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WEAK AND CLASSICAL SOLUTIONS OF THE TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC EQUATIONS

Dedicated to Professor Shōzō Koshi on his sixtieth birthday

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Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary $\partial\Omega$. In $Q_T:=\Omega\times(0, T)$, we consider the following magnetohydrodynamic equations for an *ideal* incompressible fluid coupled with magnetic field:

$$(*)\quad \begin{aligned} &\partial_t u+(u, \nabla)u-(B, \nabla)B+\nabla((1/2)|B|^2)+\nabla\pi=f && \text{in } Q_T\,,\\ &\partial_t B-\Delta B+(u, \nabla)B-(B, \nabla)u=0 && \text{in } Q_T\,,\\ &\operatorname{div} u=0\,, \qquad \operatorname{div} B=0 && \text{in } Q_T\,,\\ &u\cdot\nu=0\,, \qquad B\cdot\nu=0 \qquad \operatorname{rot} B=0 && \text{on } \partial\Omega\times(0, T)\,,\\ &u\big|_{t=0}=u_0\,, \qquad B\big|_{t=0}=B_0\,. \end{aligned}$$

Here $u=u(x, t)=(u^1(x, t), u^2(x, t))$, $B=B(x, t)=(B^1(x, t), B^2(x, t))$ and $\pi=\pi(x, t)$ denote the unknown velocity field of the fluid, magnetic field and pressure of the fluid, respectively; $f=f(x, t)=(f^1(x, t), f^2(x, t))$ denotes the given external force, $u_0=u_0(x)=(u_0^1(x), u_0^2(x))$ and $B_0=B_0(x)=(B_0^1(x), B_0^2(x))$ denote the given initial data and ν denotes the unit outward normal on $\partial\Omega$.

The first purpose of this paper is to show the existence and uniqueness of a *global weak solution* of $(*)$ without restriction on the data. In case B is identically equal to zero, i.e., in the case of the Euler equations, such a problem for *global weak* and *classical solutions* was solved by Bardos [1] and Kato [8], respectively. (Kikuchi [9] extended the result of Kato [8] in an exterior domain.) Using the energy method developed by Bardos [1], we can obtain a *global weak solution* in our case.

Our second purpose is to show the existence and uniqueness of a *local classical solution* of $(*)$. Although the method of characteristic curves for the vorticity equation plays an important role in a *global classical solution* of the two-dimensional Euler equations, such a method seems to give us only a *local classical solution* of $(*)$ because of the additional terms $(B, \nabla)B$ and $(u, \nabla)B-(B, \nabla)u$. Our result on classical solutions, however, can be regarded as a generalization of that of Kato [8] in some sense.

We shall devoted Section 1 to preliminaries and definition of a weak solution of

(∗). Two main theorems will then be stated. Sections 2 and 3 will be devoted to the proofs of the main theorems.

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1. Results.

1.1. Notation. Let us introduce some function spaces. $C_{0,\sigma}^{\infty}(\Omega)$ denotes the set of all C^{∞}-real vector-valued functions $\phi=(\phi^1, \phi^2)$ with compact support in Ω such that $\operatorname{div}\phi=0$. H is the completion of $C_{0,\sigma}^{\infty}(\Omega)$ with respect to the L^2-norm $\|\ \|$; $(\ ,\ )$ denotes the L^2-inner product. V denotes the set of all vector-valued functions u in $H^1(\Omega)$ with $\operatorname{div} u=0$ in Ω and $u\cdot\nu=0$ on $\partial\Omega$. Equipped with the norm $|\ |$:

$$|u|^2=\|\operatorname{rot} u\|^2+\|u\|^2\,,$$

V is a Hilbert space. Here and hereafter, we shall use the notations $\operatorname{rot} u$ for a vector function $u=(u^1, u^2)$ and $\operatorname{rot}\psi$ for a scalar function ψ representing $\operatorname{rot} u=\partial u^2/\partial x_1-\partial u^1/\partial x_2$ and $\operatorname{rot}\psi=(\partial\psi/\partial x_2, -\partial\psi/\partial x_1)$, respectively. By Duvaut-Lions [3, Chapter 7, Theorem 6.1], we have

$$\|u\|_{H^1(\Omega)}\leqq C(\Omega)|u| \qquad \text{for all} \quad u\in V. \tag{1.1}$$

Hence the norm $|\ |$ is equivalent to the one usually induced from $H^1(\Omega)$ and V is compactly imbedded into H.

If X is a Hilbert space, then $L^p(0, T; X)$ $(1\leqq p<\infty)$ denotes the set of all measurable functions $u(t)$ with values in X such that $\int_0^T\|u(t)\|_X^p dt<\infty$ ($\|\ \|_X$ is the norm in X). $L^{\infty}(0, T; X)$ denotes the set of all essentially bounded (with respect to the norm of X) measurable functions of t with values in X. In the case of $X=L^2(\Omega)$, we denote by $\|\ \|_{2,p}$ and $\|\ \|_{2,\infty}$ the norms in $L^p(0, T; L^2(\Omega))$ and $L^{\infty}(0, T; L^2(\Omega))$, respectively.

Let $C^m([0, T]; X)$ denote the set of all X-valued m-times continuously differentiable functions of t $(0\leqq t\leqq T)$. $C_0^m([0, T); X)$ is the set of all X-valued m-times continuously differentiable functions on $[0, T)$ with compact support in $[0, T)$.

$C^{k+\alpha}(\bar{\Omega})$ for an integer $k\geqq 0$ and $0\leqq\alpha<1$ denotes the usual Hölder space of continuous functions on $\bar{\Omega}$. $|\ |_{k+\alpha}$ denotes the norm in $C^{k+\alpha}(\bar{\Omega})$. $C^{k,j}(\bar{Q}_T)$ for integers $k, j\geqq 0$ is the set of all functions ϕ for which all the $\partial_x^q\partial_t^r\phi$ exist and are continuous on $\bar{Q}_T$ for $0\leqq|q|\leqq k$, $0\leqq r\leqq j$. $C^{k+\alpha,j+\beta}(\bar{Q}_T)$ for integers $k,j\geqq 0$ and $0\leqq\alpha,\beta<1$ is the subset of $C^{k,j}(\bar{Q}_T)$ containing all functions ϕ for which all the $\partial_x^q\partial_t^r\phi$, $0\leqq|q|\leqq k$, $0\leqq r\leqq j$, are Hölder continuous with exponents α in x and β in t. If

$$\begin{aligned}K^{\alpha,\beta}(\phi)=&\sup\{|\phi(x,t)-\phi(x',t)|/|x-x'|^{\alpha};\ (x,t),(x',t)\in\bar{Q}_T,\ |x-x'|<1\}\\&+\sup\{|\phi(x,t)-\phi(x,t')|/|t-t'|^{\beta};\ (x,t),(x,t')\in\bar{Q}_T,\ |t-t'|<1\}\,,\end{aligned}$$

we define the norm $|\ \ |_{k+\alpha,j+\beta}$ in $C^{k+\alpha,j+\beta}(\bar{Q}_T)$ by

$$|\phi|_{k+\alpha,j+\beta}=\sup_{(x,t)\in\bar{Q}_T}\sum_{\substack{|q|\leq k\\ r\leq j}}|\partial_x^q\partial_t^r\phi(x,t)|+\sum_{|q|=k}K^{\alpha,\beta}(\partial_x^q\partial_t^j\phi)\,.$$

For the spaces of vector-valued functions, we shall use the bold-faced letters analogously.

Throughout this paper, $C, C_1, C_2, \cdots$ will denote positive constants which may be different in each occurrence. In particular, we shall denote by $C=C(*, \cdots, *)$ the constant depending only on the quantities appearing in the parentheses.

1.2. Definitions and results. Our definition of a weak solution of $(*)$ is as follows:

Definition 1.1. Let $u_0\in H$, $B_0\in H$ and $f\in L^2(0, T; L^2(\Omega))$. A pair of measurable vector functions u and B on Q_T is called a *weak solution* of $(*)$ if

(i) $u\in L^\infty(0, T; H)\cap L^2(0, T; V)$, $B\in L^\infty(0, T; H)\cap L^2(0, T; V)$;

(ii) $$\int_0^T\{-(u,\partial_t\Phi)+((u, \nabla)u-(B, \nabla)B, \Phi)\}dt=(u_0, \Phi(0))+\int_0^T(f, \Phi)dt\,,$$

$$\int_0^T\{-(B, \partial_t\Phi)+(\mathrm{rot}\, B, \mathrm{rot}\, \Phi)+((u, \nabla)B-(B, \nabla)u, \Phi)\}dt=(B_0, \Phi(0))$$

for all $\Phi\in C_0^1([0, T); V)$.

Concerning the uniqueness of weak solutions of $(*)$, we have:

Proposition 1.1. *There exists at most one weak solution of* $(*)$. *If* $\{u, B\}$ *is a weak solution of* $(*)$, *after a suitable redefinition of* $u(t)$ *and* $B(t)$ *on a set of measure zero of the time interval* $[0, T]$, *we have* $u\in C([0, T]; H)$ *and* $B\in C([0, T]; H)$.

Since the proof of this proposition is parallel to that of Temam [16, Chapter 3, Theorem 3.2], we omit it.

Our result on the existence of a weak solution now reads as follows:

Theorem 1. *Let* $u_0\in V$, $B_0\in V$ *and* $f\in L^2(0, T; L^2(\Omega))$ *with* $\mathrm{rot} f\in L^2(0, T; L^2(\Omega))$. *Then there exists a weak solution* $\{u, B\}$ *of* $(*)$ *such that* $u\in L^\infty(0, T; V)\cap C([0, T]; H)$ *and* $B\in L^2(0, T; \boldsymbol{H}^2(\Omega))\cap C([0, T]; V)$.

We next proceed to our result on classical solutions. To this end, we make the following assumptions on the domain Ω and the given data u_0, B_0 and f.

Assumption 1. The boundary $\partial\Omega$ of Ω consists of $m+1$ sufficiently smooth, simple closed curves $S_0, S_1, \cdots, S_m$, where S_j $(j=1, \cdots, m)$ are inside S_0 and outside one another.

Günter [7, 1., p. 122] refers to the above assumption as "Case J".

ASSUMPTION 2. $u_0 \in C^{1+\theta}(\bar{\Omega})$, $B_0 \in C^{2+\theta}(\bar{\Omega})$ and $f \in C^{1+\theta,0}(\bar{Q}_T)$ hold for some $0<\theta<1$. Moreover, u_0 and B_0 satisfy the conditions $\operatorname{div} u_0 = 0$, $\operatorname{div} B_0 = 0$ in Ω and $u_0 \cdot \nu = 0$, $B_0 \cdot \nu = 0$ on $\partial\Omega$.

Our result on the existence and uniqueness of classical solutions reads as follows:

THEOREM 2. *Under the assumptions* 1 *and* 2, *there is a positive number* $C_* = C_*(\Omega, T, |u_0|_{1+\theta}, |f|_{1+\theta,0})$ *such that if* $|B_0|_{2+\theta} \leqq C_*$, *there exists a solution* $\{u, B, \pi\} \in C^{1,1}(\bar{Q}_T) \times C^{2,1}(\bar{Q}_T) \times C^{1,0}(\bar{Q}_T)$ *of* (*). *Such a solution is unique up to addition to* π *of an arbitrary function of* t.

REMARK 1.1. (i) Taking $B_0 = 0$ in Ω, we have the result of Kato [8].
(ii) Our construction of the solution of Theorem 2 ensures us that $u \in C^{1+\theta',1}(\bar{Q}_T)$ and $B \in C^{2+\theta',(2+\theta')/2}(\bar{Q}_T)$ for some $\theta' \in (0, \theta)$.

2. Existence of a global weak solution; Proof of Theorem 1.

2.1. The operator A. For the proof of Theorem 1, we shall use the Galerkin method. In order to make use of a special basis, we introduce the operator A from $D(A)$ to H as

$$Au = (-\Delta + 1)u = \operatorname{rot}(\operatorname{rot} u) + u$$

for $u \in D(A) = \{u \in H^2(\Omega);\ u \cdot \nu = 0,\ \operatorname{rot} u = 0 \text{ on } \partial\Omega\} \cap H$. See Miyakawa [13, Lemma 3.3]. Then we have:

PROPOSITION 2.1. 1. *A coincides with the positive self-adjoint operator on H defined by a positive quadratic form* $a(\cdot,\cdot)$ *on* $V \times V$;

$$a(u, v) = (\operatorname{rot} u, \operatorname{rot} v) + (u, v), \qquad u, v \in V.$$

This implies

$$V = D(A^{1/2}), \qquad \|A^{1/2}u\|^2 = \|\operatorname{rot} u\|^2 + \|u\|^2 \quad \textit{for} \quad u \in D(A^{1/2}). \tag{2.1}$$

2. *Zero is not an eigenvalue of* A.
3. *There is a constant* $C = C(\Omega)$ *such that*

$$\|u\|_{H^2(\Omega)} \leqq C(\|\Delta u\| + \|u\|) \qquad \textit{for all} \quad u \in D(A). \tag{2.2}$$

Indeed, 1 is easy. 2 is a consequence of (2.1). 3 follows from Georgescu [5, Theorem 3.2.3]. See also Sermange-Temam [14, p. 642, (2.8)].

By Proposition 2.1, we see that the operator A possesses a complete orthonormal system $\{\phi_j\}_{j=1}^{\infty}$ of H of eigenfunctions:

$$\phi_j \in D(A), \quad A\phi_j = \lambda_j \phi_j, \quad \lambda_j > 0, \quad \lambda_j \to +\infty, \quad j \to \infty;$$

$$(\operatorname{rot} \phi_j, \operatorname{rot} u) + (\phi_j, u) = \lambda_j (\phi_j, u) \qquad \text{for all} \quad u \in V. \tag{2.3}$$

2.2. PROOF OF THEOREM 1. We shall use $\{\phi_j\}_{j=1}^{\infty}$ defined in (2.3) as a basis of Galerkin approximation. For every integer m, we define $\{u_m, B_m\}=\{u_m(x, t), B_m(x, t)\}$ as

$$u_m(x, t)=\sum_{j=1}^{m} g_{jm}(t)\phi_j(x), \qquad B_m(x, t)=\sum_{j=1}^{m} h_{jm}(t)\phi_j(x)$$

and we may choose $\{g_{jm}\}_{j=1}^{m}$ and $\{h_{jm}\}_{j=1}^{m}$ satisfying the following equations:

$$(u'_m(t), \phi_j)+((u_m(t), \nabla)u_m(t)-(B_m(t), \nabla)B_m(t), \phi_j)=(f(t), \phi_j),$$

$$(B'_m(t), \phi_j)+(\operatorname{rot} B_m(t), \operatorname{rot}\phi_j)+((u_m(t), \nabla)B_m(t)-(B_m(t), \nabla)u_m(t), \phi_j)=0, \tag{2.4}$$

$$j=1, \cdots, m,$$

$$u_m(0)=\sum_{j=1}^{m}(u_0, \phi_j)\phi_j, \quad B_m(0)=\sum_{j=1}^{m}(B_0, \phi_j)\phi_j. \tag{2.5}$$

As is well-known, there is $T_m>0$ such that (2.4) with (2.5) has a unique solution on $[0, T_m)$. Moreover, the following *a priori* estimate guarantees that $T_m=T$.

Energy estimates: After multiplying the first and the second equation of (2.4) by $g_{jm}(t)$ and $h_{jm}(t)$, respectively, we add these equations. By integration over $(0, t)$, we get

$$\begin{aligned} &\|u_m(t)\|^2+\|B_m(t)\|^2+2\int_0^t \|\operatorname{rot} B_m(s)\|^2 ds \\ &\leqq \|u_0\|^2+\|B_0\|^2+\int_0^t \|u_m(s)\|^2 ds+\int_0^t \|f(s)\|^2 ds. \end{aligned} \tag{2.6}$$

Here we used the identities $((u, \nabla)v, v)=0$ and $((u, \nabla)v, w)=-((u, \nabla)w, v)$ for $u, v, w\in V$. Hence by the same technique as that used in the proof of Gronwall's inequality, we have

$$\|u_m(t)\|^2+\|B_m(t)\|^2+2\int_0^t \|\operatorname{rot} B_m(s)\|^2 ds\leqq e^T(\|u_0\|^2+\|B_0\|^2+\|f\|_{2,2}^2), \tag{2.7}$$

for all $t\in[0, T]$.

Estimates of the derivatives of higher order: By (2.3), we see that the equalities

$$(u, \lambda_j\phi_j)=(u, A\phi_j)=(\operatorname{rot} u, \operatorname{rot}\phi_j)+(u, \phi_j)$$

hold for all $u\in V$. Hence multiplying the first and the second equation of (2.4) by λ_j, we have

$$(\operatorname{rot} u'_m, \operatorname{rot}\phi_j)+(u'_m, \phi_j)+((u_m, \nabla)u_m-(B_m, \nabla)B_m, A\phi_j)=(f, A\phi_j),$$

$$(\operatorname{rot} B'_m, \operatorname{rot}\phi_j)+(B'_m, \phi_j)+(\operatorname{rot}(\operatorname{rot} B_m), A\phi_j)+((u_m, \nabla)B_m-(B_m, \nabla)u_m, A\phi_j)=0$$

$$(j=1, \cdots, m).$$

Proceeding as we did in deriving (2.6), we obtain

$$\begin{aligned} &(1/2)(d/dt)(\|\operatorname{rot} u_m\|^2+\|u_m\|^2+\|\operatorname{rot} B_m\|^2+\|B_m\|^2)+\|\Delta B_m\|^2+\|\operatorname{rot} B_m\|^2 \\ &\quad+((u_m, \nabla)u_m-(B_m, \nabla)B_m, \operatorname{rot}(\operatorname{rot} u_m)+u_m) \end{aligned}$$

$$+((u_m,\nabla)B_m-(B_m,\nabla)u_m,\operatorname{rot}(\operatorname{rot}B_m)+B_m)=(f,\operatorname{rot}(\operatorname{rot}u_m)+u_m)\,.$$

Taking into account $\operatorname{rot}u_m=0$, $\operatorname{rot}B_m=0$ on $\partial\Omega$, after integration by parts we get

$$\begin{aligned}(2.8)\qquad &\|\omega_m(t)\|^2+\|u_m(t)\|^2+\|J_m(t)\|^2+\|B_m(t)\|^2+2\int_0^t(\|\Delta B_m\|^2+\|J_m\|^2)ds\\ &+4\int_0^t((\partial B_m^2/\partial x_2)Du_m+(\partial u_m^1/\partial x_1)DB_m,J_m)ds\\ &=\|\omega_m(0)\|^2+\|u_m(0)\|^2+\|J_m(0)\|^2+\|B_m(0)\|^2+2\int_0^t\{(\operatorname{rot}f,\omega_m)+(f,u_m)\}ds\,,\end{aligned}$$

where $\omega_m=\operatorname{rot}u_m$, $J_m=\operatorname{rot}B_m$, $Du_m=\partial u_m^1/\partial x_2+\partial u_m^2/\partial x_1$ and $DB_m=\partial B_m^1/\partial x_2+\partial B_m^2/\partial x_1$. Here we used the equalities $((u_m,\nabla)\omega_m,\omega_m)=((u_m,\nabla)J_m,J_m)=0$ and $((B_m,\nabla)J_m,\omega_m)=-((B_m,\nabla)\omega_m,J_m)$.

Now, let us investigate the sixth term on the left hand side of (2.8). By the Hölder inequality, the Gagliardo-Nirenberg inequality (Tanabe [15, Chapter 1, Lemma 1.2.1]), (1.1) and (2.2), we have

$$\begin{aligned}|((\partial B_m^2/\partial x_2)Du_m,J_m)|&\leqq\|\partial B_m^2/\partial x_2\|_{L^4(\Omega)}\|Du_m\|\,\|J_m\|_{L^4(\Omega)}\\ &\leqq C\|\nabla B_m\|^{1/2}\|B_m\|_{H^2(\Omega)}^{1/2}\|J_m\|^{1/2}\|\nabla J_m\|^{1/2}\|Du_m\|\\ &\leqq C\|B_m\|_{H^1(\Omega)}\|B_m\|_{H^2(\Omega)}\|Du_m\|\\ &\leqq C(\|B_m\|+\|J_m\|)(\|\Delta B_m\|+\|B_m\|)(\|u_m\|+\|\omega_m\|)\,,\\ |((\partial u_m^1/\partial x_1)DB_m,J_m)|&\leqq\|\partial u_m^1/\partial x_1\|\,\|DB_m\|_{L^4(\Omega)}\|J_m\|_{L^4(\Omega)}\\ &\leqq C\|\nabla u_m\|\,\|\nabla B_m\|^{1/2}\|B_m\|_{H^2(\Omega)}^{1/2}\|J_m\|^{1/2}\|\nabla J_m\|^{1/2}\\ &\leqq C\|\nabla u_m\|\,\|B_m\|_{H^1(\Omega)}\|B_m\|_{H^2(\Omega)}\\ &\leqq C(\|B_m\|+\|J_m\|)(\|\Delta B_m\|+\|B_m\|)(\|u_m\|+\|\omega_m\|)\,,\end{aligned}$$

where $C=C(\Omega)$ is a constant independent of m. Hence by the Schwarz inequality and (2.7), we get for any $\varepsilon>0$

$$\begin{aligned}(2.9)\qquad &\left|\int_0^t((\partial B_m^2/\partial x_2)Du_m+(\partial u_m^1/\partial x_1)DB_m,J_m)ds\right|\\ &\leqq C\varepsilon\int_0^t\|\Delta B_m\|^2ds+C(\varepsilon^{-1}+1)\{(1+\|B_m\|_{2,\infty})^2(1+\|u_m\|_{2,\infty})^2T\\ &\quad+(1+\|B_m\|_{2,\infty})^2(1+\|u_m\|_{2,\infty})^2\int_0^t\|J_m\|^2ds\\ &\quad+(1+\|B_m\|_{2,\infty})^2\int_0^t\|\omega_m\|^2ds+\int_0^t\|J_m\|^2\|\omega_m\|^2ds\}\end{aligned}$$

$$\leqq C_1\varepsilon\int_0^t \|\Delta B_m\|^2 ds + C_1(\varepsilon^{-1}+1)\int_0^t (1+\|J_m\|^2)\|\omega_m\|^2 ds + C_1(\varepsilon^{-1}+1),$$

where $C_1 = C_1(\Omega, T, \|u_0\|, \|B_0\|, \|f\|_{2,2})$ is a constant independent of m. Substituting (2.9) into (2.8) and then taking $\varepsilon = 1/2C_1$, we have

$$\|\omega_m(t)\|^2 + \|J_m(t)\|^2 + \int_0^t \|\Delta B_m(s)\|^2 ds \leqq \|\operatorname{rot} u_0\|^2 + \|\operatorname{rot} B_0\|^2 + C_2 + C_2\int_0^t (1+\|J_m(s)\|^2 + \|\operatorname{rot} f(s)\|^2)\|\omega_m(s)\|^2 ds, \tag{2.10}$$

where $C_2 = C_2(\Omega, T, \|u_0\|, \|B_0\|, \|f\|_{2,2})$ is a constant independent of m. By application of Gronwall's technique as in the derivation of (2.7), we see that

$$\begin{aligned} &\|\omega_m(t)\|^2 + \|J_m(t)\|^2 + \int_0^t \|\Delta B_m(s)\|^2 ds \\ &\leqq (\|\operatorname{rot} u_0\|^2 + \|\operatorname{rot} B_0\|^2 + C_2)\exp\left(C_2\int_0^t (1+\|J_m(s)\|^2 + \|\operatorname{rot} f(s)\|_{2,2}^2) ds\right) \\ &\leqq C_3 = C_3(\Omega, T, |u_0|, |B_0|, \|f\|_{2,2}, \|\operatorname{rot} f\|_{2,2}) \quad \text{(by (2.7))} \end{aligned} \tag{2.11}$$

for all $t \in [0, T]$, where C_3 is a constant independent of m.

Taking into account (1.1) and (2.2), we can deduce from (2.7) and (2.11) that *the sequence* $\{u_m\}_{m=1}^\infty$ *remains in a bounded set of* $L^\infty(0, T; V)$ *and that the sequence* $\{B_m\}_{m=1}^\infty$ *remains in a bounded set of* $L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$. Hence there exist a subsequence of $\{u_m, B_m\}$, which we denote by the same letter, and functions $u \in L^\infty(0, T; V)$ and $B \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$ such that

$$\begin{aligned} u_m &\to u \quad \textit{weakly-star in} \quad L^\infty(0, T; V), \\ B_m &\to B \quad \textit{weakly-star in} \quad L^\infty(0, T; V), \\ &\qquad\quad \textit{weakly in} \quad L^2(0, T; H^2(\Omega)). \end{aligned} \tag{2.12}$$

Moreover by (2.4) and (2.11), we see that for each fixed j, the families $\{(u_m(t), \phi_j)\}_{m=1}^\infty$ and $\{(B_m(t), \phi_j)\}_{m=1}^\infty$ form uniformly bounded and equicontinuous families of continuous functions on $[0, T]$, respectively (see, e.g., Ladyzhenskaya [10, p. 175]). Hence by the Ascoli-Arzera theorem and the usual diagonal argument, there exist subsequences $\{u_{m_i}(t)\}$ and $\{B_{m_i}(t)\}$ of $\{u_m(t)\}$ and $\{B_m(t)\}$ which converge to some $\bar{u}(t)$ and $\bar{B}(t)$, uniformly in $t \in [0, T]$ in the weak topology of H, respectively. Clearly $u = \bar{u}$ and $B = \bar{B}$. For simplicity, we shall assume that the original sequences u_m and B_m converge to u and B, respectively.

By means of the techniques of the Friedrichs inequality (Courant-Hilbert [2, p. 519]) and (1.1), we have

$$u_m \to u \quad \textit{strongly in} \quad L^2(Q_T)^2, \qquad B_m \to B \quad \textit{strongly in} \quad L^2(Q_T)^2. \tag{2.13}$$

Now by the routine passage to the limit (see, e.g., Temam [16]), we can deduce from (2.12) and (2.13) that $\{u, B\}$ is a weak solution of $(*)$.

To complete the proof of Theorem 1, it remains to show that $B \in C([0, T]; V)$. Since $u \in L^\infty(0, T; V)$, $B \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; V)$, we get by the Gagliardo-Nirenberg inequality and the continuous imbedding $H^2(\Omega) \subset L^\infty(\Omega)$

$$\begin{aligned}\|(u, \nabla)B-(B, \nabla)u\| &\leqq \|(u, \nabla)B\| + \|(B, \nabla)u\| \leqq \|u\|_{L^4(\Omega)}\|\nabla B\|_{L^4(\Omega)} + \|B\|_{L^\infty(\Omega)}\|\nabla u\| \\ &\leqq C\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla B\|^{1/2}\|B\|^{1/2}_{H^2(\Omega)} + C\|B\|_{H^2(\Omega)}\|\nabla u\| \\ &\leqq C\|u\|_{L^\infty(0,T;V)}\|B\|^{1/2}_{L^\infty(0,T;V)}\|B\|^{1/2}_{H^2(\Omega)} + C\|u\|_{L^\infty(0,T;V)}\|B\|_{H^2(\Omega)}\,.\end{aligned}$$

This implies $(u, \nabla)B-(B, \nabla)u = \mathrm{rot}(B \wedge u) \in L^2(0, T; H)$. Hence by the second identity of Definition 1.1 (ii), we see that $B' \in L^2(0, T; H)$. Therefore, it follows from Lions-Magenes [12, p. 19, Theorem 3.1] that $B \in C([0, T]; V)$.

3. Existence of a local classical solution; Prood of Theorem 2. In this section, we shall show the existence of a local classical solution by using the Schauder fixed point theorem as in Kato [8] and Kikuchi [9].

3.1. Construction of the flow u.

LEMMA 3.1. *Under the assumption* 1, *there exist* $u^{(k)} \in C^{1+\mu}(\bar{\Omega})$ $(k=1, \cdots, m)$ *for some* $\mu>0$ *satisfying the following properties*:

(i) $\mathrm{div}\, u^{(k)}=0\,,\quad \mathrm{rot}\, u^{(k)}=0$ *in* Ω, $\quad u^{(k)}\cdot \nu=0$ *on* $\partial\Omega$; $\quad (k=1, \cdots, m)$

(ii) $\displaystyle\int_{S_j} u^{(k)}\cdot \tau dS=0$ *if* $j \neq k\,,\quad \displaystyle\int_{S_k} u^{(k)}\cdot \tau dS=1\,,\quad (j=0, \cdots, m, k=1, \cdots, m)$

where τ *denotes the unit tangent vector on* $\partial\Omega$ *and* dS *denotes the line element of* $\partial\Omega$.

PROOF. It follows from Günter [7, p. 206, p. 209 (58)] that there exist m linearly independent functions $\psi^{(k)} \in C^{1+\mu}(\partial\Omega)$ $(k=1, \cdots, m)$ satisfying the following properties (1), (2), (3):

(1) $\displaystyle\int_{S_j} \psi^{(k)} dS=0$ if $j \neq k\,,\quad \displaystyle\int_{S_k} \psi^{(k)} dS=1;\quad (j=0, \cdots, m, k=1, \cdots, m)$

(2) $\displaystyle \psi^{(k)}(x)=(1/\pi)\int_{\partial\Omega} \psi^{(k)}(\xi)(\partial/\partial\nu_x)\log(1/|x-\xi|)d_\xi S$ for $x \in \partial\Omega$; $\quad (k=1, \cdots, m)$

(3) For each $k=1, \cdots, m$, the function $\int_{\partial\Omega}\psi^{(k)}(\xi)\log(1/|x-\xi|)d_\xi S$ on $\boldsymbol{R}^2$ is constant outside Ω.

Then the desired $u^{(k)}$ $(k=1, \cdots, m)$ are defined by

$$u^{(k)}(x)=\mathrm{rot}_x\{(1/2\pi)\int_{\partial\Omega} \psi^{(k)}(\xi)\log(1/|x-\xi|)d_\xi S\}.$$

Since the proof that such $u^{(k)}$ ($k=1, \cdots, m$) have the properties (i) and (ii) is parallel to that of Kikuchi [9, Lemma 1.5], we may omit details.

Now let us define a function space $S_\alpha(M, N)$ for $M>0$, $N>0$ and $0<\alpha<\mathrm{Min.}\{\theta, \mu\}$ by

$$S_\alpha(M, N)=\{\phi\in C^{\alpha,\alpha}(\bar{Q}_T);\ |\phi|_{0,0}\leqq M,\ K^{\alpha,\alpha}(\phi)\leqq N\}.$$

For the notation, see Subsection 1.1. For $\phi\in S_\alpha(M, N)$, let us define a map $F_1:\phi\to u$ by

$$u(t)=\mathrm{rot}\, G\phi(\cdot, t)+\sum_{k=1}^{m}\lambda_k(t)u^{(k)},$$

where

$$\lambda_k(t)=\int_{S_k}u_0\cdot\tau dS+\int_0^t\int_{S_k}f(\xi, \sigma)\cdot\tau d_\xi S d\sigma-\int_{S_k}\mathrm{rot}\, G\phi(\cdot, t)\cdot\tau dS. \tag{3.1}$$

Here, $\{u^{(k)}\}_{k=1}^m$ are as in Lemma 3.1 and G denotes the Green operator of $-\Delta$ with zero Dirichlet boundary condition on $\partial\Omega$.

LEMMA 3.2. *For $\phi\in S_\alpha(M, N)$, we have $u=F_1\phi\in C^{1+\alpha,\alpha^-}(\bar{Q}_T)$ for any $0<\alpha^-<\alpha$, $\mathrm{div}\, u=0$ in Ω and $u\cdot\nu=0$ on $\partial\Omega$. Moreover, there is a positive constant $C_4=C_4(\Omega, T, |u_0|_0, |f|_{0,0}, M, N)$ such that $|u|_{1+\alpha,\alpha^-}\leqq C_4$.*

PROOF. Set $u=u_1+u_2$, where $u_1=\mathrm{rot}\, G\phi$ and $u_2=\sum_{k=1}^m\lambda_k u^{(k)}$. By Assumption 2 and Lemma 3.1, it is easy to see that the assertion of this lemma holds for u_2. Let us prove the assertion for u_1. By the Schauder estimate of $-\Delta$ (see, e.g., Gilbarg-Trudinger [6, Chapter 4]), there is a constant $C=C(\Omega, \alpha)$ such that

$$\begin{aligned}(3.2)\qquad &\sup_{(x,t)\in\bar{Q}_T}|u_1(x,t)|+\sup_{(x,t)\in\bar{Q}_T}|\nabla u_1(x,t)|\\ &\quad+\sup\{|\nabla u_1(x,t)-\nabla u_1(x',t)|/|x-x'|^\alpha;\ (x,t), (x',t)\in\bar{Q}_T,\ |x-x'|<1\}\\ &\leqq\sup_{t\in[0,T]}|u_1(\cdot,t)|_{1+\alpha}\leqq C\sup_{t\in[0,T]}|\phi(\cdot,t)|_\alpha\leqq C|\phi|_{\alpha,\alpha}.\end{aligned}$$

Similarly, for $x\in\bar{\Omega}$, $t, t'\in[0, T]$ with $|t-t'|<1$, the inequalities

$$\begin{aligned}&|u_1(x,t)-u_1(x,t')|+|\nabla u_1(x,t)-\nabla u_1(x,t')|\\ &\leqq|u_1(\cdot,t)-u_1(\cdot,t')|_1\leqq C|\phi(\cdot,t)-\phi(\cdot,t')|_r\end{aligned}$$

hold for any $0<r<\alpha$. Using the argument of Kato [8, Lemma 1.2], we have

$$|\phi(\cdot,t)-\phi(\cdot,t')|_r\leqq 2|\phi|_{\alpha,\alpha}|t-t'|^{\alpha(1-r/\alpha)}$$

and hence

$$\begin{aligned}(3.3)\qquad &\sup\{|u_1(x,t)-u_1(x,t')|/|t-t'|^{\alpha^-};\ (x,t), (x,t')\in\bar{Q}_T,\ |t-t'|<1\}\\ &\quad+\sup\{|\nabla u_1(x,t)-\nabla u_1(x,t')|/|t-t'|^{\alpha^-};\ (x,t), (x,t')\in\bar{Q}_T,\ |t-t'|<1\}\\ &\leqq C|\phi|_{\alpha,\alpha}\end{aligned}$$

holds with $\alpha^- := \alpha(1 - r/\alpha)$. It follows from (3.2) and (3.3) that u_1 has the desired property.

3.2. Construction of the magnetic field B. In this subsection, we shall solve the following equations for the magnetic field B:

$$
\text{(M.E.)} \qquad
\begin{aligned}
&\partial_t B - \Delta B + (u, \nabla)B - (B, \nabla)u = 0 && \text{in } Q_T\,, \\
&\operatorname{div} B = 0 && \text{in } Q_T\,, \\
&B \cdot \nu = 0\,, \quad \operatorname{rot} B = 0 && \text{on } \partial\Omega \times (0, T)\,, \\
&B|_{t=0} = B_0\,,
\end{aligned}
$$

where u is the flow constructed in the preceding subsection. To this end, we shall transform (M.E.) to the equations for a scalar potential of B. Let us first consider the following system of equations of parabolic type:

$$
\text{(P.S.)} \qquad
\begin{aligned}
&\partial_t \bar{B} - \Delta \bar{B} + (u, \nabla)\bar{B} - (\bar{B}, \nabla)u = 0 && \text{in } Q_T\,, \\
&\bar{B} \cdot \nu = 0\,, \quad \operatorname{rot} \bar{B} = 0 && \text{on } \partial\Omega \times (0, T)\,, \\
&\bar{B}|_{t=0} = \bar{B}_0\,.
\end{aligned}
$$

We define a weak solution of (P.S.) as follows:

DEFINITION 3.1. Let $\bar{B}_0 \in L^2(\Omega)$ and $u, \nabla u \in C^{\alpha,\alpha/2}(\bar{Q}_T)$. Let $H^1_N(\Omega) = \{\phi \in H^1(\Omega);\ \phi \cdot \nu = 0 \text{ on } \partial\Omega\}$. A measurable vector function $\bar{B}$ on Q_T is called a weak solution of (P.S.) if

(i) $\bar{B} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_N(\Omega))$;

(ii) $\displaystyle\int_0^T \{-(\bar{B}, \partial_t \Phi) + (\operatorname{rot} \bar{B}, \operatorname{rot} \Phi) + (\operatorname{div} \bar{B}, \operatorname{div} \Phi) + ((u, \nabla)\bar{B} - (\bar{B}, \nabla)u, \Phi)\}dt = (\bar{B}_0, \Phi(0))$

for all $\Phi \in C^1_0([0, T); H^1_N(\Omega))$.

In the above definition, for a smooth solution $\bar{B}$, we have by integration by parts

$$
\begin{aligned}
(-\Delta \bar{B}, \Phi) &= (\operatorname{rot}(\operatorname{rot} \bar{B}) - \nabla(\operatorname{div} \bar{B}), \Phi) \\
&= \int_\Omega \operatorname{rot} \bar{B} \operatorname{rot} \Phi dx - \int_{\partial\Omega} (\operatorname{rot} \bar{B})\nu \wedge \Phi dS + \int_\Omega \operatorname{div} \bar{B} \operatorname{div} \Phi dx - \int_{\partial\Omega} (\operatorname{div} \bar{B})\Phi \cdot \nu dS \\
&= (\operatorname{rot} \bar{B}, \operatorname{rot} \Phi) + (\operatorname{div} \bar{B}, \operatorname{div} \Phi)\,,
\end{aligned}
$$

since $\operatorname{rot} \bar{B} = 0$, $\Phi \cdot \nu = 0$ on $\partial\Omega$.

Since (P.S.) is a system of linear equations for $\bar{B}$, it is not difficult to see the following:

PROPOSITION 3.1. *Suppose that* $\bar{B}_0 \in L^2(\Omega)$ *and* $u, \nabla u \in C^{\alpha,\alpha/2}(\bar{Q}_T)$. *Then there exists a unique weak solution* $\bar{B}$ *of* (P.S.).

In order to solve the equations for a scalar potential of B, we need the following:

LEMMA 3.3. *Let B_0 be as in the assumption 2. Then the boundary value problem*

$$-\Delta\psi_0 = \operatorname{rot} B_0 \quad in \quad \Omega\,, \qquad \psi_0 = 0 \quad on \quad \partial\Omega$$

has a unique solution ψ_0 in $C^{3+\theta}(\bar{\Omega})$. Moreover, there is a constant $C_5 = C_5(\Omega, \theta)$ with $|\psi_0|_{3+\theta} \leqq C_5 |B_0|_{2+\theta}$.

For the proof, see, for example, Gilbarg-Trudinger [6].

LEMMA 3.4. *Let u and ψ_0 be as in the preceding subsection and Lemma 3.3, respectively. Then there exists a unique scalar function ψ in $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ such that*

$$\text{(P.E.)} \qquad \begin{aligned} &\partial_t\psi - \Delta\psi + (u, \nabla)\psi = 0 && in \quad Q_T\,, \\ &\psi = 0 && on \quad \partial\Omega \times (0, T)\,, \\ &\psi|_{t=0} = \psi_0. \end{aligned}$$

Since $u \in C^{1+\alpha,\alpha/2}(\bar{Q}_T)$ by Lemma 3.2, the assertion of this lemma follows from a general theory of parabolic equations. See, for example, Ladyzhenskaya-Solonnikov-Ural'ceva [11, p. 320, Theorem 5.2].

We can now show the existence of a *regular* solution of (M.E.).

LEMMA 3.5. *Let ψ be as in Lemma 3.4. Then $B = \operatorname{rot}\psi$ is in $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ and satisfies the equations* (M.E.). *Moreover, there is a positive constant $C_6 = C_6(\Omega, T, \alpha, |u_0|_0, |f|_{0,0}, M, N)$ such that $|B|_{2+\alpha,(2+\alpha)/2} \leqq C_6 |B_0|_{2+\theta}$.*

PROOF. To begin with, suppose that $B = \operatorname{rot}\psi$ is a *weak solution* of (P.S.) with the initial data B_0. Since $B_0 \in C^{2+\theta}(\bar{\Omega})$ by Assumption 2 and since $u, \nabla u \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ with $|u|_{1+\alpha,\alpha^-} \leqq C_4$ by Lemma 3.2, we can deduce from Ladyzhenskaya-Solonnikov-Ural'ceva [11, p. 616, Theorem 10.1] by taking $b=1, r=2, s_1=s_2=0, t_1=t_2=2, \sigma_1=-2, \sigma_2=-1, \rho_1=\rho_2=-2$ and $l=\alpha$ that there exists a unique solution $\bar{B}$ of (P.S.) in $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ with the initial data $\bar{B}_0$ replaced by B_0. Moreover, we see such $\bar{B}$ is subject to the inequality

$$|\bar{B}|_{2+\alpha,(2+\alpha)/2} \leqq C_6 |B_0|_{2+\theta}\,.$$

Since such $\bar{B}$ is clearly a weak solution of (P.S.) with the initial data B_0, Proposition 3.1 enables us to assert $B = \bar{B}$. Taking into account the fact that div(rot) is identically equal to zero, we have the desired result.

Now it suffices to prove that $B = \operatorname{rot}\psi$ is a weak solution of (P.S.) with the initial data B_0. Since $\psi|_{\partial\Omega\times(0,T)} = 0$, we have $B\cdot\nu = \operatorname{rot}\psi\cdot\nu = \partial\psi/\partial\tau = 0$ ($\partial/\partial\tau$; tangential derivation) on $\partial\Omega \times (0, T)$ and clearly $B \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_N(\Omega))$.

Concerning that initial condition, we have $\operatorname{rot}\psi(0) = \operatorname{rot}\psi_0 = B_0$. Indeed, the vector function $V := \operatorname{rot}\psi_0 - B_0$ is in $C^{2+\theta}(\bar{\Omega})$ and satisfies $\operatorname{div} V = 0$ in Ω and $V\cdot\nu =$

$\partial\psi_0/\partial\tau - B_0\cdot\nu = 0$ on $\partial\Omega$. Hence by the well-known decomposition theorem of solenoidal vector fields on Ω (see Kato [8, p. 193, (1.13)]), V can be written as $V = \operatorname{rot} G(\operatorname{rot} V) + \nabla p$ for some $p \in C^\infty(\bar{\Omega})$. Moreover, since $\operatorname{rot} V = -\Delta\psi_0 - \operatorname{rot} B_0 = 0$ in Ω by Lemma 3.3, such p must satisfy $\Delta p = 0$ in Ω and $\partial p/\partial\nu = 0$ on $\partial\Omega$. Therefore $p = $const. and $V = 0$, as we wished to show.

Finally, we may show the identity (ii) in Definition 3.1 for B with $\bar{B}_0$ replaced by B_0. It follows from (P.E.) that

$$\int_0^T (\partial_t\psi + \operatorname{rot} B + (u, \nabla)\psi, \operatorname{rot}\Phi)dt = 0 \tag{3.4}$$

for all $\Phi \in C_0^1([0, T); \boldsymbol{H}_N^1(\Omega))$. By integration by parts we get

$$\begin{aligned}\int_0^T (\partial_t\psi, \operatorname{rot}\Phi)dt &= -\int_0^T (\psi, \operatorname{rot}\partial_t\Phi)dt - (\psi(0), \operatorname{rot}\Phi(0)) \\ &= -\int_0^T (\operatorname{rot}\psi, \partial_t\Phi)dt - \int_0^T\int_{\partial\Omega} \psi(\partial_t\Phi\wedge\nu)dSdt - (\operatorname{rot}\psi_0, \Phi(0)) \\ &\quad -\int_{\partial\Omega}\psi_0(\Phi(0)\wedge\nu)dS = -\int_0^T (B, \partial_t\Phi)dt - (B_0, \Phi(0)), \end{aligned} \tag{3.5}$$

$$\int_0^T ((u, \nabla)\psi, \operatorname{rot}\Phi)dt = \int_0^T (\operatorname{rot}((u, \nabla)\psi), \Phi)dt + \int_0^T\int_{\partial\Omega} (u, \nabla)\psi(\Phi\wedge\nu)dSdt.$$

Since $\psi = 0$ on $\partial\Omega$, $\nabla\psi$ is perpendicular to $\partial\Omega$ and hence $(u, \nabla)\psi = 0$ on $\partial\Omega$. Thus the second integrand above is equal to zero. Moreover since $\operatorname{div} u = 0$, we have $\operatorname{rot}((u, \nabla)\psi) = (u, \nabla)\operatorname{rot}\psi - (\operatorname{rot}\psi, \nabla)u$. Therefore

$$\int_0^T ((u, \nabla)\psi, \operatorname{rot}\Phi)dt = \int_0^T ((u, \nabla)B - (B, \nabla)u, \Phi)dt. \tag{3.6}$$

Since $\operatorname{div} B = 0$, it follows from (3.4), (3.5) and (3.6) that $B = \operatorname{rot}\psi$ satisfies the equation which we wished to prove. This completes the proof.

Lemma 3.5 enables us to define a map

$$F_2 : \boldsymbol{C}^{1+\alpha,\alpha/2}(\bar{Q}_T) \to \boldsymbol{C}^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$$

by $B = F_2 u$.

3.3. Vorticity equation. Applying rot to both sides of the first equation of (*), we get

$$\begin{aligned} &\partial_t\omega + (u, \nabla)\omega = (B, \nabla)J + \operatorname{rot} f \quad \text{in} \quad Q_T, \\ &\omega(0) = \omega_0, \end{aligned} \tag{V.E.}$$

where $\omega = \operatorname{rot} u$, $J = \operatorname{rot} B$ and $\omega_0 = \operatorname{rot} u_0$. We shall consider (V.E.) as the initial value problem for ω.

Let u and B be as in the preceding subsections. For a weak solution ω of (V.E.) we give the following definition:

$$(3.7)\qquad \omega(x,t)=\omega_0(U_{0,t}(x))+\int_0^t (B,\nabla)J(U_{s,t}(x),s)ds+\int_0^t \operatorname{rot} f(U_{s,t}(x),s)ds\,,$$

where $U_{s,t}(x)$ is the solution of the initial value problem of the ordinary differential equation

$$dU_{s,t}(x)/ds=u(U_{s,t}(x),s)\,,$$
$$U_{t,t}(x)=x\in\Omega\,.$$

As is well known, if ω_0, $(B,\nabla)J$ and $\operatorname{rot} f$ are in C^1, then ω defined by (3.7) is a classical solution of (V.E.).

REMARK 3.1. (i) Since $u^{(k)}\in C^{1+\mu}(\bar\Omega)$ $(k=1,\cdots,m)$ and since $|\lambda_k(t)|\leqq C_7(\Omega, T,|u|_0,|f|_{0,0},M)$ for all $t\in[0,T]$ $(k=1,\cdots,m)$ (see Kato [8, Lemma 1.4]), it follows from Kato [8, Lemma 2.6] that there are positive constants $C_8=C_8(\Omega,M)$ and $\delta=\delta(\Omega,T,M)$ *independent of N* such that

$$|U_{s,t}(x)-U_{s',t'}(x')|\leqq C_8(|x-x'|^\delta+|s-s'|^\delta+|t-t'|^\delta)$$

for $|x-x'|\leqq 1$, $|s-s'|\leqq 1$, $|t-t'|\leqq 1$.

(ii) There is a positive constant $C_9=C_9(\Omega,T,|u_0|_0,|f|_{0,0},M,N)$ such that

$$|U_{s,t}(x)-U_{s',t'}(x')|\leqq C_9(|x-x'|+|s-s'|+|t-t'|)$$

for $|x-x'|\leqq 1$, $|s-s'|\leqq 1$, $|t-t'|\leqq 1$. In comparison with the inequality in (i), we can choose $\delta=1$, but the constant C_9 may *depend on N*.

Let us show, for example, $|U_{s,t}(x)-U_{s,t}(x')|\leqq C_9|x-x'|$ for $x,x'\in\bar\Omega$ and $0\leqq t\leqq s$. Taking $x(s)=U_{s,t}(x)$ and $x'(s)=U_{s,t}(x')$, we have $|d(x(s)-x'(s))/ds|=|u(x(s),s)-u(x'(s),s)|\leqq|u|_{1,0}|x(s)-x'(s)|$. Hence $|x(s)-x'(s)|\leqq|x-x'|+|u|_{1,0}\int_t^s|x(\tau)-x'(\tau)|d\tau$. By the Gronwall inequality and Lemma 3.2, we get $|x(s)-x'(s)|\leqq e^{|u|_{1,0}T}|x-x'|\leqq C_9|x-x'|$, which implies the desired result when $t=t'$ and $s=s'$. Since the proof in another case is parallel to that of Kato [8, Lemma 2.6, (ii), (iii)], we may omit it.

(iii) For any $\Phi\in C^1(\bar\Omega)$, ω satisfies the identity

$$d/dt(\omega(t),\Phi)=(\omega(t),(u(t),\nabla)\Phi)+((B(t),\nabla)J(t)+\operatorname{rot} f(t),\Phi)\,.$$

LEMMA 3.6. *There are positive constants* $\alpha^*=\alpha^*(\Omega,T,\theta,M)$, $C_{10}=C_{10}(\Omega,T,\theta,M)$ *independent of* N *and* $C_{11}=C_{11}(\Omega,T,\theta,|u_0|_0,|f|_{0,0},M,N)$ *such that* $\omega\in C^{\alpha^*,\alpha^*}(\bar Q_T)$ *and*

$$(3.8)\qquad |\omega|_{0,0}\leqq|u_0|_1+T|f|_{1,0}+C_{11}|B_0|^2_{2+\theta},$$

$$(3.9)\qquad K^{\alpha^*,\alpha^*}(\omega)\leqq C_{10}(|u_0|_{1+\theta}+|f|_{1+\theta,0})+C_{11}|B_0|^2_{2+\theta}\,.$$

PROOF. Since $U_{s,t}(\cdot)$ is a one-to-one measure preserving map of $\bar{\Omega}$ onto itself (see Kato [8, Lemma 2.3]), (3.8) is an immediate consequence of Lemma 3.5. Let ω_1, ω_2 and ω_3 be

$$\omega_1(x, t)=\omega_0(U_{0,t}(x)), \qquad \omega_2(x, t)=\int_0^t \operatorname{rot} f(U_{s,t}(x), s)ds$$

and

$$\omega_3(x, t)=\int_0^t (B, \nabla)J(U_{s,t}(x), s)ds .$$

By Remark 3.1 (i), we get

$$\begin{aligned}|\omega_1(x, t)-\omega_1(x', t')| &\leqq |\omega_0(U_{0,t}(x))-\omega_0(U_{0,t}(x'))|+|\omega_0(U_{0,t}(x'))-\omega_0(U_{0,t'}(x'))| \\ &\leqq |u_0|_{1+\theta}(|U_{0,t}(x)-U_{0,t}(x')|^{\theta}+|U_{0,t}(x')-U_{0,t'}(x')|^{\theta}) \\ &\leqq 2C_8^{\theta}|u_0|_{1+\theta}(|x-x'|^{\theta\delta}+|t-t'|^{\theta\delta}) .\end{aligned}$$

Taking $\alpha^*=\theta\delta$ ($\alpha^*=\alpha^*(\Omega, T, \theta, M)$), we obtain

$$K^{\alpha^*,\alpha^*}(\omega_1)\leqq C_{10}|u_0|_{1+\theta} . \tag{3.10}$$

Similarly it follows that

$$K^{\alpha^*,\alpha^*}(\omega_2)\leqq C_{10}|f|_{1+\theta,0} . \tag{3.11}$$

By Lemma 3.5 with α replaced by α^* and Remark 3.1 (ii), we have for $t>t'$

$$\begin{aligned}|\omega_3(x, t)-\omega_3(x', t')| &\leqq \int_0^t |(B, \nabla)J(U_{s,t}(x), s)-(B, \nabla)J(U_{s,t}(x'), s)|\,ds \\ &\quad +\int_0^t |(B, \nabla)J(U_{s,t}(x'), s)-(B, \nabla)J(U_{s,t'}(x'), s)|\,ds \\ &\quad +\left|\int_{t'}^t (B, \nabla)J(U_{s,t'}(x'), s)\,ds\right| \\ &\leqq C_9^{\alpha^*}\int_0^t |(B, \nabla)J|_{\alpha^*,0}(|x-x'|^{\alpha^*}+|t-t'|^{\alpha^*})ds+|(B, \nabla)J|_{0,0}|t-t'| \\ &\leqq C_9^{\alpha^*}C_6^2(T+1)|B_0|_{2+\theta}^2(|x-x'|^{\alpha^*}+|t-t'|^{\alpha^*}+|t-t'|) .\end{aligned}$$

Hence we get

$$K^{\alpha^*,\alpha^*}(\omega_3)\leqq C_{11}|B_0|_{2+\theta}^2 . \tag{3.12}$$

Then (3.9) follows from (3.10), (3.11) and (3.12). This completes the proof.

Lemma 3.6 enables us to define a map

$$F_3\colon C^{1+\alpha^*,\alpha^*/2}(\bar{Q}_T)\times C^{2+\alpha^*,(2+\alpha^*)/2}(\bar{Q}_T)\to C^{\alpha^*,\alpha^*}(\bar{Q}_T)$$

by $\omega=F_3(u, B)$, where ω is as in (3.7).

3.4. Application of the fixed point theorem. We take two positive numbers M and N and exponent α^* as follows:

$$M>|u_0|_1+T|f|_{1,0}\,,\qquad N>C_{10}(\Omega, T, \theta, M)(|u_0|_{1+\theta}+|f|_{1+\theta,0})\,,$$
$$\alpha^*=\alpha^*(\Omega, T, \theta, M)\,,$$

where C_{10} and α^* are as in Lemma 3.6. For such M, N and α^*, we define a subset $S_{\alpha^*}(M, N)$ of continuous functions on $\bar{Q}_T$ as in Subsection 3.1. Clearly $S_{\alpha^*}(M, N)$ is a compact convex subset in the Banach space $C(\bar{Q}_T)$. Moreover, we define a map F on $S_{\alpha^*}(M, N)$ by

$$F\phi=F_3(F_1\phi, F_2(F_1\phi))\qquad \text{for}\quad \phi\in S_{\alpha^*}(M, N)$$

with α replaced by α^* in the context of the preceding subsections. Then it follows from Lemmas 3.2, 3.5 and 3.6 that F maps $S_{\alpha^*}(M, N)$ into $C^{\alpha^*,\alpha^*}(\bar{Q}_T)$. More precisely, by (3.8) and (3.9) we have the following:

LEMMA 3.7. *There are two numbers* $M=M(\Omega, T, |u_0|_1, |f|_{1,0})$ *and* $N=N(\Omega, T, |u_0|_{1+\theta}, |f|_{1+\theta,0})$, *positive exponent* $\alpha^*=\alpha^*(\Omega, T, |u_0|_1, |f|_{1,0})$ *and constant* $C_*=C_*(\Omega, T, |u_0|_{1+\theta}, |f|_{1+\theta,0})$ *such that if* $|B_0|_{2+\theta}\leqq C_*$, *then* F *maps* $S_{\alpha^*}(M, N)$ *into itself.*

In order to apply the Schauder fixed point theorem, we need:

LEMMA 3.8. *Under the condition of Lemma* 3.7, F *is continuous on* $S_{\alpha^*}(M, N)$ *with respect to the topology of* $C(\bar{Q}_T)$.

PROOF. Let $\phi_n, \phi\in S_{\alpha^*}(M, N)$, $n=1, 2, \cdots$ and $|\phi_n-\phi|_{0,0}\to 0$ as $n\to\infty$. Let $u_n=F_1\phi_n$, $u=F_1\phi$, $B_n=F_2u_n$, $B=F_2u$, $\omega_n=F_3(u_n, B_n)$, $\omega=F_3(u, B)$ and let $U^n_{s,t}(x)$ and $U_{s,t}(x)$ be the solutions of $dU^n_{s,t}(x)/ds=u_n(U^n_{s,t}(x), s)$, $U^n_{t,t}(x)=x$ and $dU_{s,t}(x)/ds=u(U_{s,t}(x), s)$, $U_{t,t}(x)=x$, respectively. Since $u_n-u=\operatorname{rot} G(\phi_n-\phi)-\sum_{k=1}^m(\int_{S_k}\operatorname{rot} G(\phi_n-\phi)\cdot\tau\, dS)u^{(k)}$ (for $u^{(k)}$, $k=1, \cdots, m$, see Lemma 3.1), we see by Kato [8, Lemma 1.4] that $|u_n-u|_{0,0}\to 0$. Then it follows from a general theory for ordinary differential equations that $U^n_{s,t}(x)\to U_{s,t}(x)$ uniformly in $x\in\bar{\Omega}$, $s, t\in[0, T]$. Hence by (3.7), it suffices to prove that

$$|\partial_x^\gamma B_n-\partial_x^\gamma B|_{0,0}\to 0\qquad \text{for}\quad |\gamma|\leqq 2\,. \tag{3.13}$$

We shall first prove that $B_n\to B$ uniformly in $\bar{Q}_T$. Let ψ_n and ψ be the scalar potentials of B_n and B defined as in Lemmas 3.4 and 3.5, respectively. Then we have

$$\partial_t\Psi_n-\Delta\Psi_n+(u_n, \nabla)\Psi_n+((u_n-u), \nabla)\psi=0\quad \text{in}\quad Q_T\,,$$
$$\Psi_n=0\qquad \text{on}\quad \partial\Omega\times(0, T)\,,$$

$$\Psi_n|_{t=0}=0\,,$$

where $\Psi_n=\psi_n-\psi$. Hence Ψ_n can be written as

$$\Psi_n(x,t)=-\int_0^t d\sigma\int_\Omega E(x,y,t-\sigma)\{(u_n,\nabla)\Psi_n(y,\sigma)+((u_n-u),\nabla)\psi(y,\sigma)\}dy\,,$$

where $E(x, y, t)$ is the fundamental solution of $\partial_t-\Delta$ with zero Dirichlet condition on $\partial\Omega$. Hence it follows from a well-known property of the fundamental solution (see, e.g., Friedman [4]) that

$$\begin{aligned}|\nabla\Psi_n(x,t)|&\leqq\int_0^t d\sigma\int_\Omega|\nabla_x E(x,y,t-\sigma)|\{|u_n(y,\sigma)||\nabla_y\Psi_n(y,\sigma)|\\&\qquad+|u_n(y,\sigma)-u(y,\sigma)||\nabla_y\psi(y,\sigma)|\}dy\\&\leqq C(T)\left\{|u_n|_{0,0}\int_0^t(t-\sigma)^{-1/2}|\nabla\Psi_n(\cdot,\sigma)|_0\,d\sigma+|\nabla\psi|_{0,0}|u_n-u|_{0,0}\right\}.\end{aligned}$$

Using Gronwall's technique, we get

$$\begin{aligned}|\nabla\Psi_n(\cdot,t)|_0&\leqq C(T)|\nabla\psi|_{0,0}|u_n-u|_{0,0}\exp\left(C(T)|u_n|_{0,0}\int_0^t(t-\sigma)^{-1/2}d\sigma\right)\\&\leqq C(T)\exp(2T^{1/2}C(T)|u_n|_{0,0})|\nabla\psi|_{0,0}|u_n-u|_{0,0}\end{aligned}$$

for all $t\in[0, T]$ and hence

$$|\nabla\Psi_n|_{0,0}\leqq C\exp(C|u_n|_{0,0})|\nabla\psi|_{0,0}|u_n-u|_{0,0}\,, \tag{3.14}$$

where C is a positive constant independent of n. Since $u_n\to u$ uniformly in $\bar{Q}_T$, we obtain from (3.14) that $|B_n-B|_{0,0}\to 0$. Moreover by the a priori estimate in Lemma 3.5, the sequence $\{B_n\}_{n=1}^\infty$ is precompact in $C^{2,1}(\bar{Q}_T)$. Hence every sequence in turn has a convergent subsequence with the limit B. Therefore the sequence $\{B_n\}_{n=1}^\infty$ itself converges to B in $C^{2,1}(\bar{Q}_T)$ and (3.13) follows. This completes the proof.

It follows from Lemmas 3.7, 3.8 and the Schauder fixed point theorem that under the condition of Lemma 3.7, there exists $\omega\in S_{\alpha^*}(M, N)$ such that $F\omega=\omega$.

3.5. Proof of Theorem 2. Let ω be the fixed point of the map F constructed in the preceding subsection. Here we shall show that the pair $u=F_1\omega$, $B=F_2(F_1\omega)$ and some scalar function π is the classical solution of $(*)$ stated in Theorem 2.

Concerning the regularity of u, we see by Kato [8, Lemmas 3.1 and 3.2] and Remark 3.1 (iii) that u, $\partial_x u$ and $\partial_t u$ are in $C(\bar{Q}_T)$. To show the existence of pressure π, we need:

Lemma 3.9. *Let v be a vector-valued function of class $C^{k,q}(\bar{Q}_T)$ $(k\geqq 0,\ q\geqq 0)$ satisfying*

$$\int_{S_j} v\cdot\tau dS=0\ (j=1,\cdots,m),\qquad \int_\Omega v\cdot\operatorname{rot}\phi\,dx=0\qquad \textit{for any}\quad \phi\in C_0^\infty(\Omega).$$

Then there exists a scalar function $\pi\in C^{k+1,q}(\bar{Q}_T)$ *such that* $v=-\nabla\pi$.

This may be regarded as a generalization of the Poincaré lemma. For the proof, see Kikuchi [9, Lemma 2.13].

LEMMA 3.10 (PROOF OF THEOREM 2). *Under the condition of Lemma* 3.7, *there exists a scalar function* $\pi\in C^{1,0}(\bar{Q}_T)$ *such that the triple* $\{u, B, \pi\}$ *is the unique solution of* (∗) *stated in Theorem* 2.

PROOF. Let $v=\partial_t u+(u,\nabla)u-(B,\nabla)B+\nabla((1/2)|B|^2)-f$. Since

$$\int_{S_j}(w,\nabla)w\cdot\tau\,dS=\int_{S_j}\nabla((1/2)|w|^2)\cdot\tau dS=0\qquad (j=1,\cdots,m)$$

for all $w\in C^1(\bar{\Omega})$ with $\operatorname{div} w=0$ and $w\cdot\tau=0$ on $\partial\Omega$, we have by Lemma 3.1 and (3.1) that

$$\int_{S_j} v\cdot\tau dS=\int_{S_j}(\partial_t u-f)\cdot\tau dS=0\qquad (j=1,\cdots,m).$$

Moreover since $\operatorname{rot} u=-\Delta G\omega=\omega$ by Lemma 3.1 (i) and since $(\operatorname{rot} u, (u,\nabla)\phi)=-((u,\nabla)u, \operatorname{rot}\phi)$ for all $\phi\in C_0^\infty(\Omega)$, we obtain from Remark 3.1 (iii)

$$\int_\Omega v\cdot\operatorname{rot}\phi\,dx=0\qquad \text{for any}\quad \phi\in C_0^\infty(\Omega).$$

Hence by Lemma 3.9, there exists a scalar function $\pi\in C^{1,0}(\bar{Q}_T)$ such that $v=-\nabla\pi$.

To prove that $\{u, B, \pi\}$ is the desired solution, it remains to show that $u|_{t=0}=u_0$. Set $w=u|_{t=0}-u_0$. Then it follows from (3.1) and (3.7) that

$$\operatorname{rot} w=\operatorname{rot} u|_{t=0}-\operatorname{rot} u_0=\omega(\cdot,0)-\omega_0=0,$$

$$\int_{S_j} w\cdot\tau dS=\int_{S_j}\operatorname{rot} Gw(\cdot,0)\cdot\tau dS+\lambda_j(0)-\int_{S_j} u_0\cdot\tau dS=0\qquad (j=1,\cdots,m).$$

Therefore by Lemma 3.9, we have $w=\nabla\eta$ for some $\eta\in C^2(\bar{\Omega})$. Since $\operatorname{div} w=0$ in Ω and $w\cdot\nu=0$ on $\partial\Omega$, such η must satisfy $\Delta\eta=0$ in Ω and $\partial\eta/\partial\nu=0$ on $\partial\Omega$. Hence $\eta=$const. and $w=0$. This completes the proof.

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