# Weak and Strong Convergence of Generalized Proximal Point Algorithms with Relaxed Parameters

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#### Abstract

In this work, we propose and study a framework of generalized proximal point algorithms associated with a maximally monotone operator. We indicate sufficient conditions on the regularization and relaxation parameters of generalized proximal point algorithms for the equivalence of the boundedness of the sequence of iterations generated by this algorithm and the non-emptiness of the zero set of the maximally monotone operator, and for the weak and strong convergence of the algorithm. Our results cover or improve many results on generalized proximal point algorithms in our references. Improvements of our results are illustrated by comparing our results with related known ones.

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## 1 Introduction

Throughout this paper,

 $\mathcal{H}$  is a real Hilbert space,

with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ . Moreover, we assume that  $\mathcal{H} \neq \{0\}$  and that  $m \in \mathbb{N} \setminus \{0\}$ , where  $\mathbb{N} = \{0, 1, 2, \ldots\}$ .

In 1976, Rockafellar, in the seminal work [9], generalized the proximal point algorithm for minimizing lower semicontinuous proper convex functions by weakening the exact minimization at each iteration and by replacing the subgradient mapping with an arbitrary maximally monotone operator. In particular, Rockafellar's proximal point algorithm solves the fundamental problem:

determine an element 
$$x \in \mathcal{H}$$
 s.t.  $0 \in A(x)$ , where  $A : \mathcal{H} \to 2^{\mathcal{H}}$  is maximally monotone, (1.1)

which includes minimization problems subject to implicit constraints, variational inequality problems, and minimax problems as special cases (see [9] and the references therein for details). For example, given a proper lower semicontinuous and convex function f, it is well-known that  $\partial f$  is maximally monotone (see, e.g., [1, Theorem 20.25]), that if there exist a closed convex subset C of  $\mathcal{H}$  and  $\xi \in \mathbb{R}$  such that the set  $\{x \in C : f(x) \leq \xi\}$  is nonempty and bounded, then  $\operatorname{zer} \partial f \neq \emptyset$  (see, e.g., [1, Theorems 11.10 and 16.3] for details) and solving (1.1) with  $A = \partial f$  is equivalent to finding the minimizer of f.

Synthesizing the work of Rockafellar [9] with that of Gol'shtein and Tret'yakov [5], Eckstein and Bertsekas in [4] proposed a generalized form of the proximal point algorithm and elaborated that the

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Douglas-Rachford splitting algorithm is a special case of the proximal point algorithm. In addition, because generally the proximal point algorithm converges weakly but not strongly (see, e.g., [6] for details), various modified proximal point algorithms were studied in many articles (see, e.g., [2], [3], [4], [8], [10], [13], [14], [15], and [16]) to obtain the strong convergence.

Henceforth,

 $A: \mathcal{H} \to 2^{\mathcal{H}}$  is maximally monotone.

Then, via [1, Proposition 20.22],  $(\forall \gamma \in \mathbb{R}_{++}) \gamma A$  is maximally monotone. In the whole work, given a point  $u \in \mathcal{H}$ , we investigate the sequence  $(x_k)_{k \in \mathbb{N}}$  of iterations generated by the *generalized proximal point algorithm with relaxed parameters*:

$$(\forall k \in \mathbb{N}) \quad x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k \mathbf{J}_{c_k A}(x_k) + \delta_k e_k, \tag{1.2}$$

where  $x_0 \in \mathcal{H}$  is the *initial point* and  $(\forall k \in \mathbb{N}) e_k \in \mathcal{H}$  is the *error term*,  $c_k \in \mathbb{R}_{++}$  is the *stepsize* or *regularization parameter*, and  $\alpha_k$ ,  $\beta_k$ ,  $\gamma_k$ , and  $\delta_k$  are the *relaxation parameters* in  $\mathbb{R}$ . For simplicity, in this work, we refer to generalized proximal point algorithms with relaxed parameters as generalized proximal point algorithms.

We compare the scheme (1.2) with some known proximal point algorithms in the literatures below.

- (i) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k = \beta_k \equiv 0 \text{ and } \gamma_k = \delta_k \equiv 1$ . Then (1.2) reduces to the proximal point algorithm devised by Rockafellar in [9].
- (ii) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0, \ \gamma_k \in [0, 2]$ , and  $\beta_k = 1 \gamma_k$ . Then (1.2) turns to the generalized proximal point algorithm developed by Eckstein and Bertsekas in [4].
- (iii) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k = \delta_k \equiv 0, \ \gamma_k \equiv \gamma$ , and  $\beta_k \equiv 1 \gamma$  where  $\gamma \in \mathbb{R}_{++}$ . Then (1.2) reduces to the generalized proximal point algorithm scheme proposed by Corman and Yuan in [3]. In particular, Corman and Yuan provided examples where the generalized proximal point algorithm scheme with  $\gamma > 2$  converges faster than that with  $\gamma \in [0, 2]$ .
- (iv) Suppose that  $u = x_0$  and  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0,1], \ \beta_k \equiv 0$ , and  $\gamma_k = \delta_k = 1 \alpha_k$ , or that  $u = x_0$  and  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0, \ \beta_k \in [0,1]$ , and  $\gamma_k = \delta_k = 1 \beta_k$ . Then (1.2) becomes the modified proximal point algorithms introduced by Xu in [13].
- (v) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0, 1], \ \beta_k \equiv 0, \ \gamma_k = 1 \alpha_k$ , and  $\delta_k \equiv 1$ . Then (1.2) turns to the contraction-proximal point algorithm introduced by Marino and Xu in [8]. Note that by some natural substitution one can easily see that the regularization method for the proximal point algorithm proposed by Xu in [14] is equivalent to the contraction-proximal point algorithm and hence a special case of the scheme (1.2) as well. Moreover, as it is verified in [14], the prox-Tikhonov algorithm of Lehdili and Moudafi [7] deals essentially with a special case of the regularization method for the proximal point algorithm of Xu in [14], which in turn shows that (1.2) also covers the prox-Tikhonov algorithm in [7].
- (vi) Suppose that  $(\forall k \in \mathbb{N}) \ \delta_k \equiv 1$  and  $\{\alpha_k, \beta_k, \gamma_k\} \subseteq ]0, 1[$  with  $\alpha_k + \beta_k + \gamma_k = 1$ . Then (1.2) deduces the contraction proximal point algorithm proposed by Yao and Noor in [15].
- (vii) Suppose that u = 0 and that  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0$  and  $\{\beta_k, \gamma_k, \delta_k\} \subseteq [0, 1[$  with  $\beta_k + \gamma_k + \delta_k = 1$ . Then (1.2) becomes the proximal point algorithm with general errors constructed by Yao and Shahzad in [16].

For the generalized proximal point algorithm conforming the recursion (1.2), the advantage of considering the range  $\mathbb{R}$  of the parameters  $(\alpha_k)_{k \in \mathbb{N}}$ ,  $(\beta_k)_{k \in \mathbb{N}}$ ,  $(\gamma_k)_{k \in \mathbb{N}}$ , and  $(\delta_k)_{k \in \mathbb{N}}$  is suggested by [3]; the

necessity of the consideration of the coefficient  $(\delta_k)_{k \in \mathbb{N}}$  preceding the error terms is illustrated by [13] and [16]; and the term *u* is motivated by [13] and [8].

The goal of this work is to explore the equivalence of the boundedness of  $(x_k)_{k\in\mathbb{N}}$  generated by (1.2) and  $\operatorname{zer} A \neq \emptyset$  and to deduce sufficient conditions for the weak and strong convergence of the scheme (1.2) for solving (1.1) when  $\operatorname{zer} A \neq \emptyset$ .

Main results of this work are the following.

- **R1:** Theorems 3.11 and 3.12 present requirements on the regularization and relaxation parameters of (1.2) for the equivalence of the non-emptiness of zer *A* and the boundedness of the sequence of iterations generated by the scheme (1.2).
- **R2:** The weak convergence of the generalized proximal point algorithms is illustrated in Theorem 4.1.
- **R3:** Theorems 4.4 and 4.5 exhibit sufficient conditions for the strong convergence of the sequence of iterations conforming the scheme (1.2).

In Remarks 4.2 and 4.7 below, we shall compare our convergence results with related known results in references mentioned above and demonstrate our improvements.

The paper is organized as follows. In Section 2, we provide some fundamental and essential results for proving the convergence of generalized proximal point algorithms. The boundedness and asymptotic regularity of the sequence of iterations generated by the generalized proximal point algorithm is elaborated in Section 3. The equivalence of the boundedness of this sequence and zer  $A \neq \emptyset$  is also established in Section 3. Convergence results are exhibited in the last section, Section 4.

We now turn to the notation used in this work. Id stands for the *identity mapping*. Denote by  $\mathbb{R}_+ := \{\lambda \in \mathbb{R} : \lambda \ge 0\}$  and  $\mathbb{R}_{++} := \{\lambda \in \mathbb{R} : \lambda > 0\}$ . Let  $\bar{x}$  be in  $\mathcal{H}$ , let  $r \in \mathbb{R}_+$ , and let  $(x_k)_{k \in \mathbb{N}}$ . be a sequence in  $\mathcal{H}$ .  $B(\bar{x};r) := \{y \in \mathcal{H} : \|y - \bar{x}\| < r\}$  and  $B[\bar{x};r] := \{y \in \mathcal{H} : \|y - \bar{x}\| \le r\}$  are the open and closed ball centered at  $\bar{x}$  with radius r, respectively. If  $(x_k)_{k \in \mathbb{N}}$  converges strongly to  $\bar{x}$ , then we denote by  $x_k \to \bar{x}$ .  $(x_k)_{k \in \mathbb{N}}$  converges weakly to  $\bar{x}$  if, for every  $y \in \mathcal{H}$ ,  $\langle x_k, y \rangle \to \langle x, y \rangle$ ; in symbols,  $x_k \rightarrow \bar{x}$ . Let C be a nonempty closed convex subset of  $\mathcal{H}$ . The projector (or projection operator) onto C is the operator, denoted by  $P_C$ , that maps every point in  $\mathcal{H}$  to its unique projection onto C.  $\iota_C$  is the *indicator function of C*, that is,  $(\forall x \in C) \iota_C(x) = 0$  and  $(\forall x \in H \setminus C) \iota_C(x) = \infty$ . Let  $f : H \to ]-\infty, \infty]$ be proper, i.e., dom  $f \neq \emptyset$ . The subdifferential of f is the set-valued operator  $\partial f : \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto$  $\{z \in \mathcal{H} : (\forall y \in \mathcal{H}) \langle z, y - x \rangle \leq f(y) - f(x)\}$ . Let  $\mathcal{D}$  be a nonempty subset of  $\mathcal{H}$  and let  $T : \mathcal{D} \to \mathcal{H}$ . Fix  $T := \{x \in \mathcal{D} : x = T(x)\}$  is the set of fixed points of T. Let  $G : \mathcal{H} \to 2^{\mathcal{H}}$  be a set-valued operator. Then G is characterized by its graph gra  $G := \{(x, w) \in \mathcal{H} \times \mathcal{H} : w \in G(x)\}$ . The inverse of G, denoted by  $G^{-1}$ , is defined through its graph gra  $G^{-1} := \{(w, x) \in \mathcal{H} \times \mathcal{H} : (x, w) \in \text{gra } G\}$ . The set of zeros of G is zer  $G := G^{-1}(0) = \{x \in \mathcal{H} : 0 \in G(x)\}$ . G is monotone if  $(\forall (x, u) \in \operatorname{gra} G)$  $(\forall (y, v) \in \operatorname{gra} G) \langle x - y, u - v \rangle \geq 0$ . *G* is *maximally monotone* if there exists no monotone operator  $B: \mathcal{H} \to 2^{\mathcal{H}}$  such that gra *B* properly contains gra *G*, i.e., for every  $(x, u) \in \mathcal{H} \times \mathcal{H}$ ,  $(x, u) \in \text{gra } G$  if and only if  $(\forall (y, v) \in \operatorname{gra} G) \langle x - y, u - v \rangle \ge 0$ .

For other notation not explicitly defined here, we refer the reader to [1].

### 2 Preliminaries

In order to facilitate our investigation in the following sections, we gather some auxiliary results in this section. The ideas of these results are frequently used in proofs of the convergence of generalized proximal point algorithms in references of this work. Clearly, results in this section are interesting in their own right and are helpful to study various generalized proximal point algorithms.

### Limits of sequences

**Fact 2.1.** [13, Lemma 2.5] Let  $(s_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{R}_+$  satisfying

$$(\forall k \in \mathbb{N}) \quad s_{k+1} \leq (1-a_k)s_k + a_kb_k + \epsilon_k,$$

where  $(a_k)_{k \in \mathbb{N}}$ ,  $(b_k)_{k \in \mathbb{N}}$ , and  $(\epsilon_k)_{k \in \mathbb{N}}$  are sequences in  $\mathbb{R}$  satisfying the conditions:

- (i)  $(a_k)_{k\in\mathbb{K}}$  is a sequence in [0, 1] such that  $\sum_{k\in\mathbb{N}} a_k = \infty$ , or equivalently,  $\prod_{k\in\mathbb{N}} (1-a_k) = 0$ ;
- (ii)  $\limsup_{k\to\infty} b_k \leq 0$ ;
- (iii)  $(\forall k \in \mathbb{N}) \epsilon_k \in \mathbb{R}_+ \text{ and } \sum_{k \in \mathbb{N}} \epsilon_k < \infty.$

Then  $\lim_{k\to\infty} s_k = 0$ .

Inspired by the proof of Fact 2.1, we obtain the following Proposition 2.3, which is critical to some results in the next sections. It is not difficult to prove that Proposition 2.3(iii) is actually equivalent to Fact 2.1. We present Proposition 2.3(iii) because comparing with Fact 2.1, Proposition 2.3(iii) is more convenient to use. The following lemma is necessary to prove Proposition 2.3.

**Lemma 2.2.** Let  $(\alpha_k)_{k \in \mathbb{N}}$  be in  $\mathbb{R}_+$  and let  $(\beta_k)_{k \in \mathbb{N}}$  be in  $\mathbb{R}$  such that  $(\forall k \in \mathbb{N}) \alpha_k + \beta_k \leq 1$ . Then

$$(\forall m \in \mathbb{N})(\forall k \in \mathbb{N}) \quad \sum_{i=m}^{m+k} \prod_{j=i+1}^{m+k} \alpha_j \beta_i \le 1 - \prod_{i=m}^{m+k} \alpha_i.$$
(2.1)

Consequently,  $(\forall m \in \mathbb{N})(\forall k \in \mathbb{N}) \sum_{i=m}^{m+k} \prod_{j=i+1}^{m+k} \alpha_j (1-\alpha_i) \leq 1 - \prod_{i=m}^{m+k} \alpha_i$ .

*Proof.* Let  $m \in \mathbb{N}$ . If k = 0, then (2.1) turns to  $\beta_m \leq 1 - \alpha_m$ , which is true by assumption.<sup>1</sup> Suppose that (2.1) holds for some  $k \in \mathbb{N}$ . Then apply the induction hypothesis in the first inequality below to observe that

$$\sum_{i=m}^{m+k+1} \prod_{j=i+1}^{m+k+1} \alpha_j \beta_i = \beta_{m+k+1} + \alpha_{m+k+1} \sum_{i=m}^{m+k} \prod_{j=i+1}^{m+k} \alpha_j \beta_i \le \beta_{m+k+1} + \alpha_{m+k+1} \left( 1 - \prod_{i=m}^{m+k} \alpha_i \right) \le 1 - \prod_{i=m}^{m+k+1} \alpha_i.$$

So, we proved (2.1) by induction. The last assertion is clear with  $(\forall k \in \mathbb{N}) \beta_k = 1 - \alpha_k$ .

**Proposition 2.3.** Let  $(t_k)_{k \in \mathbb{N}}$  and  $(\alpha_k)_{k \in \mathbb{N}}$  be sequences in  $\mathbb{R}_+$ , and let  $(\beta_k)_{k \in \mathbb{N}}$ ,  $(\gamma_k)_{k \in \mathbb{N}}$ , and  $(\omega_k)_{k \in \mathbb{N}}$  be sequences in  $\mathbb{R}$  such that

$$(\forall k \in \mathbb{N}) \quad t_{k+1} \le \alpha_k t_k + \beta_k \omega_k + \gamma_k. \tag{2.2}$$

The following statements hold.

- (i) Suppose that  $\limsup_{k\to\infty} \alpha_k < 1$  and  $M := \sup_{k\in\mathbb{N}} (\beta_k \omega_k + \gamma_k) < \infty$ . Then  $(t_k)_{k\in\mathbb{N}}$  is bounded.
- (ii) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0,1], \ \alpha_k + \beta_k \leq 1, \ and \ \omega_k \in \mathbb{R}_+, \ that \ \hat{\omega} := \sup_{k \in \mathbb{N}} \omega_k < \infty, \ and \ that (\forall k \in \mathbb{N}) \ \alpha_k + \gamma_k \leq 1 \ or \ \sum_{k \in \mathbb{N}} |\gamma_k| < \infty. \ Then \ (t_k)_{k \in \mathbb{N}} \ is \ bounded.$
- (iii) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0,1] \ and \ \beta_k \in [0,1] \ with \ \alpha_k + \beta_k \leq 1 \ and \ \sum_{k \in \mathbb{N}} (1-\alpha_k) = \infty, \ that \lim \sup_{k \to \infty} \omega_k \leq 0, \ and \ that \ \sum_{k \in \mathbb{N}} |\gamma_k| < \infty. \ Then \ \lim_{k \to \infty} t_k = 0.$
- (iv) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0, 1[ and \ \beta_k \in \mathbb{R}_+ with \sup_{k \in \mathbb{N}} \frac{\beta_k}{1 \alpha_k} < \infty and \sum_{k \in \mathbb{N}} (1 \alpha_k) = \infty$ , that  $\limsup_{k \to \infty} \omega_k \leq 0$ , and that  $\sum_{k \in \mathbb{N}} |\gamma_k| < \infty$ . Then  $\lim_{k \to \infty} t_k = 0$ .

<sup>&</sup>lt;sup>1</sup>As is the custom, in the whole work, we use the empty sum convention and empty product convention, that is, given a sequence  $(t_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}$ , for every *m* and *n* in  $\mathbb{N}$  with m > n, we have  $\sum_{i=m}^{n} t_i = 0$  and  $\prod_{i=m}^{n} t_i = 1$ .

*Proof.* Based on (2.2), by induction, it is easy to get that

$$(\forall m \in \mathbb{N})(\forall k \in \mathbb{N}) \quad t_{m+k} \le \prod_{j=m}^{m+k-1} \alpha_j t_m + \sum_{i=m}^{m+k-1} \prod_{j=i+1}^{m+k-1} \alpha_j \beta_i \omega_i + \sum_{i=m}^{m+k-1} \prod_{j=i+1}^{m+k-1} \alpha_j \gamma_i.$$
(2.3)

(i): Because  $\limsup_{k\to\infty} \alpha_k < 1$ , there exists  $\hat{\alpha} \in \mathbb{R}_{++}$  and  $N \in \mathbb{N}$  such that  $\limsup_{k\to\infty} \alpha_k < \hat{\alpha} < 1$ and  $(\forall k \ge N) \ \alpha_k \le \hat{\alpha}$ . This and the assumption that  $(\forall k \in \mathbb{N}) \ \alpha_k \in \mathbb{R}_+$  ensure that

$$(\forall k \in \mathbb{N}) \quad \sum_{i=N}^{N+k} \prod_{j=i+1}^{N+k} \alpha_j \le \sum_{i=N}^{N+k} \prod_{j=i+1}^{N+k} \hat{\alpha} = \sum_{i=N}^{N+k} \hat{\alpha}^{N+k-i} = \sum_{j=0}^k \hat{\alpha}^j \le (1-\hat{\alpha})^{-1},$$

which, combining with (2.3), entails that

$$\begin{aligned} (\forall k \in \mathbb{N}) \quad t_{N+k+1} &\leq \prod_{j=N}^{N+k} \alpha_j t_N + \sum_{i=N}^{N+k} \prod_{j=i+1}^{N+k} \alpha_j \left(\beta_i \omega_i + \gamma_i\right) \\ &\leq t_N + (1-\hat{\alpha})^{-1} \max\{M, 0\} < \infty. \end{aligned}$$

(ii): In view of (2.3),

$$(\forall k \in \mathbb{N}) \quad t_{k+1} \le \prod_{j=0}^k \alpha_j t_0 + \sum_{i=0}^k \prod_{j=i+1}^k \alpha_j \beta_i \omega_i + \sum_{i=0}^k \prod_{j=i+1}^k \alpha_j \gamma_i.$$
(2.4)

Because  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0, 1]$ , we know that  $(\forall k \in \mathbb{N}) \prod_{i=0}^k \alpha_i \in [0, 1]$  and  $1 - \prod_{i=0}^k \alpha_i \in [0, 1]$ . Then combine Lemma 2.2 with the assumption to get that

$$(\forall k \in \mathbb{N}) \quad \sum_{i=0}^{k} \prod_{j=i+1}^{k} \alpha_{j} \beta_{i} \omega_{i} \le \left(1 - \prod_{i=0}^{k} \alpha_{i}\right) \hat{\omega} \le \hat{\omega}.$$
(2.5)

If  $(\forall k \in \mathbb{N}) \alpha_k + \gamma_k \leq 1$ , then by Lemma 2.2,  $(\forall k \in \mathbb{N}) \sum_{i=0}^k \prod_{j=i+1}^k \alpha_j \gamma_i \leq 1 - \prod_{i=0}^k \alpha_i \leq 1$ , which, combining with (2.4) and (2.5), forces that  $(\forall k \in \mathbb{N}) t_{k+1} \leq t_0 + \hat{\omega} + 1 < \infty$ .

On the other hand, if  $\sum_{i \in \mathbb{N}} |\gamma_i| < \infty$ , then  $(\forall k \in \mathbb{N}) \sum_{i=0}^k \prod_{j=i+1}^k \alpha_j \gamma_i \le \sum_{i=0}^k |\gamma_i| \le \sum_{i \in \mathbb{N}} |\gamma_i| < \infty$ . Combine this with (2.4) and (2.5) to get that  $(\forall k \in \mathbb{N}) t_{k+1} \le t_0 + \hat{\omega} + \sum_{i \in \mathbb{N}} |\gamma_i| < \infty$ .

Hence, in both cases,  $(t_k)_{k \in \mathbb{N}}$  is bounded.

(iii): Let  $\epsilon \in \mathbb{R}_{++}$ . Because  $\limsup_{k \to \infty} \omega_k \leq 0$  and  $\sum_{k \in \mathbb{N}} |\gamma_k| < \infty$ , there exists  $N \in \mathbb{N}$  such that

$$(\forall k \ge N) \quad \omega_k \le \epsilon \quad \text{and} \quad \sum_{i=k}^{\infty} |\gamma_i| < \epsilon.$$
 (2.6)

Taking (2.3) and Lemma 2.2 into account, we establish that

$$\begin{aligned} (\forall k \in \mathbb{N} \setminus \{0\}) \quad t_{N+k} &\leq \prod_{j=N}^{N+k-1} \alpha_j t_N + \sum_{i=N}^{N+k-1} \prod_{j=i+1}^{N+k-1} \alpha_j \beta_i \omega_i + \sum_{i=N}^{N+k-1} \prod_{j=i+1}^{N+k-1} \alpha_j \gamma_i \\ &\leq \prod_{j=N}^{N+k-1} \alpha_j t_N + \left(1 - \prod_{j=N}^{N+k-1} \alpha_j\right) \epsilon + \sum_{i=N}^{N+k-1} |\gamma_i| \\ &\leq \prod_{j=N}^{N+k-1} \alpha_j t_N + \epsilon + \epsilon, \end{aligned}$$

which implies that  $\limsup_{k\to\infty} t_k \leq 2\epsilon$ , since  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0,1]$  and  $\sum_{i\in\mathbb{N}}(1-\alpha_i) = \infty$  imply that  $\prod_{k\in\mathbb{N}} \alpha_k = 0$  and that  $\lim_{k\to\infty} \prod_{j=N}^{N+k-1} \alpha_j = 0$ . Because  $\epsilon \in \mathbb{R}_{++}$  is chosen arbitrarily, and  $(t_k)_{k\in\mathbb{N}}$  is in  $\mathbb{R}_+$ , we obtain that  $\lim_{k\to\infty} t_k = 0$ .

(iv): Because  $(\forall k \in \mathbb{N}) \xrightarrow{\beta_k}{1-\alpha_k} \in \mathbb{R}_+$  with  $\sup_{k \in \mathbb{N}} \frac{\beta_k}{1-\alpha_k} < \infty$  and  $\limsup_{k \to \infty} \omega_k \le 0$ , it is easy to prove that  $\limsup_{k \to \infty} \frac{\beta_k}{1-\alpha_k} \omega_k \le 0$ . Moreover, inasmuch as (2.2),

$$(\forall k \in \mathbb{N}) \quad t_{k+1} \leq \alpha_k t_k + \beta_k \omega_k + \gamma_k \leq \alpha_k t_k + (1 - \alpha_k) \frac{\beta_k}{1 - \alpha_k} \omega_k + |\gamma_k|.$$

So the required result follows easily from Fact 2.1.

**Fact 2.4.** [11, Lemma 2.2] Let  $(u_k)_{k \in \mathbb{N}}$  and  $(v_k)_{k \in \mathbb{N}}$  be bounded sequences in  $\mathcal{H}$  and let  $(\alpha_k)_{k \in \mathbb{K}}$  be a sequence in [0, 1] with  $0 < \liminf_{k \to \infty} \alpha_k \le \limsup_{k \to \infty} \alpha_k < 1$ . Suppose that

$$(\forall k \in \mathbb{N}) \ u_{k+1} = \alpha_k v_k + (1 - \alpha_k) u_k \quad and \quad \limsup_{i \to \infty} (\|v_{i+1} - v_i\| - \|u_{i+1} - u_i\|) \le 0.$$

Then  $\lim_{k\to\infty} \|v_k - u_k\| = 0.$ 

The existence of the limit  $\lim_{k\to\infty} a_k$  in the following Fact 2.5 was directly used in proofs of [4], [8], [9], [13] and many other papers on the convergence of proximal point algorithms. For completeness, we present a detailed proof below.

**Fact 2.5.** Let  $(a_k)_{k\in\mathbb{N}}$  and  $(b_k)_{k\in\mathbb{N}}$  be sequences in  $\mathbb{R}_+$  such that  $\sum_{k\in\mathbb{N}} b_k < \infty$  and

$$(\forall k \in \mathbb{N}) \quad a_{k+1} \le a_k + b_k. \tag{2.7}$$

*Then*  $\lim_{k\to\infty} a_k = \liminf_{k\to\infty} a_k \in \mathbb{R}_+$ .

*Proof.* Let  $\epsilon \in \mathbb{R}_{++}$ . Because  $\sum_{k \in \mathbb{N}} b_k < \infty$  and  $(b_k)_{k \in \mathbb{N}}$  is in  $\mathbb{R}_+$ , there exists  $K_1 \in \mathbb{N}$  such that

$$(\forall k \ge K_1) \quad \sum_{i=k}^{\infty} b_i < \frac{\epsilon}{3}.$$
(2.8)

Denote by  $\bar{a} := \liminf_{k \to \infty} a_k$ . By the definition of liminf, there exists a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  of  $(a_k)_{k \in \mathbb{N}}$  such that  $a_{n_k} \to \bar{a} \in \mathbb{R}_+$ . Then, there exists  $K_2 \in \mathbb{N}$  such that

$$(\forall k \ge K_2) \quad |a_{n_k} - \bar{a}| < \frac{\epsilon}{3}.$$
(2.9)

Employ the definition of lim inf again to know that there exists  $K_3 \in \mathbb{N}$  such that

$$(\forall k \ge K_3) \quad a_k > \bar{a} - \frac{\epsilon}{3}. \tag{2.10}$$

Set  $K := \max\{K_1, K_2, K_3\}$ . Then

$$(\forall i > n_K) \quad \bar{a} - \frac{\epsilon}{3} \stackrel{(2.10)}{<} a_i \stackrel{(2.7)}{\leq} a_{i-1} + b_{i-1} \stackrel{(2.7)}{\leq} \cdots \stackrel{(2.7)}{\leq} a_{n_K} + \sum_{j=n_K}^{i-1} b_j \stackrel{(2.8)(2.9)}{\leq} \bar{a} + \frac{2\epsilon}{3},$$

which implies the desired result.

**Fact 2.6.** [8, Lemma 2.5] *Let* x and y be in  $\mathcal{H}$ . Then  $||x + y||^2 \le ||x||^2 + 2\langle y, x + y \rangle \le ||x||^2 + 2||y|| ||x + y||$  and  $||x + y||^2 \le ||x||^2 + ||y|| (2||x|| + ||y||)$ .

### Maximally monotone operators

**Definition 2.7.** [1, Definition 23.1] Let  $G : \mathcal{H} \to 2^{\mathcal{H}}$  and let  $\gamma \in \mathbb{R}_{++}$ . The *resolvent of* G is

$$\mathbf{J}_G = (\mathbf{Id} + G)^{-1}$$

and the *Yosida approximation of G of index*  $\gamma$  is

$${}^{\gamma}G = \frac{1}{\gamma}(\mathrm{Id} - \mathrm{J}_{\gamma G}).$$

**Definition 2.8.** [1, Definition 4.1] Let *D* be a nonempty subset of  $\mathcal{H}$  and let  $T : D \to \mathcal{H}$ . Then *T* is

- (i) firmly nonexpansive if  $(\forall x \in D) (\forall y \in D) ||Tx Ty||^2 + ||(Id T)x (Id T)y||^2 \le ||x y||^2$ ;
- (ii) nonexpansive if  $(\forall x \in D) \ (\forall y \in D) \ \|Tx Ty\| \le \|x y\|$ .

Remember that throughout this work,

 $A: \mathcal{H} \to 2^{\mathcal{H}}$  is maximally monotone.

The following properties of the resolvent and Yosida approximation of maximally monotone operators are fundamental to our analysis later and will be frequently used in the next sections.

- **Fact 2.9.** (i) [1, Proposition 20.38(ii)] gra *A* is sequentially closed in  $\mathcal{H}^{weak} \times \mathcal{H}^{strong}$ , *i.e.*, for every sequence  $(x_k, u_k)_{k \in \mathbb{N}}$  in gra *A* and every  $(x, u) \in \mathcal{H} \times \mathcal{H}$ , if  $x_k \rightharpoonup x$  and  $u_k \rightarrow u$ , then  $(x, u) \in \text{gra } A$ .
  - (ii) [1, Proposition 23.7(i)]  $(\forall \gamma \in \mathbb{R}_{++})$   $(\forall x \in \mathcal{H}) \frac{1}{\gamma}(x J_{\gamma A}x) = {}^{\gamma}A(x) \in A(J_{\gamma A}x)$ , that is,  $(J_{\gamma A}x, {}^{\gamma}A(x)) \in \operatorname{gra} A$ .
- (iii) [1, Proposition 23.10]  $J_A$  is full domain, single-valued, and firmly nonexpansive.
- (iv) [1, Proposition 23.38] Let  $\gamma \in \mathbb{R}_{++}$ . Then  $\operatorname{zer} A = \operatorname{Fix} J_{\gamma A} = \operatorname{zer}^{\gamma} A$ .
- (v) [1, Proposition 23.39] zer A is closed and convex.

**Fact 2.10.** [8, Lemma 2.4] Let  $\lambda$  and  $\mu$  be in  $\mathbb{R}_{++}$ . Then

$$(\forall x \in \mathcal{H}) \quad J_{\lambda A}(x) = J_{\mu A}\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_{\lambda A}x\right).$$

The following Fact 2.11(i) is used in the proof of [8, Theorem 3.6].

**Fact 2.11.** Let  $\lambda$  and  $\mu$  be in  $\mathbb{R}_{++}$ . Set  $T_{\lambda} := 2 J_{\lambda A} - \text{Id}$  and  $T_{\mu} := 2 J_{\mu A} - \text{Id}$ . Then the following hold.

- (i)  $(\forall x \in \mathcal{H}) T_{\lambda}x = T_{\mu} \left(\frac{\mu}{\lambda}x + \left(1 \frac{\mu}{\lambda}\right)J_{\lambda A}x\right) + \left(1 \frac{\mu}{\lambda}\right)(J_{\lambda A}(x) x).$
- (ii)  $(\forall x \in \mathcal{H}) \|T_{\lambda}(x) T_{\mu}(x)\| \le |1 \frac{\mu}{\lambda}| \|T_{\lambda}(x) x\|.$

*Proof.* (i): The desired result follows immediately from Fact 2.10 and the definitions of  $T_{\lambda}$  and  $T_{\mu}$ .

(ii): Notice that, via Fact 2.9(iii) and [1, Proposition 4.4],  $T_{\lambda}$  and  $T_{\mu}$  are nonexpansive. Hence, the required inequality follows easily from (i).

**Fact 2.12.** [8, Lemma 3.3] Let c and  $\bar{c}$  be in  $\mathbb{R}_{++}$  with  $\bar{c} \leq c$ . Then  $(\forall x \in \mathcal{H}) ||J_{\bar{c}A}(x) - x|| \leq 2 ||J_{cA}(x) - x||$ . In particular, for every sequences  $(y_k)_{k \in \mathbb{N}}$  in  $\mathcal{H}$  and  $(c_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}_{++}$  such that  $(\forall k \in \mathbb{N})$   $\bar{c} \leq c_k$ , we have

$$(\forall k \in \mathbb{N}) \quad ||\mathbf{J}_{\bar{c}A}(y_k) - y_k|| \le 2 ||\mathbf{J}_{c_kA}(y_k) - y_k||.$$

### Sets of zeroes

The technique of the following proof was used in [9, Theorem 1] to prove the uniqueness of the weak sequential cluster point of the sequence of iterations generated by Rockafellar's proximal point algorithm. According to Rockafellar's remark, one similar uniqueness argument was used by B. Martinet in 1970 and it was suggested to Martinet by H. Brézis.

**Proposition 2.13.** Let  $(y_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$ . Set  $\Omega$  as the set of all weak sequential cluster points of  $(y_k)_{k \in \mathbb{N}}$ . Suppose that for every  $z \in \Omega$ , the limit  $\lim_{k \to \infty} ||y_k - z||$  exists. Then there is at most one element in  $\Omega$ , that is, there cannot be more than one weak sequential cluster point of  $(y_k)_{k \in \mathbb{N}}$ .

*Proof.* Suppose to the contrary that there exist  $z_1$  and  $z_2$  in  $\Omega$  with  $z_1 \neq z_2$ . Then based on the assumption, there exist  $q_1$  and  $q_2$  in  $\mathbb{R}_+$  such that  $(\forall i \in \{1,2\}) \lim_{k\to\infty} ||y_k - z_i|| = q_i$ . Note that for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|y_k - z_2\|^2 &= \|y_k - z_1\|^2 + 2\langle y_k - z_1, z_1 - z_2 \rangle + \|z_1 - z_2\|^2 \text{ and } \\ \|y_k - z_1\|^2 &= \|y_k - z_2\|^2 + 2\langle y_k - z_2, z_2 - z_1 \rangle + \|z_2 - z_1\|^2, \end{aligned}$$

which imply, respectively, that

$$2\langle y_k - z_1, z_1 - z_2 \rangle = \|y_k - z_2\|^2 - \|y_k - z_1\|^2 - \|z_1 - z_2\|^2 \to q_2^2 - q_1^2 - \|z_1 - z_2\|^2, \text{ and}$$
(2.11a)

$$2\langle y_k - z_2, z_2 - z_1 \rangle = \|y_k - z_1\|^2 - \|y_k - z_2\|^2 - \|z_2 - z_1\|^2 \to q_1^2 - q_2^2 - \|z_1 - z_2\|^2.$$
(2.11b)

On the other hand,  $\{z_1, z_2\} \subseteq \Omega$  forces that once  $\lim_{k\to\infty} \langle y_k - z_1, z_1 - z_2 \rangle$  and  $\lim_{k\to\infty} \langle y_k - z_2, z_2 - z_1 \rangle$  exist, these two limits must be 0. Combine this with (2.11) and  $z_1 \neq z_2$  to deduce that

$$q_2^2 - q_1^2 = ||z_1 - z_2||^2 > 0$$
 and  $q_1^2 - q_2^2 = ||z_1 - z_2||^2 > 0$ 

which is absurd. Therefore, the desired result holds.

The following result is inspired by the proof of [9, Theorem 1], which shows the weak convergence of Rockafellar's proximal point algorithm.

**Proposition 2.14.** Let  $r \in \mathbb{R}_{++}$ . Define  $\tilde{A} := A + \partial \iota_{B[0;r]}$ . The following assertions hold.

(i) 
$$(\forall x \in \mathcal{H}) \partial \iota_{B[0;r]} x = \begin{cases} \{0\}, & \text{if } \|x\| < r; \\ \mathbb{R}_+ x, & \text{if } \|x\| = r; \\ \varnothing, & \text{if } \|x\| > r, \end{cases}$$
 and  $\tilde{A}x = \begin{cases} A(x), & \text{if } \|x\| < r; \\ A(x) + \mathbb{R}_+ x, & \text{if } \|x\| = r; \\ \varnothing, & \text{if } \|x\| > r. \end{cases}$ 

- (ii) Suppose that dom  $A \cap B(0; r) \neq \emptyset$ . Then the following hold.
  - (a)  $\tilde{A}$  is maximally monotone. Consequently,  $(\forall \gamma \in \mathbb{R}_{++}) J_{\gamma \tilde{A}} : \mathcal{H} \to \mathcal{H}$  is full-domain and firmly nonexpansive.
  - (b)  $\operatorname{zer} \tilde{A} \neq \emptyset$ .
  - (c) Let  $\gamma \in \mathbb{R}_{++}$  and let  $x \in \mathcal{H}$ . If  $x \in J_{\gamma A}^{-1}(B(0;r))$  or  $x \in J_{\gamma \tilde{A}}^{-1}(B(0;r))$ , then  $J_{\gamma \tilde{A}} x = J_{\gamma A} x$ . Consequently,  $B(0;r) \cap \operatorname{zer} A = B(0;r) \cap \operatorname{zer} \tilde{A}$ .
  - (d) If zer  $\tilde{A}$  is not a singleton, then zer  $A \neq \emptyset$ .

*Proof.* (i): The explicit formula of  $\partial \iota_{B[0;r]}$  is a direct result from [1, Examples 6.39 and 16.13], which immediately implies the formula of  $\tilde{A}$ .

(ii)(a): Clearly, because B[0;r] is a nonempty closed and convex set, we have that  $\iota_{B[0;r]}$  is a proper lower semicontinuous and convex function. Then the required results are guaranteed by [1, Theorem 20.25, Corollary 25.5(ii), and Proposition 23.10(iii)].

(ii)(b): According to (i), dom  $\tilde{A} \subseteq B[0;r]$  is bounded. Hence, the desired result is immediate from the maximal monotonicity of  $\tilde{A}$  and [1, Proposition 23.36(iii)].

(ii)(c): If  $x \in J_{\gamma A}^{-1}(B(0;r))$ , i.e.,  $J_{\gamma A}(x) \in B(0;r)$ , then, via Definition 2.7,

$$\begin{aligned} J_{\gamma A}(x) &= (\mathrm{Id} + \gamma A)^{-1}(x) \Leftrightarrow x \in \mathrm{J}_{\gamma A}(x) + \gamma A \left( \mathrm{J}_{\gamma A}(x) \right) \\ &\Leftrightarrow x \in \mathrm{J}_{\gamma A}(x) + \gamma \tilde{A} \left( \mathrm{J}_{\gamma A}(x) \right) \quad (\text{by } \mathrm{J}_{\gamma A}(x) \in B(0; r) \text{ and (i)}) \\ &\Leftrightarrow \mathrm{J}_{\gamma A}(x) \in (\mathrm{Id} + \gamma \tilde{A})^{-1}(x) = \mathrm{J}_{\gamma \tilde{A}}(x) \\ &\Leftrightarrow \mathrm{J}_{\gamma A}(x) = \mathrm{J}_{\gamma \tilde{A}}(x). \quad (\mathrm{J}_{\gamma \tilde{A}} \text{ is single-valued}) \end{aligned}$$

On the other hand, switch *A* and  $\tilde{A}$  in the proof above to obtain that  $x \in J_{\gamma \tilde{A}}^{-1}(B(0;r))$  implies  $J_{\gamma \tilde{A}}(x) = J_{\gamma A}(x)$ . Hence, the first required result is true.

In addition, for every  $y \in B(0;r) \cap \operatorname{zer} A$ , by Fact 2.9(iv),  $J_{\gamma A}(y) = y \in B(0;r)$ , which, combining with the result proved above, entails that  $y = J_{\gamma A}(y) = J_{\gamma \tilde{A}}(y) \in \operatorname{Fix} J_{\gamma \tilde{A}} = \operatorname{zer} \tilde{A}$ . Hence,  $B(0;r) \cap$  $\operatorname{zer} A \subseteq B(0;r) \cap \operatorname{zer} \tilde{A}$ . Moreover, applying the similar technique, we obtain that  $B(0;r) \cap \operatorname{zer} \tilde{A} \subseteq$  $B(0;r) \cap \operatorname{zer} A$ . Altogether,  $B(0;r) \cap \operatorname{zer} A = B(0;r) \cap \operatorname{zer} \tilde{A}$ .

(ii)(d): Suppose that  $\{x, y\} \subseteq \operatorname{zer} \tilde{A}$  with  $x \neq y$ . If ||x|| < r or ||y|| < r, then, via (ii)(c),  $\emptyset \neq B(0;r) \cap \operatorname{zer} \tilde{A} \subseteq \operatorname{zer} A$ .

Suppose that ||x|| = r and ||y|| = r. Notice that, due to (ii)(a) and Fact 2.9(v), zer  $\tilde{A}$  is closed and convex. Let  $\alpha \in [0, 1[$ . Then based on [1, Corollary 2.15],

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha) \|y\|^2 - \alpha(1-\alpha) \|x-y\|^2 < r^2,$$

which leads to  $\alpha x + (1 - \alpha)y \in B(0; r) \cap \operatorname{zer} \tilde{A} \subseteq \operatorname{zer} A$  by (ii)(c).

Altogether, the proof is complete.

**Proposition 2.15.** Let  $(y_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$  and let  $(c_k)_{k \in \mathbb{N}}$  be in  $\mathbb{R}_{++}$ . Suppose that  $(y_k)_{k \in \mathbb{N}}$  and  $(J_{c_k A} y_k)_{k \in \mathbb{N}}$  are bounded. Set  $\Omega$  as the set of all weak sequential cluster points of  $(y_k)_{k \in \mathbb{N}}$ . Then there exists  $r \in \mathbb{R}_{++}$  such that  $\tilde{A} := A + \partial_{l_B[0;r]}$  is a maximally monotone operator and that

 $\operatorname{zer} \tilde{A} \neq \varnothing$ ,  $(\Omega \cap \operatorname{zer} \tilde{A}) \subseteq \operatorname{zer} A$ , and  $(\forall k \in \mathbb{N}) J_{c_k A} y_k = J_{c_k \tilde{A}} y_k$ .

*Proof.* Because  $(y_k)_{k \in \mathbb{N}}$  and  $(J_{c_k A} y_k)_{k \in \mathbb{N}}$  are bounded, there exists  $r \in \mathbb{R}_{++}$  such that

$$(\forall k \in \mathbb{N}) \quad \|y_k\| \le \frac{r}{2} \quad \text{and} \quad \|\mathbf{J}_{c_k A} y_k\| \le \frac{r}{2},$$

$$(2.12)$$

which, due to [1, Lemmas 2.42 and 2.45], implies that  $\emptyset \neq \Omega \subseteq B[0; \frac{r}{2}] \subseteq B(0; r)$ .

Set  $\tilde{A} := A + \partial \iota_{B[0;r]}$ . In view of Fact 2.9(ii),  $(\forall k \in \mathbb{N}) \frac{1}{c_k} (y_k - J_{c_k A} y_k) = {}^{c_k} A(y_k) \in A(J_{c_k A} y_k)$ , which, by (2.12), yields that  $(\forall k \in \mathbb{N}) J_{c_k A} y_k \in B(0; r) \cap \text{dom } A$ . Combine this with Proposition 2.14(ii) to entail that  $\tilde{A}$  is maximally monotone and that  $\text{zer } \tilde{A} \neq \emptyset$ ,  $\Omega \cap \text{zer } \tilde{A} \subseteq B(0; r) \cap \text{zer } \tilde{A} \subseteq \text{zer } A$ , and  $(\forall k \in \mathbb{N}) J_{c_k A} y_k = J_{c_k \tilde{A}} y_k$ .

The result of Proposition 2.16 under the condition (i) was also proved in proofs of [9, Theorem 1] and [4, Theorem 3] for related proximal point algorithms by applying Fact 2.9(ii) and employing the definition of maximal monotonicity. In addition, the idea of the proof of Proposition 2.16 under the hypothesis (ii) with t = 1 was adopted in the Step 2 of the proof of [13, Theorem 5.1].

**Proposition 2.16.** Let  $(y_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$  and let  $(c_k)_{k \in \mathbb{N}}$  be in  $\mathbb{R}_{++}$ . Set  $\Omega$  as the set of all weak sequential cluster points of  $(y_k)_{k \in \mathbb{N}}$ . Suppose that one of the following holds.

(i)  $\bar{c} := \inf_{k \in \mathbb{N}} c_k > 0$  and  $y_k - J_{c_k A}(y_k) \to 0$ .

(ii)  $c_k \to \infty$ ,  $(y_k)_{k \in \mathbb{N}}$  is bounded, and there exists  $t \in \mathbb{N}$  such that  $y_{k+t} - J_{c_k A}(y_k) \to 0$ .

*Then*  $\Omega \subseteq \operatorname{zer} A$ *.* 

*Proof.* If  $\Omega = \emptyset$ , then the desired inclusion is trivial. Suppose that  $\Omega \neq \emptyset$ . Take  $\bar{y} \in \Omega$ , that is, there exists a subsequence  $(y_{k_i})_{i \in \mathbb{N}}$  of  $(y_k)_{k \in \mathbb{N}}$  such that  $y_{k_i} \rightharpoonup \bar{y}$ .

Assume that (i) holds. Then Fact 2.12 and the assumption that  $y_k - J_{c_kA}(y_k) \rightarrow 0$  imply that  $J_{\bar{c}A}(y_{k_i}) - y_{k_i} \rightarrow 0$ . Therefore, by Fact 2.9(iii)&(iv) and [1, Corollary 4.28], we conclude that  $\bar{y} \in \text{Fix } J_{\bar{c}A} = \text{zer } A$ .

Assume that (ii) holds. Clearly, the boundedness of  $(y_k)_{k \in \mathbb{N}}$  and the convergence  $y_{k+t} - J_{c_k A}(y_k) \to 0$ imply that  $(J_{c_k A}(y_k))_{k \in \mathbb{N}}$  is bounded and that

$$J_{c_{k_i-t}A}(y_{k_i-t}) = y_{k_i} - \left(y_{k_i} - J_{c_{k_i-t}A}(y_{k_i-t})\right) \rightharpoonup \bar{y}.$$
(2.13)

Moreover, as a consequence of Fact 2.9(ii),

$$(\forall i \in \mathbb{N}) \quad \left( \mathcal{J}_{c_{k_i-t}A}(y_{k_i-t}), \frac{1}{c_{k_i-t}} \left( y_{k_i-t} - \mathcal{J}_{c_{k_i-t}A}(y_{k_i-t}) \right) \right) \in \operatorname{gra} A.$$
 (2.14)

Because  $c_k \to \infty$  and the boundedness of  $(y_k)_{k \in \mathbb{N}}$  and  $(J_{c_k A}(y_k))_{k \in \mathbb{N}}$  yield  $\frac{1}{c_{k_i-t}} \left( y_{k_i-t} - J_{c_{k_i-t}A}(y_{k_i-t}) \right) \to 0$ , combine (2.13), (2.14), and Fact 2.9(i) to establish that  $(\bar{y}, 0) \in \text{gra } A$ , i.e.,  $\bar{y} \in \text{zer } A$ .

Altogether, the required result is correct, since  $\bar{y} \in \Omega$  is arbitrary.

### Asymptotic regularity and convergence

Given a sequence  $(y_k)_{k \in \mathbb{N}}$  in  $\mathcal{H}$  and a sequence  $(c_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}_{++}$ , we say the *asymptotic regularity holds for*  $(y_k)_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$ , if  $y_k - J_{c_k A} y_k \to 0$ . We shall see that the asymptotic regularity plays an important role in the proof of the convergence of generalized proximal point algorithms.

**Proposition 2.17.** Suppose that  $\operatorname{zer} A \neq \emptyset$ . Let  $p \in \operatorname{zer} A$  and let  $(y_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$ . Suppose that  $\lim_{k\to\infty} \|y_k - p\|$  exists in  $\mathbb{R}_+$  and that  $y_{k+t} - J_{c_kA} y_k \to 0$  for some  $t \in \mathbb{N}$ . Then  $y_k - J_{c_kA} y_k \to 0$  and  $y_k - y_{k+t} \to 0$ .

Proof. Taking Fact 2.9(iii)&(iv) into account and employing Definition 2.8(i), we observe that

$$(\forall k \in \mathbb{N}) \quad ||\mathbf{J}_{c_k A} y_k - p||^2 + ||y_k - \mathbf{J}_{c_k A} y_k||^2 \le ||y_k - p||^2,$$

which yields that for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|y_{k} - \mathbf{J}_{c_{k}A} y_{k}\|^{2} - \|y_{k} - p\|^{2} + \|y_{k+t} - p\|^{2} &\leq - \|\mathbf{J}_{c_{k}A} y_{k} - p\|^{2} + \|y_{k+t} - p\|^{2} \\ &= \langle y_{k+t} - p - (\mathbf{J}_{c_{k}A} y_{k} - p), y_{k+t} - p + (\mathbf{J}_{c_{k}A} y_{k} - p) \rangle \\ &\leq \|y_{k+t} - \mathbf{J}_{c_{k}A} y_{k}\| \left( \|y_{k+t} - p\| + \|y_{k} - p\| \right), \end{aligned}$$

where in the last inequality we use the Cauchy-Schwarz inequality, the nonexpansiveness of  $J_{c_kA}$ , and  $p \in \operatorname{zer} A = \operatorname{Fix} J_{c_kA}$ . Hence,

$$(\forall k \in \mathbb{N}) \quad ||y_k - J_{c_k A} y_k||^2 \le ||y_k - p||^2 - ||y_{k+t} - p||^2 + ||y_{k+t} - J_{c_k A} y_k|| (||y_{k+t} - p|| + ||y_k - p||),$$

which ensures  $y_k - J_{c_kA} y_k \to 0$ , since the existence of  $\lim_{k\to\infty} ||y_k - p||$  yields  $||y_k - p||^2 - ||y_{k+t} - p||^2 \to 0$  and the boundedness of  $(||y_{k+t} - p|| + ||y_k - p||)_{k\in\mathbb{N}}$ .

Moreover, in consideration of  $(\forall k \in \mathbb{N}) ||y_k - y_{k+t}|| \le ||y_k - J_{c_kA} y_k|| + ||J_{c_kA} y_k - y_{k+t}||$ , we reach the last required convergence by using  $y_k - J_{c_kA} y_k \to 0$  and  $y_{k+t} - J_{c_kA} y_k \to 0$ .

The following result will play an essential role to prove the weak convergence of the generalized proximal point algorithm.

**Fact 2.18.** [1, Lemma 2.47] Let  $(y_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$  and let C be a nonempty subset of  $\mathcal{H}$ . Suppose that every weak sequential cluster point of  $(y_k)_{k \in \mathbb{N}}$  belongs to C, that is,  $\Omega((y_k)_{k \in \mathbb{N}}) \subseteq C$ , and that  $(\forall z \in C) \lim_{k \to \infty} ||y_k - z||$  exists in  $\mathbb{R}_+$ . Then  $(y_k)_{k \in \mathbb{N}}$  converges weakly to a point in C.

The following Proposition 2.19 is inspired by the Step 2 of the proof of [13, Theorem 5.1]. The following result is critical to prove the strong convergence of generalized proximal point algorithms.

**Proposition 2.19.** Let  $(y_k)_{k \in \mathbb{N}}$  be a bounded sequence in  $\mathcal{H}$  and let  $u \in \mathcal{H}$ . Set  $\Omega$  as the set of all weak sequential cluster points of  $(y_k)_{k \in \mathbb{N}}$ . Suppose that  $\Omega \subseteq \text{zer } A$ . Then

$$\limsup_{k\to\infty} \langle u - \mathcal{P}_{\operatorname{zer} A} \, u, y_k - \mathcal{P}_{\operatorname{zer} A} \, u \rangle \leq 0.$$

*Proof.* By the definition of lim sup, there exists a subsequence  $(y_{k_i})_{i \in \mathbb{N}}$  of  $(y_k)_{k \in \mathbb{N}}$  such that

$$\limsup_{k \to \infty} \langle u - \mathcal{P}_{\operatorname{zer} A} \, u, y_k - \mathcal{P}_{\operatorname{zer} A} \, u \rangle = \lim_{i \to \infty} \langle u - \mathcal{P}_{\operatorname{zer} A} \, u, y_{k_i} - \mathcal{P}_{\operatorname{zer} A} \, u \rangle \,. \tag{2.15}$$

Because  $(y_k)_{k \in \mathbb{N}}$  is bounded, without loss of generality (otherwise take a subsequence of  $(y_{k_i})_{i \in \mathbb{N}}$ ), we assume that  $y_{k_i} \rightharpoonup \bar{y}$  for some  $\bar{y} \in \Omega \subseteq \text{zer } A$ . Hence, due to [1, Proposition 3.16] and Fact 2.9(v),

$$\lim_{i \to \infty} \left\langle u - \mathcal{P}_{\operatorname{zer} A} \, u, y_{k_i} - \mathcal{P}_{\operatorname{zer} A} \, u \right\rangle = \left\langle u - \mathcal{P}_{\operatorname{zer} A} \, u, \bar{y} - \mathcal{P}_{\operatorname{zer} A} \, u \right\rangle \le 0. \tag{2.16}$$

Combine (2.15) and (2.16) to obtain the required inequality.

# 3 Generalized proximal point algorithms

Recall that

 $A: \mathcal{H} \to 2^{\mathcal{H}}$  is a maximally monotone operator.

In the rest of this work,  $u \in H$  and  $x_0 \in H$  are arbitrary but fixed, and the generalized proximal point algorithm is generated by conforming the following recursion:

$$(\forall k \in \mathbb{N}) \quad x_{k+1} = \alpha_k u + \beta_k x_k + \gamma_k J_{c_k A}(x_k) + \delta_k e_k, \tag{3.1}$$

where  $(\forall k \in \mathbb{N}) e_k \in \mathcal{H}, c_k \in \mathbb{R}_{++}$ , and  $\{\alpha_k, \beta_k, \gamma_k, \delta_k\} \subseteq \mathbb{R}$ . From now on,

 $\Omega$  is the set of all weak sequential cluster points of  $(x_k)_{k \in \mathbb{N}}$ .

In this section, we investigate the boundedness and asymptotic regularity of  $(x_k)_{k \in \mathbb{N}}$ ; after that, we demonstrate the equivalence of the boundedness of  $(x_k)_{k \in \mathbb{N}}$  and zer  $A \neq \emptyset$ .

### **Boundedness**

**Lemma 3.1.** Set  $(\forall k \in \mathbb{N})$   $T_k := 2 J_{c_k A} - Id$ . Then  $(\forall k \in \mathbb{N})$   $T_k$  is nonexpansive and Fix  $T_k = \text{zer } A$ . *Moreover,* 

$$(\forall k \in \mathbb{N}) \quad x_{k+1} = \left(\beta_k + \frac{\gamma_k}{2}\right) x_k + \frac{\gamma_k}{2} T_k(x_k) + \alpha_k u + \delta_k e_k.$$

*Proof.* Based on Fact 2.9(iii)&(iv) and [1, Proposition 4.4],  $(\forall k \in \mathbb{N}) T_k$  is nonexpansive and Fix  $T_k$  = Fix  $J_{c_kA}$  = zer A. In consideration of (3.1),

$$\begin{aligned} (\forall k \in \mathbb{N}) \quad x_{k+1} &= \alpha_k u + \beta_k x_k + \gamma_k J_{c_k A}(x_k) + \delta_k e_k \\ &= \alpha_k u + \beta_k x_k + \frac{1}{2} \gamma_k (x_k + T_k x_k) + \delta_k e_k \\ &= \left(\beta_k + \frac{\gamma_k}{2}\right) x_k + \frac{\gamma_k}{2} T_k(x_k) + \alpha_k u + \delta_k e_k, \end{aligned}$$

which implies directly the desired equality.

The following inequalities will be used frequently later.

**Lemma 3.2.** Let  $p \in \text{zer } A$ . Set  $(\forall k \in \mathbb{N})$   $T_k := 2 J_{c_k A} - \text{Id.}$  Then the following hold.

(i) 
$$(\forall k \in \mathbb{N}) ||x_{k+1} - p|| \le (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|) ||x_k - p|| + ||\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p||$$

(ii) Denote by  $(\forall k \in \mathbb{N}) \ \xi_k := (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2, \ \phi_k := 1 - \beta_k - \gamma_k, \ \varphi_k := 1 - \alpha_k - \beta_k - \gamma_k, \ F(k) := \|\delta_k e_k - \varphi_k u\|, \ and \ G(k) := F(k) + 2 \|(\beta_k + \frac{\gamma_k}{2})(x_k - p) + \frac{\gamma_k}{2}(T_k(x_k) - p) + \phi_k(u - p)\|.$ Then  $(\forall k \in \mathbb{N}) \|x_{k+1} - p\|^2 \le \xi_k \|x_k - p\|^2 + 2\phi_k \langle u - p, x_{k+1} - p - \delta_k e_k + \varphi_k u \rangle + F(k)G(k).$ 

(iii) Set 
$$(\forall k \in \mathbb{N}) \ M(k) := \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|$$
. Suppose  $\inf_{k \in \mathbb{N}} \gamma_k(\beta_k + \gamma_k) \ge 0$ . Then  $(\forall k \in \mathbb{N}) \|x_{k+1} - p\|^2 \le (\beta_k + \gamma_k)^2 \|x_k - p\|^2 - \gamma_k(2\beta_k + \gamma_k) \|x_k - J_{c_kA} x_k\|^2 + 2M(k) \|x_{k+1} - p\|$ .

(iv)  $(\forall k \in \mathbb{N}) \|x_{k+1} - p\|^2 \le (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 \|x_k - p\|^2 + 2\langle \alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) p, x_{k+1} - p \rangle$ . *Proof.* Let  $k \in \mathbb{N}$ .

(i): In view of Lemma 3.1,

$$\begin{split} \|x_{k+1} - p\| &= \left\| \left( \beta_k + \frac{\gamma_k}{2} \right) x_k + \frac{\gamma_k}{2} T_k(x_k) + \alpha_k u + \delta_k e_k - p \right\| \\ &= \left\| \left( \beta_k + \frac{\gamma_k}{2} \right) (x_k - p) + \frac{\gamma_k}{2} (T_k(x_k) - p) + \alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) p \right\| \\ &\leq \left| \beta_k + \frac{\gamma_k}{2} \right| \|x_k - p\| + \left| \frac{\gamma_k}{2} \right| \|T_k(x_k) - p\| + \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) p\| \\ &\leq \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) \|x_k - p\| + \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) p\| , \end{split}$$

where in the last inequality we used the nonexpansiveness of  $T_k$  and the fact that  $p \in \operatorname{zer} A = \operatorname{Fix} T_k$ .

(ii): Applying Lemma 3.1 in the first equality and the last inequality, and employing Fact 2.6 in the first and second inequalities, we obtain that

$$\begin{aligned} \|x_{k+1} - p\|^{2} \\ &= \left\| \left( \beta_{k} + \frac{\gamma_{k}}{2} \right) (x_{k} - p) + \frac{\gamma_{k}}{2} (T_{k}(x_{k}) - p) + (1 - \beta_{k} - \gamma_{k}) (u - p) + \delta_{k} e_{k} - (1 - \alpha_{k} - \beta_{k} - \gamma_{k}) u \right\|^{2} \\ &\leq \left\| \left( \beta_{k} + \frac{\gamma_{k}}{2} \right) (x_{k} - p) + \frac{\gamma_{k}}{2} (T_{k}(x_{k}) - p) + \phi_{k} (u - p) \right\|^{2} + F(k)G(k) \\ &\leq \left\| \left( \beta_{k} + \frac{\gamma_{k}}{2} \right) (x_{k} - p) + \frac{\gamma_{k}}{2} (T_{k}(x_{k}) - p) \right\|^{2} + 2\phi_{k} \langle u - p, x_{k+1} - p - \delta_{k} e_{k} + \phi_{k} u \rangle + F(k)G(k) \\ &\leq \left( \left| \beta_{k} + \frac{\gamma_{k}}{2} \right| + \left| \frac{\gamma_{k}}{2} \right| \right)^{2} \|x_{k} - p\|^{2} + 2\phi_{k} \langle u - p, x_{k+1} - p - \delta_{k} e_{k} + \phi_{k} u \rangle + F(k)G(k). \end{aligned}$$
(iii): According to Fact 2.9(iii)&(iv) and [1, Proposition 4.4],

 $\left\langle x_{k}-p, \mathbf{J}_{c_{k}A}(x_{k})-x_{k}\right\rangle = -\left\langle x_{k}-p, (\mathrm{Id}-\mathbf{J}_{c_{k}A})(x_{k})-(\mathrm{Id}-\mathbf{J}_{c_{k}A})p\right\rangle$ (3.3a)

$$\leq - \| (\mathrm{Id} - \mathbf{J}_{c_k A})(\mathbf{x}_k) - (\mathrm{Id} - \mathbf{J}_{c_k A})p \|^2$$
(3.3b)

$$= - \|x_k - \mathbf{J}_{c_k A} x_k\|^2.$$
(3.3c)

Utilizing  $\inf_{k \in \mathbb{N}} \gamma_k(\beta_k + \gamma_k) \ge 0$  in the last inequality, we establish that

$$\begin{aligned} \|x_{k+1} - p\|^{2} \\ \stackrel{\textbf{(3.1)}}{=} \|(\beta_{k} + \gamma_{k})x_{k} + \gamma_{k}\left(J_{c_{k}A}(x_{k}) - x_{k}\right) + \alpha_{k}u + \delta_{k}e_{k} - p\|^{2} \\ &= \|(\beta_{k} + \gamma_{k})(x_{k} - p) + \gamma_{k}\left(J_{c_{k}A}(x_{k}) - x_{k}\right) + \alpha_{k}u + \delta_{k}e_{k} - (1 - \beta_{k} - \gamma_{k})p\|^{2} \\ &\leq \|(\beta_{k} + \gamma_{k})(x_{k} - p) + \gamma_{k}\left(J_{c_{k}A}(x_{k}) - x_{k}\right)\|^{2} + 2M(k)\|x_{k+1} - p\| \quad \text{(by Fact 2.6)} \\ &= (\beta_{k} + \gamma_{k})^{2}\|x_{k} - p\|^{2} + \gamma_{k}^{2}\|J_{c_{k}A}(x_{k}) - x_{k}\|^{2} + 2\gamma_{k}(\beta_{k} + \gamma_{k})\left\langle x_{k} - p, J_{c_{k}A}(x_{k}) - x_{k}\right\rangle + 2M(k)\|x_{k+1} - p\| \\ \stackrel{\textbf{(3.3)}}{\leq} (\beta_{k} + \gamma_{k})^{2}\|x_{k} - p\|^{2} - \gamma_{k}(2\beta_{k} + \gamma_{k})\|x_{k} - J_{c_{k}A}x_{k}\|^{2} + 2M(k)\|x_{k+1} - p\| . \end{aligned}$$

(iv): Apply Lemma 3.1 and Fact 2.6 in the following first equality and first inequality, respectively, and employ the nonexpansiveness of  $T_k$  and the fact that  $p = T_k(p)$  in the second inequality to observe that

$$\begin{aligned} \|x_{k+1} - p\|^{2} \\ &= \left\| \left( \beta_{k} + \frac{\gamma_{k}}{2} \right) x_{k} + \frac{\gamma_{k}}{2} T_{k}(x_{k}) + \alpha_{k} u + \delta_{k} e_{k} - p \right\|^{2} \\ &= \left\| \left( \beta_{k} + \frac{\gamma_{k}}{2} \right) (x_{k} - p) + \frac{\gamma_{k}}{2} (T_{k}(x_{k}) - p) + \alpha_{k} u + \delta_{k} e_{k} - (1 - \beta_{k} - \gamma_{k}) p \right\|^{2} \\ &\leq \left\| \left( \beta_{k} + \frac{\gamma_{k}}{2} \right) (x_{k} - p) + \frac{\gamma_{k}}{2} (T_{k}(x_{k}) - p) \right\|^{2} + 2 \left\langle \alpha_{k} u + \delta_{k} e_{k} - (1 - \beta_{k} - \gamma_{k}) p, x_{k+1} - p \right\rangle \\ &\leq \left( \left| \beta_{k} + \frac{\gamma_{k}}{2} \right| + \left| \frac{\gamma_{k}}{2} \right| \right)^{2} \|x_{k} - p\|^{2} + 2 \left\langle \alpha_{k} u + \delta_{k} e_{k} - (1 - \beta_{k} - \gamma_{k}) p, x_{k+1} - p \right\rangle. \end{aligned}$$

Altogether, the proof is complete.

We present sufficient conditions for the boundedness of  $(x_k)_{k \in \mathbb{N}}$  in the remaining subsection.

Note that if  $(\forall k \in \mathbb{N}) \{\alpha_k, \beta_k, \gamma_k\} \subseteq [0, 1]$  with  $\alpha_k + \beta_k + \gamma_k = 1$  (which is the case in many publications on generalized proximal point algorithms), then based on Proposition 3.3(ii) or Proposition 3.3(iii), we deduce the classical statement:  $\operatorname{zer} A \neq \emptyset$  and  $\sum_{k \in \mathbb{N}} \|\delta_k e_k\| < \infty$  imply the boundedness of  $(x_k)_{k \in \mathbb{N}}$ .

**Proposition 3.3.** Suppose that  $\operatorname{zer} A \neq \emptyset$  and that one of the following holds.

(i) 
$$\limsup_{k\to\infty} \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) < 1$$
,  $\sup_{k\in\mathbb{N}} |\alpha_k| < \infty$ , and  $\sup_{k\in\mathbb{N}} \|\delta_k e_k\| < \infty$ .

(ii)  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ , and the following hold:

- (a)  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1 \text{ or } \sum_{i \in \mathbb{N}} |\alpha_i| < \infty;$
- (b)  $\left[\left(\forall k \in \mathbb{N} \left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right| + \left|\delta_k\right| \le 1 \text{ and } \sup_{i \in \mathbb{N}} \|e_i\| < \infty\right] \text{ or } \sum_{i \in \mathbb{N}} \|\delta_i e_i\| < \infty;$
- (c)  $(\forall k \in \mathbb{N}) \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| + \left| 1 \beta_k \gamma_k \right| \le 1 \text{ or } \sum_{i \in \mathbb{N}} \left| 1 \beta_i \gamma_i \right| < \infty.$

(iii)  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1, \sum_{k \in \mathbb{N}} |1 - \alpha_k - \beta_k - \gamma_k| < \infty, and \sum_{k \in \mathbb{N}} ||\delta_k e_k|| < \infty.$ 

(iv)  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| + |\delta_k| \leq 1$ ,  $\sup_{k \in \mathbb{N}} ||e_k|| < \infty$ ,  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k - \delta_k| < \infty$ , and  $\sum_{k \in \mathbb{N}} |\alpha_k| < \infty$ .

*Then*  $(x_k)_{k \in \mathbb{N}}$  *is bounded.* 

*Proof.* Let  $p \in \text{zer } A$ . In view of Lemma 3.2(i),

$$(\forall k \in \mathbb{N}) \quad \|x_{k+1} - p\| \le \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right) \|x_k - p\| + \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|.$$
(3.4)

(i): Note that  $(\forall k \in \mathbb{N}) |1 - \beta_k - \gamma_k| \le 1 + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|$  and that  $\limsup_{i \to \infty} (|\beta_i + \frac{\gamma_i}{2}| + |\frac{\gamma_i}{2}|) < 1$  implies the boundedness of  $(|\beta_i + \frac{\gamma_i}{2}| + |\frac{\gamma_i}{2}|)_{i \in \mathbb{N}}$  and  $(|1 - \beta_i - \gamma_i|)_{i \in \mathbb{N}}$ . The desired result is clear from (3.4) and Proposition 2.3(i) with  $(\forall k \in \mathbb{N})$   $t_k = ||x_k - p||, \alpha_k = |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|, \beta_k \equiv 0, \omega_k \equiv$ and  $\gamma_k = \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|.$ 

(ii): Clearly, there are eight cases to prove and it suffices to show the boundedness of  $(||x_k - p||)_{k \in \mathbb{N}}$ in each case. We prove only the following three cases and omit the similar proof of the remaining ones. *Case 1*: Suppose that  $\sum_{i \in \mathbb{N}} |\alpha_i| < \infty$ ,  $\sum_{i \in \mathbb{N}} \|\delta_i e_i\| < \infty$ , and  $\sum_{i \in \mathbb{N}} |1 - \beta_i - \gamma_i| < \infty$ .

Recall (3.4) and apply Proposition 2.3(ii) with  $(\forall k \in \mathbb{N})$   $t_k = ||x_k - p||, \alpha_k = |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|, \beta_k \equiv 0$ ,  $\omega_k \equiv 0$ , and  $\gamma_k = \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|$  to obtain the required boundedness of  $(\|x_k - p\|)_{k \in \mathbb{N}}$ . Case 2: Suppose that  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1, |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| + |\delta_k| \le 1, \sup_{i \in \mathbb{N}} ||e_i|| < 1$  $\infty$ , and  $\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right| + \left|1 - \beta_k - \gamma_k\right| \le 1$ .

Denote by  $(\forall k \in \mathbb{N}) \xi_k := \max\{|\alpha_k|, |\delta_k|, |1 - \beta_k - \gamma_k|\}$ . In view of the assumption above,

$$(\forall k \in \mathbb{N}) \quad \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| + \xi_k \le 1.$$
 (3.5)

On the other hand, clearly (3.4) forces

$$\begin{aligned} (\forall k \in \mathbb{N}) \quad \|x_{k+1} - p\| &\leq \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) \|x_k - p\| + |\alpha_k| \|u\| + \|\delta_k e_k\| + |1 - \beta_k - \gamma_k| \|p\| \\ &\leq \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) \|x_k - p\| + \xi_k \left( \|u\| + \|p\| + \sup_{i \in \mathbb{N}} \|e_i\| \right), \end{aligned}$$

which, connecting with (3.5) and applying Proposition 2.3(ii) with  $(\forall k \in \mathbb{N}) t_k = ||x_k - p||, \alpha_k =$  $|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|, \beta_k = \xi_k, \omega_k = ||u|| + ||p|| + \sup_{i \in \mathbb{N}} ||e_i||$ , and  $\gamma_k \equiv 0$ , guarantees the boundedness of  $(||x_k - p||)_{k \in \mathbb{N}}.$ 

*Case 3*: Suppose that  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ ,  $\sum_{i \in \mathbb{N}} \|\delta_i e_i\| < \infty$ , and  $|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| + |\frac{$  $|1 - \beta_k - \gamma_k| \le 1.$ 

Denote by  $(\forall k \in \mathbb{N}) \eta_k := \max\{|\alpha_k|, |1 - \beta_k - \gamma_k|\}$ . Similarly with the proof of Case 2, we observe that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| + \eta_k \leq 1$ , and that

$$(\forall k \in \mathbb{N}) \quad \|x_{k+1} - p\| \le \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) \|x_k - p\| + |\alpha_k| \|u\| + \|\delta_k e_k\| + |1 - \beta_k - \gamma_k| \|p\| \\ \le \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) \|x_k - p\| + \eta_k \left( \|u\| + \|p\| \right) + \|\delta_k e_k\| ,$$

which, applying Proposition 2.3(ii) with  $(\forall k \in \mathbb{N})$   $t_k = ||x_k - p||, \alpha_k = |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|, \beta_k = \eta_k$  $\omega_k = ||u|| + ||p||$ , and  $\gamma_k = ||\delta_k e_k||$ , ensures the boundedness of  $(||x_k - p||)_{k \in \mathbb{N}}$ .

(iii)&(iv): As a consequence of (3.4), for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|x_{k+1} - p\| &\leq \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right) \|x_k - p\| + |\alpha_k| \|u - p\| + |1 - \alpha_k - \beta_k - \gamma_k| \|p\| + \|\delta_k e_k\|; \quad (3.6a) \\ \|x_{k+1} - p\| &\leq \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right) \|x_k - p\| + |\delta_k| \|e_k - p\| + |1 - \beta_k - \gamma_k - \delta_k| \|p\| + |\alpha_k| \|u\|. \quad (3.6b) \end{aligned}$$

Hence, we obtain (iii) (resp. (iv)) by invoking (3.6a) (resp. (3.6b)) and applying Proposition 2.3(ii) with  $(\forall k \in \mathbb{N}) t_k = ||x_k - p||, \alpha_k = |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|, \beta_k = \alpha_k \text{ (resp. } \beta_k = \delta_k), \omega_k = ||u - p|| \text{ (resp. } \omega_k = \delta_k)$  $||e_k - p||$ ), and  $\gamma_k = |1 - \alpha_k - \beta_k - \gamma_k| ||p|| + ||\delta_k e_k||$  (resp.  $\gamma_k = |1 - \beta_k - \gamma_k - \delta_k| ||p|| + |\alpha_k| ||u||$ ). Altogether, the proof is complete.

The following result is motivated by the Step 1 in the proof of [2, Theorem 1].

**Proposition 3.4.** Suppose that  $\operatorname{zer} A \neq \emptyset$ , that  $(\forall k \in \mathbb{N}) \ \alpha_k \in ]0,1]$  and  $\alpha_k + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1$ , and that  $\frac{\delta_k e_k}{\alpha_k} \to 0$  and  $\frac{1-\alpha_k-\beta_k-\gamma_k}{\alpha_k} \to 0$ . Then  $(x_k)_{k\in\mathbb{N}}$  is bounded.

*Proof.* Let  $p \in \text{zer } A$ . Set  $(\forall k \in \mathbb{N})$   $T_k := 2 \operatorname{J}_{c_k A} - \operatorname{Id}$ . Because  $\frac{\delta_k e_k}{\alpha_k} \to 0$  and  $\frac{1 - \alpha_k - \beta_k - \gamma_k}{\alpha_k} \to 0$ , there exists  $M \in \mathbb{R}_{++}$  such that

$$(\forall k \in \mathbb{N}) \quad \|x_0 - p\| \le M \text{ and } \|u - p\| + \left\|\frac{\delta_k e_k}{\alpha_k}\right\| + \left|\frac{1 - \alpha_k - \beta_k - \gamma_k}{\alpha_k}\right| \|p\| \le M.$$

We prove

$$(\forall k \in \mathbb{N}) \quad \|x_k - p\| \le 2M. \tag{3.7}$$

by induction below.

The basic case of (3.7) follows immediately from the definition of *M*. Suppose that (3.7) holds for some  $k \in \mathbb{N}$ . Employ Lemma 3.2(iv) in the first inequality and use the assumption  $(\forall k \in \mathbb{N}) \alpha_k + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1$  in the second inequality below to observe that

$$\begin{aligned} \|x_{k+1} - p\|^{2} \\ &\leq \left(\left|\beta_{k} + \frac{\gamma_{k}}{2}\right| + \left|\frac{\gamma_{k}}{2}\right|\right)^{2} \|x_{k} - p\|^{2} + 2\langle \alpha_{k}u + \delta_{k}e_{k} - (1 - \beta_{k} - \gamma_{k}) p, x_{k+1} - p\rangle \\ &= \left(\left|\beta_{k} + \frac{\gamma_{k}}{2}\right| + \left|\frac{\gamma_{k}}{2}\right|\right)^{2} \|x_{k} - p\|^{2} + 2\alpha_{k} \left\langle u - p + \frac{\delta_{k}e_{k}}{\alpha_{k}} - \frac{1 - \alpha_{k} - \beta_{k} - \gamma_{k}}{\alpha_{k}} p, x_{k+1} - p\right\rangle \\ &\leq (1 - \alpha_{k})^{2} \|x_{k} - p\|^{2} + 2\alpha_{k}M \|x_{k+1} - p\|, \end{aligned}$$

which, utilizing the induction hypothesis in the inequality below, entails that

$$||x_{k+1} - p||^2 \le 4 (1 - \alpha_k)^2 M^2 + 2\alpha_k M ||x_{k+1} - p||.$$

This guarantees that

$$(||x_{k+1} - p|| - \alpha_k M)^2 \le 4 (1 - \alpha_k)^2 M^2 + (\alpha_k M)^2.$$

Moreover, the inequality above ensures that

$$\|x_{k+1} - p\| \le \alpha_k M + \left(4\left(1 - \alpha_k\right)^2 M^2 + (\alpha_k M)^2\right)^{\frac{1}{2}} \le M\left(\alpha_k + (2\left(1 - \alpha_k\right) + \alpha_k)\right) = 2M.$$

Therefore, (3.7) holds, which ensures the desired boundedness of  $(x_k)_{k \in \mathbb{N}}$ .

### Asymptotic regularity

In this section, we shall provide sufficient conditions for  $x_k - J_{c_kA} x_k \to 0$  or  $\Omega \subseteq \text{zer } A$ .

**Proposition 3.5.** Suppose that  $(x_k)_{k \in \mathbb{N}}$  is bounded. Then the following assertions hold.

- (i) Suppose that one of the following holds.
  - (a)  $\operatorname{zer} A \neq \emptyset$ .
  - (b)  $\liminf_{k\to\infty} |\gamma_k| > 0$ ,  $\sup_{k\in\mathbb{N}} |\alpha_k| < \infty$ ,  $\sup_{k\in\mathbb{N}} |\beta_k| < \infty$ , and  $\sup_{k\in\mathbb{N}} \|\delta_k e_k\| < \infty$ .

Then  $(\mathbf{J}_{c_kA} \mathbf{x}_k)_{k \in \mathbb{N}}$  is bounded.

- (ii) Suppose that  $\alpha_k \to 0$ ,  $\beta_k \to 0$ ,  $\gamma_k \to 1$ , and  $\delta_k e_k \to 0$ . Then  $x_{k+1} J_{c_k A} x_k \to 0$ .
- (iii) Suppose that  $c_k \to \infty$ ,  $\alpha_k \to 0$ ,  $\beta_k \to 0$ ,  $\gamma_k \to 1$ , and  $\delta_k e_k \to 0$ . Then  $x_{k+1} J_{c_k A} x_k \to 0$  and  $\emptyset \neq \Omega \subseteq \text{zer } A$ .

*Proof.* (i): If zer  $A \neq \emptyset$ , then via Fact 2.9(iii)&(iv), for every  $p \in \text{zer } A$ ,

$$(\forall k \in \mathbb{N}) \quad \|\mathbf{J}_{c_k A} x_k\| - \|p\| \le \|\mathbf{J}_{c_k A} x_k - p\| = \|\mathbf{J}_{c_k A} x_k - \mathbf{J}_{c_k A} p\| \le \|x_k - p\|.$$

Hence, the boundedness of  $(x_k)_{k \in \mathbb{N}}$  implies the boundedness of  $(J_{c_k A} x_k)_{k \in \mathbb{N}}$ .

Assume (i)(b) holds. Then  $\hat{\alpha} := \sup_{k \in \mathbb{N}} |\alpha_k| < \infty$ ,  $\hat{\beta} := \sup_{k \in \mathbb{N}} |\beta_k| < \infty$ ,  $L := \sup_{k \in \mathbb{N}} \|\delta_k e_k\| < \infty$ , and  $M := \sup_{k \in \mathbb{N}} \|x_k\| < \infty$ . Take  $\bar{\gamma} \in \mathbb{R}_{++}$  such that  $0 < \bar{\gamma} < \liminf_{k \to \infty} |\gamma_k|$ . Then there exists  $N \in \mathbb{N}$  such that

$$(\forall k \ge N) \quad \left\| \mathbf{J}_{c_k A} \, x_k \right\| \stackrel{\textbf{(3.1)}}{=} \left\| \frac{1}{\gamma_k} \left( x_{k+1} - \alpha_k u - \beta_k x_k - \delta_k e_k \right) \right\| \le \frac{1}{\bar{\gamma}} \left( M + \hat{\alpha} \| u \| + \hat{\beta} M + L \right),$$

which shows the boundedness of  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$ .

(ii): Due to (i),  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$  is bounded. Then apply (3.1) to deduce that

$$(\forall k \in \mathbb{N}) \quad ||x_{k+1} - J_{c_k A} x_k|| \le ||\alpha_k u + \beta_k x_k + \delta_k e_k|| + |\gamma_k - 1| ||J_{c_k A} (x_k)||,$$

which, by  $\|\alpha_k u + \beta_k x_k + \delta_k e_k\| + |\gamma_k - 1| \|J_{c_k A}(x_k)\| \to 0$ , necessitates that  $x_{k+1} - J_{c_k A} x_k \to 0$ .

(iii): In view of [1, Lemma 2.45], the boundedness of  $(x_k)_{k \in \mathbb{N}}$  ensures  $\Omega \neq \emptyset$ . Hence, the required result is clear from (ii) and Proposition 2.16(ii).

The idea of the following proof is motivated by the proof of [9, Theorem 1].

**Proposition 3.6.** Suppose that  $(x_k)_{k \in \mathbb{N}}$  is bounded and that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ ,  $\sum_{k \in \mathbb{N}} |\alpha_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} |\delta_k e_k| < \infty$ , and  $\gamma_k \to 1$ . Then the following statements hold.

(i) 
$$x_k - J_{c_k A} x_k \rightarrow 0$$
.

(ii) If  $\inf_{k\in\mathbb{N}} c_k > 0$  or  $c_k \to \infty$ , then  $\emptyset \neq \Omega \subseteq \operatorname{zer} A$ .

*Proof.* (i): Note that  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k| < \infty$  and  $\gamma_k \to 1$  entail  $\beta_k \to 0$ . In view of Proposition 3.5(i), our assumptions guarantee that  $(J_{c_k A} x_k)_{k \in \mathbb{N}}$  is bounded. Then by Proposition 2.15, there exists a maximally monotone operator  $\tilde{A} : \mathcal{H} \to 2^{\mathcal{H}}$  such that

$$\operatorname{zer} \tilde{A} \neq \varnothing, \quad (\Omega \cap \operatorname{zer} \tilde{A}) \subseteq \operatorname{zer} A, \quad \text{and} \quad (\forall k \in \mathbb{N}) \ \operatorname{J}_{c_k A} x_k = \operatorname{J}_{c_k \tilde{A}} x_k.$$
(3.8)

Let  $p \in \text{zer } \tilde{A}$ . Because  $\tilde{A}$  is maximally monotone, via Lemma 3.2(i) and (3.8),

$$(\forall k \in \mathbb{N}) \quad \|x_{k+1} - p\| \le \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right) \|x_k - p\| + \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|,$$

which, combining with the assumption and Fact 2.5, guarantees that  $\lim_{k\to\infty} ||x_k - p||$  exists in  $\mathbb{R}_+$ .

Using the maximal monotonicity of  $\tilde{A}$  again and noticing that  $(\forall k \in \mathbb{N}) c_k \in \mathbb{R}_{++}$ , via Fact 2.9(iii)&(iv) and Definition 2.8(i), we observe that  $(\forall k \in \mathbb{N}) \|J_{c_k \tilde{A}} x_k - p\|^2 + \|(\mathrm{Id} - J_{c_k \tilde{A}}) x_k\|^2 \le \|x_k - p\|^2$ , which implies that for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \left\| x_{k} - \mathbf{J}_{c_{k}\tilde{A}} x_{k} \right\|^{2} &- \|x_{k} - p\|^{2} + \|x_{k+1} - p\|^{2} \leq - \left\| \mathbf{J}_{c_{k}\tilde{A}} x_{k} - p \right\|^{2} + \|x_{k+1} - p\|^{2} \\ &= \left\langle x_{k+1} - p + p - \mathbf{J}_{c_{k}\tilde{A}} x_{k}, x_{k+1} - p + \mathbf{J}_{c_{k}\tilde{A}} x_{k} - p \right\rangle \\ &\leq \left\| x_{k+1} - \mathbf{J}_{c_{k}\tilde{A}} x_{k} \right\| \left( \|x_{k+1} - p\| + \left\| \mathbf{J}_{c_{k}\tilde{A}} x_{k} - p \right\| \right). \end{aligned}$$

Hence,

$$(\forall k \in \mathbb{N}) \quad \left\| x_k - \mathbf{J}_{c_k \tilde{A}} x_k \right\|^2 \le \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \|x_{k+1} - \mathbf{J}_{c_k \tilde{A}} x_k\| \left( \|x_{k+1} - p\| + \|\mathbf{J}_{c_k \tilde{A}} x_k - p\| \right).$$

This together with Proposition 3.5(ii), the existence of  $\lim_{k\to\infty} ||x_k - p||$ , and the boundedness of  $(x_k)_{k\in\mathbb{N}}$ and  $(J_{c_k\tilde{A}} x_k)_{k\in\mathbb{N}}$  leads to  $x_k - J_{c_k\tilde{A}} x_k \to 0$ , which, due to (3.8), forces  $x_k - J_{c_kA} x_k \to 0$ .

(ii): Note that the boundedness of  $(x_k)_{k \in \mathbb{N}}$  forces  $\Omega((x_k)_{k \in \mathbb{N}}) \neq \emptyset$ . Furthermore, based on (i), the required inclusion follows immediately from the assumption and Proposition 2.16(i)&(ii).

The following result is inspired by the proof of [17, Theorem 4] which improves the strong convergence of the regularization method for the proximal point algorithm in [14, Theorem 3.3].

**Proposition 3.7.** Suppose that  $(x_k)_{k \in \mathbb{N}}$  is bounded and that  $(\forall k \in \mathbb{N}) \beta_k + \gamma_k \leq 1, \alpha_k \to 0$ ,  $\limsup_{k \to \infty} |\beta_k| < 1, 1 - \alpha_k - \beta_k - \gamma_k \to 0, 0 < \liminf_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} \leq \limsup_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} < 1, \delta_k e_k \to 0, and 1 - \frac{c_k}{c_{k+1}} \to 0$ . Then the following hold.

- (i)  $(\mathbf{J}_{c_kA} \mathbf{x}_k)_{k \in \mathbb{N}}$  is bounded.
- (ii)  $x_k J_{c_k A}(x_k) \rightarrow 0.$
- (iii) If  $\inf_{k \in \mathbb{N}} c_k > 0$  or  $c_k \to \infty$ , then  $\emptyset \neq \Omega \subseteq \operatorname{zer} A$ .

*Proof.* (i): According to our assumption, it is easy to see that

$$\begin{split} \frac{1}{2} \liminf_{k \to \infty} |\gamma_k| &= \liminf_{k \to \infty} \left| 1 - \beta_k - \frac{\gamma_k}{2} - (1 - \alpha_k - \beta_k - \gamma_k) - \alpha_k \right| \\ &\geq \liminf_{k \to \infty} \left| 1 - \beta_k - \frac{\gamma_k}{2} \right| - \limsup_{k \to \infty} |1 - \alpha_k - \beta_k - \gamma_k| - \limsup_{k \to \infty} |\alpha_k| \\ &= \liminf_{k \to \infty} \left| 1 - \beta_k - \frac{\gamma_k}{2} \right| > 0. \end{split}$$

This combined with our assumptions and Proposition 3.5(i) entails the boundedness of  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$ .

(ii): Denote by  $(\forall k \in \mathbb{N}) \eta_k := 1 - \beta_k - \frac{\gamma_k}{2}$ . Inasmuch as  $0 < \liminf_{i \to \infty} \eta_i \le \limsup_{i \to \infty} \eta_i < 1$  and  $\limsup_{k \to \infty} |\beta_k| < 1$ , without loss of generality, we assume that

$$(\forall k \in \mathbb{N}) \quad \eta_k \in ]0,1] \quad \text{and} \quad |\beta_k| < 1,$$

which, in connection with  $(\forall k \in \mathbb{N}) \beta_k + \gamma_k \leq 1$ , implies that

$$(\forall k \in \mathbb{N})$$
  $1 + \frac{\gamma_k}{2} = \left(1 - \beta_k - \frac{\gamma_k}{2}\right) + \beta_k + \gamma_k \le 2$  and  $\frac{\gamma_k}{2} + 1 \ge 1 - \beta_k \ge 1 - |\beta_k| > 0.$ 

Hence,  $(\forall k \in \mathbb{N}) \gamma_k \in [-2, 2]$  and  $(\gamma_i)_{i \in \mathbb{N}}$  is bounded.

Set  $(\forall k \in \mathbb{N})$   $T_k := 2 J_{c_k A} - Id$ . Bearing Lemma 3.1 in mind, we observe that

$$(\forall k \in \mathbb{N}) \quad x_{k+1} = (1 - \eta_k) x_k + \eta_k y_k, \tag{3.9}$$

where  $(\forall k \in \mathbb{N}) y_k := \frac{1}{\eta_k} \left( \frac{\gamma_k}{2} T_k(x_k) + \alpha_k u + \delta_k e_k \right)$ . Note that for every  $k \in \mathbb{N}$ ,

$$\|y_{k+1} - y_k\| \le \left\|\frac{\gamma_{k+1}}{2\eta_{k+1}} T_{k+1}(x_{k+1}) - \frac{\gamma_k}{2\eta_k} T_k(x_k)\right\| + \left|\frac{\alpha_{k+1}}{\eta_{k+1}} - \frac{\alpha_k}{\eta_k}\right\| \|u\| + \left\|\frac{\delta_{k+1}}{\eta_{k+1}} e_{k+1} - \frac{\delta_k}{\eta_k} e_k\right\|.$$
 (3.10)

Moreover, apply Fact 2.11(ii) and recall that  $(\forall k \in \mathbb{N}) T_k$  is nonexpansive in the following second inequality to see that for every  $k \in \mathbb{N}$ ,

$$\left\|\frac{\gamma_{k+1}}{2\eta_{k+1}}T_{k+1}(x_{k+1}) - \frac{\gamma_k}{2\eta_k}T_k(x_k)\right\|$$
(3.11a)

$$\leq \frac{|\gamma_{k+1}|}{2\eta_{k+1}} \|T_{k+1}(x_{k+1}) - T_k(x_{k+1})\| + \frac{|\gamma_{k+1}|}{2\eta_{k+1}} \|T_k(x_{k+1}) - T_k(x_k)\| + \left|\frac{\gamma_{k+1}}{2\eta_{k+1}} - \frac{\gamma_k}{2\eta_k}\right| \|T_k(x_k)\|$$
(3.11b)

$$\leq \frac{|\gamma_{k+1}|}{2\eta_{k+1}} \left| 1 - \frac{c_k}{c_{k+1}} \right| \|T_{k+1}(x_{k+1}) - x_{k+1}\| + \frac{|\gamma_{k+1}|}{2\eta_{k+1}} \|x_{k+1} - x_k\| + \left|\frac{\gamma_{k+1}}{2\eta_{k+1}} - \frac{\gamma_k}{2\eta_k}\right| \|T_k(x_k)\|.$$
(3.11c)

Because  $(\forall k \in \mathbb{N}) |\beta_k| < 1$  and  $\beta_k + \gamma_k \leq 1$ , for every  $k \in \mathbb{N}$ , if  $\gamma_{k+1} \leq 0$ , then  $|\gamma_{k+1}| + \gamma_{k+1} + 2\beta_{k+1} = 2\beta_{k+1} \leq 2$ ; otherwise,  $|\gamma_{k+1}| + \gamma_{k+1} + 2\beta_{k+1} = 2(\gamma_{k+1} + \beta_{k+1}) \leq 2$ . This together with the equivalence  $(\forall k \in \mathbb{N}) \frac{|\gamma_{k+1}|}{2\eta_{k+1}} = \frac{|\gamma_{k+1}|}{2-2\beta_{k+1}-\gamma_{k+1}} \leq 1 \Leftrightarrow |\gamma_{k+1}| + \gamma_{k+1} + 2\beta_{k+1} \leq 2$  implies that  $(\forall k \in \mathbb{N}) \frac{|\gamma_{k+1}|}{2\eta_{k+1}} \leq 1$ . Then combine (3.10) and (3.11) to obtain that

$$(\forall k \in \mathbb{N}) \quad ||y_{k+1} - y_k|| \le ||x_{k+1} - x_k|| + \Lambda(k),$$
 (3.12)

where  $(\forall k \in \mathbb{N}) \Lambda(k) := \frac{|\gamma_{k+1}|}{2\eta_{k+1}} \left| 1 - \frac{c_k}{c_{k+1}} \right| \|T_{k+1}(x_{k+1}) - x_{k+1}\| + \left| \frac{\gamma_{k+1}}{2\eta_{k+1}} - \frac{\gamma_k}{2\eta_k} \right| \|T_k(x_k)\| + \left| \frac{\alpha_{k+1}}{\eta_{k+1}} - \frac{\alpha_k}{\eta_k} \right| \|u\| + \left\| \frac{\delta_{k+1}}{\eta_{k+1}} e_{k+1} - \frac{\delta_k}{\eta_k} e_k \right\|.$ 

Note that the boundedness of  $(x_k)_{k \in \mathbb{N}}$  and  $(J_{c_k A} x_k)_{k \in \mathbb{N}}$  implies that  $(||T_{k+1}(x_{k+1}) - x_{k+1}||)_{k \in \mathbb{N}}$  and  $(||T_k(x_k)||)_{k \in \mathbb{N}}$  are bounded. Combine this with  $0 < \lim \inf_{k \to \infty} \eta_k$  and the boundedness of  $(\gamma_k)_{k \in \mathbb{N}}$  to deduce the boundedness of  $(y_k)_{k \in \mathbb{N}}$ .

In addition, by some easy algebra, it is not difficult to verify that

$$\begin{split} \left| \frac{\gamma_{k+1}}{2\eta_{k+1}} - \frac{\gamma_k}{2\eta_k} \right| &= \frac{1}{2\eta_{k+1}\eta_k} \left| \gamma_{k+1}(1 - \beta_k) - \gamma_k(1 - \beta_{k+1}) \right|;\\ \gamma_{k+1}(1 - \beta_k) - \gamma_k(1 - \beta_{k+1}) &= \gamma_{k+1} \left( 1 - \alpha_k - \beta_k - \gamma_k \right) - \gamma_k \left( 1 - \alpha_{k+1} - \beta_{k+1} - \gamma_{k+1} \right) + \gamma_{k+1}\alpha_k - \gamma_k\alpha_{k+1};\\ \left| \frac{\alpha_{k+1}}{\eta_{k+1}} - \frac{\alpha_k}{\eta_k} \right| &\leq \frac{|\alpha_{k+1}| + |\alpha_k|}{\eta_{k+1}\eta_k};\\ \left\| \frac{\delta_{k+1}}{\eta_{k+1}} e_{k+1} - \frac{\delta_k}{\eta_k} e_k \right\| &\leq \frac{\|\delta_{k+1}e_{k+1}\| + \|\delta_k e_k\|}{\eta_{k+1}\eta_k}, \end{split}$$

which, connecting with the assumption, yields  $\lim_{k\to\infty} \Lambda(k) = 0$ . This and (3.12) necessitate

$$\limsup_{k\to\infty} \|y_{k+1} - y_k\| - \|x_{k+1} - x_k\| \le 0.$$

Employing (3.9) and applying Fact 2.4 with  $(\forall k \in \mathbb{N}) u_k = x_k$ ,  $\alpha_k = \eta_k$ , and  $v_k = y_k$ , we know that the inequality above leads to

$$y_k - x_k \to 0$$
 and  $||x_{k+1} - x_k|| \stackrel{(3.9)}{=} \eta_k ||y_k - x_k|| \to 0.$  (3.13)

Notice that the assumptions  $\limsup_{k\to\infty} |\beta_k| < 1$  and  $(\forall k \in \mathbb{N}) |\beta_k| < 1$  ensure the boundedness of  $\left(\frac{1}{1-|\beta_k|}\right)_{k\in\mathbb{N}}$ . Furthermore, for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|x_{k} - J_{c_{k}A} x_{k}\| &\leq \|x_{k} - x_{k+1}\| + \|x_{k+1} - J_{c_{k}A} x_{k}\| \\ &\stackrel{(3.1)}{=} \|x_{k} - x_{k+1}\| + \|\alpha_{k}u + \beta_{k}x_{k} + \gamma_{k}J_{c_{k}A}(x_{k}) + \delta_{k}e_{k} - J_{c_{k}A} x_{k}\| \\ &\leq \|x_{k} - x_{k+1}\| + |\alpha_{k}| \|u - J_{c_{k}A} x_{k}\| + |\beta_{k}| \|x_{k} - J_{c_{k}A} x_{k}\| + \|\delta_{k}e_{k}\| + |\varphi_{k}| \|J_{c_{k}A} x_{k}\|. \end{aligned}$$

The inequalities above ensure that

$$(\forall k \in \mathbb{N}) \quad \|x_k - \mathbf{J}_{c_k A} x_k\| \le \frac{1}{1 - |\beta_k|} \left( \|x_k - x_{k+1}\| + |\alpha_k| \|u - \mathbf{J}_{c_k A} x_k\| + \|\delta_k e_k\| + |\varphi_k| \|\mathbf{J}_{c_k A} x_k\| \right),$$

which, employing (3.13) and the assumption, guarantees that  $x_k - J_{c_kA} x_k \rightarrow 0$ . (iii): This is clear from (ii) and Proposition 2.16.

The following Proposition 3.8 is inspired by [8, Lemma 3.2]. Moreover, if  $(\forall k \in \mathbb{N}) \alpha_k \equiv 0, \delta_k \equiv 1, \gamma_k \in ]0, 2[$ , and  $\beta_k = 1 - \gamma_k$ , then Proposition 3.8(i)&(ii)&(iii) reduce to [8, Lemma 3.2].

**Proposition 3.8.** Suppose that  $\operatorname{zer} A \neq \emptyset$ , that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1$  and  $\gamma_k(\beta_k + \gamma_k) \geq 0$ , and that  $\sum_{i \in \mathbb{N}} |\alpha_i| < \infty$ ,  $\sum_{i \in \mathbb{N}} |1 - \beta_i - \gamma_i| < \infty$ , and  $\sum_{i \in \mathbb{N}} |\delta_i e_i| < \infty$ . Then the following statements hold.

(i) 
$$\sum_{k=0}^{\infty} \gamma_k (2\beta_k + \gamma_k) \|x_k - \mathbf{J}_{c_k A} x_k\|^2 < \infty$$
.

(ii) If 
$$\sum_{k \in \mathbb{N}} \gamma_k (2\beta_k + \gamma_k) = \infty$$
 and  $\inf_{k \in \mathbb{N}} \gamma_k (2\beta_k + \gamma_k) \ge 0$ , then  $\liminf_{k \to \infty} ||x_k - J_{c_k A}(x_k)|| = 0$ .

- (iii) Suppose that  $\liminf_{k\to\infty} \gamma_k(2\beta_k + \gamma_k) > 0$ . Then  $x_k J_{c_kA}(x_k) \to 0$ .
- (iv) Suppose that  $\liminf_{k\to\infty} \gamma_k(2\beta_k + \gamma_k) > 0$  and that  $\inf_{k\in\mathbb{N}} c_k > 0$  or  $c_k \to \infty$ . Then  $\emptyset \neq \Omega \subseteq \operatorname{zer} A$ .

*Proof.* The assumption and Proposition 3.3(ii) imply the boundedness of  $(x_k)_{k \in \mathbb{N}}$ .

(i): Let  $p \in \text{zer } A$ . Set  $(\forall k \in \mathbb{N}) M(k) := \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|$ . According to Lemma 3.2(iii), for every  $k \in \mathbb{N}$ ,  $\gamma_k (2\beta_k + \gamma_k) \|x_k - J_{c_k A} x_k\|^2 \le (\beta_k + \gamma_k)^2 \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + 2M(k) \|x_{k+1} - p\| \le \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + 2M(k) \|x_{k+1} - p\|$ , which, combining with the assumption, derives

$$\sum_{i=0}^{k} \gamma_i (2\beta_i + \gamma_i) \left\| x_i - \mathbf{J}_{c_i A} \, x_i \right\|^2 \le \|x_0 - p\|^2 + 2\sum_{i=0}^{k} M(i) \, \|x_{i+1} - p\| \le \|x_0 - p\|^2 + 2L_1 L_2 < \infty,$$

where  $L_1 := \sup_{k \in \mathbb{N}} ||x_k - p|| < \infty$  and  $L_2 := \sum_{k \in \mathbb{N}} M(k) < \infty$ . This verifies (i).

(ii): According to the assumption and (i),

$$\infty > \sum_{k=0}^{\infty} \gamma_k (2\beta_k + \gamma_k) \left\| x_k - \mathsf{J}_{c_k A} x_k \right\|^2 \ge \liminf_{k \to \infty} \left\| x_k - \mathsf{J}_{c_k A} (x_k) \right\| \sum_{k \in \mathbb{N}} \gamma_k (2\beta_k + \gamma_k),$$

which, noticing  $\sum_{k \in \mathbb{N}} \gamma_k (2\beta_k + \gamma_k) = \infty$ , forces  $\liminf_{k \to \infty} ||x_k - J_{c_k A}(x_k)|| = 0$ .

(iii): As a consequence of  $\eta := \liminf_{k \to \infty} \gamma_k (2\beta_k + \gamma_k) > 0$ , there exists  $N \in \mathbb{N}$  such that  $(\forall k \ge N) \gamma_k (2\beta_k + \gamma_k) \ge \frac{\eta}{2} > 0$ . Hence, for every  $k \ge N$ 

$$\sum_{i=0}^{k} \gamma_i (2\beta_i + \gamma_i) \|x_i - \mathbf{J}_{c_i A} x_i\|^2 \ge \sum_{i=0}^{N-1} \gamma_i (2\beta_i + \gamma_i) \|x_i - \mathbf{J}_{c_i A} x_i\|^2 + \frac{\eta}{2} \sum_{i=N}^{k} \|x_i - \mathbf{J}_{c_i A} x_i\|^2.$$

Combine this with (i) to obtain that  $\sum_{i=N}^{\infty} ||x_i - J_{c_iA} x_i||^2 < \infty$ , which yields  $x_k - J_{c_kA}(x_k) \to 0$ . (iv): This is immediate from (iii) and Proposition 2.16.

The following proof is motivated by [8, Theorem 3.6].

**Proposition 3.9.** Suppose that  $\operatorname{zer} A \neq \emptyset$ , that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1$ ,  $\gamma_k(\beta_k + \gamma_k) \geq 0$ ,  $\gamma_k(2\beta_k + \gamma_k) \geq 0$ , and  $|\beta_k + \gamma_k| |\gamma_k| \leq \max\{1 - |\beta_k|, 2 - 2 |\beta_k + \frac{\gamma_k}{2}|\}$ , that  $\sum_{k \in \mathbb{N}} |\alpha_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} \gamma_k(2\beta_k + \gamma_k) = \infty$ , and  $\sum_{k \in \mathbb{N}} |\delta_k e_k| < \infty$ , and that  $\overline{c} := \inf_{k \in \mathbb{N}} c_k > 0$  and  $\sum_{k \in \mathbb{N}} |c_{k+1} - c_k| < \infty$ . Then  $x_k - J_{c_k A}(x_k) \to 0$  and  $\emptyset \neq \Omega \subseteq \operatorname{zer} A$ .

*Proof.* Note that, via Proposition 3.3(ii) and Proposition 3.8(ii), our assumptions force that  $(x_k)_{k \in \mathbb{N}}$  is bounded and that

$$\liminf_{k \to \infty} \left\| x_k - \mathcal{J}_{c_k A} \, x_k \right\| = 0. \tag{3.14}$$

Combining this with Proposition 2.16(i) and the assumption  $\bar{c} := \inf_{k \in \mathbb{N}} c_k > 0$ , we know that it suffices to show that  $\lim_{k\to\infty} ||x_k - J_{c_kA} x_k|| = 0$ .

Set  $(\forall k \in \mathbb{N})$   $T_k := 2 J_{c_k A} - \text{Id.}$  Denote by  $(\forall k \in \mathbb{N})$   $s_k := ||x_k - J_{c_k A} x_k||$ . Then

$$(\forall k \in \mathbb{N}) \quad ||x_k - T_k x_k|| = 2 ||x_k - J_{c_k A}(x_k)|| = 2s_k.$$
 (3.15)

Due to Proposition 3.5(i),  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$  and  $(T_k x_k)_{k \in \mathbb{N}}$  are bounded. Hence,  $\hat{s} := \sup_{k \in \mathbb{N}} s_k < \infty$ . In view of Lemma 3.1,  $(\forall k \in \mathbb{N}) T_k$  is nonexpansive and  $x_{k+1} = (\beta_k + \frac{\gamma_k}{2}) x_k + \frac{\gamma_k}{2} T_k(x_k) + \alpha_k u + \delta_k e_k$ , which ensures that for every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|x_{k+1} - T_{k+1}x_{k+1}\| \\ &= \left\| \left( \beta_k + \frac{\gamma_k}{2} \right) (x_k - T_k x_k) + (\beta_k + \gamma_k) (T_k(x_k) - T_{k+1}x_{k+1}) + \alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) T_{k+1}x_{k+1} \right\| \\ &\leq \left| \beta_k + \frac{\gamma_k}{2} \right| \|x_k - T_k x_k\| + |\beta_k + \gamma_k| \|T_k(x_k) - T_{k+1}x_{k+1}\| + \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) T_{k+1}x_{k+1}\| . \end{aligned}$$

Set  $(\forall k \in \mathbb{N})$   $F_1(k) := \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)T_{k+1}x_{k+1}\|$ . Then we establish that for every  $k \in \mathbb{N}$ ,

$$\|x_{k+1} - T_{k+1}x_{k+1}\| \le \left|\beta_k + \frac{\gamma_k}{2}\right| \|x_k - T_kx_k\| + |\beta_k + \gamma_k| \|T_k(x_k) - T_{k+1}x_{k+1}\| + F_1(k).$$
(3.16)

Similarly, via (3.1), we get that for every  $k \in \mathbb{N}$ ,

$$\left\| x_{k+1} - \mathbf{J}_{c_{k+1}A} \, x_{k+1} \right\| \le |\beta_k| \, \left\| x_k - \mathbf{J}_{c_kA} \, x_k \right\| + |\beta_k + \gamma_k| \, \left\| \mathbf{J}_{c_kA} \, x_k - \mathbf{J}_{c_{k+1}A} \, x_{k+1} \right\| + F_2(k), \tag{3.17}$$

where  $(\forall k \in \mathbb{N})$   $F_2(k) := \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) J_{c_{k+1}A} x_{k+1}\|$ . Furthermore, using (3.1) again, we observe that for every  $k \in \mathbb{N}$ ,

$$\|x_{k+1} - x_k\| = \|\alpha_k u + \beta_k x_k + \gamma_k J_{c_k A}(x_k) + \delta_k e_k - x_k\| \le |\gamma_k| \|x_k - J_{c_k A} x_k\| + G_1(k),$$
(3.18a)  
$$\|x_{k+1} - J_{c_k A} x_k\| = \|\alpha_k u + \beta_k x_k + \gamma_k J_{c_k A}(x_k) + \delta_k e_k - J_{c_k A} x_k\| \le |\beta_k| \|x_k - J_{c_k A} x_k\| + G_2(k),$$
(3.18b)

where  $G_1(k) := \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) x_k\|$  and  $G_2(k) := \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k) J_{c_k A} x_k\|$ . Let  $k \in \mathbb{N}$ . We have exactly the following two cases.

*Case 1*:  $c_k \leq c_{k+1}$ . Then invoking Fact 2.11(i) in the following equality, using the nonexpansiveness of  $T_k$  in the first inequality, and employing  $||x_k - J_{c_{k+1}A} x_{k+1}|| \leq ||x_k - x_{k+1}|| + ||x_{k+1} - J_{c_{k+1}A} x_{k+1}||$  in the second inequality below, we get that

$$\begin{aligned} \|T_{k}(x_{k}) - T_{k+1}(x_{k+1})\| \\ &= \left\| T_{k}(x_{k}) - T_{k} \left( \frac{c_{k}}{c_{k+1}} x_{k+1} + \left( 1 - \frac{c_{k}}{c_{k+1}} \right) J_{c_{k+1}A} x_{k+1} \right) \right\| + \left( 1 - \frac{c_{k}}{c_{k+1}} \right) \left\| x_{k+1} - J_{c_{k+1}A} x_{k+1} \right\| \\ &\leq \frac{c_{k}}{c_{k+1}} \left\| x_{k} - x_{k+1} \right\| + \left( 1 - \frac{c_{k}}{c_{k+1}} \right) \left\| x_{k} - J_{c_{k+1}A} x_{k+1} \right\| + \left( 1 - \frac{c_{k}}{c_{k+1}} \right) \left\| x_{k+1} - J_{c_{k+1}A} x_{k+1} \right\| \\ &\leq \|x_{k} - x_{k+1}\| + 2 \left( 1 - \frac{c_{k}}{c_{k+1}} \right) \left\| x_{k+1} - J_{c_{k+1}A} x_{k+1} \right\| \\ \end{aligned}$$

$$\begin{aligned} \text{(3.18a)} \\ &\leq \|\gamma_{k}\| \left\| x_{k} - J_{c_{k}A} x_{k} \right\| + G_{1}(k) + 2 \left( 1 - \frac{c_{k}}{c_{k+1}} \right) \left\| x_{k+1} - J_{c_{k+1}A} x_{k+1} \right\|. \end{aligned}$$

Applying this result, (3.15), and (3.16) in the first inequality below and employing  $|\beta_k + \gamma_k| \le |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$  and  $2|\beta_k + \frac{\gamma_k}{2}| + |\beta_k + \gamma_k| |\gamma_k| \le 2(|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|) \le 2$  in the second inequality below, we get that

$$2s_{k+1} \stackrel{(3.15)}{=} \|x_{k+1} - T_{k+1}x_{k+1}\| \le 2 \left|\beta_k + \frac{\gamma_k}{2}\right| s_k + |\beta_k + \gamma_k| \left(|\gamma_k| s_k + G_1(k) + 2\left(1 - \frac{c_k}{c_{k+1}}\right) s_{k+1}\right) + F_1(k) \le 2s_k + 2\left(1 - \frac{c_k}{c_{k+1}}\right) s_{k+1} + F_1(k) + G_1(k),$$

which implies that

$$\frac{s_{k+1}}{c_{k+1}} \le \frac{s_k}{c_k} + \frac{F_1(k) + G_1(k)}{2\bar{c}}.$$
(3.19)

*Case 2*:  $c_{k+1} < c_k$ . We claim that

$$s_{k+1} \le s_k + \left(1 - \frac{c_{k+1}}{c_k}\right)s_k + F_1(k) + F_2(k) + G_1(k) + G_2(k).$$
 (3.20)

*Case 2.1*: Assume that  $|\beta_k + \gamma_k| |\gamma_k| \le 1 - |\beta_k|$ , i.e.,  $|\beta_k| + |\beta_k + \gamma_k| |\gamma_k| \le 1$ . Applying Fact 2.10 in the first equality, utilizing the nonexpansiveness of  $J_{c_kA}$  in the first inequality, and employing (3.18a) and (3.18b) in the second inequality, we deduce that

$$\begin{split} & \left\| J_{c_{k}A} x_{k} - J_{c_{k+1}A} x_{k+1} \right\| \\ &= \left\| J_{c_{k+1}A} \left( \frac{c_{k+1}}{c_{k}} x_{k} + \left( 1 - \frac{c_{k+1}}{c_{k}} \right) J_{c_{k}A} x_{k} \right) - J_{c_{k+1}A} x_{k+1} \right\| \\ &\leq \frac{c_{k+1}}{c_{k}} \left\| x_{k} - x_{k+1} \right\| + \left( 1 - \frac{c_{k+1}}{c_{k}} \right) \left\| J_{c_{k}A} x_{k} - x_{k+1} \right\| \\ &\leq |\gamma_{k}| \frac{c_{k+1}}{c_{k}} \left\| x_{k} - J_{c_{k}A} x_{k} \right\| + |\beta_{k}| \left( 1 - \frac{c_{k+1}}{c_{k}} \right) \left\| x_{k} - J_{c_{k}A} x_{k} \right\| + \frac{c_{k+1}}{c_{k}} G_{1}(k) + \left( 1 - \frac{c_{k+1}}{c_{k}} \right) G_{2}(k). \end{split}$$

Combine this with (3.17) and some easy algebra to get that

$$\begin{split} s_{k+1} &\leq |\beta_k| \, s_k + |\beta_k + \gamma_k| \left( \left( |\gamma_k| \frac{c_{k+1}}{c_k} + |\beta_k| \left( 1 - \frac{c_{k+1}}{c_k} \right) \right) s_k + \frac{c_{k+1}}{c_k} G_1(k) + \left( 1 - \frac{c_{k+1}}{c_k} \right) G_2(k) \right) + F_2(k) \\ &\leq (|\beta_k| + |\beta_k + \gamma_k| |\gamma_k|) \, s_k + |\beta_k + \gamma_k| \left( 1 - \frac{c_{k+1}}{c_k} \right) (|\beta_k| - |\gamma_k|) \, s_k + G_1(k) + G_2(k) + F_2(k) \\ &\leq s_k + \left( 1 - \frac{c_{k+1}}{c_k} \right) s_k + G_1(k) + G_2(k) + F_2(k), \end{split}$$

which, using  $|\beta_k| - |\gamma_k| \le |\beta_k + \gamma_k| \le |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$  and the assumption  $|\beta_k| + |\beta_k + \gamma_k| |\gamma_k| \le 1$  in the last inequality, verifies (3.20).

*Case 2.2:* Assume that  $(\forall k \in \mathbb{N}) |\beta_k + \gamma_k| |\gamma_k| \le 2 - 2 |\beta_k + \frac{\gamma_k}{2}|$ , i.e.,  $2 |\beta_k + \frac{\gamma_k}{2}| + |\beta_k + \gamma_k| |\gamma_k| \le 2$ . Similarly with the proof of Case 1 above, we have that

$$\|T_k(x_k) - T_{k+1}(x_{k+1})\| \le |\gamma_k| \|x_k - J_{c_kA} x_k\| + G_1(k) + 2\left(1 - \frac{c_{k+1}}{c_k}\right) \|x_k - J_{c_kA} x_k\|.$$
(3.21)

Invoking (3.16), (3.15), and (3.21) in the first inequality and using the assumptions  $|\beta_k + \gamma_k| \le |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$  and  $2|\beta_k + \frac{\gamma_k}{2}| + |\beta_k + \gamma_k| |\gamma_k| \le 2$  in the last inequality below, we observe that

$$2s_{k+1} \stackrel{(3.15)}{=} ||x_{k+1} - T_{k+1}x_{k+1}|| \\ \leq 2 \left| \beta_k + \frac{\gamma_k}{2} \right| s_k + |\beta_k + \gamma_k| \left( \left( |\gamma_k| + 2\left(1 - \frac{c_{k+1}}{c_k}\right) \right) s_k + G_1(k) \right) + F_1(k) \\ \leq \left( 2 \left| \beta_k + \frac{\gamma_k}{2} \right| + |\beta_k + \gamma_k| |\gamma_k| \right) s_k + 2 \left| \beta_k + \gamma_k \right| \left( 1 - \frac{c_{k+1}}{c_k} \right) s_k + F_1(k) + G_1(k) \\ \leq 2s_k + 2 \left( 1 - \frac{c_{k+1}}{c_k} \right) s_k + F_1(k) + G_1(k),$$

which confirming (3.20) as well.

Therefore, in both subcases, the claim is true and we have that

$$\frac{s_{k+1}}{c_{k+1}} \le \frac{s_k}{c_{k+1}} + \frac{c_k - c_{k+1}}{c_k c_{k+1}} s_k + \frac{F_1(k) + F_2(k) + G_1(k) + G_2(k)}{2\bar{c}}$$
(3.22a)

$$= \frac{s_k}{c_k} + s_k \left( \frac{1}{c_{k+1}} - \frac{1}{c_k} + \frac{c_k - c_{k+1}}{c_k c_{k+1}} \right) + \frac{F_1(k) + F_2(k) + G_1(k) + G_2(k)}{2\bar{c}}$$
(3.22b)

$$=\frac{s_k}{c_k} + \frac{c_k - c_{k+1} + c_k - c_{k+1}}{c_k c_{k+1}} s_k + \frac{F_1(k) + F_2(k) + G_1(k) + G_2(k)}{2\bar{c}}$$
(3.22c)

$$\leq \frac{s_k}{c_k} + 2\frac{c_k - c_{k+1}}{\bar{c}^2}\hat{s} + \frac{F_1(k) + F_2(k) + G_1(k) + G_2(k)}{2\bar{c}}.$$
(3.22d)

Furthermore, by assumptions,  $\sum_{k \in \mathbb{N}} |c_{k+1} - c_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} F_1(k) < \infty$ ,  $\sum_{k \in \mathbb{N}} F_2(k) < \infty$ ,  $\sum_{k \in \mathbb{N}} G_1(k) < \infty$ , and  $\sum_{k \in \mathbb{N}} G_2(k) < \infty$ . Hence, applying Fact 2.5, (3.19), and (3.22), we obtain that in both cases,  $\lim_{k\to\infty} \frac{s_k}{c_k}$  exists in  $\mathbb{R}_+$ . Clearly, (3.14) and (3.15) necessitate  $\lim_{k\to\infty} \inf_{k\to\infty} s_k = 0$ . These results imply that  $\lim_{k\to\infty} \frac{s_k}{c_k} = \lim_{k\to\infty} \inf_{k\to\infty} \frac{s_k}{c_k} \leq \lim_{k\to\infty} \inf_{k\to\infty} \frac{s_k}{c_k} \leq \lim_{k\to\infty} \inf_{k\to\infty} \frac{s_k}{c_k} = \lim_{k\to\infty} \sup_{k\to\infty} s_k = \lim_{k\to\infty} \sup_{k\to\infty} \frac{s_k}{c_k} c_k \leq \lim_{k\to\infty} \frac{s_k}{c_k} \sup_{k\in\mathbb{N}} c_k = 0$  since

$$\sum_{i\in\mathbb{N}}|c_{i+1}-c_i|<\infty\Rightarrow \sup_{k\in\mathbb{N}}c_k<\infty,$$

Recall that, via (3.14) and (3.15),  $\liminf_{k\to\infty} s_k = \liminf_{k\to\infty} \|x_k - \mathbf{J}_{c_k A} x_k\| = 0$ .

Altogether,  $\lim_{k\to\infty} ||x_k - J_{c_kA} x_k|| = \lim_{k\to\infty} s_k = 0.$ 

The following result is inspired by the proof of [8, Theorem 4.1].

**Proposition 3.10.** Suppose that  $\operatorname{zer} A \neq \emptyset$  and  $(x_k)_{k \in \mathbb{N}}$  is bounded, that  $\sum_{i \in \mathbb{N}} |\alpha_{i+1} - \alpha_i| < \infty$  or  $(\forall k \in \mathbb{N})$  $|\gamma_k| \neq 1$  with  $\lim_{k \to \infty} \frac{|\alpha_{k+1} - \alpha_k|}{1 - |\gamma_{k+1}|} = 0$ , that  $(\forall k \in \mathbb{N}) |\gamma_k| \in [0, 1]$  with  $\alpha_i + \gamma_i \to 1$ ,  $\sum_{i \in \mathbb{N}} (1 - |\gamma_i|) = \infty$ , and  $\sum_{i \in \mathbb{N}} |(\alpha_{i+1} + \gamma_{i+1}) - (\alpha_i + \gamma_i)| < \infty$ , that  $\alpha_k \to 0$ ,  $\sum_{k \in \mathbb{N}} |\beta_k| < \infty$ , and  $\sum_{k \in \mathbb{N}} |\delta_k e_k|| < \infty$ , and that  $\overline{c} := \inf_{k \in \mathbb{N}} c_k > 0$  and  $\sum_{k \in \mathbb{N}} |c_{k+1} - c_k| < \infty$ . Then  $x_k - J_{c_k A}(x_k) \to 0$  and  $\emptyset \neq \Omega \subseteq \operatorname{zer} A$ .

*Proof.* Because  $\bar{c} = \inf_{k \in \mathbb{N}} c_k > 0$ , via Proposition 2.16(i), it suffices to prove  $x_k - J_{c_k A}(x_k) \to 0$ .

Because  $(x_k)_{k \in \mathbb{N}}$  is bounded and  $\operatorname{zer} A \neq \emptyset$ , due to Proposition 3.5(i),  $(J_{c_k A} x_k)_{k \in \mathbb{N}}$  is bounded. Notice that for every  $k \in \mathbb{N}$ ,  $||x_k - J_{c_k A} x_k|| \le ||x_k - x_{k+1}|| + ||x_{k+1} - J_{c_k A} x_k||$  and that

$$\|x_{k+1} - J_{c_k A} x_k\| \stackrel{(3.1)}{\leq} \|\alpha_k (u - J_{c_k A} x_k)\| + |\alpha_k + \gamma_k - 1| \|J_{c_k A} x_k\| + \|\beta_k x_k\| + \|\delta_k e_k\| \to 0.$$

Hence, it remains to show that  $x_k - x_{k+1} \rightarrow 0$ .

Clearly,  $\sum_{k \in \mathbb{N}} |c_{k+1} - c_k| < \infty$  leads to  $\hat{c} := \sup_{k \in \mathbb{N}} c_k < \infty$ . In view of Fact 2.10, for every  $k \in \mathbb{N}$ ,

$$\left\| J_{c_{k+1}A} x_{k+1} - J_{c_kA} x_k \right\| = \left\| J_{c_kA} \left( \frac{c_k}{c_{k+1}} x_{k+1} + \left( 1 - \frac{c_k}{c_{k+1}} \right) J_{c_{k+1}A} x_{k+1} \right) - J_{c_kA} x_k \right\|$$
(3.23a)

$$\leq \frac{c_k}{c_{k+1}} \|x_{k+1} - x_k\| + \left|1 - \frac{c_k}{c_{k+1}}\right| \left\|J_{c_{k+1}A} x_{k+1} - x_k\right\|.$$
(3.23b)

Set  $(\forall k \in \mathbb{N} \setminus \{0\}) M(k) := |\gamma_k| |c_k - c_{k-1}| ||J_{c_kA} x_k - x_{k-1}|| + \hat{c} |\alpha_k + \gamma_k - \alpha_{k-1} - \gamma_{k-1}| ||J_{c_{k-1}A} x_{k-1}|| + \hat{c} ||\beta_k x_k - \beta_{k-1} x_{k-1} + \delta_k e_k - \delta_{k-1} e_{k-1}||.$  Based on the assumption, it is clear that  $\sum_{k=1}^{\infty} M(k) < \infty$ . Due to (3.1),  $(\forall k \in \mathbb{N} \setminus \{0\}), ||x_{k+1} - x_k|| \le |\alpha_k - \alpha_{k-1}| ||u - J_{c_{k-1}A} x_{k-1}|| + |\gamma_k| ||J_{c_kA} x_k - J_{c_{k-1}A} x_{k-1}|| + |\alpha_k + \gamma_k - \alpha_{k-1} - \gamma_{k-1}| ||J_{c_{k-1}A} x_{k-1}|| + ||\beta_k x_k - \beta_{k-1} x_{k-1} + \delta_k e_k - \delta_{k-1} e_{k-1}||,$  which, via (3.23), implies

$$c_{k} \|x_{k+1} - x_{k}\| \le |\gamma_{k}| c_{k-1} \|x_{k} - x_{k-1}\| + \hat{c} |\alpha_{k} - \alpha_{k-1}| \left\| u - J_{c_{k-1}A} x_{k-1} \right\| + M(k).$$
(3.24)

If  $\sum_{i \in \mathbb{N}} |\alpha_{i+1} - \alpha_i| < \infty$  (resp.  $(\forall k \in \mathbb{N}) |\gamma_k| \neq 1$  with  $\lim_{k \to \infty} \frac{|\alpha_{k+1} - \alpha_k|}{1 - |\gamma_{k+1}|} = 0$ ) is satisfied, then, by (3.24), applying Proposition 2.3(iii) with  $(\forall k \in \mathbb{N} \setminus \{0\}) t_k = c_{k-1} ||x_k - x_{k-1}||$ ,  $\alpha_k = |\gamma_i|$ ,  $\beta_k = \omega_k \equiv 0$ , and  $\gamma_k = \hat{c} |\alpha_k - \alpha_{k-1}| ||u - J_{c_{k-1}A} x_{k-1}|| + M(k)$  (resp.  $\beta_k = 1 - |\gamma_k|$ ,  $\omega_k = \frac{\hat{c} |\alpha_k - \alpha_{k-1}|}{1 - |\gamma_k|} ||u - J_{c_{k-1}A} x_{k-1}||$ , and  $\gamma_k = M(k)$ ), we obtain that  $\lim_{k \to \infty} c_k ||x_{k+1} - x_k|| = 0$ , which, in connection with  $\inf_{k \in \mathbb{N}} c_k > 0$ , guarantees that  $x_{k+1} - x_k \to 0$ . Therefore, the proof is complete.

### Equivalence of boundedness and non-emptiness of sets of zeroes

Theorem 3.11(i) reduces to the equivalence proved in [9, Theorem 1] when  $(\forall k \in \mathbb{N}) \gamma_k = \delta_k \equiv 1$  and  $\alpha_k = \beta_k \equiv 0$  and  $\inf_{k \in \mathbb{N}} c_k > 0$ .

**Theorem 3.11.** Suppose that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ , that  $\sum_{k \in \mathbb{N}} |\alpha_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k| < \infty$ , and  $\sum_{k \in \mathbb{N}} |\delta_k e_k| < \infty$ , and that one of the following statements holds.

- (i) Suppose that  $\gamma_k \to 1$  and that  $\inf_{k \in \mathbb{N}} c_k > 0$  or  $c_k \to \infty$ .
- (ii) Suppose that  $(\forall k \in \mathbb{N}) \gamma_k(\beta_k + \gamma_k) \ge 0$ , that  $\liminf_{k\to\infty} \gamma_k(2\beta_k + \gamma_k) > 0$ , and that  $\inf_{k\in\mathbb{N}} c_k > 0$ or  $c_k \to \infty$ . Assume that  $\liminf_{k\to\infty} |\gamma_k| > 0$  and  $\sup_{k\in\mathbb{N}} |\beta_k| < \infty$ .
- (iii) Suppose that  $(\forall k \in \mathbb{N}) \gamma_k(\beta_k + \gamma_k) \ge 0$ ,  $\gamma_k(2\beta_k + \gamma_k) \ge 0$ , and  $|\beta_k + \gamma_k| |\gamma_k| \le \max\{1 |\beta_k|, 2 2|\beta_k + \frac{\gamma_k}{2}|\}$ , that  $\sum_{k \in \mathbb{N}} \gamma_k(2\beta_k + \gamma_k) = \infty$ , and that  $\inf_{k \in \mathbb{N}} c_k > 0$  and  $\sum_{k \in \mathbb{N}} |c_{k+1} c_k| < \infty$ . Assume that  $\liminf_{k \to \infty} |\gamma_k| > 0$  and  $\sup_{k \in \mathbb{N}} |\beta_k| < \infty$ .

*Then* zer  $A \neq \emptyset$  *if and only if*  $(x_k)_{k \in \mathbb{N}}$  *is bounded.* 

*Proof.* If zer  $A \neq \emptyset$ , then combine Proposition 3.3(ii) with the global assumptions to deduce the boundedness of  $(x_k)_{k \in \mathbb{N}}$ .

Suppose that  $(x_k)_{k \in \mathbb{N}}$  is bounded. Due to Proposition 3.6, (i) necessitates zer  $A \neq \emptyset$ .

Suppose that (ii) or (iii) holds. Then, via Proposition 3.5(i),  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$  is bounded. Moreover, apply Proposition 2.15 to ensure that there exists  $r \in \mathbb{R}_{++}$  such that  $\tilde{A} := A + \partial \iota_{B[0;r]}$  is a maximally monotone operator and that

$$\operatorname{zer} \tilde{A} \neq \varnothing, \quad (\Omega \cap \operatorname{zer} \tilde{A}) \subseteq \operatorname{zer} A, \quad \text{and} \quad (\forall k \in \mathbb{N}) \ \operatorname{J}_{c_k A} x_k = \operatorname{J}_{c_k \tilde{A}} x_k.$$
 (3.25)

If (ii) (resp. (iii)) holds, then apply Proposition 3.8(iv) (resp. Proposition 3.9) with *A* replaced by  $\tilde{A}$  to obtain that  $\emptyset \neq \Omega \subseteq \operatorname{zer} \tilde{A}$ , which, in connection with (3.25), establishes that  $\emptyset \neq \Omega = (\Omega \cap \operatorname{zer} \tilde{A}) \subseteq \operatorname{zer} A$ .

Theorem 3.12. Assume that one of the following is satisfied.

- (I)  $\limsup_{k\to\infty} \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right) < 1$ ,  $\sup_{k\in\mathbb{N}} |\alpha_k| < \infty$ , and  $\sup_{k\in\mathbb{N}} \|\delta_k e_k\| < \infty$ .
- (II)  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ , and the following hold:
  - (a)  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1 \text{ or } \sum_{i \in \mathbb{N}} |\alpha_i| < \infty;$
  - (b)  $\left[\left(\forall k \in \mathbb{N} \left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right| + \left|\delta_k\right| \le 1 \text{ and } \sup_{i \in \mathbb{N}} \|e_i\| < \infty\right] \text{ or } \sum_{i \in \mathbb{N}} \|\delta_i e_i\| < \infty;$
  - (c)  $(\forall k \in \mathbb{N}) \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| + \left| 1 \beta_k \gamma_k \right| \le 1 \text{ or } \sum_{i \in \mathbb{N}} \left| 1 \beta_i \gamma_i \right| < \infty.$
- (III)  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1, \sum_{k \in \mathbb{N}} |1 \alpha_k \beta_k \gamma_k| < \infty, and \sum_{k \in \mathbb{N}} ||\delta_k e_k|| < \infty.$
- (IV)  $\sup_{k \in \mathbb{N}} \|e_k\| < \infty$ ,  $\sum_{k \in \mathbb{N}} |\alpha_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} |1 \beta_k \gamma_k \delta_k| < \infty$ , and  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| + |\frac{\gamma$
- (V)  $(\forall k \in \mathbb{N}) \ \alpha_k \in ]0,1]$  and  $\alpha_k + \left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right| \le 1, \frac{\delta_k e_k}{\alpha_k} \to 0$ , and  $\frac{1-\alpha_k \beta_k \gamma_k}{\alpha_k} \to 0$ .

Then the following statements hold.

- (i) zer  $A \neq \emptyset$  implies the boundedness of  $(x_k)_{k \in \mathbb{N}}$ .
- (ii) Suppose additionally that one of the following holds.
  - (a)  $c_k \to \infty$ ,  $\alpha_k \to 0$ ,  $\beta_k \to 0$ ,  $\gamma_k \to 1$ , and  $\delta_k e_k \to 0$ .
  - (b) Suppose that  $(\forall k \in \mathbb{N}) \ \beta_k + \gamma_k \leq 1, \ \alpha_k \to 0$ ,  $\limsup_{k \to \infty} |\beta_k| < 1, \ 1 \alpha_k \beta_k \gamma_k \to 0$ ,  $0 < \liminf_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} \leq \limsup_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} < 1$ ,  $\delta_k e_k \to 0$ , and  $1 - \frac{c_k}{c_{k+1}} \to 0$ , and that  $\inf_{k \in \mathbb{N}} c_k > 0$  or  $c_k \to \infty$ .
  - (c) Suppose that  $\sum_{i\in\mathbb{N}} |\alpha_{i+1} \alpha_i| < \infty$  or  $(\forall k \in \mathbb{N}) |\gamma_k| \neq 1$  with  $\lim_{k\to\infty} \frac{|\alpha_{k+1} \alpha_k|}{1 |\gamma_{k+1}|} = 0$ , that  $(\forall k \in \mathbb{N}) |\gamma_k| \in [0, 1]$ , that  $\alpha_k \to 0$  and  $\alpha_k + \gamma_k \to 1$ , that  $\sum_{k\in\mathbb{N}} |(\alpha_{k+1} + \gamma_{k+1}) (\alpha_k + \gamma_k)| < \infty$ ,  $\sum_{k\in\mathbb{N}} |\beta_k| < \infty$ ,  $\sum_{k\in\mathbb{N}} |\alpha_k| = \infty$ , and  $\sum_{k\in\mathbb{N}} |\delta_k e_k| < \infty$ , and that  $\inf_{k\in\mathbb{N}} c_k > 0$  and  $\sum_{k\in\mathbb{N}} |c_{k+1} c_k| < \infty$ .

*Then* zer  $A \neq \emptyset$  *if and only if*  $(x_k)_{k \in \mathbb{N}}$  *is bounded.* 

*Proof.* (i): This is clear from Proposition 3.3 and Proposition 3.4.

(ii): In view of (i), it remains to prove that the boundedness of  $(x_k)_{k \in \mathbb{N}}$  leads to zer  $A \neq \emptyset$ .

In the rest of the proof we assume that  $(x_k)_{k \in \mathbb{N}}$  is bounded. Then via [1, Lemma 2.45],  $\Omega \neq \emptyset$ . If (ii)(a) (resp. (ii)(b)) is satisfied, then  $\emptyset \neq \Omega \subseteq \text{zer } A$  follows immediately from Proposition 3.5(iii) (resp. Proposition 3.7(iii)).

Suppose that (ii)(c) is satisfied. Due to Proposition 3.5(i),  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$  is bounded. Then apply Proposition 2.15 to ensure that there exists  $r \in \mathbb{R}_{++}$  such that  $\tilde{A} := A + \partial \iota_{B[0;r]}$  is a maximally monotone operator and that

$$\operatorname{zer} \tilde{A} \neq \varnothing, \quad (\Omega \cap \operatorname{zer} \tilde{A}) \subseteq \operatorname{zer} A, \quad \text{and} \quad (\forall k \in \mathbb{N}) \ \operatorname{J}_{c_k A} x_k = \operatorname{J}_{c_k \tilde{A}} x_k.$$
 (3.26)

Furthermore, apply Proposition 3.10 with *A* replaced by  $\tilde{A}$  to obtain that  $\emptyset \neq \Omega \subseteq \operatorname{zer} \tilde{A}$ , which, connecting with (3.26), establishes that  $\emptyset \neq \Omega = (\Omega \cap \operatorname{zer} \tilde{A}) \subseteq \operatorname{zer} A$ .

# 4 Convergence of generalized proximal point algorithms

Because convergence implies boundedness, based on the equivalence of the boundedness of  $(x_k)_{k \in \mathbb{N}}$  and zer  $A \neq \emptyset$  shown in Theorems 3.11 and 3.12, to study the convergence of  $(x_k)_{k \in \mathbb{N}}$ , we always assume zer  $A \neq \emptyset$ .

We first uphold our general assumptions and notations that

 $A: \mathcal{H} \to 2^{\mathcal{H}}$  is a maximally monotone operator with zer  $A \neq \emptyset$ ,

that  $u \in \mathcal{H}$  and  $x_0 \in \mathcal{H}$  are arbitrary but fixed, and that

$$(\forall k \in \mathbb{N}) \quad x_{k+1} = \alpha_k u + \beta_k x_k + \gamma_k \mathbf{J}_{c_k A}(x_k) + \delta_k e_k, \tag{4.1}$$

where  $(\forall k \in \mathbb{N}) e_k \in \mathcal{H}, c_k \in \mathbb{R}_{++}$ , and  $\{\alpha_k, \beta_k, \gamma_k, \delta_k\} \subseteq \mathbb{R}$ . Recall that

 $\Omega$  is the set of all weak sequential cluster points of  $(x_k)_{k \in \mathbb{N}}$ .

### Weak convergence

**Theorem 4.1.** Suppose that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ , that  $\sum_{k \in \mathbb{N}} |\alpha_k| < \infty$ ,  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k| < \infty$ , and  $\sum_{k \in \mathbb{N}} \|\delta_k e_k\| < \infty$ , and that one of the following statements holds.

- (i) Assume that  $\gamma_k \to 1$  and that  $\inf_{k \in \mathbb{N}} c_k > 0$  or  $c_k \to \infty$ .
- (ii) Assume that  $\limsup_{k\to\infty} |\beta_k| < 1$  and  $0 < \liminf_{k\to\infty} 1 \beta_k \frac{\gamma_k}{2} \le \limsup_{k\to\infty} 1 \beta_k \frac{\gamma_k}{2} < 1$ , that  $1 \frac{c_k}{c_{k+1}} \to 0$ , and that  $\inf_{k\in\mathbb{N}} c_k > 0$  or  $c_k \to \infty$ .
- (iii) Assume that  $\inf_{k\in\mathbb{N}} \gamma_k(\beta_k + \gamma_k) \ge 0$  and  $\liminf_{k\to\infty} \gamma_k(2\beta_k + \gamma_k) > 0$ , and that  $\inf_{k\in\mathbb{N}} c_k > 0$  or  $c_k \to \infty$ .
- (iv) Assume that  $\inf_{k\in\mathbb{N}} \gamma_k(\beta_k + \gamma_k) \ge 0$  and  $\inf_{k\in\mathbb{N}} \gamma_k(2\beta_k + \gamma_k) \ge 0$ , that  $\sum_{k\in\mathbb{N}} \gamma_k(2\beta_k + \gamma_k) = \infty$ , that  $(\forall k \in \mathbb{N}) |\beta_k + \gamma_k| |\gamma_k| \le \max\{1 |\beta_k|, 2 2 |\beta_k + \frac{\gamma_k}{2}|\}$ , and that  $\inf_{k\in\mathbb{N}} c_k > 0$  and  $\sum_{k\in\mathbb{N}} |c_{k+1} c_k| < \infty$ .

*Then*  $(x_k)_{k \in \mathbb{N}}$  *converges weakly to a point in* zer *A*.

*Proof.* According to Lemma 3.2(i),

$$(\forall p \in \operatorname{zer} A)(\forall k \in \mathbb{N}) \quad \|x_{k+1} - p\| \le \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right) \|x_k - p\| + \|\alpha_k u + \delta_k e_k - (1 - \beta_k - \gamma_k)p\|,$$

which, combining with the global assumptions and Fact 2.5, ensures that  $(\forall p \in \text{zer } A) \lim_{k \to \infty} ||x_k - p||$  exists in  $\mathbb{R}_+$  and that  $(x_k)_{k \in \mathbb{N}}$  is bounded.

Therefore, via Fact 2.18, it suffices to prove  $\Omega \subseteq \text{zer } A$  under the assumption (i), (ii), (iii), or (iv). If (i) is true, the desired inclusion is immediate from Proposition 3.6.

Note that  $(\forall k \in \mathbb{N}) \ \beta_k + \gamma_k \le |\beta_k + \gamma_k| \le |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|; \sum_{k \in \mathbb{N}} |\alpha_k| < \infty \text{ and } \sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k| < \infty \text{ imply that } \alpha_k \to 0, 1 - \beta_k - \gamma_k \to 0, \text{ and } 1 - \alpha_k - \beta_k - \gamma_k \to 0. \text{ As a consequence of Proposition 3.7(iii),}$ (ii) implies  $\Omega \subseteq \text{zer } A$ .

In addition, it is easy to see that the required inclusion is also immediate from (iii) and Proposition 3.8(iv) (or from (iv) and Proposition 3.9).

**Remark 4.2.** We compare Theorem 4.1 with related existed results on the weak convergence of generalized proximal point algorithms below.

- (i) Suppose that  $(\forall k \in \mathbb{N}) \ \alpha_k = \beta_k \equiv 0 \text{ and } \gamma_k = \delta_k \equiv 1, \text{ and that } \inf_{k \in \mathbb{N}} c_k > 0 \text{ and } \sum_{k \in \mathbb{N}} \|e_k\| < \infty.$ Then Theorem 4.1(i) reduces to the weak convergence proved in [9, Theorem 1].
- (ii) The relaxed proximal point algorithm presented in [13, Algorithm 5.2] is a special case of the scheme (4.1) with  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0, \ \beta_k \in [0, 1[$ , and  $\gamma_k = \delta_k = 1 \beta_k$ . Because in the Step 2 of the proof of [13, Theorem 5.2], the author requires "repeating the proof of the second part of step 2 of Theorem 5.1" and in the Step 2 of [13, Theorem 5.1],  $\beta_k \to 0$  is a critical assumption, we assume  $\beta_k \to 0$  is a necessary assumption of [13, Theorem 5.2]. Therefore, Theorem 4.1(i) is also a generalized result of [13, Theorem 5.2] which requires that  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0, \ \beta_k \in [0, 1 \delta]$  for some  $\delta \in [0, 1[, \beta_k \to 0, \text{ and } \gamma_k = \delta_k = 1 \beta_k$ , and that  $c_k \to \infty$  and  $\sum_{k \in \mathbb{N}} \|e_k\| < \infty$ .
- (iii) Consider (4.1) with  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0, \ \beta_k \in [0, 1], \ \gamma_k = 1 \beta_k, \ \text{and} \ \delta_k \equiv 1$ . In this case  $(\forall k \in \mathbb{N}) \ \eta_k := 1 \beta_k \frac{\gamma_k}{2} = \frac{\gamma_k}{2}, \ \text{so} \ 0 < \liminf_{i \to \infty} \eta_i \leq \limsup_{i \to \infty} \eta_i < 1$  follows immediately from  $0 < \liminf_{i \to \infty} \gamma_i \leq \limsup_{i \to \infty} \gamma_i < 2$ . Moreover, it is easy to see that  $\overline{c} := \inf_{k \in \mathbb{N}} c_k > 0$  and  $c_{k+1} c_k \to 0$  imply that  $1 \frac{c_k}{c_{k+1}} \to 0$ , since  $\left|1 \frac{c_k}{c_{k+1}}\right| = \left|\frac{c_{k+1} c_k}{c_{k+1}}\right| \leq \frac{|c_{k+1} c_k|}{\overline{c}}$ . Hence, we know that Theorem 4.1(ii) improves [15, Theorem 3.2].
- (iv) Note that if  $(\forall k \in \mathbb{N}) \gamma_k \in \mathbb{R}_+$  and  $\beta_k = 1 \gamma_k$  such that  $0 < \bar{\gamma} := \inf_{k \in \mathbb{N}} \gamma_k \le \hat{\gamma} := \sup_{k \in \mathbb{N}} \gamma_k < 2$ , then  $\inf_{k \in \mathbb{N}} \gamma_k (\beta_k + \gamma_k) = \inf_{k \in \mathbb{N}} \gamma_k \ge 0$  and  $\liminf_{k \to \infty} \gamma_k (2\beta_k + \gamma_k) = \liminf_{k \to \infty} \gamma_k (2 \gamma_k) \ge \lim_{k \to \infty} \inf_{k \to \infty} \gamma_k \lim_{k \to \infty} 2 \gamma_k \ge \bar{\gamma} (2 \hat{\gamma}) > 0$ . Therefore, we see that Theorem 4.1(iii) covers [4, Theorem 3] in which the assumptions  $(\forall k \in \mathbb{N}) \alpha_k \equiv 0, \gamma_k \in \mathbb{R}_+$  with  $0 < \bar{\gamma} = \inf_{k \in \mathbb{N}} \gamma_k \le \hat{\gamma} = \sup_{k \in \mathbb{N}} \gamma_k < 2, \beta_k = 1 \gamma_k$ , and  $\delta_k \equiv \gamma_k$ ,  $\inf_{k \in \mathbb{N}} c_k > 0$ , and  $\sum_{k \in \mathbb{N}} \|e_k\| < \infty$  are required.
- (v) Note that the generalized proximal point algorithms studied in [8, Section 3] are (4.1) satisfying that  $0 < \inf_{k \in \mathbb{N}} c_k \leq \sup_{k \in \mathbb{N}} c_k < \infty$  and that  $(\forall k \in \mathbb{N}) \ \alpha_k \equiv 0, \ \gamma_k \in ]0, 2[, \ \beta_k = 1 \gamma_k, \text{ and } \delta_k \equiv 1$ . In this case, the conditions  $\inf_{k \in \mathbb{N}} \gamma_k(\beta_k + \gamma_k) \ge 0$ ,  $\inf_{k \in \mathbb{N}} \gamma_k(2\beta_k + \gamma_k) \ge 0$ ,  $\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right| \le 1$ , and  $|\beta_k + \gamma_k| |\gamma_k| \le \max\{1 |\beta_k|, 2 2 |\beta_k + \frac{\gamma_k}{2}|\}$  hold trivially. Moreover, since  $(\forall k \in \mathbb{N}) \ \gamma_k(2 \gamma_k) = \gamma_k(2\beta_k + \gamma_k)$ , thus  $\sum_k \gamma_k(2 \gamma_k) = \infty$  is exactly  $\sum_{i \in \mathbb{N}} \gamma_i(2\beta_k + \gamma_k) = \infty$ . Therefore, we know that Theorem 4.1(iv) covers [8, Theorems 3.5 and 3.6].

### Strong convergence

Note that based on Fact 2.9(v), the maximal monotoneness of *A* and the assumption zer  $A \neq \emptyset$  above guarantee that the projection  $P_{\text{zer }A} u$  is well-defined. In this subsection, we shall specify sufficient conditions on coefficients of (4.1) for  $x_k \rightarrow P_{\text{zer }A} u$ . Notice that when u = 0 (resp.  $u = x_0$ ) in (4.1), the strong limit  $P_{\text{zer }A} u$  of  $(x_k)_{k \in \mathbb{N}}$  is the minimum-norm solution in zer *A* (resp. the closest point to the initial point  $x_0$  onto zer *A*).

**Proposition 4.3.** Suppose that  $(x_k)_{k \in \mathbb{N}}$  is bounded and that  $\Omega \subseteq \text{zer } A$ . Suppose that  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1$  with  $\sum_{i \in \mathbb{N}} 1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 = \infty$ , and that one of the following holds.

- (i)  $(\forall k \in \mathbb{N}) \ \beta_k + \gamma_k \ge 0 \text{ and } \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right)^2 \beta_k \gamma_k \le 0, \ \sum_{k \in \mathbb{N}} \left| 1 \alpha_k \beta_k \gamma_k \right| < \infty, \text{ and } \sum_{k \in \mathbb{N}} \left\| \delta_k e_k \right\| < \infty.$
- (ii)  $(\forall k \in \mathbb{N}) \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| < 1$ ,  $\sup_{k \in \mathbb{N}} \frac{1 \beta_k \gamma_k}{1 \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right)^2} < \infty$ ,  $\sum_{k \in \mathbb{N}} \left| 1 \alpha_k \beta_k \gamma_k \right| < \infty$ , and  $\sum_{k \in \mathbb{N}} \left\| \delta_k e_k \right\| < \infty$ .
- (iii)  $(\forall k \in \mathbb{N}) \alpha_k \in [0, 1] \text{ and } \alpha_k + (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 \leq 1, \sum_{k \in \mathbb{N}} |1 \alpha_k \beta_k \gamma_k| < \infty, \sum_{k \in \mathbb{N}} ||\delta_k e_k|| < \infty.$

- (iv)  $(\forall k \in \mathbb{N}) \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| < 1 \text{ and } \alpha_k \ge 0, \sup_{k \in \mathbb{N}} \frac{\alpha_k}{1 \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right)^2} < \infty, \sum_{k \in \mathbb{N}} \left| 1 \alpha_k \beta_k \gamma_k \right| < \infty, \text{ and } \sum_{k \in \mathbb{N}} \left\| \delta_k e_k \right\| < \infty.$
- (v)  $(\forall k \in \mathbb{N}) \ \alpha_k \in ]0,1] \ and \ \alpha_k + \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right)^2 \le 1, \ \frac{\delta_k e_k}{\alpha_k} \to 0, \ and \ \frac{1-\alpha_k \beta_k \gamma_k}{\alpha_k} \to 0.$

(vi)  $(\forall k \in \mathbb{N}) |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| < 1 \text{ and } \alpha_k \in \mathbb{R}_{++}, \sup_{k \in \mathbb{N}} \frac{\alpha_k}{1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2} < \infty, \frac{\delta_k e_k}{\alpha_k} \to 0, \text{ and } \frac{1 - \alpha_k - \beta_k - \gamma_k}{\alpha_k} \to 0.$ 

*Then*  $x_k \to P_{\operatorname{zer} A} u$ .

*Proof.* Because  $\Omega \subseteq \text{zer } A$  and  $(x_k)_{k \in \mathbb{N}}$  is bounded, due to Proposition 2.19, we get that

$$\limsup_{k \to \infty} \langle u - \mathcal{P}_{\operatorname{zer} A} \, u, \, x_k - \mathcal{P}_{\operatorname{zer} A} \, u \rangle \le 0. \tag{4.2}$$

Set  $p := P_{\text{zer}A} u$  and  $(\forall k \in \mathbb{N}) T_k := 2J_{c_kA} - \text{Id.}$  Since  $\text{zer} A \neq \emptyset$ , due to Proposition 3.5(i), we know that  $(J_{c_kA} x_k)_{k \in \mathbb{N}}$  and  $(T_k x_k)_{k \in \mathbb{N}}$  are bounded.

Denote by  $(\forall k \in \mathbb{N}) \ \xi_k := \left(\left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right|\right)^2, \ \phi_k := 1 - \beta_k - \gamma_k, \ \varphi_k := 1 - \alpha_k - \beta_k - \gamma_k, \ F(k) := \|\delta_k e_k - \varphi_k u\|, \ \text{and} \ G(k) := F(k) + 2 \left\|\left(\beta_k + \frac{\gamma_k}{2}\right)(x_k - p) + \frac{\gamma_k}{2}(T_k(x_k) - p) + \phi_k(u - p)\right\|.$  Because  $p = P_{\text{zer } A} u \in \text{zer } A$ , via Lemma 3.2(ii)&(iv), we have that for every  $k \in \mathbb{N}$ ,

$$\|x_{k+1} - p\|^2 \le \xi_k \|x_k - p\|^2 + 2\phi_k \langle u - p, x_{k+1} - p - \delta_k e_k + \phi_k u \rangle + F(k)G(k);$$
(4.3a)

$$\|x_{k+1} - p\|^{2} \leq \xi_{k} \|x_{k} - p\|^{2} + 2\alpha_{k} \langle u - p, x_{k+1} - p \rangle + 2 \langle \delta_{k} e_{k} - \varphi_{k} p, x_{k+1} - p \rangle.$$
(4.3b)

We separate the remaining proof into the following three cases.

*Case 1*: Assume (i) or (ii) is satisfied. Note that  $(\forall k \in \mathbb{N}) \ 0 \le 1 - \beta_k - \gamma_k \le 1 \Leftrightarrow 0 \le \beta_k + \gamma_k \le 1$ and that  $(\forall k \in \mathbb{N}) \ \beta_k + \gamma_k \le |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ . Exploiting  $\sum_{k \in \mathbb{N}} |1 - \alpha_k - \beta_k - \gamma_k| < \infty$  and  $\sum_{k \in \mathbb{N}} ||\delta_k e_k|| < \infty$ , we know that  $\varphi_k \to 0$ ,  $\delta_k e_k \to 0$ , and  $\langle u - p, \delta_k e_k - \varphi_k u \rangle \to 0$ , that  $\sup_{k \in \mathbb{N}} G(k) < \infty$ , and that  $\sum_{k \in \mathbb{N}} F(k)G(k) < \infty$ . Moreover, combining  $\varphi_k \to 0$  and  $\delta_k e_k \to 0$  with (4.2), we observe that

$$\limsup_{k \to \infty} 2 \langle u - p, x_{k+1} - p - \delta_k e_k + \varphi_k u \rangle \le 0.$$
(4.4)

If (i) (resp. (ii)) holds, then using (4.3a) and applying Proposition 2.3(iii) (resp. Proposition 2.3(iv)) with  $(\forall k \in \mathbb{N})$   $t_k = ||x_k - p||^2$ ,  $\alpha_k = \xi_k$ ,  $\beta_k = \phi_k$ ,  $\omega_k = 2 \langle u - p, x_{k+1} - p - \delta_k e_k + \phi_k u \rangle$ , and  $\gamma_k = F(k)G(k)$ , we obtain the required convergence.

*Case 2*: Assume (iii) or (iv) is true. If (iii) (resp. (iv)) holds, employing (4.3b) and applying Proposition 2.3(iii) (resp. Proposition 2.3(iv)) with  $(\forall k \in \mathbb{N})$   $t_k = ||x_k - p||^2$ ,  $\alpha_k = \xi_k$ ,  $\beta_k = \alpha_k$ ,  $\omega_k = 2 \langle u - p, x_{k+1} - p \rangle$ , and  $\gamma_k = 2 \langle \delta_k e_k - \varphi_k p, x_{k+1} - p \rangle$ , we get the required convergence.

*Case 3*: Assume (v) or (vi) is true. In view of (4.3b), for every  $k \in \mathbb{N}$ ,

$$\|x_{k+1} - p\|^2 \le \xi_k \|x_k - p\|^2 + 2\alpha_k \left( \langle u - p, x_{k+1} - p \rangle + \left\langle \frac{\delta_k e_k}{\alpha_k} - \frac{\varphi_k}{\alpha_k} p, x_{k+1} - p \right\rangle \right).$$

$$(4.5)$$

Because  $(x_k)_{k \in \mathbb{N}}$  is bounded,  $\frac{\delta_k e_k}{\alpha_k} \to 0$  and  $\frac{1-\alpha_k - \beta_k - \gamma_k}{\alpha_k} \to 0$  yield  $\left\langle \frac{\delta_k e_k}{\alpha_k} - \frac{1-\alpha_k - \beta_k - \gamma_k}{\alpha_k} p, x_{k+1} - p \right\rangle \to 0$ . This in connection with (4.2) leads to

$$\limsup_{k\to\infty} 2\left(\langle u-p, x_{k+1}-p\rangle + \left\langle \frac{\delta_k e_k}{\alpha_k} - \frac{\varphi_k}{\alpha_k}p, x_{k+1}-p\right\rangle\right) \leq 0.$$

If (v) (resp. (vi)) is true, then utilizing (4.5) and applying Proposition 2.3(iii) (resp. Proposition 2.3(iv)) with  $(\forall k \in \mathbb{N}) t_k = ||x_k - p||^2$ ,  $\alpha_k = \xi_k$ ,  $\beta_k = \alpha_k$ ,  $\omega_k = 2\left(\langle u - p, x_{k+1} - p \rangle + \left\langle \frac{\delta_k e_k}{\alpha_k} - \frac{\varphi_k}{\alpha_k} p, x_{k+1} - p \right\rangle \right)$ , and  $\gamma_k \equiv 0$ , we get the desired convergence.

Altogether, the proof is complete.

Note that Propositions 3.3 and 3.4 provide sufficient conditions for the boundedness of  $(x_k)_{k \in \mathbb{N}}$ , that Propositions 3.5 to 3.10 specify conditions for  $\Omega \subseteq \text{zer } A$ , and that Proposition 4.3 present conditions for the strong convergence under the assumptions of the boundedness of  $(x_k)_{k \in \mathbb{N}}$  and  $\Omega \subseteq \text{zer } A$ . Clearly, combining these results, we are able to deduce many sufficient conditions for the strong convergence of the generalized proximal point algorithm generated by the scheme (4.1). For simplicity, we present only some easy sufficient conditions for the strong convergence below.

**Theorem 4.4.** Suppose that  $\sum_{i \in \mathbb{N}} 1 - \left( \left| \beta_k + \frac{\gamma_k}{2} \right| + \left| \frac{\gamma_k}{2} \right| \right)^2 = \infty$  and that one of the following holds.

- (i) Assume that  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ ,  $\beta_k + \gamma_k \ge 0$ , and  $(|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 \beta_k \gamma_k \le 0$ , that  $\sum_{k \in \mathbb{N}} |1 \alpha_k \beta_k \gamma_k| < \infty$  and  $\sum_{k \in \mathbb{N}} |\delta_k e_k| < \infty$ , and that  $\alpha_k \to 0$ ,  $\beta_k \to 0$ ,  $\gamma_k \to 1$ , and  $c_k \to \infty$ .
- (ii) Assume that  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1$ ,  $\beta_k + \gamma_k \ge 0$ , and  $(|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 \beta_k \gamma_k \le 0$ , that  $\sum_{k \in \mathbb{N}} |1 \alpha_k \beta_k \gamma_k| < \infty$  and  $\sum_{k \in \mathbb{N}} ||\delta_k e_k|| < \infty$ , that  $\alpha_k \to 0$ ,  $\limsup_{k \to \infty} |\beta_k| < 1$ ,  $0 < \liminf_{k \to \infty} 1 \beta_k \frac{\gamma_k}{2} \le \limsup_{k \to \infty} 1 \beta_k \frac{\gamma_k}{2} < 1$ , and  $1 \frac{c_k}{c_{k+1}} \to 0$ , and that  $\inf_{k \in \mathbb{N}} c_k > 0$  or  $c_k \to \infty$ .
- (iii) Assume that  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1, \ \beta_k + \gamma_k \geq 0, \ (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 \beta_k \gamma_k \leq 0, \ |\gamma_k| \in [0,1], \ that \ \sum_{k \in \mathbb{N}} |(\alpha_{k+1} + \gamma_{k+1}) (\alpha_k + \gamma_k)| < \infty, \ \sum_{k \in \mathbb{N}} |1 \alpha_k \beta_k \gamma_k| < \infty, \ \sum_{k \in \mathbb{N}} |\beta_k| < \infty, \ \sum_{k \in \mathbb{N}} |\beta_k| < \infty, \ \sum_{k \in \mathbb{N}} (1 |\gamma_k|) = \infty, \ and \ \sum_{k \in \mathbb{N}} |\delta_k e_k|| < \infty, \ that \ \sum_{k \in \mathbb{N}} |\alpha_{k+1} \alpha_k| < \infty \ or \ (\forall k \in \mathbb{N}) \ |\gamma_k| \neq 1 \ with \ \lim_{k \to \infty} \frac{|\alpha_{k+1} \alpha_k|}{1 |\gamma_{k+1}|} = 0, \ that \ \alpha_k \to 0 \ and \ \alpha_k + \gamma_k \to 1, \ and \ that \ \inf_{k \in \mathbb{N}} c_k > 0 \ and \ \sum_{i \in \mathbb{N}} |c_{k+1} c_k| < \infty.$
- (iv) Assume that  $(\forall k \in \mathbb{N}) \ \alpha_k \in ]0,1]$  and  $\alpha_k + \left|\beta_k + \frac{\gamma_k}{2}\right| + \left|\frac{\gamma_k}{2}\right| \le 1$ , and that  $\alpha_k \to 0$ ,  $\beta_k \to 0$ ,  $\gamma_k \to 1$ ,  $\frac{\delta_k e_k}{\alpha_k} \to 0$ ,  $\frac{1-\alpha_k \beta_k \gamma_k}{\alpha_k} \to 0$ , and  $c_k \to \infty$ .

Then  $x_k \to P_{\operatorname{zer} A} u$ .

*Proof.* If (i) (resp. (ii) or (iii)) holds, the boundedness of  $(x_k)_{k \in \mathbb{N}}$  comes from Proposition 3.3(iii);  $\Omega \subseteq$  zer *A* follows from Proposition 3.5(iii) (resp. Proposition 3.7(iii) or Proposition 3.10); and the strong convergence is clear from Proposition 4.3(i).

If (iv) is satisfied, employing Proposition 3.4 and Proposition 3.5(iii), we establish the boundedness of  $(x_k)_{k \in \mathbb{N}}$  and  $\Omega \subseteq \text{zer } A$ , respectively. Then the strong convergence follows immediately from Proposition 4.3(v).

Theorem 4.5(i) is inspired by the proof of [16, Theorem 3.1] and illustrates that comparing with the closest point to the initial point  $x_0$  onto zer A, the minimum-norm solution in zer A is easier to find in some circumstances.

**Theorem 4.5.** Denote by  $(\forall k \in \mathbb{N})$   $\xi_k := |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|$ . Suppose that  $(\forall k \in \mathbb{N})$   $\xi_k < 1$ , that  $\sum_{i \in \mathbb{N}} 1 - \xi_k^2 = \infty$ , that  $\limsup_{k \to \infty} |\beta_k| < 1$ ,  $0 < \liminf_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} \le \limsup_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} < 1$ ,  $\frac{\delta_k e_k}{1 - \beta_k - \gamma_k} \to 0$ , and  $1 - \frac{c_k}{c_{k+1}} \to 0$ , that  $\sup_{k \in \mathbb{N}} \frac{1 - \beta_k - \gamma_k}{1 - \xi_k^2} < \infty$ , and that  $\inf_{k \in \mathbb{N}} c_k > 0$  or  $c_k \to \infty$ . Suppose further that one of the following holds.

- (i) u = 0,  $\sum_{k \in \mathbb{N}} |1 \beta_k \gamma_k \delta_k| < \infty$ ,  $\sup_{k \in \mathbb{N}} ||e_k|| < \infty$ ,  $(\forall k \in \mathbb{N}) |\xi_k| \le 1$  and  $\alpha_k \equiv 0$ , and  $1 \beta_k \gamma_k \to 0$ .
- (ii)  $\sup_{k\in\mathbb{N}} \|e_k\| < \infty$ ,  $\sum_{k\in\mathbb{N}} |\alpha_k| < \infty$ ,  $\sum_{k\in\mathbb{N}} |1-\beta_k-\gamma_k-\delta_k| < \infty$ ,  $\frac{1-\alpha_k-\beta_k-\gamma_k}{1-\beta_k-\gamma_k} \to 0$ , and  $(\forall k\in\mathbb{N})$  $\xi_k + |\delta_k| \le 1$ .
- (iii)  $(\forall k \in \mathbb{N}) |\alpha_k| + \xi_k \leq 1, \alpha_k \to 0, \frac{1-\alpha_k \beta_k \gamma_k}{1-\beta_k \gamma_k} \to 0, \sum_{k \in \mathbb{N}} \|\delta_k e_k\| < \infty, and \sum_{k \in \mathbb{N}} |1-\alpha_k \beta_k \gamma_k| < \infty.$

*Then*  $x_k \to P_{\operatorname{zer} A} u$ .

*Proof.* We separate the proof into the following three steps.

*Step* 1: If the assumption (i) or (ii) (resp. (iii)) holds, then combining the assumptions with Proposition 3.3(iv) (resp. Proposition 3.3(iii)) and Proposition 3.7(iii), we deduce that  $(x_k)_{k \in \mathbb{N}}$  is bounded and that  $\Omega \subseteq$  zer *A*.

*Step 2*: Denote by  $(\forall k \in \mathbb{N}) \phi_k := 1 - \beta_k - \gamma_k, \phi_k := 1 - \alpha_k - \beta_k - \gamma_k$ ,

$$F(k) := \|\delta_k e_k - \varphi_k u\| \text{ and } G(k) := F(k) + 2 \left\| \left( \beta_k + \frac{\gamma_k}{2} \right) (x_k - p) + \frac{\gamma_k}{2} (T_k(x_k) - p) + \phi_k (u - p) \right\|.$$

Note that  $(\forall k \in \mathbb{N}) \beta_k + \gamma_k \le |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| < 1$ . Because  $p := P_{\operatorname{zer} A} u \in \operatorname{zer} A$ , due to Lemma 3.2(ii), we have that for every  $k \in \mathbb{N}$ ,

$$\|x_{k+1} - p\|^2 \le \xi_k^2 \|x_k - p\|^2 + 2\phi_k \langle u - p, x_{k+1} - p - \delta_k e_k + \phi_k u \rangle + F(k)G(k).$$
(4.6)

Because  $\frac{\delta_k e_k}{1-\beta_k-\gamma_k} \to 0$  and  $\sup_{k \in \mathbb{N}} G(k) < \infty$ , any one of the assumptions (i), (ii), and (iii) ensures that for every  $k \in \mathbb{N}$ ,

$$\frac{F(k)G(k)}{\phi_k} = \left\| \frac{\delta_k e_k}{\phi_k} - \frac{\varphi_k}{\phi_k} u \right\| G(k) \le \left( \frac{\|\delta_k e_k\|}{1 - \beta_k - \gamma_k} + \frac{|\varphi_k|}{1 - \beta_k - \gamma_k} \|u\| \right) \sup_{k \in \mathbb{N}} G(k) \to 0,$$

which, connecting with Proposition 2.19, entails that

$$\limsup_{k\to\infty} 2\langle u-p, x_{k+1}-p-\delta_k e_k+\varphi_k u\rangle+\frac{F(k)G(k)}{\varphi_k}\leq 0.$$

*Step 3*: Considering (4.6) and applying Proposition 2.3(iv) with  $(\forall k \in \mathbb{N}) t_k = ||x_k - p||^2$ ,  $\alpha_k = \xi_k^2$ ,  $\beta_k = \phi_k$ ,  $\omega_k = 2 \langle u - p, x_{k+1} - p - \delta_k e_k + \phi_k u \rangle + \frac{F(k)G(k)}{\phi_k}$ , and  $\gamma_k \equiv 0$ , we obtain the required results.

Notice that none of the sums of

$$\alpha_k + \beta_k + \gamma_k, \quad \beta_k + \gamma_k, \quad \text{or} \quad \beta_k + \gamma_k + \delta_k$$

in the particular examples of Example 4.6 is 1. Therefore, none of the examples in Example 4.6 is covered in previous papers in the literature.

**Example 4.6.** Suppose that zer  $A \neq \emptyset$  and that one of the following holds.

- (i)  $(\forall k \in \mathbb{N}) \ \alpha_k = \frac{1}{k+3}, \ \beta_k = \frac{1}{k+2}, \ \gamma_k = \frac{k}{k+2}, \ \delta_k \equiv 1, \ c_k = k, \ \text{and} \ \|e_k\| \leq \frac{1}{(k+2)^2}.$  (Employ Theorem 4.4(i).)
- (ii) u = 0,  $(\forall k \in \mathbb{N}) \alpha_k \equiv 0$ ,  $\beta_k = \gamma_k \equiv \frac{k+1}{2(k+2)}$ ,  $\delta_k = \frac{1}{k+3}$ ,  $c_k \equiv c \in \mathbb{R}_{++}$ , and  $e_k \to 0$ . (Apply Theorem 4.5(i).)
- (iii)  $(\forall k \in \mathbb{N}) \ \alpha_k = \frac{1}{k+3}, \ \beta_k = \gamma_k = \frac{k}{2(k+2)}, \ \delta_k \equiv 1, \ c_k \equiv c \in \mathbb{R}_{++}, \ \text{and} \ \|e_k\| \leq \frac{1}{(k+2)^2}.$  (Adopt Theorem 4.5(iii).)

Then  $x_k \to P_{\operatorname{zer} A} u$ .

To end this paper, we compare our conditions for the strong convergence of generalized proximal point algorithm with that of related references in the remark below.

**Remark 4.7.** (i) The modified proximal point algorithm in [13, Algorithm 5.1] is (4.1) with  $u = x_0$  and  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0,1], \ \beta_k = 0$ , and  $\gamma_k = \delta_k = 1 - \alpha_k$ . In this case, the conditions  $(\forall k \in \mathbb{N}) \ |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \le 1, \ \beta_k + \gamma_k \ge 0, \ \text{and} \ (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 - \beta_k - \gamma_k \le 0, \ \sum_{k \in \mathbb{N}} |1 - \alpha_k - \beta_k - \gamma_k| < \infty, \ \text{and} \ \beta_k \to 0 \ \text{hold trivially; moreover, } \alpha_k \to 0 \ \text{implies that} \ \gamma_k \to 1; \ \text{furthermore, since} \ (\forall k \in \mathbb{N}) \ 1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 = 1 - (1 - \alpha_k)^2 \ge \alpha_k, \ \text{thus } \sum_{k \in \mathbb{N}} \alpha_k = \infty \ \text{necessitates} \ \sum_{i \in \mathbb{N}} 1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 = \infty.$ 

Hence, it is clear that Theorem 4.4(i) covers [13, Theorem 5.1].

(ii) The contraction proximal point algorithm in [15] is (4.1) satisfying that  $(\forall k \in \mathbb{N}) \{\alpha_k, \beta_k, \gamma_k\} \subseteq$  ]0,1[ with  $\alpha_k + \beta_k + \gamma_k = 1$  and  $\delta_k \equiv 1$ . In this case, it is trivial that  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1, \beta_k + \gamma_k \geq 0, (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 - \beta_k - \gamma_k \leq 0, \text{ and } |1 - \alpha_k - \beta_k - \gamma_k| = 0. \text{ Since } (\forall k \in \mathbb{N}) 1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 \geq \alpha_k, \text{ thus } \sum_{k \in \mathbb{N}} \alpha_k = \infty \text{ leads to } \sum_{i \in \mathbb{N}} 1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 = \infty.$ Moreover, because  $(\forall k \in \mathbb{N}) 1 - \beta_k - \frac{\gamma_k}{2} = \frac{1}{2} + \frac{\alpha_k}{2} - \frac{\beta_k}{2}$ , the assumptions  $\alpha_k \to 0$  and  $0 < \lim \inf_{k \to \infty} \beta_k \leq \lim \sup_{k \to \infty} \beta_k < 1$  in [15, Theorem 3.3] necessitate  $0 < \lim \inf_{i \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} \leq \lim \sup_{i \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} \leq 1$ . In addition, it is easy to see that  $c_{k+1} - c_k \to 0$  and  $\inf_{k \in \mathbb{N}} c_k > 0$  guarantee  $1 - \frac{c_k}{c_{k+1}} \to 0$ .

Hence, Theorem 4.4(ii) generalizes [15, Theorem 3.3].

(iii) As stated in [2, Page 637], the expression  $(\forall k \in \mathbb{N}) y_{k+1} = J_{c_k A} ((1 - \alpha_k)y_k + \alpha_k u + e_k)$  can be rewrite as  $(\forall k \in \mathbb{N}) x_{k+1} = \alpha_{k+1}u + (1 - \alpha_{k+1}) J_{c_k A}(x_k) + e_{k+1}$ , where  $(\forall k \in \mathbb{N}) x_k := (1 - \alpha_k)y_k + \alpha_k u + e_k$ . Hence, (4.1) with  $(\forall k \in \mathbb{N}) \alpha_k \in ]0, 1[$ ,  $\beta_k = 0, \gamma_k = 1 - \alpha_k$ , and  $\delta_k = 1$  is the generalized proximal point algorithm studied in [2], [14], and [17]. Note that in this case the conditions  $(\forall k \in \mathbb{N}) |\alpha_k| + |\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| \leq 1, \beta_k + \gamma_k \geq 0$ , and  $(|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 - \beta_k - \gamma_k \leq 0, \sum_{k \in \mathbb{N}} |1 - \alpha_k - \beta_k - \gamma_k| < \infty$ , and  $\limsup_{k \to \infty} |\beta_k| < 1$  hold trivially; because  $(\forall k \in \mathbb{N}) 1 - \beta_k - \frac{\gamma_k}{2} = 1 - \frac{1}{2}(1 - \alpha_k) = \frac{1 - \alpha_k}{2}, \alpha_k \to 0$  implies directly  $\lim_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} = \frac{1}{2} \in ]0, 1[$ ; furthermore, similarly with our statement in (i) above,  $\sum_{k \in \mathbb{N}} \alpha_k = \infty$  implies  $\sum_{i \in \mathbb{N}} 1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2 = \infty$ .

Therefore, we know that Theorem 4.4(ii) and Theorem 4.4(iv) improve [17, Theorem 4] and [2, Theorem 1], respectively. Moreover, because actually [17, Theorem 4] refines [14, Theorem 3.3], Theorem 4.4(ii) naturally improves [14, Theorem 3.3] as well.

- (iv) The contraction-proximal point algorithm is (4.1) with  $(\forall k \in \mathbb{N}) \ \alpha_k \in [0,1], \ \beta_k = 0, \ \gamma_k = 1 \alpha_k$ , and  $\delta_k = 1$ . Repeating some analysis presented in (iii) and noticing that now  $(\forall k \in \mathbb{N}) | (\alpha_{k+1} + \gamma_{k+1}) (\alpha_k + \gamma_k)| = 0$ ,  $|\beta_k| = 0$ , and  $\frac{|\alpha_{k+1} \alpha_k|}{1 |\gamma_{k+1}|} = \left|\frac{\alpha_{k+1} \alpha_k}{\alpha_{k+1}}\right| = \left|1 \frac{\alpha_k}{\alpha_{k+1}}\right|$ , we observe easily that Theorem 4.4(iii) covers [8, Theorem 4.1].
- (v) The proximal point algorithm with error terms studied in [16] is (4.1) such that u = 0,  $(\forall k \in \mathbb{N})$   $\alpha_k \equiv 0$  and  $\{\beta_k, \gamma_k, \delta_k\} \subseteq [0, 1[$  with  $\beta_k + \gamma_k + \delta_k = 1$ . In this case, it is trivial that  $(\forall k \in \mathbb{N})$   $|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| < 1$  and  $|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}| + |\delta_k| \leq 1$ , and that  $\sum_{k \in \mathbb{N}} |1 - \beta_k - \gamma_k - \delta_k| < \infty$ . In this case,  $\delta_k \to 0$  and  $e_k \to 0$  imply that  $\sup_{k \in \mathbb{N}} \frac{1 - \beta_k - \gamma_k}{1 - (|\beta_k + \frac{\gamma_k}{2}| + |\frac{\gamma_k}{2}|)^2} < \infty$ ,  $\sup_{k \in \mathbb{N}} ||e_k - p|| < \infty$ , and  $\frac{\delta_k e_k}{1 - \beta_k - \gamma_k} \to 0$ ; moreover, since  $(\forall k \in \mathbb{N})$   $1 - \beta_k - \frac{\gamma_k}{2} = \frac{1}{2} - \frac{\beta_k}{2} + \frac{\delta_k}{2}$ , thus  $\delta_k \to 0$  and  $0 < \lim \inf_{i \to \infty} \beta_i \leq \limsup_{i \to \infty} \beta_i < 1$  necessitate that  $0 < \liminf_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} \leq \limsup_{k \to \infty} 1 - \beta_k - \frac{\gamma_k}{2} < 1$ .

Therefore, Theorem 4.5 generalizes [16, Theorem 3.1]

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