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WEAK AMENABILITY OF MODULE EXTENSIONS OF BANACH ALGEBRAS

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ABSTRACT. We start by discussing general necessary and sufficient conditions for a module extension Banach algebra to be n-weakly amenable, for $n = 0, 1, 2, \cdots$. Then we investigate various special cases. All these case studies finally provide us with a way to construct an example of a weakly amenable Banach algebra which is not 3-weakly amenable. This answers an open question raised by H. G. Dales, F. Ghahramani and N. Grønbæk.

INTRODUCTION

Suppose that $\mathfrak{A}$ is a Banach algebra, and that X is a Banach $\mathfrak{A}$-bimodule. A *derivation* from $\mathfrak{A}$ into X is a linear operator D: $\mathfrak{A} \to X$ satisfying

$$D(ab) = D(a)b + aD(b) \quad (a, b \in \mathfrak{A}).$$

A derivation D is *inner* if there is $x_0 \in X$ such that $D(a) = ax_0 - x_0a$ for $a \in \mathfrak{A}$. The quotient space $\mathcal{H}^1(\mathfrak{A}, X)$ of all continuous derivations from $\mathfrak{A}$ into X modulo the subspace of inner derivations is called the *first cohomology group* of $\mathfrak{A}$ with coefficients in X. A Banach algebra $\mathfrak{A}$ is said to be *amenable* if $\mathcal{H}^1(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule X; here X^* denotes the Banach dual module of X. The algebra $\mathfrak{A}$ is said to be *weakly amenable* if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$, and is called *$n$-weakly amenable*, for an integer $n \geq 0$, if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$, where $\mathfrak{A}^{(n)}$ is the n-th dual module of $\mathfrak{A}$ when $n \geq 1$, and is $\mathfrak{A}$ itself when $n = 0$. The algebra $\mathfrak{A}$ is said to be *permanently weakly amenable* if it is n-weakly amenable for all $n \geq 1$.

The concept of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [22] (see also [7], [9], [11]–[16], [21] and [24]). Dales, Ghahramani and Grønbæk initiated the study of n-weak amenability of Banach algebras in their recent paper [10], where they revealed many important properties of this sort of Banach algebra. An interesting problem concerning this class of Banach algebras is the relation between n-weak amenability and m-weak amenability for different integers n and m. For instance, if $\mathfrak{A}$ is a commutative Banach algebra, then the assertion that $\mathfrak{A}$ is weakly amenable is equivalent to saying that it is permanently weakly amenable ([1, Theorem 1.5]); but, for noncommutative Banach algebras, things are different—we only know that $(n + 2)$-weak amenability implies n-weak

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amenability for $n \geq 1$ ([10, Proposition 1.2]), and weak amenability does not imply 2-weak amenability ([10, Theorems 5.1 and 5.2]). After investigating varieties of classical Banach algebras, Dales, Ghahramani and Grønbæk raised and left open the following question in [10]: Does weak amenability imply 3-weak amenability?

This paper is designed to answer the preceding question. We will construct a counterexample to the question. For this purpose, we study n-weak amenability of the module extension Banach algebra $\mathfrak{A} \oplus X$, the l_1-direct sum of a Banach algebra $\mathfrak{A}$ and a nonzero Banach $\mathfrak{A}$-module X with the algebra product defined as follows:

$$(a, x) \cdot (b, y) = (ab,\ ay + xb) \quad (a, b \in \mathfrak{A},\ x, y \in X).$$

Some aspects of algebras of this form have been discussed in [2] and [10]. We choose this class of Banach algebras to investigate for the preceding question because this class is neither too small nor is it too large; it contains permanently weakly amenable Banach algebras (see Section 6), and it contains no amenable Banach algebras due to [8, Lemma 2.7], since X is a complemented nilpotent ideal in the algebra. If $\mathfrak{A}$ has both left and right approximate identities and they are also, respectively, left and right approximate identities for X, then $\mathfrak{A} \oplus X$ cannot be pointwise approximately biprojective (see [30]). The class of module extension Banach algebras also includes the natural triangular Banach algebra whose amenability has been investigated in [12]. We will give some comment on the latter algebra in Section 2.

This paper is organized as follows: in Section 1 we study the construction of module actions of $2m$-th dual algebras on $2m$-th dual modules. This extends the corresponding discussion in [10]. In Section 2 we give the main theorems which deal with the necessary and sufficient conditions for $\mathfrak{A} \oplus X$ to be n-weakly amenable. Section 3 discusses various techniques for lifting derivations. These will be applied in Section 4 to give the proofs of the main theorems. Sections 5 and 6 deal with the special cases of $X = \mathfrak{A}$, $\mathfrak{A}^*$ and X_0, where X_0 denotes an $\mathfrak{A}$-bimodule with the right module action trivial. In Section 7, we first discuss the condition for $\mathfrak{A} \oplus (X_1 \dot{+} X_2)$ to be weakly amenable, where $\dot{+}$ denotes the l_1 direct sum (of modules). Then, we give an example of a weakly amenable Banach algebra of this form and prove that it is not 3-weakly amenable. This finally answers the preceding open question in the negative.

Since $(\mathfrak{A} \oplus X)^* = (0 \oplus X)^\perp \dot{+} (\mathfrak{A} \oplus 0)^\perp$, where $\dot{+}$ denotes the direct $\mathfrak{A}$-module l_∞-sum, and $(0 \oplus X)^\perp$ (respectively, $(\mathfrak{A} \oplus 0)^\perp$) is isometrically isomorphic to $\mathfrak{A}^*$ (respectively, X^*) as $\mathfrak{A}$-bimodules, for convenience, in this paper we simply identify the corresponding terms and write:

$$(\mathfrak{A} \oplus X)^* = \mathfrak{A}^* \dot{+} X^*.$$

Similarly, we will identify the underlying space of the n-th conjugate $(\mathfrak{A} \oplus X)^{(n)}$ with $\mathfrak{A}^{(n)} \dot{+} X^{(n)}$. The sum is an l_1-sum when n is even and is an l_∞-sum when n is odd.

1. Bimodule actions of $\mathfrak{A}^{(2m)}$ on $X^{(2m)}$

Suppose that $\mathfrak{A}$ is a Banach algebra, and X is a Banach $\mathfrak{A}$-bimodule. According to [10, pp. 27 and 28], X^{**} is a Banach $\mathfrak{A}^{**}$-bimodule, where $\mathfrak{A}^{**}$ is equipped with the first Arens product. The module actions are successively defined as follows.

First, for $x \in X$, $f \in X^*$, $\phi \in X^{**}$ and $u \in \mathfrak{A}^{**}$, define ϕf, $fx \in \mathfrak{A}^*$ and $uf \in X^*$ by

$$\langle a,\ \phi f\rangle = \langle fa,\ \phi\rangle, \quad \langle a,\ fx\rangle = \langle xa,\ f\rangle \quad (a \in \mathfrak{A}),$$
$$\langle x,\ uf\rangle = \langle fx,\ u\rangle \quad (x \in X).$$

Then, for $\phi \in X^{**}$ and $u \in \mathfrak{A}^{**}$, define $u\phi$, $\phi u \in X^{**}$ by

$$\langle f,\ u\phi\rangle = \langle \phi f,\ u\rangle, \quad \langle f,\ \phi u\rangle = \langle uf,\ \phi\rangle \quad (f \in X^*).$$

These give the left and right $\mathfrak{A}^{**}$-module actions on X^{**}. Also, the definition for uf with $u \in \mathfrak{A}^{**}$ and $f \in X^*$ gives a left Banach $\mathfrak{A}^{**}$-module action on X^*. When $u = a \in \mathfrak{A}$, all the above $\mathfrak{A}^{**}$-module actions agree with the $\mathfrak{A}$-module actions on the corresponding dual modules X^* and X^{**}. Moreover, it is readily seen that, with these module actions, the first Arens product on $(\mathfrak{A} \oplus X)^{**}$ may be represented by

$$(u, \phi) \cdot (v, \psi) = (uv, u\psi + \phi v) \quad (u, v \in \mathfrak{A}^{**},\ \phi, \psi \in X^{**}).$$

Viewing $\mathfrak{A}^{(2m)}$ as a new $\mathfrak{A}$ and $X^{(2m)}$ as a new X, the preceding procedure will successively define $X^{(2m+2)}$ as a Banach $\mathfrak{A}^{(2m+2)}$-bimodule. Here, and throughout the paper, the first Arens product is consistently assumed on each $\mathfrak{A}^{(2n)}$. Since some relations arising from the procedure are important for later use, we now give the definition in detail as follows.

Suppose that the bimodule action of $\mathfrak{A}^{(2m)}$ on $X^{(2m)}$ has been defined, where $m \geq 1$. Then in a natural way, $X^{(2m+k)}$, $k \geq 1$, is a Banach $\mathfrak{A}^{(2m)}$-bimodule with the module multiplications $u\Lambda$ and $\Lambda u \in X^{(2m+k)}$, for $\Lambda \in X^{(2m+k)}$ and $u \in \mathfrak{A}^{(2m)}$, defined by

$$\langle \gamma,\ u\Lambda\rangle = \langle \gamma u,\ \Lambda\rangle, \quad \langle \gamma,\ \Lambda u\rangle = \langle u\gamma,\ \Lambda\rangle \quad (\gamma \in X^{(2m+k-1)}).$$

If $u = a \in \mathfrak{A}$, these module actions coincide with $\mathfrak{A}$-module actions on $X^{(2m+k)}$.

Then, for $F \in X^{(2m+1)}$ and $\Phi \in X^{(2m+2)}$, define $F\Phi$, $\Phi F \in \mathfrak{A}^{(2m+1)}$ by

$$\langle u,\ F\Phi\rangle = \langle F,\ \Phi u\rangle \ (= \langle uF,\ \Phi\rangle)$$

and

$$\langle u,\ \Phi F\rangle = \langle Fu,\ \Phi\rangle \ (= \langle F,\ u\Phi\rangle) \quad (u \in \mathfrak{A}^{(2m)}).$$

Throughout this paper, for a Banach space Y and an element $y \in Y$, $\hat{y}$ always denotes the image of y in Y^{**} under the canonical mapping. When $F \in X^{(2m+1)}$ and $\phi \in X^{(2m)}$, we denote $F\hat{\phi}$ by $F\phi$ and $\hat{\phi}F$ by ϕF. It is easy to check that

$$\langle u,\ F\phi\rangle = \langle \phi u,\ F\rangle, \quad \langle u,\ \phi F\rangle = \langle u\phi,\ F\rangle \quad \text{for } u \in \mathfrak{A}^{(2m)}. \tag{1.1}$$

By using the canonical image of F or Φ in the appropriate $2l$-th dual space of the space that it belongs to, we can then signify a meaning for $F\Phi$ and ΦF for every $F \in X^{(2m+1)}$ and $\Phi \in X^{(2n)}$; they are elements of $\mathfrak{A}^{(2k+1)}$, where $k = \max\{m, n-1\}$.

Now for $\mu \in \mathfrak{A}^{(2m+2)}$ and $F \in X^{(2m+1)}$, we define $\mu F \in X^{(2m+1)}$ by

$$\langle \phi,\ \mu F\rangle = \langle F\phi,\ \mu\rangle \quad (\phi \in X^{(2m)}).$$

This actually defines a left Banach $\mathfrak{A}^{(2m+2)}$-module action on $X^{(2m+1)}$.

Finally, for $\mu \in \mathfrak{A}^{(2m+2)}$ and $\Phi \in X^{(2m+2)}$, define $\mu\Phi$, $\Phi\mu \in X^{(2m+2)}$ by

$$\langle F,\ \mu\Phi\rangle = \langle \Phi F,\ \mu\rangle, \quad \langle F,\ \Phi\mu\rangle = \langle \mu F,\ \Phi\rangle \quad (F \in X^{(2m+1)}).$$

These finally define the $\mathfrak{A}^{(2m+2)}$-module actions on $X^{(2m+2)}$ and, therefore, complete our definition.

If $\lim u_\alpha = \mu$ in $\sigma(\mathfrak{A}^{(2m+2)},\ \mathfrak{A}^{(2m+1)})$ and $\lim \phi_\beta = \Phi$ in $\sigma(X^{(2m+2)},\ X^{(2m+1)})$, where $(u_\alpha) \subset \mathfrak{A}^{(2m)}$ and $(\phi_\beta) \subset X^{(2m)}$, and $\sigma(Y^*,\ Y)$ denotes the weak* topology on Y^*, then

$$\mu\Phi = \lim_\alpha \lim_\beta u_\alpha \phi_\beta, \quad \Phi\mu = \lim_\beta \lim_\alpha \phi_\beta u_\alpha \quad \text{in } \sigma(X^{(2m+2)},\ X^{(2m+1)}).$$

For $\mu \in \mathfrak{A}^{(2m+2)}$ and $\phi \in X^{(2m)}$, since $\mu\phi = \mu\hat{\phi}$, $\phi\mu = \hat{\phi}\mu$, we have

$$\langle F,\ \mu\phi\rangle = \langle \phi F,\ \mu\rangle, \quad \langle F,\ \phi\mu\rangle = \langle F\phi,\ \mu\rangle \quad (F \in X^{(2m+1)}). \tag{1.2}$$

One can also easily check the relations

$$u\hat{f} = \hat{u}\hat{f} = (uf)\hat{}\,,$$
$$\hat{f}\hat{\phi} = (f\phi)\hat{}\,, \quad \hat{\phi}\hat{f} = (\phi f)\hat{}\,,$$
$$\hat{u}\hat{\phi} = (u\phi)\hat{}\,, \quad \hat{\phi}\hat{u} = (\phi u)\hat{}\,,$$

where $f \in X^{(2m-1)}$, $\phi \in X^{(2m)}$ and $u \in \mathfrak{A}^{(2m)}$ $(m \geq 1)$. Therefore, each product agrees with those previously defined.

Concerning dual module morphisms, we have the following.

Lemma 1.1. *Suppose that X and Y are Banach $\mathfrak{A}$-bimodules. Then, for every continuous $\mathfrak{A}$-bimodule morphism $\tau\colon X \to Y$ and for each $m \geq 1$, $\tau^{(2m)}\colon X^{(2m)} \to Y^{(2m)}$, the $2m$-th dual operator of τ is an $\mathfrak{A}^{(2m)}$-bimodule morphism.*

Proof. It suffices to prove the lemma in the case where $m = 1$. However, for this simple case, the proof is straightforward if we note that τ^{**} is weak*-weak* continuous. □

In the following, to avoid involving unnecessarily complicated notation, for an element y in a Banach space Y, we will use the same notation y to represent its canonical image in any of the $2m$-th dual spaces $Y^{(2m)}$.

Take $\mathfrak{A}^{(n)} \dotplus X^{(n)}$ as the underlying space of $(\mathfrak{A} \oplus X)^{(n)}$. From induction, by using the relations in (1.1) and (1.2), one can verify that the $(\mathfrak{A} \oplus X)$-bimodule actions on $(\mathfrak{A} \oplus X)^{(n)}$ are formulated as follows:

$$(a, x) \cdot (a^{(n)}, x^{(n)}) = \begin{cases} (aa^{(n)} + xx^{(n)},\ ax^{(n)}), & \text{if } n \text{ is odd;} \\ (aa^{(n)},\ ax^{(n)} + xa^{(n)}), & \text{if } n \text{ is even,} \end{cases} \tag{1.3}$$

and

$$(a^{(n)}, x^{(n)}) \cdot (a, x) = \begin{cases} (a^{(n)}a + x^{(n)}x,\ x^{(n)}a), & \text{if } n \text{ is odd;} \\ (a^{(n)}a,\ a^{(n)}x + x^{(n)}a), & \text{if } n \text{ is even,} \end{cases} \tag{1.4}$$

where $(a, x) \in \mathfrak{A} \oplus X$ and $(a^{(n)}, x^{(n)}) \in \mathfrak{A}^{(n)} \dotplus X^{(n)} = (\mathfrak{A} \oplus X)^{(n)}$.

2. Main theorems

Suppose that $\mathfrak{A}$ is a Banach algebra, and X is a Banach $\mathfrak{A}$-bimodule. For n-weak amenability of the Banach algebra $\mathfrak{A} \oplus X$, we have the following main results, whose proofs will be given in Section 4.

Theorem 2.1. *For $m \geq 0$, $\mathfrak{A} \oplus X$ is $(2m + 1)$-weakly amenable if and only if the following conditions hold:*

1. *$\mathfrak{A}$ is $(2m + 1)$-weakly amenable;*
2. *$\mathcal{H}^1(\mathfrak{A},\ X^{(2m+1)}) = \{0\}$;*

3. *for every continuous* $\mathfrak{A}$*-bimodule morphism* $\Gamma\colon X \to \mathfrak{A}^{(2m+1)}$*, there is* $F \in X^{(2m+1)}$ *such that* $aF - Fa = 0$ *for* $a \in \mathfrak{A}$ *and* $\Gamma(x) = xF - Fx$ *for* $x \in X$*;*
4. *the only continuous* $\mathfrak{A}$*-bimodule morphism* $T\colon X \to X^{(2m+1)}$ *for which* $xT(y) + T(x)y = 0$ $(x, y \in X)$ *in* $\mathfrak{A}^{(2m+1)}$ *is* $T = 0$.

Theorem 2.2. *For* $m \geq 0$, $\mathfrak{A} \oplus X$ *is* $2m$*-weakly amenable if and only if the following conditions hold:*

1. *the only continuous derivations* $D\colon \mathfrak{A} \to \mathfrak{A}^{(2m)}$ *for which there is a continuous operator* $T\colon X \to X^{(2m)}$ *such that* $T(ax) = D(a)x + aT(x)$ *and* $T(xa) = xD(a) + T(x)a$ $(a \in \mathfrak{A},\ x \in X)$ *are the inner derivations;*
2. $\mathcal{H}^1(\mathfrak{A}, X^{(2m)}) = \{0\}$*;*
3. *the only continuous* $\mathfrak{A}$*-bimodule morphism* $\Gamma\colon X \to \mathfrak{A}^{(2m)}$ *for which* $x\Gamma(y) + \Gamma(x)y = 0$ $(x, y \in X)$ *in* $X^{(2m)}$ *is zero;*
4. *for every continuous* $\mathfrak{A}$*-bimodule morphism* $T\colon X \to X^{(2m)}$*, there exists* $u \in \mathfrak{A}^{(2m)}$ *for which* $au = ua$ *for* $a \in \mathfrak{A}$ *and* $T(x) = xu - ux$ *for* $x \in X$.

Remark 2.3. A simple calculation shows that, when $m = 0$, condition 3 in Theorem 2.1 is equivalent to the following:

3^0. there is no nonzero continuous $\mathfrak{A}$-bimodule morphism $\Gamma\colon X \to \mathfrak{A}^*$.

For the general case, condition 3 in Theorem 2.1 is equivalent to the following:

3^m. if $\Gamma\colon X \to \mathfrak{A}^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism, then $\Gamma(X) \subset \mathfrak{A}^\perp$ and there is $G \in X^{(2m+1)} \cap X^\perp$ for which $aG - Ga = 0$ in $X^{(2m+1)}$ $(a \in \mathfrak{A})$ and $\Gamma(x) = xG - Gx$ $(x \in X)$.

Proposition 2.4. *Suppose that condition* 4 *of Theorem 2.1 holds for an* $m \geq 0$*. Then,* $\operatorname{span}(\mathfrak{A}X + X\mathfrak{A})$ *is dense in* X*.*

Proof. Assume, towards a contradiction, that $\operatorname{span}(\mathfrak{A}X + X\mathfrak{A})$ is not dense in X. Take a nonzero element $F \in X^* \cap (\mathfrak{A}X + X\mathfrak{A})^\perp$, and define $T\colon X \to X^*$ by

$$T(x) = F(x)F.$$

Since $F|_{\mathfrak{A}X + X\mathfrak{A}} = 0$, it is easy to see that T is a nonzero, continuous $\mathfrak{A}$-bimodule morphism and that $\mathfrak{A}T(X) = T(X)\mathfrak{A} = \{0\}$. Also, for $x, y \in X$, we have $xT(y) = T(x)y = 0$ in $\mathfrak{A}^*$ since $T(X) \subset (\mathfrak{A}X)^\perp \cap (X\mathfrak{A})^\perp$. This shows that condition 4 of Theorem 2.1 does not hold for $m = 0$. So it does not hold for all $m \geq 0$. This is a contradiction. ∎

Corollary 2.5. *For* $m = 0$*, condition* 4 *in Theorem 2.1 is equivalent to the following*:

4^0. $\operatorname{span}(\mathfrak{A}X + X\mathfrak{A})$ *is dense in* X *and there is no nonzero* $\mathfrak{A}$*-bimodule morphism* $T\colon X \to X^*$ *satisfying* $\langle x, T(y)\rangle + \langle y, T(x)\rangle = 0$ *for* $x, y \in X$*.*

Proof. Suppose that condition 4 in Theorem 2.1 holds. From the preceding proposition, $\operatorname{span}(\mathfrak{A}X + X\mathfrak{A})$ is dense in X. If the $\mathfrak{A}$-bimodule morphism $T\colon X \to X^*$ satisfies

$$\langle x, T(y)\rangle + \langle y, T(x)\rangle = 0 \quad \text{for } x, y \in X,$$

then, for every $a \in \mathfrak{A}$,

$$\langle a, xT(y) + T(x)y\rangle = \langle ax, T(y)\rangle + \langle y, T(ax)\rangle = 0.$$

This shows that $xT(y) + T(x)y = 0$ for $x, y \in X$. Therefore, $T = 0$ and so 4^0 holds.

Conversely, if 4^0 holds, and $T\colon X \to X^*$ is a continuous $\mathfrak{A}$-bimodule morphism satisfying $xT(y)+T(x)y=0$ in $\mathfrak{A}^*$, then, for every $x = ax_1 + x_2 b \in \mathfrak{A}X + X\mathfrak{A}$ and $y \in X$, we have

$$\langle x, T(y)\rangle + \langle y, T(x)\rangle = \langle a, x_1T(y) + T(x_1)y\rangle + \langle b, T(y)x_2 + yT(x_2)\rangle = 0.$$

Since $\operatorname{span}(\mathfrak{A}X + X\mathfrak{A})$ is dense in X, this implies that $\langle x, T(y)\rangle + \langle y, T(x)\rangle = 0$ for all $x, y \in X$. Hence $T = 0$, and so condition 4 of Theorem 2.1 holds for $m = 0$. $\square$

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, and let $\mathcal{M}$ be a Banach $\mathfrak{A},\mathfrak{B}$-module. The algebra $\mathcal{T}$ with the triangular matrix structure

$$\mathcal{T} = \begin{pmatrix} \mathfrak{A} & \mathcal{M} \\ 0 & \mathfrak{B} \end{pmatrix}$$

is called a triangular Banach algebra. The sum and product on $\mathcal{T}$ are given by the usual 2×2 matrix operations and obvious internal module actions. The norm on $\mathcal{T}$ is

$$\left\| \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right\| = \|a\|_{\mathfrak{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathfrak{B}}.$$

Denote by $\mathfrak{A}\dot{+}\mathfrak{B}$ the direct l_1-sum Banach algebra of $\mathfrak{A}$ and $\mathfrak{B}$, and view $\mathcal{M}$ as an $(\mathfrak{A}\dot{+}\mathfrak{B})$-bimodule with the module actions given by

$$(a,b)\cdot m = am, \quad m\cdot(a,b) = mb, \quad a \in \mathfrak{A},\ b \in \mathfrak{B},\ m \in M.$$

Then $\mathcal{T}$ is isometrically isomorphic to the module extension Banach algebra $(\mathfrak{A}\dot{+}\mathfrak{B}) \oplus \mathcal{M}$. With this setting and some calculations, one sees that Theorems 2.1 and 2.2 imply some main results in [12]. For instance, if $\mathfrak{A}$ and $\mathfrak{B}$ are unital and $\mathcal{M}$ is a unital $\mathfrak{A},\mathfrak{B}$-module, then $\mathcal{T}$ is weakly amenable if and only if both $\mathfrak{A}$ and $\mathfrak{B}$ are weakly amenable. In fact, the condition can be weakened further to the following: there exist a bounded approximate identity of $\mathfrak{A}$ and a bounded approximate identity of $\mathfrak{B}$ that are also, respectively, left and right approximate identities for $\mathcal{M}$.

3. Lifting derivations

In this section we give several lemmas concerning the lifting of derivations (and module morphisms) from $\mathfrak{A}$ (or X) into $\mathfrak{A}^{(n)}$ or $X^{(n)}$ to derivations from $\mathfrak{A} \oplus X$ into $(\mathfrak{A} \oplus X)^{(n)}$.

Lemma 3.1. *Suppose that $\Gamma\colon X \to \mathfrak{A}^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism. Then $\overline{\Gamma}\colon \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$, defined by*

$$\overline{\Gamma}((a,x)) = (\Gamma(x), 0),$$

is a continuous derivation. The derivation $\overline{\Gamma}$ is inner if and only if there exists $F \in X^{(2m+1)}$ such that $aF - Fa = 0$ and $\Gamma(x) = xF - Fx$ for $a \in \mathfrak{A}$ and $x \in X$.

Proof. It is straightforward to check that $\overline{\Gamma}$ is a continuous derivation. Noting that $(\Gamma(x),\ 0) = \overline{\Gamma}((0,x))$ and $\overline{\Gamma}((a,0)) = (0,0)$, one can also see easily that the element $F \in \mathfrak{A}^{(2m+1)}$ described in the lemma exists if $\overline{\Gamma}$ is inner.

Conversely, if such an element F exists, then

$$\overline{\Gamma}((a,x)) = (\Gamma(x), 0) = (xF - Fx,\ aF - Fa) = (a,x)\cdot(0,F) - (0,F)\cdot(a,x),$$

showing that $\overline{\Gamma}$ is inner. $\square$

A similar proof gives the following lemma.

Lemma 3.2. *Suppose that $T\colon X \to X^{(2m)}$ is a (continuous) $\mathfrak{A}$-bimodule morphism. Then $\overline{T}\colon \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}$, defined by*

$$\overline{T}((a,x)) = (0, T(x)),$$

is a continuous derivation. The derivation $\overline{T}$ is inner if and only if there exists $u \in \mathfrak{A}^{(2m)}$ such that $ua = au$ for $a \in \mathfrak{A}$, and $T(x) = xu - ux$ for all $x \in X$.

Concerning dual operators we have the following.

Lemma 3.3. *Suppose that $k > 0$ is an integer, and that $D\colon \mathfrak{A} \to X^{(k)}$ is a (continuous) derivation. Then, for every integer $m \geq 0$, $D^{(2m+1)}\colon X^{(k+2m+1)} \to \mathfrak{A}^{(2m+1)}$, the $(2m+1)$-th dual operator of D, satisfies*

$$D^{(2m+1)}(aF) = aD^{(2m+1)}(F) - (D(a)F)|_{\mathfrak{A}^{(2m)}},$$
$$D^{(2m+1)}(Fa) = D^{(2m+1)}(F)a - (FD(a))|_{\mathfrak{A}^{(2m)}},$$

for $a \in \mathfrak{A}$ and $F \in X^{(k+2m+1)}$.

Proof. The lemma is true for $m = 0$ because

$$\langle b,\ D^*(aF)\rangle = \langle D(b)a,\ F\rangle = \langle D(ba) - bD(a),\ F\rangle = \langle b,\ aD^*(F) - D(a)F\rangle$$

and

$$\langle b,\ D^*(Fa)\rangle = \langle aD(b),\ F\rangle = \langle D(ab) - D(a)b,\ F\rangle = \langle b,\ D^*(F)a - FD(a)\rangle,$$

for $a, b \in \mathfrak{A}$ and $F \in X^{(k+1)}$.

For $m > 0$, from Proposition 1.7 of [10], $D^{(2m)}\colon \mathfrak{A}^{(2m)} \to X^{(k+2m)}$ is a (continuous) derivation; here we take the first Arens product in each $\mathfrak{A}^{(2m)}$. Then, the above shows that $D^{(2m+1)} = (D^{(2m)})^*:\ X^{(k+2m+1)} \to (\mathfrak{A}^{(2m)})^*$ satisfies

$$D^{(2m+1)}(uF) = uD^{(2m+1)}(F) - (D^{(2m)}(u)F)|_{\mathfrak{A}^{(2m)}}$$

and

$$D^{(2m+1)}(Fu) = D^{(2m+1)}(F)u - (FD^{(2m)}(u))|_{\mathfrak{A}^{(2m)}},$$

for $u \in \mathfrak{A}^{(2m)}$ and $F \in X^{(k+2m+1)}$. In particular, when $u = a \in \mathfrak{A}$, these give the formulae in the lemma. ☐

Lemma 3.4. *Let m be an integer. Suppose that D: $\mathfrak{A} \to X^{(2m+1)}$ is a (continuous) derivation. Then $\overline{D}$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$, defined by*

$$\overline{D}((a,x)) = (-D^{(2m+1)}(x),\ D(a)) \quad \textit{for } (a,x) \in \mathfrak{A} \oplus X,$$

is also a (continuous) derivation. Moreover,

1. *if $\overline{D}$ is inner, then so is D;*
2. *if D is inner, then there exists a (continuous) derivation $\widetilde{D}$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\widetilde{D}((a,0)) = 0$ $(a \in \mathfrak{A})$ and for which $\overline{D} - \widetilde{D}$ is inner.*

Proof. For $a, b \in \mathfrak{A}$ and $x, y \in X$, we have, from Lemma 3.3,

$$\begin{aligned}
\overline{D}((a,x)\cdot(b,y)) &= \overline{D}\left((ab,\ ay+xb)\right) = \Big(-D^{(2m+1)}(ay+xb),\ D(ab)\Big)\\
&= \Big(-[aD^{(2m+1)}(y) - (D(a)y)|_{\mathfrak{A}^{(2m)}}\\
&\qquad + D^{(2m+1)}(x)b - (xD(b))|_{\mathfrak{A}^{(2m)}}],\ D(a)b + aD(b)\Big)\\
&= \Big(-[aD^{(2m+1)}(y) - D(a)y + D^{(2m+1)}(x)b - xD(b)],\ D(a)b + aD(b)\Big)\\
&= \Big(-aD^{(2m+1)}(y) + xD(b),\ aD(b)\Big) + \Big(-D^{(2m+1)}(x)b + D(a)y,\ D(a)b\Big)\\
&= (a,x)\cdot\Big(-D^{(2m+1)}(y),\ D(b)\Big) + \Big(-D^{(2m+1)}(x),\ D(a)\Big)\cdot(b,y)\\
&= (a,x)\cdot\overline{D}((b,y)) + \overline{D}((a,x))\cdot(b,y).
\end{aligned}$$

Therefore, $\overline{D}$ is a (continuous) derivation.

If $\overline{D}$ is inner, then, for some $u \in \mathfrak{A}^{(2m+1)}$ and $F \in X^{(2m+1)}$, we have

$$\overline{D}((a,x)) = (a,x)\cdot(u,F) - (u,F)\cdot(a,x).$$

Thus,

$$(0, D(a)) = \overline{D}((a,0)) = (a,0)\cdot(u,F) - (u,F)\cdot(a,0) = (au - ua,\ aF - Fa).$$

This shows that $D(a) = aF - Fa$ for all $a \in \mathfrak{A}$, and hence D is inner.

Conversely, if D is inner, then there exists $F \in X^{(2m+1)}$ such that $D(a) = aF - Fa$ for $a \in \mathfrak{A}$. Let $T\colon X \to \mathfrak{A}^{(2m+1)}$ be defined by

$$T(x) = -D^{(2m+1)}(x) - (xF - Fx) \quad (x \in X),$$

and let $\overline{T}\colon \mathfrak{A}\oplus X \to (\mathfrak{A}\oplus X)^{(2m+1)}$ be defined by

$$\overline{T}((a,x)) = (T(x), 0) \quad ((a,x) \in \mathfrak{A}\oplus X).$$

Then

$$(\overline{D} - \overline{T})((a,x)) = (xF - Fx,\ aF - Fa) = (a,x)\cdot(0,F) - (0,F)\cdot(a,x)$$

for $(a,x) \in \mathfrak{A}\oplus X$. Therefore, $\overline{D} - \overline{T}$ is an inner derivation. This in turn implies that $\overline{T}$ is a (continuous) derivation. So $\widetilde{D} = \overline{T}$ satisfies all the requirements. This completes the proof. $\square$

If D is a (continuous) derivation from $\mathfrak{A}$ into $\mathfrak{A}^{(2m+1)}$, $m \geq 0$, we define $\overline{D}$: $\mathfrak{A}\oplus X \to (\mathfrak{A}\oplus X)^{(2m+1)}$ by

$$\overline{D}((a,x)) = (D(a),\ 0).$$

If D is a (continuous) derivation from $\mathfrak{A}$ into $X^{(2m)}$, $m \geq 0$, we define $\overline{D}$: $\mathfrak{A}\oplus X \to (\mathfrak{A}\oplus X)^{(2m)}$ by

$$\overline{D}((a,x)) = (0,\ D(a)).$$

If $T\colon X \to \mathfrak{A}^{(2m)}$ is a (continuous) $\mathfrak{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ in $X^{(2m)}$ for $x, y \in X$, we define $\overline{T}$: $\mathfrak{A}\oplus X \to (\mathfrak{A}\oplus X)^{(2m)}$ by

$$\overline{T}((a,x)) = (T(x),\ 0).$$

Finally, if $T\colon X \to X^{(2m+1)}$ is a (continuous) $\mathfrak{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ for $x, y \in X$, we define $\overline{T}$: $\mathfrak{A}\oplus X \to (\mathfrak{A}\oplus X)^{(2m+1)}$ by

$$\overline{T}((a,x)) = (0,\ T(x)).$$

Then, straightforward calculations yield the following result.

Lemma 3.5. *The operators $\overline{D}$ and $\overline{T}$ defined above are (continuous) derivations. Furthermore, the derivation $\overline{D}$ is inner if and only if D is inner, and $\overline{T}$ is inner if and only if $T = 0$.*

4. Proofs of the main theorems

We first prove Theorem 2.1.

Proof. Denote by Δ_1 the projection from $(\mathfrak{A} \oplus X)^{(2m+1)}$ onto $\mathfrak{A}^{(2m+1)}$ with kernel $X^{(2m+1)}$. Let Δ_2 be the projection $id - \Delta_1$: $(\mathfrak{A} \oplus X)^{(2m+1)} \to X^{(2m+1)}$, and let τ_1: $\mathfrak{A} \to \mathfrak{A} \oplus X$ be the inclusion mapping (i.e., $\tau_1(a) = (a, 0)$). Then Δ_1 and Δ_2 are $\mathfrak{A}$-bimodule morphisms, and τ_1 is an algebra homomorphism.

We now prove the sufficiency in Theorem 2.1. Suppose that conditions 1–4 hold. Suppose also that D: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. Then $D \circ \tau_1$: $\mathfrak{A} \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. This implies that $\Delta_1 \circ D \circ \tau_1$: $\mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ and $\Delta_2 \circ D \circ \tau_1$: $\mathfrak{A} \to X^{(2m+1)}$ are continuous derivations. By conditions 1 and 2, they are inner. Therefore, $D \circ \tau_1$ is inner. From Lemmas 3.5 and 3.4,

$$\overline{D \circ \tau_1} = \overline{\Delta_1 \circ D \circ \tau_1} + \overline{\Delta_2 \circ D \circ \tau_1} : \ \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$$

is a continuous derivation, and there is a continuous derivation $\widetilde{D}$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\widetilde{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$ and such that $\overline{D \circ \tau_1} - \widetilde{D}$ is inner. On the other hand,

$$\begin{aligned}(D - \overline{D \circ \tau_1})((a, 0)) &= D((a, 0)) - \overline{D \circ \tau_1}((a, 0)) \\ &= D \circ \tau_1(a) - D \circ \tau_1(a) = 0 \quad (a \in \mathfrak{A}).\end{aligned}$$

Let $\widehat{D} = D - \overline{D \circ \tau_1} + \widetilde{D}$. Then $\widehat{D}$ is a continuous derivation from $\mathfrak{A} \oplus X$ into $(\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\widehat{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$. Moreover,

$$\widehat{D}((0, ax)) = \widehat{D}((a, 0) \cdot (0, x)) = (a, 0) \cdot \widehat{D}((0, x)) = a\widehat{D}((0, x)) \quad (a \in \mathfrak{A}, x \in X),$$

and

$$\widehat{D}((0, xa)) = \widehat{D}((0, x) \cdot (a, 0)) = \widehat{D}((0, x))a \quad (a \in \mathfrak{A},\ x \in X).$$

Denote by τ_2: $X \to \mathfrak{A} \oplus X$ the inclusion mapping given by $\tau_2(x) = (0, x)$ $(x \in X)$. Then $\widehat{D} \circ \tau_2$: $X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism. From condition 3, there exists $F \in X^{(2m+1)}$ for which $\Delta_1 \circ \widehat{D} \circ \tau_2(x) = xF - Fx$, and $aF - Fa = 0$ for $x \in X$ and $a \in \mathfrak{A}$. Since

$$\begin{aligned}(0, 0) = \widehat{D}((0, 0)) &= \widehat{D}((0, x) \cdot (0, y)) \\ &= \widehat{D}((0, x)) \cdot (0, y) + (0, x) \cdot \widehat{D}((0, y)) \\ &= ([\Delta_2 \circ \widehat{D}((0, x))]y,\ 0) + (x[\Delta_2 \circ \widehat{D}((0, y))],\ 0) \\ &= ([\Delta_2 \circ \widehat{D} \circ \tau_2(x)]y + x[\Delta_2 \circ \widehat{D} \circ \tau_2(y)],\ 0),\end{aligned}$$

we have

$$(\Delta_2 \circ \widehat{D} \circ \tau_2(x))y + x(\Delta_2 \circ \widehat{D} \circ \tau_2(y)) = 0 \quad (x, y \in X).$$

From condition 4, $\Delta_2 \circ \widehat{D} \circ \tau_2 = 0$. Thus,

$$\begin{aligned}\widehat{D}((a,x)) &= \widehat{D}((0,x)) = \widehat{D} \circ \tau_2(x) \\ &= (\Delta_1 \circ \widehat{D} \circ \tau_2(x),\ \Delta_2 \circ \widehat{D} \circ \tau_2(x)) \\ &= (xF - Fx,\ 0) = (a,x) \cdot (0,F) - (0,F) \cdot (a,x).\end{aligned}$$

We have that $\widehat{D}$ is inner. Thus $D = \widehat{D} + (\overline{D \circ \tau_1} - \widetilde{D})$ is inner. This proves that $\mathfrak{A} \oplus X$ is $(2m+1)$-weakly amenable.

Necessity: Suppose that $\mathfrak{A} \oplus X$ is $(2m+1)$-weakly amenable. Then from Lemmas 3.5 and 3.4, $\mathcal{H}^1(\mathfrak{A},\ \mathfrak{A}^{(2m+1)}) = \{0\}$ and $\mathcal{H}^1(\mathfrak{A},\ X^{(2m+1)}) = \{0\}$. Therefore, conditions 1 and 2 hold. Furthermore, Lemma 3.1 gives condition 3, and Lemma 3.5 shows that condition 4 holds. This completes the proof of Theorem 2.1. □

We now turn to the proof of Theorem 2.2.

Proof. Denote by τ_1 and τ_2 the inclusion mappings described in the preceding proof from, respectively, $\mathfrak{A}$ and X into $\mathfrak{A} \oplus X$, and denote by Δ_1 and Δ_2 the natural projections from $(\mathfrak{A} \oplus X)^{(2m)}$ onto $\mathfrak{A}^{(2m)}$ and $X^{(2m)}$, respectively. These are $\mathfrak{A}$-bimodule morphisms.

To prove the sufficiency we assume that conditions 1–4 in Theorem 2.2 hold. Let D: $(\mathfrak{A} \oplus X) \to (\mathfrak{A} \oplus X)^{(2m)}$ be a continuous derivation. Then $\Delta_1 \circ D \circ \tau_1$: $\mathfrak{A} \to \mathfrak{A}^{(2m)}$ and $\Delta_2 \circ D \circ \tau_1$: $\mathfrak{A} \to X^{(2m)}$ are continuous derivations.

Claim 1: $\Delta_1 \circ D \circ \tau_2$: $X \to \mathfrak{A}^{(2m)}$ is trivial.

Let $\Gamma = \Delta_1 \circ D \circ \tau_2$. To prove claim 1, by condition 3 it suffices to show that Γ is an $\mathfrak{A}$-bimodule morphism satisfying $x\Gamma(y) + \Gamma(x)y = 0$ in $X^{(2m)}$ for $x, y \in X$. In fact,

$$\begin{aligned}0 = D((0,0)) = D((0,x) \cdot (0,y)) &= D((0,x)) \cdot (0,y) + (0,x) \cdot D((0,y)) \\ &= (0, \Gamma(x)y) + (0, x\Gamma(y)).\end{aligned}$$

Thus, $x\Gamma(y) + \Gamma(x)y = 0$. On the other hand,

$$\begin{aligned}\Gamma(ax) &= \Delta_1 \circ D((0,ax)) = \Delta_1 \circ D((a.0) \cdot (0,x)) \\ &= \Delta_1 \left(D((a,0)) \cdot (0,x) + (a,0) \cdot D((0,x))\right) \\ &= \Delta_1 \left((a,0) \cdot D((0,x))\right) = \Delta_1(aD \circ \tau_2(x)) = a\Gamma(x).\end{aligned}$$

Similarly, $\Gamma(xa) = \Gamma(x)a$ and so Γ is an $\mathfrak{A}$-bimodule morphism. Therefore, claim 1 is true.

Now let $T = \Delta_2 \circ D \circ \tau_2$: $X \to X^{(2m)}$, and set $D_1 = \Delta_1 \circ D \circ \tau_1$: $\mathfrak{A} \to \mathfrak{A}^{(2m)}$.

Claim 2: $T(ax) = D_1(a)x + aT(x)$ and $T(xa) = xD_1(a) + T(x)a$ for $a \in \mathfrak{A}$ and $x \in X$.

In fact, from claim 1,

$$\begin{aligned}(0, T(ax)) = D((0,ax)) = D((a,0) \cdot (0,x)) &= D((a,0)) \cdot (0,x) + (a,0) \cdot D((0,x)) \\ &= (0, D_1(a)x) + a(0, T(x)) = (0, D_1(a)x + aT(x)).\end{aligned}$$

Similarly, $(0, T(xa)) = (0, xD_1(a) + T(x)a)$, for $a \in \mathfrak{A}$ and $x \in X$. Thus, claim 2 is true.

Therefore, by condition 1, $D_1 = \Delta_1 \circ D \circ \tau_1$ is inner. Suppose that $u \in \mathfrak{A}^{(2m)}$ satisfies $D_1(a) = au - ua$ for $a \in \mathfrak{A}$. Let T_1: $X \to X^{(2m)}$ be defined by $T_1(x) = xu - ux$

for $x \in X$. Then $T - T_1\colon\ X \to X^{(2m)}$ is a continuous $\mathfrak{A}$-bimodule morphism. In fact, from claim 2, for $a \in \mathfrak{A}$ and $x \in X$,

$$\begin{aligned}(T - T_1)(ax) &= T(ax) - T_1(ax) = (D_1(a)x + aT(x)) - (axu - uax)\\ &= (au - ua)x + aT(x) - (axu - uax)\\ &= a(ux - xu) + aT(x) = a(T - T_1)(x).\end{aligned}$$

Similarly, $T - T_1$ is a right $\mathfrak{A}$-module morphism. From condition 4, there is a $v \in \mathfrak{A}^{(2m)}$ such that $av = va$ for $a \in \mathfrak{A}$, and $(T - T_1)(x) = xv - vx$ for $x \in X$. From Lemma 3.2, we have that

$$\overline{T - T_1}:\ (a, x) \mapsto (0, (T - T_1)(x)), \quad \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}$$

is an inner derivation.

Since $\Delta_2 \circ D \circ \tau_1\colon \mathfrak{A} \to X^{(2m)}$ is a continuous derivation, it is inner by condition 2. From Lemma 3.5,

$$\overline{\Delta_2 \circ D \circ \tau_1}:\ (a, x) \mapsto (0, \Delta_2 \circ D \circ \tau_1(a)), \quad \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}$$

is also inner. Using claim 1, we now have

$$\begin{aligned}D((a, x)) &= (D_1(a), \quad \Delta_2 \circ D \circ \tau_1(a) + T(x))\\ &= \overline{\Delta_2 \circ D \circ \tau_1}((a, x)) + (\overline{T - T_1})((a, x)) + (D_1(a), T_1(x)).\end{aligned}$$

Since

$$(D_1(a), T_1(x)) = (au - ua, xu - ux) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x),$$

for $a \in \mathfrak{A}$ and $x \in X$, it gives an inner derivation from $\mathfrak{A} \oplus X$ into $(\mathfrak{A} \oplus X)^{(2m)}$. Hence as a sum of three inner derivations, D is inner. This shows that under conditions 1–4 of Theorem 2.2, $\mathfrak{A} \oplus X$ is $2m$-weakly amenable.

Now we prove the necessity. Suppose that $\mathfrak{A} \oplus X$ is $2m$-weakly amenable. Let D: $\mathfrak{A} \to \mathfrak{A}^{(2m)}$ be a continuous derivation with the property given in condition 1. Then $\overline{D}$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}$ defined by

$$\overline{D}((a, x)) = (D(a), T(x)),\ (a, x) \in \mathfrak{A} \oplus X,$$

is a continuous derivation and hence is inner. This implies that D is inner, and so condition 1 holds. The other conditions are consequences of Lemma 3.5 and Lemma 3.2.

The proof is complete. ∎

5. The algebras $\mathfrak{A} \oplus \mathfrak{A}$ and $\mathfrak{A} \oplus \mathfrak{A}^*$

In this and the following section we consider several concrete cases. This section deals mainly with the two cases $X = \mathfrak{A}$ and $X = \mathfrak{A}^*$ as Banach $\mathfrak{A}$-bimodules.

We first note that, if $\mathfrak{A}$ is not amenable, then there is a Banach $\mathfrak{A}$-bimodule X such that $\mathcal{H}^1(\mathfrak{A}, X^*) \neq \{0\}$. From Theorem 2.1, for this X, $\mathfrak{A} \oplus X$ is not weakly amenable. In fact, the Banach algebra $\mathfrak{A} \oplus X$ is never weakly amenable when $X = \mathfrak{A}^*$, as implied in the following proposition.

Proposition 5.1. *Suppose that $\mathfrak{A}$ is a Banach algebra. Then $\mathfrak{A} \oplus \mathfrak{A}^*$ is not n-weakly amenable for every $n \geq 0$.*

Proof. From Proposition 1.2 of [10], it suffices to prove the cases of $n = 0$, $n = 1$ and $n = 2$. Note that condition 3^0 does not hold, because the identity mapping from X $(= \mathfrak{A}^*)$ onto $\mathfrak{A}^*$ is a nonzero, continuous $\mathfrak{A}$-bimodule morphism. So the proposition is true for $n = 1$.

For $n = 2m$ with $m = 0$ or $m = 1$, if condition 4 in Theorem 2.2 holds for $X = \mathfrak{A}^*$, then the operator T described in this condition has the property that $T(f) \in \mathfrak{A}^\perp$ for $f \in X$. In fact, for $a \in \mathfrak{A}$, we have

$$\langle a,\ T(f)\rangle = \langle a,\ f \cdot u - u \cdot f\rangle = \langle af - fa,\ u\rangle = \langle f,\ ua - au\rangle = 0.$$

But $X = \mathfrak{A}^*$ certainly does not annihilate $\mathfrak{A}$. So, as $\mathfrak{A}$-bimodule morphisms, the identity mapping (in the case $m = 0$) from X onto X and the inclusion mapping (in the case $m = 1$) from X into X^{**} do not satisfy condition 4. Consequently, $\mathfrak{A} \oplus \mathfrak{A}^*$ is not $2m$-weakly amenable for $m = 0$ and 1. □

Now we consider the case that $X = \mathfrak{A}$. To avoid any confusion, from now on, when we regard $\mathfrak{A}$ as an $\mathfrak{A}$-bimodule, we will use the notation A instead of $\mathfrak{A}$. If $X = A$, condition 4 in Theorem 2.2 never holds for any integer m (the canonical embedding is a nonzero morphism). It turns out that $\mathfrak{A} \oplus A$ is never $2m$-weakly amenable for any $m \geq 0$. If $\mathfrak{A}$ is commutative, for the same reason we can conclude more as in the next proposition. Recall that an $\mathfrak{A}$-bimodule X is symmetric if $ax = xa$ for $a \in \mathfrak{A}$ and $x \in X$.

Proposition 5.2. *Suppose that $\mathfrak{A}$ is a commutative Banach algebra. Then for every nonzero, symmetric $\mathfrak{A}$-bimodule X, $\mathfrak{A} \oplus X$ is not $2m$-weakly amenable.*

Proof. Let X be symmetric. Then $xu = ux$ for $u \in \mathfrak{A}^{(2m)}$ and $x \in X$. Since the canonical embedding from X into $X^{(2m)}$ is a nontrivial $\mathfrak{A}$-bimodule morphism, condition 4 in Theorem 2.2 does not hold for such a module X. □

But $\mathfrak{A} \oplus A$ can be weakly amenable. Before giving an example let us go through some relation identities for corresponding elements of $A^{(n)}$ and $\mathfrak{A}^{(n)}$. Suppose that $\phi \in A^{(n)}$. We denote the same element in $\mathfrak{A}^{(n)}$ by $\tilde{\phi}$.

Lemma 5.3. *Suppose that $\mathfrak{A}$ is a Banach algebra, and let $m \geq 0$. Then, for $\phi, \psi \in A^{(2m)}$ and $F \in A^{(2m+1)}$, we have*

$$(\tilde{\phi}\psi)^\sim = \tilde{\phi}\tilde{\psi} = (\phi\tilde{\psi})^\sim, \quad \phi F = (\tilde{\phi}F)^\sim = \tilde{\phi}\widetilde{F}, \quad F\phi = (F\tilde{\phi})^\sim = \widetilde{F}\tilde{\phi}.$$

Proof. It is straightforward to check the identities for the case $m = 0$. Then, an induction on m completes the proof for the general case. □

A special case of Lemma 5.3 is the following group of identities which will be used in the proof of the next theorem:

$$(a\phi)^\sim = a\tilde{\phi}, \quad (\phi a)^\sim = \tilde{\phi}a,$$
$$xF = (\tilde{x}F)^\sim = \tilde{x}\widetilde{F}, \quad Fx = (F\tilde{x})^\sim = \widetilde{F}\tilde{x},$$

where $a \in \mathfrak{A}$, $x \in A$, $\phi \in A^{(2m)}$ and $F \in A^{(2m+1)}$. From these identities, we also see that, for $X = A$ and $m \geq 0$, condition 3 in Theorem 2.1 holds if and only if there is no nonzero $\mathfrak{A}$-bimodule morphism T from A into $A^{(2m+1)}$, and that, if this is the case, then condition 4 holds automatically. Moreover, with $X = A$, conditions 1 and 2 of Theorem 2.1 are the same.

Theorem 5.4. *For a Banach algebra $\mathfrak{A}$:*

1. *if span$\{ab-ba;\ a,b\in\mathfrak{A}\}$ is not dense in $\mathfrak{A}$, then $\mathfrak{A}\oplus A$ is not weakly amenable;*
2. *if span$\{ab-ba;\ a,b\in\mathfrak{A}\}$ is dense in $\mathfrak{A}$, then $\mathfrak{A}\oplus A$ is weakly amenable, provided that $\mathfrak{A}$ is weakly amenable and has a bounded approximate identity.*

Proof. By condition 1 of Theorem 2.1, without loss of generality, we can assume that $\mathfrak{A}$ is weakly amenable for both cases. If span$\{ab-ba;\ a,b\in\mathfrak{A}\}$ is not dense in $\mathfrak{A}$, then there exists $f\in\mathfrak{A}^*$ such that $f\neq 0$ and $\langle ab-ba,\ f\rangle=0$ for $a,b\in\mathfrak{A}$. So $af=fa$ for $a\in\mathfrak{A}$. Then $T\colon\ A\to\mathfrak{A}^*$, defined by

$$T(x)=\tilde{x}f=f\tilde{x},$$

is an $\mathfrak{A}$-bimodule morphism. According to Proposition 1.3 of [10], $\mathfrak{A}^2$, the linear span of all product elements ab, $a,b\in\mathfrak{A}$, is dense in $\mathfrak{A}$. So there are $a,b\in\mathfrak{A}$ such that $\langle ab,\ f\rangle\neq 0$. This implies that $T\neq 0$. Therefore, condition 3^0 does not hold. As a consequence, $\mathfrak{A}\oplus A$ is not weakly amenable.

If span$\{ab-ba;\ a,b\in\mathfrak{A}\}$ is dense in $\mathfrak{A}$, and $\mathfrak{A}$ has a bounded approximate identity (e_i), then, for every given continuous $\mathfrak{A}$-bimodule morphism $T\colon\ A\to\mathfrak{A}^*$, we have $T(a)=af=fa$, where f is a weak* cluster point of $(T(e_i))$. Therefore, $\langle ab-ba,\ f\rangle=0$ for all $a,b\in\mathfrak{A}$. This shows that $f=0$ and hence $T=0$. Thus conditions 3 and 4 in Theorem 2.1 hold for $m=0$. The other two conditions hold automatically for $m=0$. So, from Theorem 2.1, the second statement of the theorem is true. □

From case 1 of Theorem 5.4 we immediately have the following corollary.

Corollary 5.5. *If $\mathfrak{A}$ is a commutative Banach algebra, then $\mathfrak{A}\oplus A$ is not weakly amenable.*

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. According to a classical result due to Halmos (Theorem 8 of [18]), every element in $B(\mathcal{H})$ can be written as a sum of two commutators (see also [4] and [5]). Together with the fact that $B(\mathcal{H})$ has an identity and, as a C^*-algebra, is weakly amenable [17], from Theorem 5.4 we see that $B(\mathcal{H})\oplus B(\mathcal{H})$ is weakly amenable. Later in this section we will see that it is in fact $(2m+1)$-weakly amenable.

Proposition 5.6. *Suppose that $V=span\{au-ua;\ u\in\mathfrak{A}^{**},a\in\mathfrak{A}\}$ is not dense in $\mathfrak{A}\mathfrak{A}^{**}+\mathfrak{A}^{**}\mathfrak{A}$ (if $\mathfrak{A}$ has an identity, this is equivalent to saying that V is not dense in $\mathfrak{A}^{**}$). Then $\mathfrak{A}\oplus A$ is not 3-weakly amenable.*

Proof. In fact, in this case $\mathfrak{A}^{**}\mathfrak{A}\nsubseteq cl(V)$, since otherwise it would follow that both $\mathfrak{A}\mathfrak{A}^{**}$ and $\mathfrak{A}^{**}\mathfrak{A}$ were in $cl(V)$, and then $cl(V)\supseteq\mathfrak{A}\mathfrak{A}^{**}+\mathfrak{A}^{**}\mathfrak{A}$, which contradicts the assumption that V is not dense in $\mathfrak{A}\mathfrak{A}^{**}+\mathfrak{A}^{**}\mathfrak{A}$.

Hence, from the Hahn-Banach Theorem, there exists $F\in\mathfrak{A}^{***}$ such that $F|_V=0$, but $F\neq 0$ on $\mathfrak{A}^{**}\mathfrak{A}$. This implies that $aF=Fa$ for all $a\in\mathfrak{A}$ and $aF\neq 0$ for some $a\in\mathfrak{A}$. Define $T\colon\ A\to\mathfrak{A}^{***}$ by $T(x)=\tilde{x}F(=F\tilde{x})$. Then, T is a non-zero, continuous $\mathfrak{A}$-bimodule morphism from A into $\mathfrak{A}^{***}$. Therefore, condition 3 in Theorem 2.1 does not hold for $m=1$. This shows that $\mathfrak{A}\oplus A$ is not 3-weakly amenable. □

Regarding the open question of whether weak amenability implies 3-weak amenability, Theorem 5.4 and Proposition 5.6 suggest that one might find a counterexample in the Banach algebras of the form $\mathfrak{A}\oplus A$. Unfortunately, $B(\mathcal{H})$ cannot be a candidate. We can see this from the next two lemmas. The following lemma is basically Theorem 8 in [18], but we have highlighted some of its features which will be useful for our purposes.

Lemma 5.7. *Suppose that $\mathcal{H}$ is an infinite-dimensional Hilbert space. Then there are two elements Q_0 and S_0 in $B(\mathcal{H})$ such that, for each $B \in B(\mathcal{H})$, there exist $P_B \in B(\mathcal{H})$ and $R_B \in B(\mathcal{H})$ with $\|P_B\| \le \|B\|$ and $\|R_B\| \le \|B\|$ for which*

$$B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B).$$

Proof. For an infinite-dimensional Hilbert space $\mathcal{H}$, there exists an isometry η: $\mathcal{H} \to \sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i$, where $\sum_{i=1}^{\infty} \dot{+}$ denotes an l_2 direct sum and each $\mathcal{H}_i$ is a copy of $\mathcal{H}$.

Let Q: $\mathcal{H} \to \sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i$ and S: $\sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i \to \sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i$ be the bounded operators given by the infinite matrices

$$Q = \begin{pmatrix} I \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ I & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \end{pmatrix}.$$

Let $Q_0 = \eta^{-1} \circ Q$ and $S_0 = \eta^{-1} \circ S \circ \eta$. Then $Q_0, S_0 \in B(\mathcal{H})$. Given an element $B \in B(\mathcal{H})$, let P: $\sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i \to \mathcal{H}$ and R: $\sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i \to \sum_{i=1}^{\infty} \dot{+} \mathcal{H}_i$ be the bounded operators given by the infinite matrices

$$P = \begin{pmatrix} B & 0 & 0 & \cdots \end{pmatrix} \qquad \text{and} \quad R = \begin{pmatrix} 0 & B & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & \cdots \\ 0 & 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}.$$

Take $P_B = P \circ \eta$ and $R_B = \eta^{-1} \circ R \circ \eta$. Then $P_B, R_B \in B(\mathcal{H})$ and $\|P_B\| \le \|B\|$, $\|R_B\| \le \|B\|$. We have that $B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B)$. ☐

The following result on the $2n$-th dual of $B(\mathcal{H})$ seems not to be known.

Lemma 5.8. *For every integer $n \ge 0$,*

$$B(\mathcal{H})^{(2n)} = \operatorname{span}\{au - ua;\ a \in B(\mathcal{H}), u \in B(\mathcal{H})^{(2n)}\}.$$

Proof. By taking weak* limits and using induction, one can show the result immediately from Lemma 5.7. ☐

Proposition 5.9. *For each integer $m \ge 0$, $B(\mathcal{H}) \oplus B(\mathcal{H})$ is $(2m+1)$-weakly amenable but is not $2m$-weakly amenable.*

Proof. First, as a C^*-algebra, $B(\mathcal{H})$ is permanently weakly amenable. So conditions 1 and 2 of Theorem 2.1 hold for $X = \mathfrak{A} = B(\mathcal{H})$ and $m \ge 0$. To show that conditions 3 and 4 also hold, it suffices to prove that every continuous $B(\mathcal{H})$-bimodule morphism T from $B(\mathcal{H})$ into $B(\mathcal{H})^{(2m+1)}$ is trivial.

In fact, letting e be the identity of $B(\mathcal{H})$ and $F = T(e)$, we have $T(a) = aF = Fa$ for all $a \in B(\mathcal{H})$. Therefore, for all $u \in B(\mathcal{H})^{(2m)}$, we have $\langle au - ua,\ F\rangle = 0$. From Lemma 5.8, this implies that $F = 0$. Hence $T = 0$. Therefore, $B(\mathcal{H}) \oplus B(\mathcal{H})$ is $(2m+1)$-weakly amenable for $m \ge 0$.

On the other hand, we have indicated in the paragraph before Proposition 5.2 that $\mathfrak{A} \oplus A$ is never $2m$-weakly amenable. So $B(\mathcal{H}) \oplus B(\mathcal{H})$ is not $2m$-weakly amenable for $m \geq 0$. This completes the proof. □

Remark 5.10. Denote by $K(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$. Using Theorem 1 of [29] one can also prove that $K(\mathcal{H}) \oplus K(\mathcal{H})$ and $B(\mathcal{H}) \oplus K(\mathcal{H})$ are $(2m+1)$- (but not $2m$-) weakly amenable. On the other hand, it is interesting to recall Proposition 2.4, which implies that $K(\mathcal{H}) \oplus B(\mathcal{H})$ is not weakly amenable.

6. The algebra $\mathfrak{A} \oplus X_0$

In this section we consider the case that the module action on one side of X is trivial. We denote by X_0 (respectively, ${}_0Y$) specifically the $\mathfrak{A}$-bimodules with right (respectively, left) module action trivial. We observe that, when $X = X_0$, conditions 3 and 4 in Theorem 2.1 are reduced, respectively, to the following:

$3'_0$. for each continuous $\mathfrak{A}$-bimodule morphism Γ: $X_0 \to \mathfrak{A}^{(2m+1)}$, there is $F \in X_0^{(2m+1)}$ such that $Fa = 0$ for $a \in \mathfrak{A}$ and $\Gamma(x) = xF$ for $x \in X_0$;

$4'_0$. $\mathfrak{A}X_0$ is dense in X_0.

Also, conditions 1, 3 and 4 in Theorem 2.2 are reduced, respectively, to the following:

$1''_0$. every continuous derivation D: $\mathfrak{A} \to \mathfrak{A}^{(2m)}$ with the property that there is a continuous operator T: $X_0 \to X_0^{(2m)}$ such that $T(ax) = D(a)x + aT(x)$ for $a \in \mathfrak{A}$ and $x \in X_0$ is inner;

$3''_0$. the only continuous $\mathfrak{A}$-bimodule morphism Γ: $X_0 \to \mathfrak{A}^{(2m)}$ satisfying $\Gamma(x)y = 0$ $(x, y \in X_0)$ in $X_0^{(2m)}$ is zero;

$4''_0$. for every continuous $\mathfrak{A}$-bimodule morphism T: $X_0 \to X_0^{(2m)}$, there exists $u \in \mathfrak{A}^{(2m)}$ such that $au = ua$ for $a \in \mathfrak{A}$ and $T(x) = ux$ for $x \in X_0$.

Proposition 6.1. *Suppose that $\mathfrak{A}$ is a $(2m+1)$-weakly amenable Banach algebra with a bounded approximate identity and satisfying that $\mathfrak{A}\mathfrak{A}^{(2m)} = \mathfrak{A}^{(2m)}$. Then, $\mathfrak{A} \oplus X_0$ is $(2m+1)$-weakly amenable if and only if $\mathfrak{A}X_0$ is dense in X_0.*

Proof. Since $\mathfrak{A}$ has a bounded approximate identity, from Proposition 1.5 of [23], condition 2 in Theorem 2.1 always holds for $X = X_0$. If $\mathfrak{A}\mathfrak{A}^{(2m)} = \mathfrak{A}^{(2m)}$, then there is no nonzero, continuous $\mathfrak{A}$-bimodule morphism T: $X_0 \to \mathfrak{A}^{(2m+1)}$, since such a morphism must satisfy $\langle au,\ T(x)\rangle = \langle u,\ T(xa)\rangle = 0$ $(a \in \mathfrak{A},\ u \in \mathfrak{A}^{(2m)})$. So condition $3'_0$ holds automatically. □

For $m = 0$, the above proposition yields the following.

Corollary 6.2. *Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra with a bounded approximate identity. Then $\mathfrak{A} \oplus X_0$ is weakly amenable if and only if $\mathfrak{A}X_0$ is dense in X_0.*

A dual result to Corollary 6.2 is as follows.

Corollary 6.3. *Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra with a bounded approximate identity. Let ${}_0Y$ be a Banach $\mathfrak{A}$-bimodule with left module action trivial. Then, $\mathfrak{A} \oplus {}_0Y$ is weakly amenable if and only if ${}_0Y\mathfrak{A}$ is dense in ${}_0Y$.*

View $\mathfrak{A}$ as a left $\mathfrak{A}$-module and then impose a trivial right $\mathfrak{A}$-module action on it. This results in a Banach $\mathfrak{A}$-bimodule. We denote it by A_0. Suppose that $\phi \in A_0^{(n)}$.

We denote the same element in $\mathfrak{A}^{(n)}$ by $\tilde{\phi}$. Similarly to Lemma 5.3, one can check that the following equalities hold:

$$(u\phi)^{\sim} = u\tilde{\phi}, \quad \phi u = 0, \quad \phi F = \tilde{\phi}\widetilde{F},$$
$$F\phi = 0, \quad uF = 0, \quad (Fu)^{\sim} = \widetilde{F}u,$$

where $u \in \mathfrak{A}^{(2m)}$, $\phi \in A_0^{(2m)}$, $F \in A_0^{(2m+1)}$ $(m \geq 0)$.

Proposition 6.4. *Suppose that $\mathfrak{A}$ is a $(2m+1)$-weakly amenable Banach algebra with a bounded approximate identity. Then $\mathfrak{A} \oplus A_0$ is $(2m+1)$-weakly amenable.*

Proof. As in the proof of Proposition 6.1, it suffices to verify conditions $3'$ and $4'$. Condition $4'_0$ holds since $\mathfrak{A}$ has a left bounded approximate identity for A_0. Let $(x_\alpha) \subset A_0$ be a net such that $(\tilde{x}_\alpha)$ is a bounded approximate identity for $\mathfrak{A}$. If Γ: $A_0 \to \mathfrak{A}^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism, we let $\widetilde{F}$ be a weak* cluster point of $(\Gamma(x_\alpha))$. Let the element in $A_0^{(2m+1)}$ corresponding to $\widetilde{F}$ be F. Then F satisfies the requirement in condition $3'_0$. $\square$

Concerning $2m$-weak amenability, we have the following.

Proposition 6.5. *Let $m \geq 1$, and suppose that $\mathfrak{A}$ is a commutative $2m$-weakly amenable Banach algebra with a bounded approximate identity. Then $\mathfrak{A} \oplus A_0$ is $2m$-weakly amenable.*

Proof. It suffices to show that conditions $3''_0$ and $4''_0$ hold. Suppose that an $\mathfrak{A}$-bimodule morphism Γ: $A_0 \to \mathfrak{A}^{(2m)}$ satisfies $\Gamma(x)y = 0$ in $A_0^{(2m)}$ $(x, y \in A_0)$. Then

$$0 = (\Gamma(x)y)^{\sim} = \Gamma(x)\tilde{y} = \tilde{y}\Gamma(x) = \Gamma(\tilde{y}x) \quad (x, y \in A_0).$$

This implies that $\Gamma(ax) = 0$ for $a \in \mathfrak{A}$ and $x \in A_0$. So $\Gamma(x) = 0$ for all $x \in A_0$. Therefore, condition $3''_0$ holds.

Assume that T: $A_0 \to A_0^{(2m)}$ is a continuous $\mathfrak{A}$-bimodule morphism. Let v be a weak* cluster point of $(T(x_i))$, where $(\tilde{x}_i)$ is a bounded approximate identity for $\mathfrak{A}$. Let $u = \tilde{v}$. Then, $T(x) = \lim T(\tilde{x}x_i) = \tilde{x}v$. However, $(\tilde{x}v)^{\sim} = \tilde{x}\tilde{v} = \tilde{x}u = u\tilde{x} = (ux)^{\sim}$. Hence $T(x) = ux$. On the other hand, $ua = au$ since $\mathfrak{A}$ is commutative. Condition $4''_0$ holds. $\square$

Although we have already had an example of a Banach algebra which is $(2m+1)$-weakly amenable but not $2m$-weakly amenable (see Proposition 5.9; another known example is the nuclear algebra $\mathcal{N}(E)$ with E a reflexive Banach space having the approximation property [10, Corollary 5.4]), we end this section by giving one more example of a weakly amenable Banach algebra which is not 2-weakly amenable.

Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra with a bounded approximate identity and satisfying that $\mathfrak{A}\mathfrak{A}^* \neq \mathfrak{A}^*\mathfrak{A}$ (an example is $\mathfrak{A} = L^1(G)$ with G a non-SIN locally compact group; see [28] and [25] for the reference of SIN groups, and Theorem 32.44 of [20] as well as [26] for the property we need here). Without loss of generality, we assume that $\mathfrak{A}\mathfrak{A}^* \nsubseteq \mathfrak{A}^*\mathfrak{A}$.

Example 6.6. For the above Banach algebra $\mathfrak{A}$, $\mathfrak{A} \oplus A_0$ is weakly amenable but is not 2-weakly amenable.

Proof. From Proposition 6.4, $\mathfrak{A} \oplus A_0$ is weakly amenable. We show that condition $3_0''$ does not hold for $m = 1$. Take a $\phi \in \mathfrak{A}^{**}$ for which $\phi|_{\mathfrak{A}\mathfrak{A}^*} = 0$ but $\phi|_{\mathfrak{A}^*\mathfrak{A}} \neq 0$ (notice that by Cohen's factorization theorem, $\mathfrak{A}\mathfrak{A}^*$ is closed in $\mathfrak{A}^*$). Then $\phi a = 0$ for all $a \in \mathfrak{A}$ and $a\phi \neq 0$ for some $a \in \mathfrak{A}$. Let T: $A_0 \to \mathfrak{A}^{**}$ be defined by $T(x) = \tilde{x}\phi$. Then T is a continuous $\mathfrak{A}$-bimodule morphism and $T \neq 0$. Since

$$(T(x)y)^{\sim} = T(x)\tilde{y} = (\tilde{x}\phi)\tilde{y} = \tilde{x}(\phi\tilde{y}) = 0,$$

we have $T(x)y = 0$ for all $x, y \in A_0$. Therefore, condition $3_0''$ is not satisfied. □

7. Weak amenability does not imply 3-weak amenability

Suppose that X_1 and X_2 are two Banach $\mathfrak{A}$-bimodules. We denote by $X_1 \dot{+} X_2$ the direct module sum of X_1 and X_2, i.e., the l_1 direct sum of X_1 and X_2 with the module actions given by $a(x_1, x_2) = (ax_1,\ ax_2)$, $(x_1, x_2)a = (x_1a,\ x_2a)$. For this module we have the following equality:

$$(x_1, x_2) \cdot (f_1^*, f_2^*) = x_1 f_1^* + x_2 f_2^* \quad \big((x_1, x_2) \in X_1 \dot{+} X_2,\ (f_1^*, f_2^*) \in (X_1 \dot{+} X_2)^*\big).$$

In this section we shall first study the weak amenability of the Banach algebra $\mathfrak{A} \oplus (X_1 \dot{+} X_2)$. Then we shall give an example of a weakly amenable Banach algebra of this form which is not 3-weakly amenable.

Lemma 7.1. *Suppose that $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$ are weakly amenable. Then the following are equivalent:*

(i) $\mathfrak{A} \oplus (X_1 \dot{+} X_2)$ *is weakly amenable;*
(ii) *there is no nonzero, continuous $\mathfrak{A}$-bimodule morphism γ: $X_1 \to X_2^*$;*
(iii) *there is no nonzero, continuous $\mathfrak{A}$-bimodule morphism η: $X_2 \to X_1^*$.*

Proof. Suppose that (i) holds. We show that (ii) also holds. Indeed, suppose that γ: $X_1 \to X_2^*$ is a continuous $\mathfrak{A}$-bimodule morphism. Consider the continuous $\mathfrak{A}$-bimodule morphism T: $X_1 \dot{+} X_2 \to (X_1 \dot{+} X_2)^*$ defined by

$$T((x_1, x_2)) = (-\gamma^*(x_2), \gamma(x_1)), \quad (x_1, x_2) \in X_1 \dot{+} X_2.$$

For $(x_1, x_2), (y_1, y_2) \in X_1 \dot{+} X_2$, and $a \in \mathfrak{A}$, we have

$$\begin{aligned}
&\langle\, a,\ (x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2)\,\rangle \\
&= \langle\, a,\ -x_1\gamma^*(y_2) + x_2\gamma(y_1)\,\rangle + \langle\, a,\ -\gamma^*(x_2)y_1 + \gamma(x_1)y_2\,\rangle \\
&= \langle\, a,\ -\gamma(x_1)y_2 + x_2\gamma(y_1)\,\rangle + \langle\, a,\ -x_2\gamma(y_1) + \gamma(x_1)y_2\,\rangle = 0.
\end{aligned}$$

So $(x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0$. Then, from condition 4 of Theorem 2.1, $T = 0$. Thus $\gamma = 0$. As a consequence, (ii) holds.

To prove that (ii) implies (iii), we suppose that η: $X_2 \to X_1^*$ is a continuous $\mathfrak{A}$-bimodule morphism. Then γ: $X_1 \to X_2^*$ defined by $\gamma = \eta^*|_{X_1}$ is a continuous $\mathfrak{A}$-bimodule morphism. Therefore, $\gamma = 0$. This implies that $\eta^* = 0$ since η^* is weak*-weak* continuous and X_1 is weak* dense in X_1^{**}. Thus, $\eta = 0$, showing that (iii) holds. Similarly, one can prove that (iii) implies (ii).

Finally, we prove that (ii) + (iii) implies (i). Because $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$ are weakly amenable, conditions 1–3 of Theorem 2.1 hold automatically for $X = X_1 \dot{+} X_2$ and $m = 0$. We show that condition 4 also holds. Suppose that T: $X \to X^*$ is a continuous $\mathfrak{A}$-bimodule morphism satisfying

$$(x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0 \quad ((x_1, x_2), (y_1, y_2) \in X).$$

Let P_i: $X^* \to X_i^*$ be the natural projections and let τ_i: $X_i \to X$ be the natural embeddings, $i = 1, 2$. Then, by taking $x_2 = y_2 = 0$ and $x_1 = y_1 = 0$ separately, we have

$$x_1 \cdot P_1 \circ T \circ \tau_1(y_1) + P_1 \circ T \circ \tau_1(x_1) \cdot y_1 = 0,$$
$$x_2 \cdot P_2 \circ T \circ \tau_2(y_2) + P_2 \circ T \circ \tau_2(x_2) \cdot y_2 = 0,$$

for all $x_i, y_i \in X_i$, $i = 1, 2$. So we have $P_i \circ T \circ \tau_i = 0$ by applying condition 4 of Theorem 2.1 to the weakly amenable Banach algebras $\mathfrak{A} \oplus X_i$, $i = 1, 2$. Furthermore, (ii) and (iii) imply that $P_1 \circ T \circ \tau_2$: $X_2 \to X_1^*$ and $P_2 \circ T \circ \tau_1$: $X_1 \to X_2^*$ are trivial. Therefore, we have $T = 0$. Condition 4 of Theorem 2.1 holds for $X = X_1 \dot{+} X_2$. From Theorem 2.1, $\mathfrak{A} \oplus (X_1 \dot{+} X_2)$ is weakly amenable. This completes the proof. ☐

Proposition 7.2. *The algebra $\mathfrak{A} \oplus (X_1 \dot{+} X_2)$ is weakly amenable if and only if both $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$ are weakly amenable and condition (ii) or condition (iii) in Lemma 7.1 holds.*

Proof. If $\mathfrak{A} \oplus (X_1 \dot{+} X_2)$ is weakly amenable, then conditions 1–4 of Theorem 2.1 hold for this algebra. It follows that these conditions also hold for the algebras $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$. So the latter two are also weakly amenable. The rest has been given in Lemma 7.1. ☐

In the remainder of the paper we focus on constructing an example of a weakly amenable Banach algebra which is not 3-weakly amenable. Recall that we always equip $\mathfrak{A}^{(2m)}$ with the first Arens product. The following lemma has been proved in [31].

Lemma 7.3. *Suppose that $\mathfrak{A}$ is a left (right) ideal in $\mathfrak{A}^{**}$. Then it is also a left (respectively, right) ideal in $\mathfrak{A}^{(2m)}$ for all $m \geq 1$.*

Suppose that $\mathfrak{B}$ is a Banach algebra and $\mathfrak{A} = \mathfrak{B}^{**}$. If $\mathfrak{B}$ is an ideal in $\mathfrak{B}^{**}$, then a natural way to make $\mathfrak{B}$ an $\mathfrak{A}$-bimodule is using (the first) Arens product to give the module actions. In this way $\mathfrak{B}^{**}$ is an $\mathfrak{A}^{**}$-bimodule. For $b \in \mathfrak{B} \subset \mathfrak{B}^{**}$ and $u \in \mathfrak{A}^{**}$, the module coupling $u \cdot b$ and $b \cdot u$ result in elements of $\mathfrak{B}^{**}$. Since $\mathfrak{B} \subset \mathfrak{B}^{(4)}$ $(= \mathfrak{A}^{**})$, we can also consider the products ub and bu in the sense of Arens in $\mathfrak{B}^{(4)}$. But, from the above lemma, $ub, bu \in \mathfrak{B} \subset \mathfrak{B}^{**}$. It is routine to check that, as elements in $\mathfrak{B}^{**}$, $u \cdot b = ub$ and $b \cdot u = bu$.

From this point on, $\mathcal{H}$ will denote an infinite-dimensional, separable Hilbert space, $B(\mathcal{H})$ will denote the Banach algebra of all bounded operators on $\mathcal{H}$, and $K(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$. It is well known that, with any Arens product, $K(\mathcal{H})^{**} = B(\mathcal{H})$ (see [27, p. 103] for details).

Lemma 7.4. *There is an element $a_0 \in B(\mathcal{H})$ such that $a_0 \notin K(\mathcal{H})$, a_0 is not right invertible in $B(\mathcal{H})$ and $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$.*

Proof. Let $(e_i)_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Let $a_0 \in B(\mathcal{H})$ be defined by

$$a_0(e_i) = \begin{cases} \frac{1}{i} e_i & \text{if } i \text{ is even;} \\ e_i & \text{if } i \text{ is odd.} \end{cases}$$

Clearly, $a_0 \notin K(\mathcal{H})$. Also, a_0 is neither right nor left invertible because any one-sided inverse of a_0 must satisfy

$$a_0^{-1}(e_i) = \begin{cases} i e_i & \text{if } i \text{ is even;} \\ e_i & \text{if } i \text{ is odd,} \end{cases}$$

which cannot be a bounded operator.

For each $n \geq 1$, denote by V_n the subspace of $\mathcal{H}$ generated by $\{e_1, e_2, \dots, e_n\}$, and let P_n be the orthogonal projection from $\mathcal{H}$ onto V_n. Then, from Corollary II.4.5 of [6], for every $k \in K(\mathcal{H})$ and $\varepsilon > 0$, there is $n = n(k, \varepsilon)$, such that $\|P_n \circ k - k\| < \varepsilon$. For this $n = n(k, \varepsilon)$, let $b_n \in B(\mathcal{H})$ be defined by

$$b_n(e_i) = \begin{cases} ie_i & \text{if } i \leq n \text{ and } i \text{ is even;} \\ e_i & \text{if } i \leq n \text{ and } i \text{ is odd;} \\ 0 & \text{for } i \geq n. \end{cases}$$

Then $a_0 \circ b_n = P_n$ and $a_0 \circ b_n \circ P_n = P_n^2 = P_n$. Let $k_n = b_n \circ P_n \circ k$. Then $k_n \in K(\mathcal{H})$, and $a_0 \circ k_n = P_n \circ k$. Also, $\|a_0 \circ k_n - k\| = \|P_n \circ k - k\| < \varepsilon$. Since $k \in K(\mathcal{H})$ and $\varepsilon \geq 0$ are arbitrarily given, this shows that $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$. □

For the element a_0 in the above lemma, $a_0 B(\mathcal{H})$ is a proper right ideal of $B(\mathcal{H})$ since the identity $1 \notin a_0 B(\mathcal{H})$. The closure of $a_0 B(\mathcal{H})$ is also a proper right ideal of $B(\mathcal{H})$ ([3, p. 46]). So there is $F \in B(\mathcal{H})^*$ such that $F \neq 0$ but $Fa_0 = 0$. Then, $FB(\mathcal{H}) \neq \{0\}$ is a right $B(\mathcal{H})$-submodule of $B(\mathcal{H})^*$. Take

$$X_0 = (K(\mathcal{H}))_0, \text{ and } {}_0Y = {}_0(cl(FB(\mathcal{H})).$$

Then we have the following example.

Example 7.5. $B(\mathcal{H}) \oplus (X_0 \dot{+} {}_0Y)$ is weakly amenable but not 3-weakly amenable.

Proof. Clearly, we have $B(\mathcal{H})X_0 = X_0$ and ${}_0YB(\mathcal{H}) = {}_0Y$. By Corollaries 6.2 and 6.3, the Banach algebras $B(\mathcal{H}) \oplus X_0$ and $B(\mathcal{H}) \oplus {}_0Y$ are weakly amenable.

Suppose that $T\colon {}_0Y \to X_0^*$ is a continuous $B(\mathcal{H})$-bimodule morphism. We prove that T is trivial. Let $f = T(F)$. Then $fa_0 = T(Fa_0) = 0$, and so $\langle a_0 K(\mathcal{H}), f \rangle = \{0\}$. We then have $f = 0$ since $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$. This shows that $T(F) = 0$ and hence $T(FB(\mathcal{H})) = \{0\}$. Thus, $T = 0$. From Proposition 7.2, $B(\mathcal{H}) \oplus (X_0 \dot{+} {}_0Y)$ is weakly amenable.

To prove that $B(\mathcal{H}) \oplus (X_0 \dot{+} {}_0Y)$ is not 3-weakly amenable, we show that it fails condition 4 of Theorem 2.1 for $m = 1$. Since

$$(X_0)^{***} = {}_0(K(\mathcal{H})^{***}) = {}_0(B(\mathcal{H})^*) \supset {}_0Y,$$

there exists a nonzero $B(\mathcal{H})$-bimodule morphism from ${}_0Y$ into $(X_0)^{***}$ (e.g., the inclusion mapping). Let $\tau\colon {}_0Y \to (X_0)^{***}$ be such a morphism, and let $\Delta\colon (X_0)^{***} \to (X_0)^*$ be the projection with the kernel $X_0^{\perp}$. Take $T = \Delta \circ \tau\colon {}_0Y \to X_0^*$. From the preceding paragraph, we have that $T = 0$. So

$$\langle x, \tau(y) \rangle = \langle x, T(y) \rangle = 0 \quad (y \in {}_0Y,\ x \in X_0).$$

Now let $\Gamma\colon X_0 \dot{+} {}_0Y \to (X_0 \dot{+} {}_0Y)^{***}$ be the continuous $B(\mathcal{H})$-bimodule morphism defined by

$$\Gamma((x, y)) = (\tau(y),\ 0).$$

Then, for $(x_1, y_1), (x_2, y_2) \in X_0 \dot{+} {}_0Y$, and $u \in B(\mathcal{H})^{**}$, we have

$$\begin{aligned} &\langle u,\ (x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) \rangle \\ &= \langle (u \cdot x_1, 0),\ (\tau(y_2), 0) \rangle + \langle (0, y_2 \cdot u),\ (\tau(y_1), 0) \rangle \\ &= \langle u \cdot x_1,\ \tau(y_2) \rangle = \langle ux_1,\ \tau(y_2) \rangle = 0. \end{aligned}$$

Here we used the fact that $u \cdot x_1 = ux_1 \in X_0$ (see the paragraph following Lemma 7.3). So

$$(x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) = 0$$

for all $(x_1, y_1), (x_2, y_2) \in X_0 \dot{+}_0 Y$. But $\Gamma \neq 0$; so condition 4 of Theorem 2.1 does not hold for $m = 1$ and $X = X_0 \dot{+}_0 Y$. As a consequence, $B(\mathcal{H}) \oplus (X_0 \dot{+}_0 Y)$ is not 3-weakly amenable. $\square$

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