

Research Article

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Weak and strong estimates for linear and multilinear fractional Hausdorff operators on the Heisenberg group

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Abstract: This paper is devoted to the weak and strong estimates for the linear and multilinear fractional Hausdorff operators on the Heisenberg group H^n . A sharp strong estimate for T_Φ^m is obtained. As an application, we derive the sharp constant for the product Hardy operator on H^n . Some weak-type (p, q) ($1 \leq p \leq \infty$) estimates for $T_{\Phi, \beta}$ are also obtained. As applications, we calculate some sharp weak constants for the fractional Hausdorff operator on the Heisenberg group. Besides, we give an explicit weak estimate for $T_{\Phi, \beta}^{m, \rightarrow}$ under some mild assumptions on Φ . We extend the results of Guo et al. [*Hausdorff operators on the Heisenberg group*, Acta Math. Sin. (Engl. Ser.) **31** (2015), no. 11, 1703–1714] to the fractional setting.

Keywords: Hausdorff operator, Heisenberg group, multilinear, sharp bound

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1 Introduction

The Hausdorff operator was first introduced in [1] with summability of number series. As is well known, the Hausdorff operator includes many famous operators such as the Hardy operator, the adjoint Hardy operator, the Cesaro operator and the Hardy-Littlewood-Polya operator (see the examples below). In particular, the Hardy operator, as a kind of very important average operator, is deeply studied by many mathematicians. Researchers have built a relatively complete and mature theory about the Hardy-type operator. Naturally, the study on the Hausdorff operator is of great significance. In recent years, the Hausdorff operator and its variations have been widely studied by many researchers. In [2,3], Chen et al. considered the boundedness properties of the Hausdorff operator on the Euclidean spaces, such as the Lebesgue spaces L^p , the Hardy spaces H^p and the Herz-type spaces. For the sake of convenience, one can refer to [4–8] for more details of the recent progress on the Hausdorff operators. In 2015, Gao et al. [9] studied the boundedness properties of the (fractional) Hausdorff operators on the Lebesgue spaces L^p with powers. It is the starting point of our research.

We first recall the definition of the classical one-dimensional Hausdorff operator. For a given locally integrable function $\Phi(y)$ defined on $\mathbb{R} = (-\infty, \infty)$, the one-dimensional Hausdorff operator is defined by

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$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(x/y)}{y} f(y) dy, \quad x \in \mathbb{R}.$$

Correspondingly, the n -dimensional Hausdorff operator (see [10]) is given by

$$H_\Psi(f)(x) = \int_{\mathbb{R}^n} \frac{\Psi(x/|y|)}{|y|^n} f(y) dy, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where Ψ is a radial function.

Note that if we take $\Phi(t) = \frac{1}{t} \chi_{(1,\infty)}(t)$ for $t > 0$ and $\Psi(t) = \frac{1}{|B(0,|t|)|} \chi_{(1,\infty)}(|t|)$, then we get the one-dimensional and the n -dimensional Hardy operator, respectively,

$$H(f)(x) = \frac{1}{x} \int_0^x f(t) dt,$$

$$\mathcal{H}(f)(x) = \frac{1}{|B(0, |x|)|} \int_{|y| < |x|} f(y) dy,$$

here and throughout the paper, $B(0, |x|)$ is an open ball centered at the origin with radius $|x|$ and $|B(0, |x|)|$ is the volume of the ball $B(0, |x|)$.

In the past few years, most work on the Hardy operator mainly focused on the n -dimensional Euclidean spaces \mathbb{R}^n , and we refer to [11–15] for many discussions on the Hardy operators. A natural question is whether the techniques for the investigation of the Hardy operator in \mathbb{R}^n can be used in different underlying spaces. It is a remarkable fact that the Heisenberg group, denoted by \mathbb{H}^n , arises in two fundamental but different settings in analysis: in connection with the Fourier transform, pseudodifferential operators and related matters [16]. Recently, Fu and Wu [17] proved a sharp estimate for the n -dimensional Hardy operator on the Heisenberg group \mathbb{H}^n . As we observed earlier, the Hausdorff operators are generalization of the Hardy operators. It is therefore natural to ask whether one can obtain the best constants for the high-dimensional and multilinear Hausdorff operators on the Heisenberg group.

2 Basic properties of the Heisenberg group

The Heisenberg group \mathbb{H}^n is $\mathbb{R}^{2n} \times \mathbb{R}$ with group law

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2 \sum_{j=1}^n (y_j x_{n+j} - x_j y_{n+j}) \right)$$

for $x = (x_1, \dots, x_{2n}, x_{2n+1}), y = (y_1, \dots, y_{2n}, y_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}$.

We can see that the identity element on \mathbb{H}^n is $0 \in \mathbb{R}^{2n+1}$ and the inverse element of x is $-x$. The Haar measure on \mathbb{H}^n coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}^1$.

\mathbb{H}^n is a homogeneous group with dilations

$$\delta_r x = (rx_1, rx_2, \dots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0.$$

For any measurable set $E \subseteq \mathbb{H}^n$, denoted by $|E|$ the measure of E , one has

$$|\delta_r(E)| = r^Q |E|, \quad d(\delta_r x) = r^Q dx,$$

where $Q = 2n + 2$ is the so-called homogeneous dimension. And we rule the norm

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2 \right)^2 + x_{2n+1}^2 \right]^{\frac{1}{4}}.$$

Then the distance on the Heisenberg group \mathbb{H}^n is defined by

$$d(x, y) = |y^{-1} \cdot x|_h.$$

For $x \in \mathbb{H}^n$, $r > 0$, the ball with center x and radius r on \mathbb{H}^n is given by

$$B(x, r) = \{y \in \mathbb{H}^n : d(x, y) < r\},$$

and its sphere is defined by

$$S(x, r) = \{y \in \mathbb{H}^n : d(x, y) = r\}.$$

Thanks to the previous definitions, we obtain immediately that

$$|B(x, r)| = |B(0, r)| = v_Q r^Q,$$

where v_Q is the volume of the unit ball $B(0, 1)$ on \mathbb{H}^n , and

$$v_Q = \frac{2\pi^{n+\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma(n+1)\Gamma\left(\frac{n+1}{2}\right)}.$$

The unit sphere $S(0, 1)$ is often simply denoted by S^{Q-1} , and the area of S^{Q-1} is $\omega_Q = Qv_Q$. The reader is referred to [16,18] for more details.

3 Estimates for multilinear Hausdorff operators on central Morrey space with power weight

In 2012, Chen et al. [5] first introduced the multilinear Hausdorff operator on Euclid spaces. The m -linear Hausdorff operator is defined as follows.

For a locally integrable function Φ defined on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$, we denote

$$H_{\Phi}^m(f_1, f_2, \dots, f_m)(x) = \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \frac{\Phi\left(\frac{x}{|y_1|}, \frac{x}{|y_2|}, \dots, \frac{x}{|y_m|}\right)}{\prod_{i=1}^m |y_i|^n} \prod_{j=1}^m f_j(y_j) dy_1 dy_2 \cdots dy_m.$$

Zhao et al. studied the m -linear Hausdorff operator on the Heisenberg group \mathbb{H}^n in [19]:

Definition 3.1. Let Φ be a locally integrable function on m -fold space $\mathbb{H}^n \times \cdots \times \mathbb{H}^n$, $m \in \mathbb{N}$. The m -linear Hausdorff operator is defined by

$$T_{\Phi}^m(f_1, \dots, f_m)(x) = \int_{\mathbb{H}^n \times \cdots \times \mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}}x, \dots, \delta_{|y_m|_h^{-1}}x)}{\prod_{i=1}^m |y_i|_h^Q} \prod_{j=1}^m f_j(y_j) dy_1 \cdots dy_m.$$

This study is based on central Morrey space with power weight, so we give the definition of central Morrey space with power weight on \mathbb{H}^n .

Definition 3.2. Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$, $\omega = \omega(x) = |x|_h^{\alpha}$, $\alpha \geq 0$. The central Morrey space with power weight $\dot{B}^{p,\lambda}(\mathbb{H}^n, |x|_h^{\alpha} dx)$ is defined by

$$\dot{B}^{p,\lambda}(\mathbb{H}^n, |x|_h^{\alpha} dx) := \{f \in L_{\text{loc}}^p(\mathbb{H}^n, |x|_h^{\alpha} dx) : \|f\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n, |x|_h^{\alpha} dx)} < +\infty\},$$

where

$$\|f\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n, |x|_h^{\alpha} dx)} = \sup_{R>0} \left(\frac{1}{\omega(B(0, R))^{1+p\lambda}} \int_{B(0, R)} |f(x)|^p |x|_h^{\alpha} dx \right)^{\frac{1}{p}}.$$

It is worth noting that, when $\lambda = -1/p$, the central Morrey space turns to be the Lebesgue space.

Our first main result is a strong estimate for the m -linear Hausdorff operators on the central Morrey space with power weight. In particular, if we replace the central Morrey space with Lebesgue space, the result is sharp.

Denote by $\vec{p} = (p_1, p_2, \dots, p_m)$ with $1 \leq p_j \leq \infty$ for $1 \leq j \leq m$.

Theorem 3.1. *Let $m \in \mathbb{N}$, $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$, $1 \leq p < \infty$, $1 < p_i < \infty$, $-1/p_i \leq \lambda_i < 0$, $i = 1, \dots, m$, $\alpha_i \geq 0$, $i = 0, 1, \dots, m$ and $1/p = 1/p_1 + \dots + 1/p_m$, $\lambda = \lambda_1 + \dots + \lambda_m$, $\alpha_0/p = \alpha_1/p_1 + \dots + \alpha_m/p_m$. If Φ is a non-negative, radial function, and*

$$C_{\Phi, \vec{p}, \vec{\alpha}} = \omega_Q^m \int_0^\infty \dots \int_0^\infty \int_0^\infty \Phi(r_1, r_2, \dots, r_m) \prod_{i=1}^m r_i^{-(\lambda_i(\alpha_i+Q)+1)} dr_1 dr_2 \dots dr_m < \infty, \tag{3.1}$$

then for $f_i \in \dot{B}^{p_i, \lambda_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)$, we have

$$\|T_\Phi^m(f_1, f_2, \dots, f_m)\|_{\dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha_0} dx)} \leq C_{\Phi, \vec{p}, \vec{\alpha}} \prod_{i=1}^m \|f_i\|_{\dot{B}^{p_i, \lambda_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)}. \tag{3.2}$$

Moreover, the constant (3.1) in (3.2) is the best possible when $\lambda_1 p_1 = \dots = \lambda_m p_m$ and $\alpha_1 = \dots = \alpha_m$.

Proof. We merely give the proof with the case $m = 2$ for the sake of clarity in writing, and the same is true for the general case $m > 2$.

Let $y_j \in \mathbb{H}^n$, $j = 1, 2$. For $y_j \neq 0$, let $y'_j = \delta_{|y_j|_h^{-1}} y_j$. For any $f_j \in \dot{B}^{p_j, \lambda_j}(\mathbb{H}^n, |x|_h^{\alpha_j} dx)$, set

$$(g_{f_j})(y_j) = \frac{1}{\omega_Q} \int_{|y'_j|_h=1} f_j(\delta_{|y_j|_h} y'_j) dy'_j.$$

It is easy to see that g_{f_j} is a radial function. By the definition of g_{f_j} ,

$$\begin{aligned} T_\Phi^2(g_{f_1}, g_{f_2})(x) &= \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}} x, \delta_{|y_2|_h^{-1}} x)}{|y_1|_h^Q |y_2|_h^Q} g_{f_1}(y_1) g_{f_2}(y_2) dy_1 dy_2 \\ &= \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}} x, \delta_{|y_2|_h^{-1}} x)}{|y_1|_h^Q |y_2|_h^Q} \prod_{j=1}^2 \frac{1}{\omega_Q} \int_{|y'_j|_h=1} f_j(\delta_{|y_j|_h} y'_j) dy'_j dy_1 dy_2 \\ &= \frac{1}{\omega_Q^2} \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}} x, \delta_{|y_2|_h^{-1}} x)}{|y_1|_h^Q |y_2|_h^Q} \prod_{j=1}^2 \int_{|y'_j|_h=1} f_j(\delta_{|y_j|_h} y'_j) dy'_j dy_1 dy_2 \\ &= \int_{|y'_1|_h=1} \int_{|y'_2|_h=1} \left(\int_0^\infty \int_0^\infty \frac{\Phi(\delta_{r_1^{-1}} x, \delta_{r_2^{-1}} x)}{r_1^Q r_2^Q} \left(\prod_{j=1}^2 f_j(\delta_{r_j} y'_j) r_j^{Q-1} \right) dr_1 dr_2 \right) dy'_1 dy'_2 \\ &= \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}} x, \delta_{|y_2|_h^{-1}} x)}{|y_1|_h^Q |y_2|_h^Q} f_1(y_1) f_2(y_2) dy_1 dy_2 = T_\Phi^2(f_1, f_2)(x). \end{aligned}$$

We use Hölder’s inequality to bound $\|g_{f_j}\|_{\dot{B}^{p_j, \lambda_j}(\mathbb{H}^n, |x|_h^{\alpha_j} dx)}$ by

$$\begin{aligned} &\|g_{f_j}\|_{\dot{B}^{p_j, \lambda_j}(\mathbb{H}^n, |x|_h^{\alpha_j} dx)} \\ &= \sup_{R>0} \left(\frac{1}{\omega(B(0, R))^{1+p_j \lambda_j}} \int_{B(0, R)} \left| \frac{1}{\omega_Q} \int_{|y'_j|_h=1} f_j(\delta_{|y_j|_h} y'_j) dy'_j \right|^{p_j} |x|_h^{\alpha_j} dx \right)^{\frac{1}{p_j}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\omega_Q} \sup_{R>0} \left(\frac{1}{\omega(B(0, R))^{1+p_j\lambda_j}} \int_{B(0,R)} \left(\int_{|y'_j|_h=1} |f_j(\delta_{|x|_h} y'_j)|^{p_j} dy'_j \right) \left(\int_{|y'_j|_h=1} dy'_j \right)^{\frac{p_j}{p_j}} |x|_h^{\alpha_j} dx \right)^{\frac{1}{p_j}} \\ &= \frac{1}{\omega_Q^{1/p_j}} \sup_{R>0} \left(\frac{1}{\omega(B(0, R))^{1+p_j\lambda_j}} \int_0^R \int_{|x'|_h=1} \left(\int_{|y'_j|_h=1} |f_j(\delta_r y'_j)|^{p_j} dy'_j \right) r^{\alpha_j+Q-1} dx' dr \right)^{\frac{1}{p_j}} \\ &= \sup_{R>0} \left(\frac{1}{\omega(B(0, R))^{1+p_j\lambda_j}} \int_{B(0,R)} |f(x)|^{p_j} |x|_h^{\alpha_j} dx \right)^{\frac{1}{p_j}} = \|f_j\|_{\dot{B}^{p_j, \lambda_j}(\mathbb{H}^n, |x|_h^{\alpha_j} dx)}. \end{aligned}$$

Hence,

$$\frac{\|T_\Phi^2(f_1, f_2)\|_{\dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha_0} dx)}}{\|f_1\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{H}^n, |x|_h^{\alpha_1} dx)} \|f_2\|_{\dot{B}^{p_2, \lambda_2}(\mathbb{H}^n, |x|_h^{\alpha_2} dx)}} \leq \frac{\|T_\Phi^2(g_{f_1}, g_{f_2})\|_{\dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha_0} dx)}}{\|g_{f_1}\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{H}^n, |x|_h^{\alpha_1} dx)} \|g_{f_2}\|_{\dot{B}^{p_2, \lambda_2}(\mathbb{H}^n, |x|_h^{\alpha_2} dx)}},$$

which implies that the operator T_Φ^2 and its restriction to the radial functions have the same operator norm on $\dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha_0} dx)$. Consequently, without loss of generality, it suffices to fulfill the proof of the theorem by assuming that $f_j \in \dot{B}^{p_j, \lambda_j}(\mathbb{H}^n, |x|_h^{\alpha_j} dx)$ ($j = 1, 2$) are non-negative radial functions.

By the coordinate transform, one can check that

$$T_\Phi^2(f_1, f_2)(x) = \omega_Q^2 \int_0^\infty \int_0^\infty \frac{\Phi(t_1, t_2)}{t_1 t_2} f_1(|x|_h/t_1) f_2(|x|_h/t_2) dt_1 dt_2.$$

We use Minkowski's integral inequality and Hölder's inequality to estimate

$$\begin{aligned} &\|T_\Phi^2(f_1, f_2)\|_{\dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha_0} dx)} \\ &= \omega_Q^2 \sup_{R>0} \left(\frac{1}{\omega(B(0, R))^{1+p\lambda}} \int_{B(0,R)} \left| \int_0^\infty \int_0^\infty \frac{\Phi(t_1, t_2)}{t_1 t_2} f_1(|x|_h/t_1) f_2(|x|_h/t_2) dt_1 dt_2 \right|^p |x|_h^{\alpha_0} dx \right)^{\frac{1}{p}} \\ &\leq \omega_Q^2 \sup_{R>0} \frac{1}{\omega(B(0, R))^{1/p+\lambda}} \int_0^\infty \int_0^\infty \frac{\Phi(t_1, t_2)}{t_1 t_2} \left(\int_{B(0,R)} |f_1(|x|_h/t_1) f_2(|x|_h/t_2)|^p |x|_h^{\alpha_0} dx \right)^{\frac{1}{p}} dt_1 dt_2 \\ &\leq \omega_Q^2 \int_0^\infty \int_0^\infty \frac{\Phi(t_1, t_2)}{t_1 t_2} t_1^{-\lambda_1(\alpha_1+Q)} t_2^{-\lambda_2(\alpha_2+Q)} dt_1 dt_2 \|f_1\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{H}^n, |x|_h^{\alpha_1} dx)} \|f_2\|_{\dot{B}^{p_2, \lambda_2}(\mathbb{H}^n, |x|_h^{\alpha_2} dx)}. \end{aligned}$$

Thus,

$$\|T_\Phi^2\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{H}^n, |x|_h^{\alpha_1} dx) \times \dot{B}^{p_2, \lambda_2}(\mathbb{H}^n, |x|_h^{\alpha_2} dx) \rightarrow \dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha_0} dx)} \leq \omega_Q^2 \int_0^\infty \int_0^\infty \Phi(t_1, t_2) t_1^{-(\lambda_1(\alpha_1+Q)+1)} t_2^{-(\lambda_2(\alpha_2+Q)+1)} dt_1 dt_2.$$

Next, we need to prove the converse inequality. When $\lambda_1 p_1 = \lambda_2 p_2 \neq -1$ and $\alpha_1 = \dots = \alpha_m$, we denote $\alpha = \alpha_1$, and take

$$f_1(x) = |x|_h^{\lambda_1(\alpha+Q)}, \quad f_2(x) = |x|_h^{\lambda_2(\alpha+Q)}.$$

It is easy to see that

$$\|f_1\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{H}^n, |x|_h^{\alpha} dx)} = (1 + \lambda_1 p_1)^{-1/p_1} \left(\frac{\alpha + Q}{\omega_Q} \right)^{\lambda_1}, \quad \|f_2\|_{\dot{B}^{p_2, \lambda_2}(\mathbb{H}^n, |x|_h^{\alpha} dx)} = (1 + \lambda_2 p_2)^{-1/p_2} \left(\frac{\alpha + Q}{\omega_Q} \right)^{\lambda_2}.$$

Then we have

$$T_{\Phi}^2(f_1, f_2)(x) = C_{\Phi, \vec{p}, a} |x|_h^{\lambda(\alpha+Q)}.$$

The aforementioned estimate gives that

$$\|T_{\Phi}^2(f_1, f_2)\|_{\dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha} dx)} = C_{\Phi, \vec{p}, a} (1 + \lambda p)^{-1/p} \left(\frac{\alpha + Q}{\omega_Q}\right)^{\lambda}.$$

Consequently, we conclude that

$$\|T_{\Phi}^2\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{H}^n, |x|_h^{\alpha_1} dx) \times \dot{B}^{p_2, \lambda_2}(\mathbb{H}^n, |x|_h^{\alpha_2} dx) \rightarrow \dot{B}^{p, \lambda}(\mathbb{H}^n, |x|_h^{\alpha} dx)} \geq \omega_Q^2 \int_0^{\infty} \int_0^{\infty} \Phi(t_1, t_2) t_1^{-(\lambda_1(\alpha+Q)+1)} t_2^{-(\lambda_2(\alpha+Q)+1)} dt_1 dt_2.$$

When $\lambda_1 p_1 = \lambda_2 p_2 = -1$ and $\alpha_1 = \dots = \alpha_m$, we also denote $\alpha = \alpha_1$, and take

$$f_{1,k}(x) = |x|_h^{\frac{\alpha+Q+\frac{1}{k}}{p_1}} \chi_{\{|x|_h \geq 1\}}(x), \quad f_{2,k}(x) = |x|_h^{\frac{\alpha+Q+\frac{1}{k}}{p_2}} \chi_{\{|x|_h \geq 1\}}(x).$$

By a similar calculation, we can get

$$\|T_{\Phi}^2\|_{L^{p_1}(\mathbb{H}^n, |x|_h^{\alpha} dx) \times L^{p_2}(\mathbb{H}^n, |x|_h^{\alpha} dx) \rightarrow L^p(\mathbb{H}^n, |x|_h^{\alpha} dx)} \geq (k^{-\frac{1}{k}})^{\frac{1}{p}} \omega_Q^2 \int_0^k \int_0^k \Phi(t_1, t_2) t_1^{\alpha k - 1} t_2^{\beta k - 1} dt_1 dt_2.$$

Consequently, letting $k \rightarrow \infty$, we conclude that

$$\|T_{\Phi}^2\|_{L^{p_1}(\mathbb{H}^n, |x|_h^{\alpha} dx) \times L^{p_2}(\mathbb{H}^n, |x|_h^{\alpha} dx) \rightarrow L^p(\mathbb{H}^n, |x|_h^{\alpha} dx)} \geq \omega_Q^2 \int_0^{\infty} \int_0^{\infty} \Phi(t_1, t_2) t_1^{\frac{\alpha+Q}{p_1} - 1} t_2^{\frac{\alpha+Q}{p_2} - 1} dt_1 dt_2.$$

This finishes the proof of Theorem 3.1. □

Remark 3.2. If we take $\alpha_1 = \dots = \alpha_m = 0$, $\lambda_i = -1/p_i$, then Theorem 3.1 covers Theorem 1.5 in [19].

To deal with the multilinear setting, we use the method of rotation which is inspired by the work of [19]. In fact, this method is very useful to compute the operator norm of many average operators. We will give an application to product Hardy-type operator.

The rectangular product operator was defined in [20] by

$$\mathcal{H}^m(f)(x) = \left(\prod_{i=1}^m \frac{1}{|B(0, |x_i|)|} \right) \int_{|y_1| < |x_1|} \dots \int_{|y_m| < |x_m|} f(y_1, \dots, y_m) dy_1 \dots dy_m,$$

where $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_m}$ with $\prod_{i=1}^m |x_i| \neq 0$. In the following, we also use $|x|_h^{\vec{\alpha}} = |x_1|_h^{\alpha_1} \dots |x_m|_h^{\alpha_m}$ and $dx = dx_1 \dots dx_m$ for short. We will consider the rectangular Hardy-type operator on \mathbb{H}^n similar to that defined in [20].

Definition 3.3. Let f be a locally integrable function on m -fold product space $\mathbb{H}^n \times \dots \times \mathbb{H}^n$. For $m \in \mathbb{N}$, the m -linear rectangular Hardy-type operator is defined by

$$\mathfrak{R}^m(f)(x) = \left(\prod_{i=1}^m \frac{1}{|B(0, |x_i|_h)|} \right) \int_{|y_1|_h < |x_1|_h} \dots \int_{|y_m|_h < |x_m|_h} f(y_1, \dots, y_m) dy_1 \dots dy_m.$$

Our next result is based on central product Morrey space with power weight on \mathbb{H}^n .

Definition 3.4. Let $1 \leq p < \infty$, $m \in \mathbb{N}^+$, $-1/p \leq \lambda_i < 0$, $\lambda = (\lambda_1, \dots, \lambda_m)$, $\omega_i(x_i) = |x_i|^{\alpha_i}$, $\alpha_i \geq 0$, $1 \leq i \leq m$. The central product Morrey space with power weights $\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)$ is defined by

$$\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx) := \{f \in L^p_{loc}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx) : \|f\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} < +\infty\},$$

where

$$\|f\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} = \sup_{R_i > 0, 1 \leq i \leq m} \left(\frac{1}{\prod_{i=1}^m \omega_i(B(0, R_i))^{1+p\lambda_i}} \int_{B(0, R_1)} \dots \int_{B(0, R_m)} |f(x)|^p |x|_{\vec{h}}^{\vec{\alpha}} dx \right)^{\frac{1}{p}}.$$

Theorem 3.3. Let $m \in \mathbb{N}$, $1 < p < \infty$, $-1/p \leq \lambda_i < 0$, $\lambda = (\lambda_1, \dots, \lambda_m)$, $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$, $0 \leq \alpha_i < Q(p - 1)$ and $i = 1, \dots, m$. If $f \in \dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)$, then we have

$$\|\mathfrak{R}^m(f)\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} \leq \left(\prod_{i=1}^m \frac{Q}{(\alpha_i + Q)\lambda_i + Q} \right) \|f\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)}. \tag{3.3}$$

Moreover, the constant $\left(\prod_{i=1}^m \frac{Q}{(\alpha_i + Q)\lambda_i + Q} \right)$ in (3.3) is sharp.

Proof of Theorem 3.3. We give the proof only in the case $m = 2$. Following the method of Theorem 3.1, we can easily obtain the upper bound

$$\|\mathfrak{R}^2\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx) \rightarrow \dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} \leq \prod_{i=1}^m \frac{Q}{(\alpha_i + Q)\lambda_i + Q}.$$

On the other hand, when $\lambda_i \neq -1/p$, $i = 1, 2$, we set

$$f(x_1, x_2) = |x_1|_{\vec{h}}^{(\alpha_1+Q)\lambda_1} |x_2|_{\vec{h}}^{(\alpha_2+Q)\lambda_2}.$$

Since

$$\mathfrak{R}^2(f)(x_1, x_2) = \frac{1}{|B(0, |x_1|_h)|} \cdot \frac{1}{|B(0, |x_2|_h)|} \int_{|y_1|_h < |x_1|_h} \int_{|y_2|_h < |x_2|_h} f(y_1, y_2) dy_1 dy_2 = \left(\prod_{i=1}^m \frac{Q}{(\alpha_i + Q)\lambda_i + Q} \right) f(x_1, x_2),$$

we obtain

$$\|\mathfrak{R}^2\|_{\dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx) \rightarrow \dot{B}^{p,\lambda}(\mathbb{H}^n \times \dots \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} \geq \left(\prod_{i=1}^m \frac{Q}{(\alpha_i + Q)\lambda_i + Q} \right).$$

When $\lambda_i = -1/p$, $i = 1, 2$, we set $0 < \varepsilon < \min \left\{ 1, \frac{(p-1)Q - \alpha_1}{p}, \frac{(p-1)Q - \alpha_2}{p} \right\}$, and

$$f_\varepsilon(x_1, x_2) = |x_1|_{\vec{h}}^{-\frac{\alpha_1+Q}{p} + \varepsilon} |x_2|_{\vec{h}}^{-\frac{\alpha_2+Q}{p} + \varepsilon} \chi_{\{|x_1|_h < 1, |x_2|_h < 1\}}(x_1, x_2).$$

By a similar calculation, we can get

$$\|\mathfrak{R}^2\|_{L^p(\mathbb{H}^n \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx) \rightarrow L^p(\mathbb{H}^n \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} \geq \left(\prod_{i=1}^2 \frac{Q}{Q - \frac{\alpha_i + Q}{p} + \varepsilon} \right).$$

Consequently, by letting $\varepsilon \rightarrow 0^+$, we obtain

$$\|\mathfrak{R}^2\|_{L^p(\mathbb{H}^n \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx) \rightarrow L^p(\mathbb{H}^n \times \mathbb{H}^n, |x|_{\vec{h}}^{\vec{\alpha}} dx)} \geq \left(\prod_{i=1}^2 \frac{Q}{-\frac{\alpha_i + Q}{p} + Q} \right).$$

We finish the proof of Theorem 3.3. □

Remark 3.4. If we take $\vec{\alpha} = \vec{0}$, $\lambda_i = -1/p$, then Theorem 3.3 covers Theorem 1.8 in ref. [19].

4 Weak-type estimates for the fractional Hausdorff operators on the Heisenberg group and applications

Now we define the fractional Hausdorff operators on \mathbb{H}^n as follows.

Definition 4.1. Let Φ be a locally integrable function on \mathbb{H}^n and $0 \leq \beta < Q$. The n -dimensional fractional Hausdorff operator is given by

$$T_{\Phi,\beta}(f)(x) = \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^{Q-\beta}} f(y) dy.$$

In [19], the authors obtained the strong-type (p, p) estimate for $\beta = 0$. We refer readers to [21,22] for more details of Hausdorff operators on the Heisenberg group. It is known that the strong-type (p, p) estimate implies the weak-type (p, p) estimate. On the other hand, it is natural to ask whether it is possible to relax the condition on the function Φ . Inspired by the result of [7], we derive a better upper bound for the weak estimate of the operator T_Φ on the Heisenberg group with the non-radial function Φ . Note that the weak estimate holds under weaker assumptions than those in the strong estimate. The best constant for weak estimate is not yet to be determined and will be investigated in the future.

To state our results, we need to introduce more notations. Let $1 \leq p \leq \infty$ and denote by p' the exponent conjugate to p , that is, let $1/p' + 1/p = 1$ with the agreement that $1/\infty = 0$.

For a non-negative measurable function $\omega = \omega(x)$ defined on \mathbb{H}^n , the weighted weak Lebesgue space $L_\omega^{p,\infty}(\mathbb{H}^n)$ is given by the set of all measurable functions f on \mathbb{H}^n satisfying

$$\|f\|_{L_\omega^{p,\infty}(\mathbb{H}^n)} := \sup_{\lambda > 0} \lambda \left(\int_{\{x \in \mathbb{H}^n : |f(x)| > \lambda\}} \omega(x) dx \right)^{1/p} < \infty.$$

For $x \in \mathbb{H}^n$, $1 \leq p, q < \infty$, we let $x' = \delta_{|x|_h^{-1}}x$, and

$$K_p(x') = \left(\int_0^\infty |\Phi(\delta_t x')|^{p'} t^{\frac{(Q+\gamma)p'}{q}-1} dt \right)^{\frac{1}{p'}}, \quad K_1(x') = \sup_{t > 0} |\Phi(\delta_t x')| t^{\frac{Q+\gamma}{q}}.$$

Our first main result is an explicit power weighted weak-type estimate for the high-dimensional Hausdorff operators on the Heisenberg group.

Theorem 4.1. Let $1 \leq p, q < \infty$, $0 \leq \beta < Q$, $\gamma > -Q$, $\alpha \in \mathbb{R}$, $\frac{\gamma + Q}{q} = \frac{\alpha + Q}{p} - \beta$. If $K_p(\cdot) \in L^\infty(S^{Q-1})$, then for $f \in L^p(\mathbb{H}^n, |x|_h^\alpha dx)$, we have

$$\|T_{\Phi,\beta}(f)\|_{L_{|x|_h^\alpha}^{q,\infty}(\mathbb{H}^n)} \leq \|K_p(\cdot)\|_{L^\infty(S^{Q-1})} \cdot \left(\frac{\omega_Q}{\gamma + Q}\right)^{\frac{1}{q}} \cdot \omega_Q^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{H}^n, |x|_h^\alpha dx)}.$$

Proof. Using the Hölder’s inequality, we have

$$|T_{\Phi,\beta}(f)(x)| = \left| \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^{Q-\beta}} f(y) dy \right| \leq \|g_x(\cdot)\|_{L^{p'}(\mathbb{H}^n)} \|f\|_{L^p(\mathbb{H}^n, |x|_h^\alpha dx)},$$

where $g_x(y) = \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^{Q-\beta}} |y|_h^{-\frac{\alpha}{p}}$, for $y \in \mathbb{H}^n$. By using the spherical coordinates and the equality $\frac{\gamma + Q}{q} = \frac{\alpha + Q}{p} - \beta$, we can get

$$\begin{aligned}
 \|g_x(\cdot)\|_{L^{p'}(\mathbb{H}^n)} &= \left(\int_{\mathbb{H}^n} \left| \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^{Q-\beta}} |y|_h^{-\frac{\alpha}{p}} \right|^{p'} dy \right)^{\frac{1}{p'}} \\
 &= \left(\int_0^\infty \int_{|y|_h=1} \left| \frac{\Phi(\delta_{r^{-1}}x)}{r^{Q-\beta}} r^{-\frac{\alpha}{p}} \right|^{p'} r^{Q-1} dr dy' \right)^{\frac{1}{p'}} \\
 &= \omega_Q^{\frac{1}{p'}} \left(\int_0^\infty |\Phi(\delta_{r^{-1}}x)|^{p'} r^{Q-(Q-\beta)p'-1-\alpha p'/p} dr \right)^{\frac{1}{p'}} \\
 &= \omega_Q^{\frac{1}{p'}} \left(\int_0^\infty |\Phi(\delta_t x)|^{p'} t^{(\alpha/p+Q-\beta)p'-Q-1} dt \right)^{\frac{1}{p'}} \\
 &= \omega_Q^{\frac{1}{p'}} |x|_h^{-\frac{y+Q}{q}} \left(\int_0^\infty |\Phi(\delta_t x')|^{p'} t^{(\alpha/p+Q-\beta)p'-Q-1} dt \right)^{\frac{1}{p'}} \\
 &\leq \omega_Q^{\frac{1}{p'}} |x|_h^{-\frac{y+Q}{q}} \|K_p(\cdot)\|_{L^\infty(S^{Q-1})}.
 \end{aligned}$$

Let $M = \omega_Q^{\frac{1}{p'}} \|K_p(\cdot)\|_{L^\infty(S^{Q-1})} \|f\|_{L^p(\mathbb{H}^n, |x|_h^\alpha dx)}$. Then $|T_{\Phi,\beta}(f)(x)| \leq M |x|_h^{-\frac{y+Q}{q}}$. So we have

$$\begin{aligned}
 \|T_{\Phi,\beta}(f)\|_{L^{\frac{q,\infty}{|x|_h^\gamma}(\mathbb{H}^n)}} &= \sup_{\lambda>0} \lambda \left(\int_{\{x \in \mathbb{H}^n : |T_{\Phi,\beta}(f)(x)| > \lambda\}} |x|_h^\gamma dx \right)^{\frac{1}{q}} \\
 &\leq \sup_{\lambda>0} \lambda \left(\int_{\{x \in \mathbb{H}^n : M |x|_h^{-\frac{y+Q}{q}} > \lambda\}} |x|_h^\gamma dx \right)^{\frac{1}{q}} \\
 &= \sup_{\lambda>0} \lambda \left(\int_{\mathbb{H}^n} \chi_{\{M |x|_h^{-\frac{y+Q}{q}} > \lambda\}}(x) |x|_h^\gamma dx \right)^{\frac{1}{q}} \\
 &= \sup_{\lambda>0} \lambda \left(\frac{\omega_Q}{y+Q} (M\lambda^{-1})^q \right)^{\frac{1}{q}} \\
 &= M \left(\frac{\omega_Q}{y+Q} \right)^{\frac{1}{q}}.
 \end{aligned}$$

Therefore, we conclude that

$$\|T_{\Phi,\beta}(f)\|_{L^{\frac{q,\infty}{|x|_h^\gamma}(\mathbb{H}^n)}} \leq \|K_p(\cdot)\|_{L^\infty(S^{Q-1})} \omega_Q^{\frac{1}{p'}} \left(\frac{\omega_Q}{y+Q} \right)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{H}^n, |x|_h^\alpha dx)}.$$

We are done. □

Remark 4.2. If we take $\alpha = \gamma$, then Theorem 4.1 covers Theorem 1 in [23].

As is well known, the Hausdorff operator includes many famous operators such as the Hardy operator and the adjoint Hardy operator. As a simple application of Theorem 4.1, we consider the fractional Hardy operator \mathcal{H}_β and its adjoint operator \mathcal{H}_β^* on \mathbb{H}^n :

$$\mathcal{H}_\beta f(x) = \frac{1}{|B(0, |x|_h)|^{1-\frac{\beta}{Q}}} \int_{|y|_h < |x|_h} f(y) dy, \quad x \in \mathbb{H}^n \setminus \{0\},$$

$$\mathcal{H}_\beta^* f(x) = \int_{B(0, |x|_h)^c} \frac{f(y)}{|B(0, |y|_h)|^{1-\frac{\beta}{Q}}} dy, \quad x \in \mathbb{H}^n \setminus \{0\}.$$

When $\beta = 0$, we often use \mathcal{H} and \mathcal{H}^* instead.

Corollary 4.3. *Let $-Q < \alpha \leq 0$ and $\gamma > -Q$. Then*

- (i) $\|\mathcal{H}\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx) \rightarrow L^1_{|x|_h^\alpha}(\mathbb{H}^n)} = 1$;
- (ii) $\|\mathcal{H}^*\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx) \rightarrow L^1_{|x|_h^\gamma}(\mathbb{H}^n)} = \frac{Q}{Q + \gamma}$.

Proof. The upper bound can be easily obtained by Theorem 4.1 when we choose $\beta = 0$ and suitable functions Φ . Therefore, it is sufficient to show that the upper bound is sharp.

(i) Taking $f_0(x) = \chi_{\{|x|_h < 1\}}(x)$. Since $-Q < \alpha \leq 0$, we can get

$$\|f_0\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx)} = \frac{\omega_Q}{Q + \alpha}.$$

Moreover, we have

$$\mathcal{H}f_0(x) = \begin{cases} |x|_h^{-Q}, & \text{if } |x|_h \geq 1; \\ 1, & \text{if } |x|_h < 1. \end{cases}$$

Therefore,

$$\|\mathcal{H}f_0\|_{L^1_{|x|_h^\alpha}(\mathbb{H}^n)} = \sup_{\lambda > 0} \int_{\{x \in \mathbb{H}^n : |\mathcal{H}f_0(x)| > \lambda\}} |x|_h^\alpha dx = \frac{\omega_Q}{Q + \alpha} \sup_{0 < \lambda \leq 1} \lambda^{-\alpha/Q}.$$

Note that $\alpha \leq 0$, we can get

$$\|\mathcal{H}f_0\|_{L^1_{|x|_h^\alpha}(\mathbb{H}^n)} = \frac{\omega_Q}{Q + \alpha} = \|f_0\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx)}.$$

So we obtain

$$\|\mathcal{H}\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx) \rightarrow L^1_{|x|_h^\alpha}(\mathbb{H}^n)} = 1.$$

(ii) Take $f_0(y) = |y|_h^{-k(Q+\gamma)} \chi_{\{|y|_h \geq 1\}}(y)$, where $k > 1$. Since $\gamma > -Q$, we can get

$$\|f_0\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx)} = \frac{\omega_Q}{(Q + \gamma)(k - 1)}.$$

Moreover, we have

$$\mathcal{H}^*f_0(x) = \begin{cases} \frac{Q|x|_h^{-k(Q+\gamma)}}{k(Q + \gamma)} < \frac{Q}{k(Q + \gamma)}, & \text{if } |x|_h > 1; \\ \frac{Q}{k(Q + \gamma)}, & \text{if } |x|_h \leq 1. \end{cases}$$

Therefore,

$$\|\mathcal{H}^*f_0\|_{L^1_{|x|_h^\alpha}(\mathbb{H}^n)} = \sup_{\lambda > 0} \int_{\{x \in \mathbb{H}^n : |\mathcal{H}^*f_0(x)| > \lambda\}} |x|_h^\alpha dx = \frac{Q\omega_Q}{(Q + \gamma)^2 k} = \frac{Q}{Q + \gamma} \left(1 - \frac{1}{k}\right) \|f_0\|_{L^1(\mathbb{H}^n, |x|_h^\alpha dx)}.$$

So we achieve the desired result by letting $k \rightarrow \infty$. □

Next, we consider the weak-type weighted estimates for the multilinear fractional Hausdorff operator $T_{\Phi, \vec{\beta}}$ in \mathbb{H}^n . $T_{\Phi, \vec{\beta}}$ is defined as follows:

Definition 4.2. Let Φ be a locally integrable function on $\mathbb{H}^n \times \dots \times \mathbb{H}^n$ and $\vec{\beta} = (\beta_1, \dots, \beta_m)$, $0 \leq \beta_i < Q$, $i = 1, \dots, m$. The multilinear fractional Hausdorff operator is given by

$$T_{\Phi, \vec{\beta}}(f_1, \dots, f_m)(x) = \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}x}, \dots, \delta_{|y_m|_h^{-1}x})}{\prod_{i=1}^m |y_i|_h^{Q-\beta_i}} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m.$$

Theorem 4.4. Let $m \in \mathbb{N}$, $1 \leq p, q < \infty$, $1 < p_i < \infty$, $\gamma > -Q$, $\vec{\beta} = (\beta_1, \dots, \beta_m)$, $0 \leq \beta_i < Q$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, m$ and $1/p = 1/p_1 + \dots + 1/p_m$, $\frac{\gamma+Q}{q} = \sum_{i=1}^m \left(\frac{\alpha_i+Q}{p_i} - \beta_i \right)$. If $K_{\vec{\beta}}(\cdot) \in L^\infty(S^{Q-1})$, then for $f_i \in L^{p_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)$, we have

$$\|T_{\Phi, \vec{\beta}}^m(f_1, f_2, \dots, f_m)\|_{L^q_{|x|_h^\gamma}(\mathbb{H}^n)} \leq \|K_{\vec{\beta}}(\cdot)\|_{L^\infty(S^{Q-1})} \cdot \left(\frac{\omega_Q}{\gamma+Q} \right)^{\frac{1}{q}} \cdot \prod_{i=1}^m \omega_Q^{\frac{1}{p_i}} \cdot \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)},$$

where $x \neq 0$, $x' = \delta_{|x|_h^{-1}x}$ and

$$K_{\vec{\beta}}(\cdot)(x') = \left(\int_0^\infty \cdots \left(\int_0^\infty \left(\int_0^\infty |\Phi(\delta_{s_1}x', \dots, \delta_{s_m}x')|^{p'_1} \right. \right. \right. \\ \left. \left. \left. \times s_1^{\left(\frac{\alpha_1}{p'_1}+Q-\beta_1\right)p'_1-Q-1} ds_1 \right)^{\frac{p'_2}{p_2}} s_2^{\left(\frac{\alpha_2}{p'_2}+Q-\beta_2\right)p'_2-Q-1} ds_2 \right)^{\frac{p'_3}{p_3}} \cdots s_m^{\left(\frac{\alpha_m}{p'_m}+Q-\beta_m\right)p'_m-Q-1} ds_m \right)^{\frac{1}{p_m}}.$$

To prove Theorem 4.4, we first prove the following lemma. Define $\vec{B}(x)$ by

$$\vec{B}(x) = \left(\int_{\mathbb{H}^n} \cdots \left(\int_{\mathbb{H}^n} \left| \left(\int_{\mathbb{H}^n} \left| \frac{\Phi(\delta_{|y_1|_h^{-1}x}, \dots, \delta_{|y_m|_h^{-1}x})}{\prod_{j=1}^m |y_j|_h^{Q-\beta_j}} |y_1|_h^{\frac{\alpha_1}{p_1}} \right|^{p'_1} dy_1 \right)^{\frac{1}{p_1}} |y_2|_h^{\frac{\alpha_2}{p_2}} \right)^{p'_2} \cdots dy_m \right)^{\frac{1}{p_m}}.$$

Lemma 4.5. Letting p, p_i and $K_{\vec{\beta}}$ be as in Theorem 4.4, we can obtain

$$\vec{B}(x) = K_{\vec{\beta}}(x') \prod_{j=1}^m \omega_Q^{\frac{1}{p_j}} \cdot |x|_h^{-\frac{\gamma+Q}{q}}.$$

Proof. For simplicity, we only present the proof in the case $m = 2$.

Using the spherical coordinates, we have

$$\vec{B}(x) = \left(\int_{\mathbb{H}^n} \left| \left(\int_{\mathbb{H}^n} \left| \frac{\Phi(\delta_{|y_1|_h^{-1}x}, \delta_{|y_2|_h^{-1}x})}{\prod_{j=1}^2 |y_j|_h^{Q-\beta_j}} |y_1|_h^{\frac{\alpha_1}{p_1}} \right|^{p'_1} dy_1 \right)^{\frac{1}{p_1}} |y_2|_h^{\frac{\alpha_2}{p_2}} \right)^{p'_2} dy_2 \right)^{\frac{1}{p_2}} \\ = \prod_{j=1}^2 \omega_Q^{1/p'_j} \left(\int_0^\infty r_2^{Q-1} \left| \left(\int_0^\infty r_1^{Q-1} \left| \frac{\Phi(\delta_{r_1^{-1}|x|_h^{-1}x'}, \delta_{r_2^{-1}|x|_h^{-1}x'})}{r_1^{Q-\beta_1} r_2^{Q-\beta_2}} r_1^{-\frac{\alpha_1}{p_1}} \right|^{p'_1} dr_1 \right)^{\frac{1}{p_1}} r_2^{-\frac{\alpha_2}{p_2}} \right)^{p'_2} dr_2 \right)^{1/p'_2} \\ = \prod_{j=1}^2 \omega_Q^{1/p'_j} \cdot \vec{b}_0(x).$$

Assume $s_1 = r_1^{-1}|x|_h, s_2 = r_2^{-1}|x|_h$, and then

$$r_1 = s_1^{-1}|x|_h, \quad r_2 = s_2^{-1}|x|_h, \quad dr_1 = -|x|_h s_1^{-2} ds_1, \quad dr_2 = -|x|_h s_2^{-2} ds_2.$$

Thus, we have

$$\begin{aligned} \widetilde{b}_0(x) &= |x|_h^{-\frac{\alpha_1+Q}{p_1}-\frac{\alpha_2+Q}{p_2}+\beta_1+\beta_2} \cdot \left(\int_0^\infty \left(\int_0^\infty |\Phi(\delta_{s_1}x', \delta_{s_2}x')|^{p_1'} s_1^{\left(\frac{\alpha_1}{p_1}+Q-\beta_1\right)p_1'-Q-1} ds_1 \right)^{\frac{p_2'}{p_1}} s_2^{\left(\frac{\alpha_2}{p_2}+Q-\beta_2\right)p_2'-Q-1} ds_2 \right)^{\frac{1}{p_2'}} \\ &= |x|_h^{-\frac{y+Q}{q}} K_{\vec{p}}(x'). \end{aligned}$$

Therefore, we finish the proof of Lemma 4.5. □

Next, we will provide the proof of Theorem 4.4.

Proof of Theorem 4.4. Using Hölder’s inequality, we can obtain

$$|T_{\Phi, \vec{\beta}}^m(f_1, f_2, \dots, f_m)(x)| \leq \widetilde{B}(x) \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)}.$$

By Lemma 4.5 and $K_{\vec{p}}(\cdot) \in L^\infty(S^{Q-1})$, there holds

$$|T_{\Phi, \vec{\beta}}^m(f_1, f_2, \dots, f_m)(x)| \leq |x|_h^{-\frac{y+Q}{q}} \cdot \|K_{\vec{p}}(\cdot)\|_{L^\infty(S^{Q-1})} \cdot \prod_{j=1}^m \omega_Q^{\frac{1}{p_j}} \cdot \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)}.$$

Thus, for any $\lambda > 0$,

$$\begin{aligned} &\|T_{\Phi, \vec{\beta}}^m(f_1, f_2, \dots, f_m)\|_{L_{|x|_h^y}^{q, \infty}(\mathbb{H}^n)} \\ &= \sup_{\lambda > 0} \lambda \left(\int_{\{|T_{\Phi, \vec{\beta}}^m(f_1, f_2, \dots, f_m)(x)| > \lambda\}} |x|_h^y dx \right)^{\frac{1}{q}} \\ &\leq \sup_{\lambda > 0} \lambda \left(\int_{\left\{ |x|_h^{-\frac{y+Q}{q}} \cdot \|K_{\vec{p}}(\cdot)\|_{L^\infty(S^{Q-1})} \cdot \prod_{j=1}^m \omega_Q^{\frac{1}{p_j}} \cdot \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)} > \lambda \right\}} |x|_h^y dx \right)^{\frac{1}{q}} \\ &= \|K_{\vec{p}}(\cdot)\|_{L^\infty(S^{Q-1})} \cdot \left(\frac{\omega_Q}{y+Q}\right)^{\frac{1}{q}} \cdot \prod_{i=1}^m \omega_Q^{\frac{1}{p_i}} \cdot \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{H}^n, |x|_h^{\alpha_i} dx)}. \end{aligned}$$

Therefore, we get the desired result. □

Remark 4.6. If we take $\vec{\beta} = \vec{0}, \vec{\alpha} = \vec{0}, q = p$, then Theorem 4.4 covers Theorem 1.6 in [19].

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