \tag{8.6}$$

The *outflow domain* $\Gamma_i^\uparrow$ and the *inflow domain* $\Gamma_i^\downarrow$ of Γ_i will be denoted by

$$\Gamma_i^\uparrow := \text{int } \left(\overline{\Gamma_i} \cup \{\overline{\Gamma'} \,:\, \Gamma' \in \mathcal{N}_i^\uparrow\}\right), \qquad \Gamma_i^\downarrow := \text{int } \left(\overline{\Gamma_i} \cup \{\overline{\Gamma'} \,:\, \Gamma' \in \mathcal{N}_i^\downarrow\}\right), \tag{8.7}$$

respectively. Then the *neighbourhood domain* Γ_i^N of Γ_i is

$$\Gamma_i^N := \text{int } \overline{\Gamma_i^\uparrow} \cup \overline{\Gamma_i^\downarrow}. \tag{8.8}$$

As we want to push wavelets on $\square$ satisfying certain boundary conditions forward to the patches Γ_i, we have to pull back the boundary conditions of Γ_i to $\square$. This can be done as follows. To each patch Γ_i assign a set

$$\mathbf{Z}^{(i)} = (Z_1^{(i)}, \ldots, Z_N^{(i)}) \subset \{0,1\}^n, \tag{8.9}$$

where the $Z_j^{(i)} \subset \{0,1\}$ are characterized by

$$\partial_{\mathbf{Z}^{(i)}} \square := \kappa_i^{-1}\left(\partial^\downarrow \Gamma_i\right) = \bigcup_{j=1}^n [0,1]^{j-1} \times Z_j^{(i)} \times [0,1]^{n-j}; \tag{8.10}$$

i.e., the $\mathbf{Z}^{(i)}$ encode the preimage of the inflow boundary $\partial^\downarrow \Gamma_i$. Analogously, $\tilde{\mathbf{Z}}^{(i)} := (\{0,1\} \setminus Z_1^{(i)}, \ldots, \{0,1\} \setminus Z_N^{(i)})$ encodes the preimage of the outflow boundary $\partial^\uparrow \Gamma_i$, denoted by $\partial_{\tilde{\mathbf{Z}}^{(i)}} \square$. With the $\varepsilon_i^a, \varepsilon_i^b$ from Remark 8.2 and

$$[0,1]_{Z_i} := \begin{cases} [0,1], & \text{if } Z = \emptyset, \\ [-\varepsilon_i^a, 1], & \text{if } Z = \{0\}, \\ [0, 1+\varepsilon_i^b], & \text{if } Z = \{1\}, \\ [-\varepsilon_i^a, 1+\varepsilon_i^b], & \text{if } Z = \{0,1\}, \end{cases} \qquad \text{for } Z \subset \{0,1\}, \tag{8.11}$$

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we define, in analogy to $\Gamma_i^{\downarrow}$,

$$\square_{\mathbf{Z}^{(i)}} := [0,1]_{Z_1^{(i)}} \times \cdots \times [0,1]_{Z_N^{(i)}} \tag{8.12}$$

and $\square_{\tilde{\mathbf{Z}}^{(i)}}$ analogously plays the part of $\Gamma_i^{\uparrow}$ in the parameter domain. Obviously, the analogue to Γ_i^N from (8.8) is $\square_i^N = [-\varepsilon_1^a, 1+\varepsilon_1^b] \times \cdots \times [-\varepsilon_n^a, 1+\varepsilon_n^b]$ defined in (8.1). This notion is needed to define wavelets on $\square$ with certain complementary boundary conditions.

8.3. **Wavelets on $\square$.** In [DS2], biorthogonal wavelet bases of $\square$ with certain complementary boundary conditions were introduced for use in [DS]. Here we only focus on the most important properties of these wavelet bases. To characterize the complementary boundary conditions mentioned above, we introduce for $\mathbf{Z}$ as in (8.9) and $\square_{\mathbf{Z}}$ from (8.12) the Sobolev spaces with built-in boundary values

$$H^m(\square)_{\mathbf{Z}} := \{f \in H^m(\square) \,:\, \chi_{\square} f \in H^m(\square_{\mathbf{Z}})\}. \tag{8.13}$$

Here $\chi_{\square} f$ denotes the extension of f by zero to $\square_{\mathbf{Z}}$.

Remark 8.3.
(1) The spaces $H^m(\square)_{\mathbf{Z}^{(i)}}$ consist of functions which vanish in the H^m-sense on the retract of the inflow boundary of Γ_i. On the retract of the outflow boundary we have free boundary conditions.
(2) The spaces $H^m(\square)_{\tilde{\mathbf{Z}}^{(i)}}$ consist of functions which vanish in H^m-sense on the retract of the outflow boundary of Γ_i. On the retract of the inflow boundary we have free boundary conditions. In this sense, $H^m(\square)_{\mathbf{Z}^{(i)}}$ and $H^m(\square)_{\tilde{\mathbf{Z}}^{(i)}}$ contain complementary boundary conditions.
(3) In [DS2], biorthogonal wavelet bases $\Psi^{\mathbf{Z}}$ of $H^m(\square)_{\mathbf{Z}^{(i)}}$, $\tilde{\Psi}^{\tilde{\mathbf{Z}}}$ of $H^m(\square)_{\tilde{\mathbf{Z}}^{(i)}}$ were constructed for arbitrary $\mathbf{Z}$. This means we have for $i = 1, \ldots, N$ wavelet bases $\Psi^{\mathbf{Z}^{(i)}}, \tilde{\Psi}^{\tilde{\mathbf{Z}}^{(i)}}$. $\Psi^{\mathbf{Z}^{(i)}}$ has zero boundary values on the inflow and $\tilde{\Psi}^{\tilde{\mathbf{Z}}^{(i)}}$ has zero boundary values on the outflow, and $\Psi^{\mathbf{Z}^{(i)}}$ and $\tilde{\Psi}^{\tilde{\mathbf{Z}}^{(i)}}$ are biorthogonal. More properties of $\Psi^{\mathbf{Z}^{(i)}}, \tilde{\Psi}^{\tilde{\mathbf{Z}}^{(i)}}$ are found in [DS2].

8.4. **Wavelets on Γ_i.** The wavelets $\Psi^{\mathbf{Z}^{(i)}}, \tilde{\Psi}^{\tilde{\mathbf{Z}}^{(i)}}$ on $\square$ introduced in Subsection 8.3 shall be pushed forward to Γ_i using the parametric mapping κ_i. The boundary conditions in the flat case (8.13) have the following analogue on the manifold:

$$\begin{aligned} H^m(\Gamma_i)^{\uparrow} &:= \{f \in H^m(\Gamma_i) \,:\, \chi_{\Gamma_i} f \in H^m(\Gamma_i^{\uparrow})\}, \\ H^m(\Gamma_i)^{\downarrow} &:= \{f \in H^m(\Gamma_i) \,:\, \chi_{\Gamma_i} f \in H^m(\Gamma_i^{\downarrow})\}. \end{aligned} \tag{8.14}$$

We define wavelets on the patches Γ_i by

$$\Psi^{\Gamma_i \downarrow} := \Psi^{\mathbf{Z}^{(i)}} \circ \kappa_i^{-1}, \qquad \tilde{\Psi}^{\Gamma_i \uparrow} := \tilde{\Psi}^{\tilde{\mathbf{Z}}^{(i)}} \circ \kappa_i^{-1}. \tag{8.15}$$

Remark 8.4. By Remark 8.3, it is clear that

$$\Psi^{\Gamma_i \downarrow} \subset H^m(\Gamma_i)^{\downarrow}, \qquad \tilde{\Psi}^{\Gamma_i \uparrow} \subset H^m(\Gamma_i)^{\uparrow}, \tag{8.16}$$

but not necessarily

$$\Psi^{\Gamma_i \downarrow} \subset H^m(\Gamma), \qquad \tilde{\Psi}^{\Gamma_i \uparrow} \subset H^m(\Gamma).$$

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In fact, when Γ_i has a nonempty outflow boundary, by Remark 8.3.1 the primal wavelets $\Psi^{\Gamma_i\downarrow}$ have free boundary conditions there. Due to the basis property Remark 8.3.3 there must be wavelets $\psi \in \Psi^{\Gamma_i\downarrow}$ that do not vanish on $\partial^{\uparrow}\Gamma_i$. But then the extension $\chi_{\Gamma_i}\psi$ by zero onto the whole of Γ is not continuous in Γ, hence $\chi_{\Gamma_i}\psi \notin H^m(\Gamma)$ for $m \geq 1$. Complementary observations hold for the dual wavelets $\tilde{\Psi}^{\Gamma_i\uparrow}$.

Remark 8.5. Due to Remark 8.4, we must extend the wavelets $\Psi^{\Gamma_i\downarrow}$ over the outflow boundary of Γ_i, and the wavelets $\tilde{\Psi}^{\Gamma_i\uparrow}$ over the inflow boundary to the whole of Γ in some appropriate way. So we need extension operators

$$E_i : H^m(\Gamma_i) \to H^m(\Gamma_i^{\uparrow}), \qquad \tilde{E}_i : H^m(\Gamma_i) \to H^m(\Gamma_i^{\downarrow}). \tag{8.17}$$

To achieve $E\psi \in H^m(\Gamma)$ for all $\psi \in \Psi^{\Gamma_i\downarrow}$, we additionally have the requirement

$$E_i f \text{ has zero boundary values on } \partial\Gamma_i^{\uparrow} \setminus \partial\Gamma_i. \tag{8.18}$$

Then for $\psi \in \Psi^{\Gamma_i\downarrow}$, we have $E_i\psi \in H^m(\Gamma)$ as ψ already has zero boundary values on the inflow boundary.

8.5. **Wavelets on Γ.** Next we extend the wavelets $\Psi^{\Gamma_i\downarrow}, \tilde{\Psi}^{\Gamma_i\uparrow}$ defined in (8.15) to the whole of Γ. We will derive extension operators $E_i : H^m(\Gamma_i) \to H^m(\Gamma_i^{\uparrow})$ satisfying the analogous property to the univariate Property (A) for the multivariate case, called property (E):

(L) *Locality*: E_i satisfies (8.18).
(CE) *Continuity*: $E_i : H^m(\Gamma_i) \to H^m(\Gamma_i^{\uparrow})$ is continuous.
(CE*) *Continuity of E_i^**: $E_i^* : H^m(\Gamma_i^{\uparrow}) \to H^m(\Gamma_i)$ satisfies

$$\|E_i^* f\|_{H^m(\Gamma_i)^{\uparrow}} \lesssim \|f\|_{H^m(\Gamma_i^{\uparrow})} \quad \text{for all } f \in H^m(\Gamma_i^{\uparrow}).$$

(CE*) means that E_i^* continuously pulls back functions living on the outflow of Γ_i back to Γ_i such that the pullback has zero boundary values on the outflow boundary.

We will define the E_i as tensor products of the Hestenes extension A defined in Section 6. Note that we could take any extension operator A satisfying (A). The constructive proof of the following lemma is given in [KS].

Lemma 8.6. *For any* $\mathbf{Z}$ *as in* (8.12) *we have an extension operator*

$$A_{\mathbf{Z}} : H^m(\square) \to H^m(\square_{\mathbf{Z}}) \tag{8.19}$$

such that $A_{\mathbf{Z}} f$ has zero boundary values on $\partial\square_{\mathbf{Z}} \setminus \partial\square$.

Now extensions of $f \in H^m(\Gamma_i)$ to $H^m(\Gamma_i^{\uparrow})$ can be defined by first pulling back functions $f \in H^m(\Gamma_i)$ to the parameter domain $\square$, applying the Hestenes extension $A_{\tilde{\mathbf{Z}}^{(i)}}$ to the pullback and pushing forward the extension again. Define the pullback mapping $\kappa_i^* : H^m(\Gamma_i) \to H^m(\square)$ to be $(\kappa_i^* v)(x) := v(\kappa_i(x))$. By Lemma 8.6 and the definition of $\mathbf{Z}$ in (8.12) we have $\operatorname{supp} A_{\tilde{\mathbf{Z}}^{(i)}} f \subset \square_{\mathbf{Z}^{(i)}} \subset \square_N^{(i)} \subset \Diamond$ and, hence, we

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can push $A_{\tilde{\mathbf{Z}}^{(i)}} f$ forward to the manifold by applying $\left(\kappa_i^{\uparrow -1}\right)^*$. Here $\kappa_i^{\uparrow} := \hat{\kappa}_i|_{\tilde{\mathbf{Z}}^{(i)}}$ is the part of $\hat{\kappa}_i$ defined in Remark 8.2 that maps to the outflow domain $\Gamma_i^{\uparrow}$. For each $i = 1, \ldots, N$ we define the Hestenes extension $E_i : H^m(\Gamma_i) \to H^m(\Gamma_i^{\uparrow})$ by

$$E_i := \left(\kappa_i^{\uparrow -1}\right)^* \circ A_{\tilde{\mathbf{Z}}^{(i)}} \circ \kappa_i^*. \tag{8.20}$$

Theorem 8.7. *E_i fulfills the requirements* (L) *and* (CE) *by Lemma* 6.3*, and we even have $E_i\psi \in H^m(\Gamma)$ for $\psi \in \Psi^{\Gamma_i\downarrow}$ by Remark* 8.5*.*

The key operators $P_i : L^2(\Gamma) \to L^2(\Gamma)$, which serve to extend the wavelets $\Psi^{\Gamma_i\downarrow}$ onto the whole of Γ, are defined inductively by $P_1 v := \chi_{\Gamma_i^{\uparrow}} E_1(v|_{\Gamma_i})$, and

$$P_i v := \chi_{\Gamma_i}^{\uparrow} E_i \left((v - \sum_{j<i} P_j v)|_{\Gamma_i} \right) \qquad \text{for } i = 2, \ldots, N. \tag{8.21}$$

In [DS] the following results are proved.

Theorem 8.8. *The adjoint P^* of P relative to the L^2 inner product* (8.2) *is given by*

$$P_i^* w = \left(I - \sum_{j<i} P_j^* \right) \chi_{\Gamma_i} E_i^*(w|_{\Gamma_i^{\uparrow}}), \tag{8.22}$$

where by (8.20) *and* (8.2) *E_i^* is given by $E_i^* = (\kappa_i^{-1})^* \left(\partial\kappa^{-1} A^*_{\tilde{\mathbf{Z}}^{(i)}} \partial\kappa_i^{\uparrow} (\kappa_i^{\uparrow})^*\right)$. Due to Lemma* 6.3*, E_i^* satisfies* (CE*) *and, hence,* (E) *is satisfied. For $f \in H^m(\Gamma_i)^{\downarrow}$ we have $P_i f = E_i f \in H^m(\Gamma)$, and for $f \in H^m(\Gamma_i)^{\uparrow}$ we have $P_i^* f \in H^m(\Gamma)$. Hence the operator P_i^* takes on the role of $\tilde{E}_i$.*

Consequently, defining wavelets $\Psi_i^{\Gamma} \subset H^m(\Gamma), \tilde{\Psi}_i^{\Gamma} \subset H^m(\Gamma)$ on Γ by

$$\Psi_i^{\Gamma} := P_i \chi_{\Gamma_i} \Psi^{\Gamma_i\downarrow}, \qquad \tilde{\Psi}_i^{\Gamma} := P_i^* \chi_{\Gamma_i} \tilde{\Psi}^{\Gamma_i\uparrow}$$

with the $\Psi^{\Gamma_i\downarrow}$ and $\tilde{\Psi}^{\Gamma_i\uparrow}$ from (8.15) is justified. With E_i satisfying (E) due to Theorem 8.7 and Theorem 8.8 it is possible following the proof in [DS] to obtain the following result, where further properties such as approximation inequalities and polynomial exactness of these bases can also be found.

Theorem 8.9. *The wavelets $\Psi^{\Gamma} := \Psi_1^{\Gamma} \cup \cdots \cup \Psi_N^{\Gamma}$, $\tilde{\Psi}^{\Gamma} := \tilde{\Psi}_1^{\Gamma} \cup \cdots \cup \tilde{\Psi}_N^{\Gamma}$ constitute biorthogonal wavelet bases of $H^m(\Gamma)$ (i.e., $\langle \Psi^{\Gamma}, \tilde{\Psi}^{\Gamma} \rangle_{\Gamma} = \mathbf{I}$), realizing norm equivalences $\|v\|^2_{H^m(\Gamma)} \sim \sum_{j=j_0}^{\infty} 2^{2jm} \|\langle v, (\Psi^{\Gamma})_j \rangle_{L^2(\Gamma)}\|^2_{\ell_2}$ for all $v \in H^m(\Gamma)$, where $(\Psi^{\Gamma})_j$ are all wavelets of level j from the set Ψ^{Γ}.*

9. Biorthogonal wavelet bases for the sphere

Finally we construct and plot biorthogonal wavelet bases for the sphere $S^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$. First we introduce a suitable quadrangulation of the sphere using six quadrangles which satisfies the property required by Remark 8.1.

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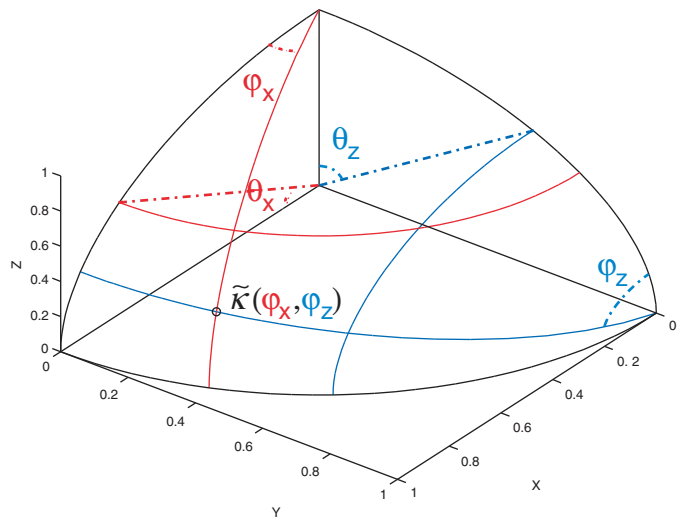

FIGURE 9. The polar coordinates (φ_x, θ_x) around the z-pole and (φ_z, θ_z) around the y-pole.

9.1. **A quadrangulation of the sphere.** One possibility for defining a quadrangulation of the sphere is to introduce the polar coordinates (φ_x, θ_x) around the z-pole and (φ_z, θ_z) around the y-pole; see Figure 9. We see that φ_x describes a positive angle towards the zx-plane and φ_z describes a positive angle towards the yz-plane. Hence these angles describe great circles $G_x(\varphi_x), G_z(\varphi_z)$, respectively, which lie in the planes E_x, E_z with normal vectors ν_x, ν_z. We easily see that

$$\begin{aligned} \nu_x &= \begin{pmatrix} \cos\varphi_x & -\sin\varphi_x & 0 \\ \sin\varphi_x & \cos\varphi_x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -\sin\varphi_x \\ \cos\varphi_x \\ 0 \end{pmatrix}, \\ \nu_z &= \begin{pmatrix} \cos\varphi_z & 0 & \sin\varphi_z \\ 0 & 1 & 0 \\ -\sin\varphi_z & 0 & \cos\varphi_z \end{pmatrix}, \qquad e_1 = \begin{pmatrix} \cos\varphi_z \\ 0 \\ -\sin\varphi_z \end{pmatrix}, \end{aligned} \tag{9.1}$$

i.e., ν_x is the unit vector e_2 rotated by the angle φ_x around the z-axis and ν_z is the unit vector e_1 rotated by the angle φ_z around the x-axis. Any point $(x, y, z) \in S^2 \setminus \{(0,0,1), (0,1,0)\}$ can be expressed uniquely by means of the polar coordinates (φ_x, θ_x), (φ_z, θ_z); i.e.,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi_x \cos\theta_x \\ \sin\varphi_x \cos\theta_x \\ \sin\theta_x \end{pmatrix} = \begin{pmatrix} \sin\varphi_z \cos\theta_z \\ \sin\theta_z \\ \cos\varphi_z \cos\theta_z \end{pmatrix}. \tag{9.2}$$

The idea of the quadrangulation introduced next is the following. We aim at finding six quadrangular patches, each centered at the positive and negative poles of each dimension, i.e., at $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$, $(0,0,-1)$. We concentrate on the patch centered at the x-pole $(1,0,0)$, called Γ_1 later. All other patches are just rotations of this patch. φ_x and φ_z will play the role of coordinates

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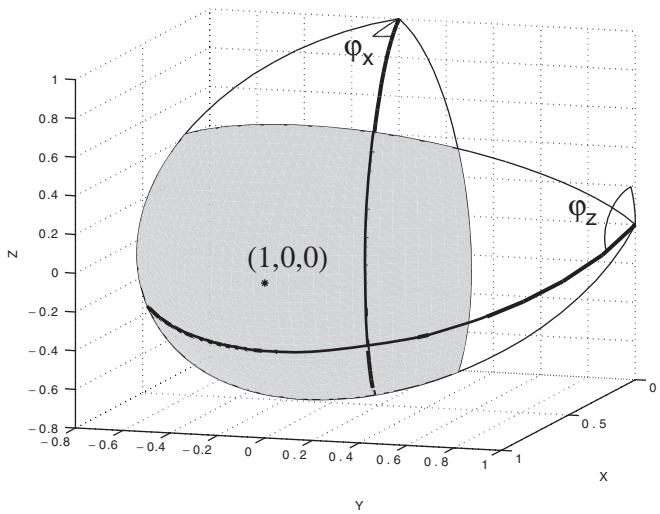

FIGURE 10. Coordinates $\tilde{\kappa}(\varphi_x, \varphi_z)$ forming Patch 1 centered at $(1, 0, 0)$.

in the parametrization $\tilde{\kappa}$, which is defined by

$$\tilde{\kappa}(\theta_x, \theta_z) := \left\{ \begin{array}{l} \text{the point } \tilde{x} \text{ with } \tilde{x}_1 > 0 \text{ where the great circles} \\ G_x(\varphi_x), G_z(\varphi_z), \text{ described by } \varphi_x, \varphi_z, \text{ meet.} \end{array} \right\}$$

The intersection point $\tilde{\kappa}(\varphi_x, \varphi_z)$ is displayed in Figure 9.

Lemma 9.1. *We have*

$$\tilde{\kappa}(\varphi_x, \varphi_z) = \frac{1}{\sqrt{\sin^2 \varphi_z + \cos^2 \varphi_x \cos^2 \varphi_z}} \begin{pmatrix} \cos \varphi_x \sin \varphi_z \\ \sin \varphi_x \sin \varphi_z \\ \cos \varphi_x \cos \varphi_z \end{pmatrix}.$$

Proof. The points where the great circles $G_x(\varphi_x)$ and $G_z(\varphi_z)$ meet are the same as the points where the intersection line of the planes E_x and E_z meet the sphere. The intersection line of the planes is given by the scalar multiples of the outer product $\nu_x \times \nu_z$ of their normals. Normalizing this outer product yields the point were the line hits the sphere and hence

$$\tilde{\kappa}(\varphi_x, \varphi_z) = \frac{\nu_x \times \nu_z}{|\nu_x \times \nu_z|} = \frac{1}{|\nu_x \times \nu_z|} \begin{pmatrix} \cos \varphi_x \sin \varphi_z \\ \sin \varphi_x \sin \varphi_z \\ \cos \varphi_x \cos \varphi_z \end{pmatrix},$$

which proves the claim after calculating the norm of the vector $\nu_x \times \nu_z$. □

Now we are ready to define the first patch Γ_1 as $\Gamma_1 := \tilde{\kappa}\left(\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \left(\frac{\pi}{4}, \frac{3}{2}\pi\right)\right)$. In Figure 10 we see Γ_1 with coordinates (φ_x, φ_z). Γ_1 is centered at $(1, 0, 0)$. By the nature of the polar coordinates it is clear that Γ_1 is exactly a sixth of the sphere. As all parametric mappings shall map from $(0, 1)^2$ we have to compose $\tilde{\kappa}$ with the linear reparametrization $\phi : (0, 1)^2 \to (-\frac{\pi}{4}, \frac{\pi}{4}) \times (\frac{\pi}{4}, \frac{3}{2}\pi)$ given by

$$\phi(x, y) = \left(\frac{\pi}{2}x - \frac{\pi}{4}, \frac{\pi}{2}y + \frac{\pi}{4}\right)$$

and define $\kappa_1 : (0, 1)^2 \to \Gamma_1$ by $\kappa_1 := \tilde{\kappa} \circ \phi$. So far it is not clear that κ_1 is invertible.

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Lemma 9.2. *The inverse of $\tilde{\kappa}$ is given by*

$$\tilde{\kappa}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \arcsin\left(\frac{y}{\cos(\arcsin(z))}\right) \\ \arcsin\left(\frac{x}{\cos(\arcsin(y))}\right) \end{pmatrix} \tag{9.3}$$

for $z \geq 0$ and

$$\tilde{\kappa}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \arcsin\left(\frac{y}{\cos(\arcsin(z))}\right) \\ \pi - \arcsin\left(\frac{x}{\cos(\arcsin(y))}\right) \end{pmatrix} \tag{9.4}$$

for $z < 0$.

Proof. According to (9.2), the point (x, y, z) has the representation $(\varphi_x, \theta_x), (\varphi_z, \theta_z)$ in polar coordinates. The inverses of the polar coordinates for $z \geq 0$ are obviously given by

$$\begin{pmatrix} \varphi_x \\ \theta_x \end{pmatrix} = \begin{pmatrix} \arcsin\left(\frac{y}{\cos\theta_x}\right) \\ \arcsin(z) \end{pmatrix}, \qquad \begin{pmatrix} \varphi_z \\ \theta_z \end{pmatrix} = \begin{pmatrix} \arcsin\left(\frac{x}{\cos\theta_z}\right) \\ \arcsin(y) \end{pmatrix},$$

respectively. Inserting $\arcsin(z)$ for θ_x, $\arcsin(y)$ for θ_z yields (9.3). For $z < 0$ the proof is analogous. □

Note that $\tilde{\kappa}$ is injective only on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, \pi)$. Still, $\tilde{\kappa}$ can be extended to $\tilde{D} := \left(-\frac{\pi}{8}, \frac{5}{8}\pi\right) \times \left(\frac{\pi}{8}, \frac{7}{8}\pi\right)$. This means κ_1 can be extended to $\hat{\kappa}_1$ onto $\phi(\tilde{D}) = \left(\frac{1}{4}, 1 + \frac{1}{4}\right)^2$. Hence, κ_1 satisfies Remark 8.2 with $\varepsilon_i^a = \varepsilon_i^b = \frac{1}{4}$ for $i = 1, 2$. The remaining patches Γ_i and their parametrizations $\kappa_i : (0, 1)^2 \to \Gamma_i$ for $i = 2, \ldots, 6$ are defined by rotating both the patches and the parametrizations by multiples of $\frac{\pi}{2}$. As we have a patch centered at each of the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$, it is convenient to identify a patch with its center.

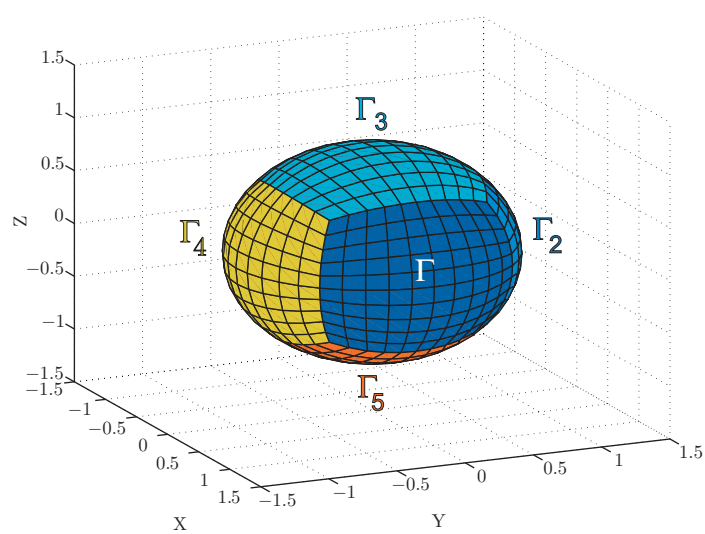

FIGURE 11. The sphere S^2 plotted by the current implementation with the visible Patches 1 through 5. Patch 6 is not visible.

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So Γ_x will denote the patch centered at x for the moment. We then see in Figure 11 that the numbering

$$\begin{aligned} \Gamma_1 &= \Gamma_{(1,0,0)}, & \Gamma_2 &:= \Gamma_{(0,1,0)}, \\ \Gamma_3 &:= \Gamma_{(0,0,1)}, & \Gamma_4 &:= \Gamma_{(0,-1,0)}, \\ \Gamma_5 &:= \Gamma_{(0,0,-1)}, & \Gamma_6 &:= \Gamma_{(-1,0,0)} \end{aligned}$$

follows the numbering rules presented in Subsection 8.1. The parametrizations of the κ_i are then given by

$$\kappa_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \kappa_1, \qquad \kappa_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \circ \kappa_1,$$

$$\kappa_4 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \kappa_1, \qquad \kappa_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \circ \kappa_1,$$

$$\kappa_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \kappa_1.$$

The plot of the sphere can be seen in Figure 11.

9.2. **Plotting the wavelets.** We can now generate plots of the full biorthogonal wavelet bases of the Sobolev spaces $H^m(S^2)$ of the sphere for arbitrary m. As the results of the optimization of the Hestenes extension in Section 7 show that the Hestenes extension is very well conditioned for $m = 1$, we will focus on this case.

There is a vast amount of basis functions already on the coarsest level. To get an idea of the number of essentially different types of wavelets, we quickly summarize the different types incorporated: Both the primal and the dual basis contain functions which are tensor products of unidirectional generators and wavelets. A number of those unidirectional generators and wavelets is adapted to the boundary, and left boundary functions are not necessarily symmetric to left boundary

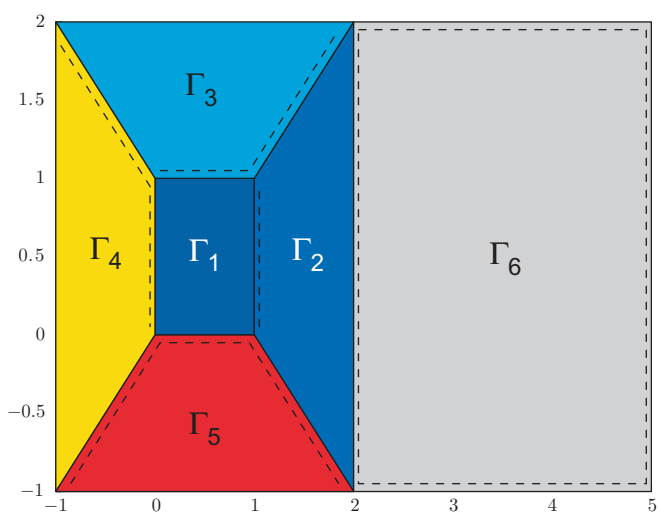

FIGURE 12. Flattened sphere with inflow boundary indicated by dashed lines.

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functions due to the varying boundary conditions. Finally there is one type of unidirectional generator and wavelet which does not intersect the boundary. This would amount to plotting at least $2 \times 4 \times 9 \times 6 = 432$ wavelets to get an idea of the wavelet bases—the factor 2 corresponds to the primal and dual functions, the factor 4 refers to the different choices of tensor products of generators and wavelets in each direction. The basis contains wavelets of type $\phi(x)\phi(y)$, $\psi(x)\phi(y)$, $\phi(x)\psi(y)$, and $\psi(x)\psi(y)$, where ϕ stands for a generator and ψ for a wavelet. The factor 9 results from the location of the plotted wavelet in the patch. Due to the possibly asymmetric boundary conditions, there is a wavelet in each corner of a patch as well as a wavelet in the middle of each edge and one central wavelet, totaling in $4 + 4 + 1 = 9$ wavelets for each patch. Finally there are six patches, explaining the last factor. Plots of all these wavelets are provided in [S].

Here we pick two wavelets which we plot in detail. The first wavelet ψ is a primal wavelet living in the corner of Γ_1 that adjoins to Γ_2 and Γ_5, both lying in the outflow domain of Γ_1. So due to Remark 8.5, this wavelet has to be extended both to Γ_2 and Γ_5 (see Figure 12 for the neighbour relations). Figure 13 shows ψ and the extension $E_1\psi$ to the outflow patches Γ_2 and Γ_5. Note that although Γ_3 and Γ_4 belong to the outflow of Γ_1 as well, the extension $E_1\psi$ is zero on those patches. Due to the support of the cutoff function incorporated in the Hestenes extension, only function values that are within a distance of $\varepsilon_i^a = \varepsilon_i^b = \frac{1}{4}$ (see Remark 8.2) of the outflow patch are regarded for the extension. The second wavelet $\tilde{\psi}$ is a dual wavelet living in the corner of Γ_3 that adjoins to Γ_2 and Γ_1, both lying in the inflow of Γ_3. As dual wavelets have free boundary conditions on the inflow (see Remark 8.4) $\tilde{\psi}$ has to be extended to these patches by P_3^*. Figure 14 shows $\tilde{\psi}$ and the extension to Γ_2, Γ_1.

FIGURE 13. A primal wavelet on Γ_1 (left) being extended to the outflow patches Γ_2 (middle) and Γ_5, Γ_2 (right).

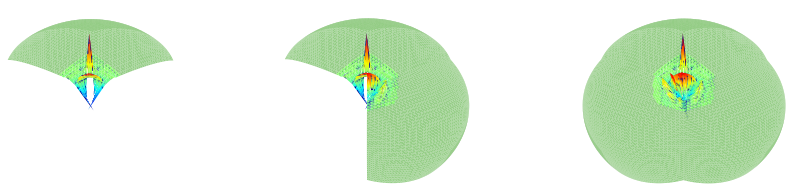

FIGURE 14. A dual wavelet on Γ_3 (above) being extended to the inflow patches Γ_2 (bottom left) and Γ_1, Γ_2 (bottom right).

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