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WEAK L_1 CHARACTERIZATIONS OF POISSON INTEGRALS, GREEN POTENTIALS, AND H^p SPACES

BY

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ABSTRACT. Our main result can be described as follows. A subharmonic function u in a suitable domain Ω in $\mathbf{R}^n$ is the difference of a Poisson integral and a Green potential if and only if u divided by the distance to $\partial\Omega$ is in weak L_1 in Ω.

Similar conditions are given for a harmonic function to be the Poisson integral of an L_p function on $\partial\Omega$. Iterated Poisson integrals in a polydisc are also considered. As corollaries, we get weak L_1 characterizations of H^p spaces of different kinds.

1. **Introduction.** A harmonic function u in, say, the unit ball U in $\mathbf{R}^n$ is the Poisson integral of a measure on ∂U if and only if the integral of $|u|$ over the sphere $\{|x| = 1 - \eta\}$ is bounded as $\eta \to 0$. In this paper, we shall prove that u is of this type precisely when $(1 - |x|)^{-1}u$ is in weak L_1 of U. If instead u is subharmonic, this last condition will characterize those u which can be written as the difference of a Poisson integral and a Green potential in U. This result carries over to arbitrary bounded domains of class $C^{(1,\alpha)}$, if $1 - |x|$ is replaced by the distance to the boundary of the domain. This can be applied to suitable powers $|u|^p$ of harmonic or holomorphic functions u, yielding weak L_1 characterizations of Poisson integrals of L_p functions on the boundary and of H^p spaces. More generally, we may have Orlicz spaces instead of L_p spaces here. In particular, a holomorphic function u in the unit disc $U \subset \mathbf{C}$ is in $H^p(U), p > 0$, if and only if $(1 - |z|)^{-1}|u|^p$ is in weak L_1, or equivalently $(1 - |z|)^{-1/p}u$ is in weak L_p.

Quite similar results hold for a half-space. In that case, we also give analogous characterizations of Poisson integrals of classes of functions and measures defined by means of weight functions on the boundary. In a polydisc, the class of n-harmonic functions which are iterated Poisson integrals of measures on the distinguished boundary has a characterization of the same type. This time, spaces slightly larger than weak L_1 are involved, and there is again a corollary about H^p spaces.

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The main tool used in the proofs is a theorem about convolutions of the kernel $|x|^{-n}$ in $\mathbf{R}^n$ which was proved in Sjögren [7]. In §3, we state this result and give a simpler proof of it. Poisson integrals for bounded domains are studied in §4, where we give the main result together with its applications to L_p and H^p spaces. In §5, we state the corresponding results for a half-space. The proofs of these theorems, which constitute §6, are analogous to those of §4, although technically more complicated. §6 is therefore less detailed. In §7, the iterated Poisson kernel of a polydisc is studied.

Some of our present results were given in the preliminary report Sjögren [10].

2. **Preliminaries.** We will work in $\mathbf{R}^n, n \geqslant 2$, and denote by $|E|$ the Lebesgue measure of $E \subset \mathbf{R}^n$. If f is a real-valued measurable function in a domain $\Omega \subset \mathbf{R}^n$, the distribution function of f is defined by

$$\lambda_f(\alpha) = |\{x \in \Omega: |f(x)| > \alpha\}|, \qquad \alpha > 0.$$

We let f^* be its decreasing rearrangement, defined by

$$f^*(t) = \inf\{\alpha: \lambda_f(\alpha) \leqslant t\}, \qquad 0 < t < |\Omega|.$$

If $E \subset \Omega$ is measurable, it is well known that

$$\int_E f\,dx \leqslant \int_0^{|E|} f^*(t)\,dt. \tag{2.1}$$

For more details about these notions, see Stein and Weiss [12, Section V: 3]. The space $\Lambda(\Omega)$, usually called weak $L_1(\Omega)$, consists of those f which satisfy $f^*(t) \leqslant \text{const} \cdot t^{-1}$, $0 < t < |\Omega|$. Letting $\|f\|_\Lambda$ be the smallest such constant, we obtain a quasi-norm on $\Lambda(\Omega)$; in fact, $\|f + g\|_\Lambda \leqslant 2(\|f\|_\Lambda + \|g\|_\Lambda)$. Equivalently, $\Lambda(\Omega)$ can be defined by the inequality $\lambda_f(\alpha) \leqslant \text{const} \cdot \alpha^{-1}$, $\alpha > 0$, with the same constant.

In the sequel, we shall denote by C many different constants, indicating if necessary which variables C depends on. The relation $f \sim g$ means $C \leqslant f/g \leqslant C$ with constants of this type (f and g are equivalent). In the rest of this section, C may be chosen to depend only on the domain Ω described below.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain of class $C^{(1,\alpha)}$. We put $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, and let $P(x,y)$ and $G(x,y)$ be the Poisson kernel and the Green's function, respectively, of Ω. If λ is a (finite Radon) measure on $\partial\Omega$, its Poisson integral is defined by

$$P\lambda(x) = \int_{\partial\Omega} P(x,y)\,d\lambda(y), \qquad x \in \Omega.$$

For suitable functions f on $\partial\Omega$, we write Pf for the Poisson integral of the measure $f dS$. Here dS is the area measure on $\partial\Omega$. From Theorem 2.3 in Widman [13], it follows that

$$P(x,y) \leqslant C\delta(x)|x-y|^{-n}. \tag{2.2}$$

The Green potential of a positive measure μ in Ω is given by

$$G\mu(x) = \int_\Omega G(x,y)\, d\mu(y).$$

This potential is superharmonic (i.e., $\not\equiv +\infty$) precisely when μ belongs to the class M_Ω of positive measures in Ω for which $G\mu < \infty$ a.e. As shown in [13], $G(x_0, x) \sim \delta(x)$ if we let x approach $\partial\Omega$ and keep $x_0 \in \Omega$ fixed. This implies that M_Ω consists of all μ for which $\int \delta\, d\mu < \infty$.

Again fixing an $x_0 \in \Omega$, we let $\Omega_\eta = \{x \in \Omega\colon\ G(x_0, x) > \eta\}$ for small $\eta > 0$, which is a family of domains approximating Ω. From the property of G just stated, it follows that $\delta(x) \sim \eta$ for $x \in \partial\Omega_\eta$. Further, the surfaces $\partial\Omega_\eta$ are uniformly of class $C^{(1,\alpha)}$, in view of [13, Theorems 2.4 and 2.5]. We denote by dS_η the area measure of $\partial\Omega_\eta$. From [13], it also follows that the Poisson kernel P_η of Ω_η satisfies

$$P_\eta(x_0, \cdot\,) \sim 1 \tag{2.3}$$

on $\partial\Omega_\eta$, so that the harmonic measure of Ω_η at x_0 is equivalent to dS_η, uniformly in η.

The Green's function of Ω_η is given by

$$G_\eta(x,y) = N(x-y) - h_x^\eta(y), \qquad x, y \in \Omega_\eta,$$

where N is the Newtonian kernel and h_x^η the harmonic function in Ω_η which equals $N(x - \cdot)$ on $\partial\Omega_\eta$. We call h_x the corresponding function in Ω; it satisfies $h_x \in C^{(1,\alpha)}(\bar\Omega)$. From now on, we fix $x \in \Omega$. It follows that the $C^{(1,\alpha)}$ norm of the restriction of $h_x^\eta - h_x$ to $\partial\Omega_\eta$ is bounded as $\eta \to 0$, and that $h_x^\eta - h_x|\partial\Omega_\eta$ together with its tangential derivatives can be made arbitrarily small, uniformly on $\partial\Omega_\eta$, by taking η small enough. But this implies that the $C^{(1)}$ norm of $h_x^\eta - h_x$ in $\bar\Omega_\eta$ tends to 0 as $\eta \to 0$, as can be seen e.g. from the Poisson representation formula of $h_x^\eta - h_x$ in terms of its restriction to $\partial\Omega_\eta$. Thus,

$$\sup_{y\in\Omega_\eta} \left|\operatorname{grad}_y\big(G_\eta(x,y) - G(x,y)\big)\right| \to 0 \quad \text{as } \eta \to 0.$$

Since P_η is a normal derivative of G_η, we conclude that $P_\eta \to P$ as $\eta \to 0$, in the following sense: For any $x \in \Omega$, we can make $|P_\eta(x,y) - P(x,z)|$ arbitrarily small by taking η small and $y \in \partial\Omega_\eta$ and $z \in \partial\Omega$ sufficiently close to each other.

3. **An auxiliary theorem**. The results of this section hold in all dimensions $n > 1$. If μ is a finite positive measure in $\mathbf{R}^n$, we put $U^\mu = r^{-n} * \mu$, where $r = |x|$ and the convolution is defined at each point in $\mathbf{R}^n$, with values in $[0, \infty]$.

DEFINITION. A closed set $F \subset \mathbf{R}^n$ is called a convolution set if $U^\mu \in \Lambda(\mathbf{R}^n)$ for any finite positive measure μ with supp $\mu \subset F$.

REMARK. If F is a convolution set, there is a constant C_F such that $\|U^\mu\|_\Lambda < C_F\|\mu\|$ for such μ. To see this, notice that if there is no such constant, we can, by positive homogeneity, find a sequence (μ_i) with $\Sigma\|\mu_i\| < \infty$ but $\sup\|U^{\mu_i}\|_\Lambda = \infty$. Considering $\mu = \Sigma\,\mu_i$, we see that F is not a convolution set.

The following geometrical characterization of convolution sets was given in Sjögren [7], and similar results for more general kernels can be found in Sjögren [8]. In the sequel, all cubes will be open and have sides parallel to the axes.

THEOREM 1. *A closed set F is a convolution set if and only if there is an $\varepsilon > 0$ with the following property: Any cube Q_1 in $\mathbf{R}^n$ contains a cube Q_2 which is disjoint with F and such that the ratio between the sides of Q_2 and Q_1 equals ε.*

In one dimension, this condition means, roughly speaking, that F is contained in a Cantor set with constant ratio. The proof we shall give is a simplification of that of [7]. The cubes in the theorem could of course be replaced by balls, but cubes are more useful in the proof.

PROOF. If the geometrical condition is not satisfied, we can, for any N, find a cube Q such that if we divide Q into N^n equal subcubes in the obvious way (divide each edge into N equal parts, etc.), then all these subcubes intersect F. Now choose a probability measure μ supported by F and having a point mass N^{-n} in each of these subcubes. Then a simple lower estimation of U^μ in Q shows that $\|U^\mu\|_\Lambda > c\log N$, with $c > 0$. By the above remark, F is therefore not a convolution set, since N is arbitrary.

Conversely, assume the condition satisfied, and let M be the smallest integer $> 2\varepsilon^{-1}$. In this proof, C will denote different constants depending only on n and ε (or M). Let μ be a positive finite measure supported by F. For $\alpha > 0$, we must prove

$$|\{x\colon U^\mu(x) > \alpha\}| < C\alpha^{-1}\|\mu\|. \tag{3.1}$$

With i_0 a negative integer, we let Q_0 be the cube of side M^{-i_0} centered at the origin. Then it is of course enough to derive an estimate for $|\{U^\mu > \alpha\} \cap Q_0|$, similar to (3.1) and uniform in i_0.

Divide Q_0 into M^n subcubes of sides M^{-i_0-1}. At least one of these subcubes will be disjoint with F, in view of the hypothesis and the choice of M. Pick such a subcube, and call it a hole of order $i_0 + 1$. Each of the remaining $M^n - 1$ subcubes is now divided into M^n cubes of sides M^{-i_0-2}. Among these, we pick one which does not intersect F, and call it a hole of order $i_0 + 2$, thus getting $M^n - 1$ such holes. The process is then continued with the remaining cubes of sides M^{-i_0-2}, and so on. In this way, we obtain holes of all orders $j \geq i_0 + 1$, which are all disjoint with F. Since there are

$(M^n - 1)^{j-i_0-1}$ holes of order j, the holes together exhaust Q_0, up to a null set.

Next, we divide each hole into subcubes called pieces, whose sides are equivalent to their distances from the boundary of the hole, as follows. Given a hole Q of order j, we divide it into M^n subcubes of sides M^{-j-1}. Among these subcubes, those whose closures do not intersect ∂Q are called pieces of order $j+1$, and the remaining ones are divided into subcubes of sides M^{-j-2}. Among these, those whose closures do not intersect ∂Q are called pieces of order $j+2$, and the remaining ones are again subdivided, and so on. We thus obtain pieces of all orders $j \geqslant j_0$, where $j_0 = i_0 + 2$. Notice that every piece is contained in precisely one hole, and that the distance from F to a piece of order j is at least M^{-j}.

LEMMA 1. *Let $x \in \mathbf{R}^n$, and take $j \geqslant j_0$ and $k \leqslant j$. The Lebesgue measure of the union of all pieces of order j whose distances from x are at most M^{-k} does not exceed $CM^{-nk}(1 - M^{-n})^{j-k}$.*

This follows from some simple geometrical considerations. The main step is the observation that those of the pieces considered which are contained in holes of order i, with $k \leqslant i < j$, have a total measure of at most

$$M^{-nk}(1 - M^{-n})^{i-k}M^{i-j}.$$

The details are left to the reader.

The number $\chi > 0$ will be determined later. Let $(P_r)_{r=1}^{\infty}$ be an enumeration of all the pieces, chosen so that the order $\bar{r}$ of P_r is nondecreasing in r. By induction, we are now going to construct sets F_r, $r = 0, 1, 2, \ldots,$ and each F_r will either be empty or a set of "forbidden" pieces lying close to P_r. Starting with $F_0 = \varnothing$, we assume F_r defined for $r < s$. If P_s intersects $\{U^\mu > \alpha\}$ and P_s does not belong to $\bigcup_{r<s} F_r$ (i.e., P_s is not forbidden), then we define $F_s = \{P_r\colon r > s \text{ and } \operatorname{dist}(P_r, P_s) \leqslant 2M^{-\bar{s}+\chi(\bar{r}-\bar{s})}\}$.

In all other cases, we put $F_s = \varnothing$.

With the F_r thus defined, we let ν be the restriction of Lebesgue measure to the union of all pieces P_r for which F_r is nonempty. Roughly speaking, this is the union of all pieces which intersect $\{U^\mu > \alpha\}$ and which are not forbidden. We claim that ν satisfies

(i) $|\{U^\mu > \alpha\} \cap Q_0| \leqslant C\|\nu\|$,
(ii) $U^\mu > \alpha/C$ on supp ν,
(iii) $U^\nu \leqslant C$ on F.

These three conditions imply

$$\begin{aligned}|\{U^\mu > \alpha\} \cap Q_0| &\leqslant C\|\nu\| \leqslant C\alpha^{-1}\int U^\mu\, d\nu \\ &= C\alpha^{-1}\int U^\nu\, d\mu \leqslant C\alpha^{-1}\|\mu\|,\end{aligned}$$

in view of Fubini's Theorem. As we noticed above, this would prove (3.1), and thus complete the proof of Theorem 1.

Before proving (i)–(iii), we make a few comments. Letting ν' be the restriction of Lebesgue measure to the set $\{U^\mu > \alpha\} \cap Q_0$, we observe that ν' satisfies (i) and (ii) but not (iii). The same is true for the restriction ν'' of Lebesgue measure to the union of all pieces intersecting $\{U^\mu > \alpha\}$, which for our purpose is equivalent to ν'. Roughly speaking, (iii) is violated because ν' and ν'' have too much mass near $\operatorname{supp}\mu \subset F$. When constructing ν, we therefore modify ν'' by taking away mass near F. This is why we introduce the sets F_r of forbidden pieces, where no mass is placed. The F_r are thus constructed so that the resulting measure tends to avoid F, and therefore satisfies (iii). Still, (i) and (ii) are preserved.

Proof of (i). The measure of $\{U^\mu > \alpha\} \cap Q_0$ is majorized by the measure of the union of all pieces intersecting this set. Such a piece is either contained in $\operatorname{supp}\nu$ or belongs to $\bigcup F_r$, by construction. Of course, $|\operatorname{supp}\nu| = \|\nu\|$, so we need only estimate the measure of the union of all pieces in each F_r. If $F_r \neq \varnothing$, the pieces of order $j, j > \bar{r}$, belonging to F_r all have a distance of at most $2M^{-\bar{r}+\chi(j-\bar{r})}$ from P_r, and thus a distance of at most $CM^{-\bar{r}+\chi(j-\bar{r})}$ from the center of P_r. Applying Lemma 1 C times, with points x suitably chosen, we see that the total measure of these pieces is at most

$$CM^{-n\bar{r}+\chi n(j-\bar{r})}(1-M^{-n})^{(1+\chi)(j-\bar{r})}.$$

This quantity must now be summed over $j > \bar{r}$. If $\chi = \chi(n, M) > 0$ is small enough, the series thus obtained will converge, with a sum dominated by $CM^{-n\bar{r}} = C|P_r|$. Summing over r, we obtain C times the total Lebesgue measure of all pieces P_r with $F_r \neq \varnothing$, i.e., $C\|\nu\|$. This is the estimate needed to end the proof of (i).

Proof of (ii). In each of the pieces forming $\operatorname{supp}\nu$, there is some point x with $U^\mu(x) > \alpha$. Since $|x-y| \leq C\operatorname{dist}(x, \operatorname{supp}\mu)$ for any y in the closure of this piece, we have $|x-z|^{-n} \leq C|y-z|^{-n}$ for $z \in \operatorname{supp}\mu$. Integrating with respect to $d\mu(z)$, we obtain (ii).

Proof of (iii). We fix $z \in F$. Let ν_j be the part of ν which is carried by pieces of order j, so that $\nu = \sum_{j=j_0}^\infty \nu_j$.

Lemma 2. *If* $\operatorname{dist}(z, \operatorname{supp}\nu_j) \geq M^{-p}$ *with* $p \leq j$, *then*

$$U^{\nu_j}(z) \leq C_0(1-M^{-n})^{j-p},$$

for some $C_0 = C_0(n, \varepsilon)$.

This is easily proved by means of Lemma 1, if we successively estimate the contributions to $U^{\nu_j}(z)$ from the pieces (of order j) whose distances from z are between M^{-p} and M^{-p+1}, between M^{-p+1} and M^{-p+2}, and so on. Notice that the lemma holds also for noninteger p, with a suitable C_0.

Inequality (iii) will result from the following lemma, where C_0 is the constant of Lemma 2.

LEMMA 3. *For any $j > j_0$, one can rearrange the sum $S_1 = \sum_{j_0}^{j} U^{\nu_k}(z)$ so that it becomes dominated term by term by the sum $S_2 = \sum_0^{j-j_0} C_0(1 - M^{-n})^{\chi k}$.*

PROOF. We use induction. The case $j = j_0$ is clear by Lemma 2 with $j = p = j_0$, so suppose the assertion holds for $j - 1$. Let $m > 0$ be the integer for which

$$C_0(1 - M^{-n})^{m+1} < U^{\nu_j}(z) < C_0(1 - M^{-n})^{m}. \tag{3.2}$$

Then $\operatorname{dist}(z, \operatorname{supp} \nu_j) < M^{m+1-j}$, because of Lemma 2. On the other hand, $\operatorname{dist}(\operatorname{supp} \nu_k, \operatorname{supp} \nu_j) > 2M^{-k+\chi(j-k)}$ for $j_0 < k < j$, by the construction of the F_r. The triangle inequality then yields

$$\operatorname{dist}(z, \operatorname{supp} \nu_k) > 2M^{-k+\chi(j-k)} - M^{m+1-j} > M^{-k+\chi(j-k)}$$

if $-k + \chi(j - k) > m + 1 - j$, in particular if $k < j - m - 1$. Lemma 2 now implies $U^{\nu_k}(z) < C_0(1 - M^{-n})^{\chi(j-k)}$ for $k = j_0, j_0 + 1, \ldots, j - m - 1$. This means that we have estimated the first $j - m - j_0$ terms of S_1 by the last $j - m - j_0$ terms of S_2. Further, the induction assumption implies that the terms $U^{\nu_k}(z)$, $k = j - m, \ldots, j - 1$, are dominated, in some order, by the first (and greatest) m terms of S_2. To complete the induction, we need only show that

$$U^{\nu_j}(z) < C_0(1 - M^{-n})^{\chi m},$$

and this is a trivial consequence of the right-hand inequality of (3.2).

Lemma 3 is proved, and thus also Theorem 1.

4. Results for bounded domains. Let Ω be a bounded domain of class $C^{(1,\alpha)}$, as in §2. If u is a subharmonic function in Ω, we put $u^+ = \max(u, 0)$ and $u^- = u^+ - u$. It is elementary that u has a representation $u = P\lambda - G\mu$ if and only if

$$\int u^+ \, dS_\eta = O(1) \qquad \text{as } \eta \to 0. \tag{4.1}$$

Here it is assumed that λ is a finite measure on $\partial\Omega$ and $\mu \in M_\Omega$. To prove this equivalence, represent u as a Poisson integral minus a Green potential in Ω_η and examine what happens when $\eta \to 0$, considering (2.3). Then use the convergence $P_\eta \to P$ as $\eta \to 0$ stated at the end of §2.

We will give another characterization of such u.

THEOREM 2. *A subharmonic function u in Ω has a representation $u = P\lambda - G\mu$, where λ is a finite measure on $\partial\Omega$ and $\mu \in M_\Omega$, if and only if $\delta^{-1}u \in \Lambda(\Omega)$.*

REMARK. From the proof we give, it can be seen that for such u

$$\|\delta^{-1}u\|_\Lambda \sim \|\lambda\| + \int_\Omega \delta\, d\mu.$$

Similar equivalences between norms and quasi-norms hold also for all our further results but will not be stated. The constants C depend only on Ω in this section.

Before proving Theorem 2, we give two corollaries.

COROLLARY 1. *Let $1 < p < \infty$. A harmonic function u in Ω is the Poisson integral of a function in $L_p(\partial\Omega)$ if and only if*

$$\delta^{-1}|u|^p \in \Lambda(\Omega). \tag{4.2}$$

Here $L_p(\partial\Omega)$ is defined by means of the area measure dS. To deduce this corollary, notice that $|u|^p$ is subharmonic. Theorem 2 says that $\int |u|^p\, dS_\eta$ is bounded as $\eta \to 0$ if and only if (4.2) holds. The rest is standard.

REMARK. Corollary 1 can be extended to Orlicz spaces. Let $\phi > 0$ be an increasing convex function on $]0,\infty[$ for which $\phi(t)/t \to \infty$ as $t \to \infty$, and assume for simplicity that there exists a constant A such that $\phi(2t) < A\phi(t)$ for large t. Then a harmonic function u in Ω is the Poisson integral of a measurable function f on $\partial\Omega$ verifying $\int\phi(|f|)\, dS < \infty$ if and only if $\delta^{-1}\phi(|u|) \in \Lambda(\Omega)$.

Next, we apply Theorem 2 to H^p spaces. For $n = 2$, the space $H^p(\Omega)$ consists of all functions u which are holomorphic in Ω and such that

$$\int |u|^p\, dS_\eta = O(1), \qquad \eta \to 0. \tag{4.3}$$

Moreover, $N(\Omega)$ is the space of holomorphic functions for which

$$\int \log^+ |u|\, dS_\eta$$

is similarly bounded. When $n > 2$, we define $H^p(\Omega)$ as the space of n-tuples of functions $u = (u_1, \ldots, u_n)$ in Ω satisfying the generalized Cauchy-Riemann equations and for which (4.3) holds true (see Stein [11, §VII: 3]). Here $|u|$ means the Euclidean norm of $(u_1, \ldots, u_n)$.

COROLLARY 2. (i) *An n-tuple $u = (u_1, \ldots, u_n)$ of functions satisfying the generalized Cauchy-Riemann equations in Ω (a holomorphic function in Ω if $n = 2$) is in $H^p(\Omega)$, $0 < p > (n-2)/(n-1)$, if and only if $\delta^{-1}|u|^p \in \Lambda(\Omega)$.*

(ii) *For $n = 2$, a holomorphic function u in Ω is in $N(\Omega)$ if and only if $\delta^{-1}\log^+|u| \in \Lambda(\Omega)$.*

In view of [11, Lemma, p. 217], the function $|u|^p$ is subharmonic for these p, and so is $\log^+|u|$ for $n = 2$. Hence, this corollary follows at once from Theorem 2.

PROOF OF THEOREM 2. We first prove that $\delta^{-1}P\lambda$ and $\delta^{-1}G\mu$ are in $\Lambda(\Omega)$, if λ and μ are as in Theorem 2. Because (2.2), we have

$$\left|\delta(x)^{-1}P\lambda(x)\right| \leq C\int|x-y|^{-n}\,d|\lambda|(y) = Cr^{-n} * |\lambda|(x).$$

But λ is supported by $\partial\Omega$, which is a convolution set by Theorem 1. Hence, $r^{-n} * |\lambda|$ and thus also $\delta^{-1}P\lambda$ are in $\Lambda(\Omega)$.

As to $\delta^{-1}G\mu$, we write

$$G\mu(x) = \int G(x,y)\,d\mu(y) = \int_{|x-y|<\delta(x)/2} + \int_{|x-y|>\delta(x)/2} = u_1(x) + u_2(x).$$

If $|x-y| < \delta(x)/2$, we easily obtain $2\delta(y)/3 < \delta(x) < 2\delta(y)$ and thus $|x-y| < \delta(y)$. Hence,

$$\int_\Omega \delta(x)^{-1}u_1(x)\,dx \leq \tfrac{3}{2}\int\delta(y)^{-1}\,d\mu(y)\int_{|x-y|<\delta(y)} G(x,y)\,dx. \tag{4.4}$$

For $n \geq 3$, we have $G(x,y) \leq C|x-y|^{2-n}$, and it follows that the inner integral in the right-hand side of (4.4) is dominated by $C\delta(y)^2$. To see that this last conclusion holds also for $n = 2$, we integrate the estimate

$$G(x,y) \leq C\left(\log|x-y|^{-1} + \log\delta(y)\right) + C \tag{4.5}$$

for $|x-y| < \delta(y)$. Inequality (4.5) can be proved by means of the expression for G in terms of the conformal mapping of Ω onto the unit disc, see, e.g., Hellwig [4, I.3.6]. (In case Ω is not simply connected, consider simply connected subdomains of Ω having part of the boundary in common with Ω.)

In both cases, we thus obtain, from (4.4), $\int_\Omega \delta^{-1}u_1\,dx \leq C\int\delta\,d\mu$. Since $\mu \in M_\Omega$, the last integral is finite, so that $\delta^{-1}u_1 \in L_1(\Omega) \subset \Lambda(\Omega)$.

To deal with u_2, we choose a Borel map $\varphi\colon \Omega \to \partial\Omega$ for which $|\varphi(x) - x| = \delta(x)$. Let $d\mu^*$ be the image of the measure $\delta d\mu$ under φ, i.e., define $d\mu^*$ by

$$\int f\,d\mu^* = \int f\circ\varphi\delta\,d\mu$$

for any continuous function f on $\partial\Omega$. Then $d\mu^*$ is a finite measure on $\partial\Omega$. If $|x-y| \geq \delta(x)/2$, we conclude from [13, Theorem 2.3] that

$$G(x,y) \leq C\delta(x)\delta(y)|x-y|^{-n} \leq C\delta(x)\delta(y)|x-\varphi(y)|^{-n},$$

since

$$|x-\varphi(y)| \leq |x-y| + \delta(y) \leq |x-y| + \delta(x) + |x-y| \leq C|x-y|.$$

Hence,

$$\delta(x)^{-1}u_2(x) \leq C\int|x-\varphi(y)|^{-n}\delta(y)\,d\mu(y) = C\int|x-z|^{-n}\,d\mu^*(z)$$

and Theorem 1 implies that $\delta^{-1}u_2 \in \Lambda(\Omega)$. Altogether then, $\delta^{-1}P\lambda$ and $\delta^{-1}G\mu$ belong to $\Lambda(\Omega)$.

Conversely, suppose u is a subharmonic function with $\delta^{-1}u \in \Lambda(\Omega)$. Then the same is true of u^+, and since we shall prove (4.1), it is no restriction to assume $u \geq 0$ and $\|\delta^{-1}u\|_\Lambda = 1$.

Fix p with $0 < p < 1$. Denoting by B the ball with center $x \in \Omega$ and radius $\delta(x)/2$, we have

$$u(x)^p \leq C|B|^{-1}\int_B u^p\,dy.$$

This is proved for harmonic functions in Fefferman and Stein [3, Lemma 2, p. 172], and their proof carries over to our case. The use of this inequality was suggested to the author by Dr. B. Dahlberg. From our hypothesis it follows that the decreasing rearrangement of $(\delta^{-1}u)^p$ is $\leq t^{-p}$, so using (2.1) we get

$$\begin{aligned} u(x)^p &\leq C\delta(x)^p|B|^{-1}\int_B (\delta^{-1}u)^p\,dy \\ &\leq C\delta(x)^p|B|^{-1}\int_0^{|B|} t^{-p}\,dt \leq C\delta(x)^p|B|^{-p}. \end{aligned}$$

This means that in Ω

$$u \leq C\delta^{1-n}. \tag{4.6}$$

Take r and η small, with $0 < r < \eta$. It is well known that u remains subharmonic if its values in Ω_η are replaced by those of the harmonic function in Ω_η which equals u on $\partial\Omega_\eta$. This implies that for $x_0 \in \Omega$,

$$\int P_\eta(x_0, y)u(y)\,dS_\eta(y) \leq \int P_r(x_0, y)u(y)\,dS_r(y),$$

and, in view of (2.3), we conclude

$$\int u\,dS_\eta \leq C\int u\,dS_r. \tag{4.7}$$

We now integrate (4.7) with respect to $r^{-1}dr$, from an $\varepsilon > 0$ to η, getting

$$\int_\varepsilon^\eta r^{-1}\,dr\int u\,dS_\eta \leq C\int_{\Omega_\varepsilon\setminus\Omega_\eta}\delta^{-1}u\,dx, \tag{4.8}$$

since $\delta \sim r$ on $\partial\Omega_r$ and the measure $dr dS_r$ is dominated by C times Lebesgue measure. From the hypothesis and (4.6), it follows that the decreasing rearrangement of the restriction of $\delta^{-1}u$ to $\Omega_\varepsilon \setminus \Omega_\eta$ is dominated by $\min(C\varepsilon^{-n}, t^{-1})$. Applying (2.1) to the last integral in (4.8), we obtain

$$(\log\varepsilon^{-1} + \log\eta)\int u\,dS_\eta \leq C\int_0^C \min(C\varepsilon^{-n}, t^{-1})\,dt \leq C\log\varepsilon^{-1} + C.$$

If we divide by $\log\varepsilon^{-1}$ and let $\varepsilon \to 0$, it follows that $\int u\,dS_\eta \leq C$, and this completes the proof of Theorem 2.

5. Results for a half-space. Let $\mathbf{R}_+^n = \{x \in \mathbf{R}^n\colon x_n > 0\}$. We write points in $\mathbf{R}_+^n$ as $x = (x', x_n)$, and identify $\partial\mathbf{R}_+^n$ with $\mathbf{R}^{n-1}$. The notations $P\lambda$ and $G\mu$ will be used as in §4. The analogue of Theorem 2 for $\mathbf{R}_+^n$ reads as follows.

THEOREM 3. *Let u be a subharmonic function in $\mathbf{R}_+^n$. A necessary and sufficient condition for u to have a representation*

$$u(x) = P\lambda(x) - G\mu(x) + cx_n$$

is that

$$(1 + |x|)^{-n} x_n^{-1} u \in \Lambda(\mathbf{R}_+^n). \tag{5.1}$$

Here $c \in \mathbf{R}$, and λ and $\mu \geqslant 0$ are such that $P\lambda$ and $G\mu$ converge, which means

$$\int_{\mathbf{R}^{n-1}} (1 + |x'|)^{-n}\, d|\lambda|(x') < \infty \quad \text{and} \quad \int_{\mathbf{R}_+^n} (1 + |x|)^{-n} x_n \, d\mu(x) < \infty.$$

Of course, the term cx_n can be interpreted as the Poisson integral of a point mass at infinity. A necessary and sufficient condition for this term to vanish is that, in addition to (5.1), $\int_{B_\varepsilon} (1 + |x|)^{-n}\, dx < \infty$ for all $\varepsilon > 0$, where $B_\varepsilon = \{x \in \mathbf{R}_+^n\colon |u| > \varepsilon x_n\}$. This is a restatement of an inequality proved by Beurling [1] for $n = 2$. A proof based on Theorem 1 can be found in Sjögren [7, Corollary 2]. For the bounded domain Ω of §4, an analogous condition for $\lambda(\{y\}) = 0$ for some fixed $y \in \partial\Omega$, if $u = P\lambda - G\mu$, is that $\int_{B_\varepsilon} |x - y|^{-n}\, dx < \infty$ for all $\varepsilon > 0$, where $B_\varepsilon = \{x \in \Omega\colon |u(x)| > \varepsilon P(x, y)\}$. See Maz'ja [5], Dahlberg [2], and Sjögren [9].

We next give a Λ characterization of Poisson integrals of measures on $\mathbf{R}^{n-1}$ which do not increase too fast at infinity. Let M_γ, $\gamma \geqslant 0$, be the class of Radon measures λ on $\mathbf{R}^{n-1}$ with $\int (1 + |x'|)^{-\gamma}\, d|\lambda|(x') < \infty$. Similarly, we denote by $L_{p,\gamma}$ the class of measurable functions f on $\mathbf{R}^{n-1}$ for which $\int (1 + |x'|)^{-\gamma} |f(x')|^p\, dx' < \infty$.

THEOREM 4. *Let u be harmonic in $\mathbf{R}_+^n$, and suppose $1 < p < \infty$. A necessary and sufficient condition for u to be the Poisson integral of a measure in M_γ, $0 \leqslant \gamma < n$, or a function in $L_{p,\gamma}$, $0 \leqslant \gamma \leqslant n$, is that $(1 + |x|)^{-\gamma} x_n^{-1} u \in \Lambda(\mathbf{R}_+^n)$ or $(1 + |x|)^{-\gamma} x_n^{-1} |u|^p \in \Lambda(\mathbf{R}_+^n)$, resp.*

As proved in [10], this result can be extended to classes of measures defined by more general weight functions.

Again, our results have corollaries about H^p spaces, of which only one will be stated. We define $H^p(\mathbf{R}_+^n)$ as in Stein [11, p. 220] and apply the case $\gamma = 0$ of Theorem 4.

COROLLARY 3. *Part* (i) *of Corollary* 2 *holds also for $\Omega = \mathbf{R}_+^n$ (and $\delta(x) = x_n$).*

Let us finally remark that Theorem 3 has an analogue in $\mathbf{R}^n$, $n \geq 3$. A subharmonic function u in $\mathbf{R}^n$ has a representation $u = \text{const} - r^{2-n} * \mu$ if and only if $(1 + |x|)^{-n}u \in \Lambda(\mathbf{R}^n)$. Here we assume $\int(1 + |x|)^{2-n}\, d\mu(x) < \infty$, so that the Newtonian potential $r^{2-n} * \mu$ of $\mu \geq 0$ is superharmonic. The proof is quite simple and will not be given.

6. **Proofs of Theorems 3 and 4.** In this section, the constants C depend only on n.

Proof of Theorem 3. Taking λ and μ as in the statement of the theorem, we must show that the functions $(1 + |x|)^{-n}x_n^{-1}P\lambda = C(1 + |x|)^{-n}r^{-n} * \lambda$ and $(1 + |x|)^{-n}x_n^{-1}G\mu$ are in $\Lambda(\mathbf{R}^n_+)$. For $x \in \mathbf{R}^n_+$ with $|x| > 1$, we write

$$r^{-n} * \lambda(x) = \int |x - y'|^{-n}\, d\lambda(y') = \int_{|y'|<|x|/2} + \int_{|x|/2<|y'|<2|x|} + \int_{2|x|<|y'|}.$$

In this last sum, we use Theorem 1 to deal with the second term, and simple estimates for the other two terms. Since the case $|x| < 1$ can be similarly treated, we conclude $(1 + |x|)^{-n}r^{-n} * \lambda \in \Lambda(\mathbf{R}^n_+)$.

As to $G\mu$, we proceed as in the proof of Theorem 2. The only essential difference is that the mapping $\varphi\colon \mathbf{R}^n_+ \to \mathbf{R}^{n-1}$ must now have the property $|\varphi(x)| \sim |x|$, in addition to $|\varphi(x) - x| \sim x_n$. We can take, e.g., $\varphi(x', x_n) = |x|x'/|x'|$ for $|x'| \neq 0$ and $\varphi(0, x_n) = (x_n, 0, \ldots, 0)$. The details are left to the reader.

Since of course $u = cx_n$ satisfies (5.1), we have proved the necessity part of Theorem 3.

For the converse, we start by studying the Poisson kernel P_r of the half-ball $H_r = \{x \in \mathbf{R}^n_+\colon |x| < r\}$. By means of a reflection, P_r can be calculated explicitly, but we only need the following properties of $P_r(x, y)$. It is understood that $x \in H_r$ and $y \in \partial H_r$.

(a) $P_r(x, y) \leq Cr^{-n-1}x_n y_n$ for $|x| < r/2$ and $y_n > 0$.

(b) $P_r(x, y) = Cr^{-n-1}x_n y_n + O(r^{-n-2}x_n y_n)$ as $r \to \infty$, uniformly for $|x| < r_0$ and $y_n > 0$, any fixed r_0.

(c) If $P(x, y) = Cx_n|x - y|^{-n}$ is the Poisson kernel of $\mathbf{R}^n_+$, then $P_r(x, y) \leq P(x, y)$ for $y_n = 0$.

(d) $P_r(x, y) \to P(x, y)$ as $r \to \infty$ for each x and y with $y_n = 0$.

Assume u is subharmonic and satisfies (5.1) with quasi-norm 1. To begin with, we consider the subharmonic function $v = u^+$. The reasoning leading to (4.6) carries over to $\mathbf{R}^n_+$, and yields

$$v(x) \leq Cx_n^{1-n}(1 + |x|)^n. \tag{6.1}$$

Let H_r^η be the translate of H_r with center at $(0, \eta)$, $\eta > 0$, and P_r^η the associated Poisson kernel. Of course,

$$v(x) \leq \int_{\partial H_r^\eta} P_r^\eta(x, y) v(y)\, dS(y)$$

for $x \in H_r^\eta$. Integrating with respect to $r^{-1}dr$, we get

$$(6.2)\quad v(x) \leq (\log R)^{-1} \int_R^{R^2} r^{-1}\, dr \int_{\partial H_r^\eta} P_r^\eta(x, y) v(y)\, dS(y) = \int_{y_n > \eta} + \int_{y_n = \eta}$$

for $x \in H_R^\eta$ and R large. Here we have separated that part of the double integral involving the values of v in $\{y_n > \eta\}$ from that dealing with $\{y_n = \eta\}$.

LEMMA 4. *With the above notations, we have for small η, large R, and $x \in H_R^\eta$, $\int_{y_n > \eta} \leq Cx_n$.*

PROOF. From (a), we see that

$$\int_{y_n > \eta} \leq C(\log R)^{-1}(x_n - \eta) \int_D r^{-1} r^{-n-1}(y_n - \eta) v(y)\, dy,$$

where $r^2 = |y'|^2 + (y_n - \eta)^2$ and $D = H_{R^2}^\eta \setminus H_R^\eta$. For $y \in D$, we have $r \sim |y| \sim |y| + 1$, and $r^{-1} \leq Cy_n^{-1}$. Hence,

$$\int_{y_n > \eta} \leq C(\log R)^{-1} x_n \int_D (y_n - \eta) r^{-1}(1 + |y|)^{-n} y_n^{-1} v(y)\, dy.$$

Put $D_j = \{y \in D: 2^{-j-1} < (y_n - \eta) r^{-1} \leq 2^{-j}\}, j = 0, 1, \ldots$. Then

$$(6.3)\quad \begin{aligned} &\int_{D_j} (y_n - \eta) r^{-1}(1 + |y|)^{-n} y_n^{-1} v(y)\, dy \\ &\qquad \leq 2^{-j} \int_{D_j} (1 + |y|)^{-n} y_n^{-1} v(y)\, dy. \end{aligned}$$

The integrand in the right-hand side of this inequality belongs to Λ and is at most $C2^{nj}R^{-n}$ in D_j, because of (6.1). Applying (2.1), we see that both sides of (6.3) are majorized by $C2^{-j}(1 + j + \log R)$, since of course $|D_j| \leq CR^{2n}$. Summing over j, we easily obtain the lemma.

LEMMA 5. *For any $R_0 > 0$, there is an $\varepsilon_0 > 0$ such that, for $0 < x_n < \varepsilon_0$,*

$$\int_{|x'| < R_0} (1 + |x'|)^{-n} v(x', x_n)\, dx' \leq C.$$

PROOF. We may assume $R_0 = 2^k$, where k is a natural number. Take $R = 2^{k+2}$, and let j be an integer with $0 \leq j \leq k$. For $\eta < x_n$ and x_n small, we obtain from (6.2)

$$(6.4)\quad \begin{aligned} &\int_{2^{j-1} < |x'| < 2^j} (1 + |x'|)^{-n} v(x', x_n)\, dx' \\ &\qquad = \int (1 + |x'|)^{-n}\, dx' \int_{y_n > \eta} + \int (1 + |x'|)^{-n}\, dx' \int_{y_n = \eta}. \end{aligned}$$

Here, the first term on the right-hand side is dominated by $C2^{-j}x_n$, because of Lemma 4. The estimate (c) will apply to the weighted average of P_r^η occurring in (6.2), so the last term in (6.4) is at most

$$C2^{-nj}(x_n-\eta)\int_{2^{j-1}<|x'|<2^j}dx'\int_{|y'|<R^2}\left(|x'-y'|^2+(x_n-\eta)^2\right)^{-n/2}v(y',\eta)\,dy'.$$

Now change the order of integration, and separate the integral over $A_j=\{y'\colon 2^{j-2}<|y'|<2^{j+1}\}$ from that over the rest of $\{|y'|<R^2\}$. After some calculations, and after summing over $1\leqslant j\leqslant k$, we get

$$\begin{aligned}\int_{|x'|<R_0}&(1+|x'|)^{-n}v(x',x_n)\,dx'\\ &\leqslant Cx_n+C\int_{|y'|<2R_0}(1+|y'|)^{-n}v(y',\eta)\,dy'\\ &\quad+CR^{2n}x_n\int_{|y'|<R^2}(1+|y'|)^{-n}v(y',\eta)\,dy'.\end{aligned}$$

(The modifications needed for the integral over $|x'|<\frac{1}{2}$ are easy.) Now integrate this inequality with respect to $\eta^{-1}d\eta$ over $[\varepsilon,x_n]$, as in (4.8). If we apply (2.1), together with (6.1) and the hypothesis, to the volume integrals thus obtained, it follows that

$$\begin{aligned}&\left(\log\varepsilon^{-1}+\log x_n\right)\int_{|x'|<R_0}(1+|x'|)^{-n}v(x',x_n)\,dx'\\ &\quad\leqslant Cx_n\left(\log\varepsilon^{-1}+\log x_n\right)+C\log\varepsilon^{-n}+C(1+x_n\log\varepsilon^{-n})g(R_0),\end{aligned}$$

for some function g. This implies Lemma 5 with $\varepsilon_0=1/g(R_0)$ (cf. the end of §4).

End of Proof of Theorem 3. As before, u is subharmonic and satisfies (5.1). We put $\mu=\Delta u$, in the sense of distributions. Picking an $x\in\mathbf{R}^n_+$ with $u(x)$ finite, we have

$$\begin{aligned}u(x)&=(\log R)^{-1}\int_R^{R^2}r^{-1}\,dr\int_{\partial H_r^\eta}P_r^\eta(x,y)u(y)\,dS(y)\\ &\quad-(\log R)^{-1}\int_R^{R^2}r^{-1}\,dr\int_{H_r^\eta}G_r^\eta(x,y)\,d\mu(y)=J_1-J_2,\end{aligned}\tag{6.5}$$

if R is large and $\eta>0$ small. Here G_r^η is the Green's function of H_r^η. Consider the positive part of J_1, i.e., the quantity obtained by replacing u by u^+ in J_1. Lemmas 4 and 5 imply that this part is bounded for $\eta<\varepsilon_0=\varepsilon_0(R)$. From (6.5), it then follows that J_2 and the negative part of J_1 are bounded for $\eta<\varepsilon_0$. By monotonic convergence, we see that $\int G(x,y)\,d\mu(y)<\infty$, and so μ is as required. Lemma 5 and the boundedness of the negative part of J_1

imply that the restriction of u to $\{y: y_n = \eta\}$ converges locally weakly to a measure λ on $\mathbf{R}^{n-1}$, for some sequence $\eta_i \to 0$. Further, λ satisfies

$$\int (1 + |y'|)^{-n} \, d|\lambda|(y') < \infty.$$

Now let $\eta \to 0$ in (6.5), through the sequence just chosen, and then $R \to \infty$. Then J_2 will tend to $G\mu(x)$, and the part of J_1 which involves $\{y_n = \eta\}$ tends to $P\lambda(x)$, in view of (c) and (d). The remaining part of J_1 therefore tends to a harmonic function h. From Lemma 4 and inequality (b), it can be seen that h must be proportional to x_n. In the limit, (6.5) therefore yields the searched-for representation of u, and Theorem 3 is proved.

As to Theorem 4, the necessity part is proved like that of Theorem 3, in the case of M_γ. For $L_{p,\gamma}$ we note that $|Pf|^p \leqslant P|f|^p$ because of Hölder's inequality, and then apply the M_γ case. The proof of the sufficiency part is a modification of the proof just given (for v we take $|u|$ and $|u|^p$, respectively).

7. **The iterated Poisson integral.** For $n \geqslant 2$, we consider the unit polydisc

$$U^n = \{z = (z_1, \dots, z_n) \in \mathbf{C}^n: |z_i| < 1, i = 1, \dots, n\}.$$

Let $\mathbf{T}^n = \{z: |z_i| = 1, i = 1, \dots, n\}$ be its distinguished boundary. If λ is a (finite Radon) measure carried by $\mathbf{T}^n$, we denote by

$$P\lambda(z) = \int_{\mathbf{T}^n} P(z_1, t_1) \dots P(z_n, t_n) \, d\lambda(t)$$

the iterated Poisson integral of λ. Here of course $t = (t_1, \dots, t_n)$, and $P(z_i, t_i)$ is the Poisson kernel of the unit disc. Then $P\lambda$ is n-harmonic in U^n, i.e., $P\lambda$ is continuous there and harmonic in any one variable z_i, if the other z_j are kept fixed. We put $\delta(z) = \Pi_i(1 - |z_i|)$ for $z \in U^n$.

To characterize iterated Poisson integrals, we need spaces which are slightly larger than Λ. For $k \geqslant 0$, we let $\Lambda^k(U^n)$ be the space of measurable real-valued functions in U^n satisfying

$$f^*(t) \leqslant \text{const} \cdot t^{-1}(\log(2 + t^{-1}))^k, \qquad 0 < t < |U^n|.$$

Equivalently, these spaces may be defined by the inequality

$$\lambda_f(\alpha) \leqslant \text{const} \cdot \alpha^{-1}(\log(2 + \alpha))^k, \qquad \alpha > 0.$$

Of course, λ_f and f^* are defined by means of $2n$-dimensional Lebesgue measure in U^n.

THEOREM 5. *Let u be an n-harmonic function in U^n. A necessary and sufficient condition for u to be the iterated Poisson integral of a measure on $\mathbf{T}^n$ or of a function in $L_p(\mathbf{T}^n)$, $1 < p < \infty$, is that $\delta^{-1}u \in \Lambda^{n-1}(U^n)$ or $\delta^{-1}|u|^p \in \Lambda^{n-1}(U^n)$, resp.*

The measure used in the definition of $L_p(\mathbf{T}^n)$ is ithe product of n ordinary arc measures on $\mathbf{T}$. We state Theorem 5 only for a polydisc, although the proof easily carries over to finite products of domains $\Omega_i \subset \mathbf{R}^{n_i}$ satisfying the hypotheses of §4.

COROLLARY 4. *A holomorphic function u in U^n is in $H^p(U^n)$, $0 < p < \infty$, if and only if $\delta^{-1}|u|^p \in \Lambda^{n-1}(U^n)$.*

Here $H^p(U^n)$ defined as in, e.g., Rudin [6, §3.4]. This corollary follows from Theorem 5 and its proof, applied to the n-subharmonic function $|u|^p$.

PROOF OF THEOREM 5. *Necessity*. The L_p result follows from the result for measures, as in the proof of Theorem 4. To prove that $\delta^{-1}P\lambda \in \Lambda^{n-1}(U^n)$ for a measure λ on $\mathbf{T}^n$, we use induction. Observe first of all that this amounts to showing that the convolution $(\Pi|z_i|^{-2}) * \lambda$ is in Λ^{n-1}. The case $n = 1$ is contained in Theorem 1, so assume the assertion holds for $n - 1$. We consider first the case when λ is given by an integrable function, $d\lambda(t) = f(\theta_1, \ldots, \theta_n)d\theta_1 \ldots d\theta_n$, where $\theta_i = \arg t_i$. We may assume $f \geqslant 0$ and $\int f(\theta)\, d\theta = 1$. (All integrals in any θ_i are taken from 0 to 2π.)

For $z \in U^n$, we have

$$\delta^{-1}P\lambda(z) = \int \Pi|z_i - t_i|^{-2} f(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n$$

$$= \int |z_1 - t_1|^{-2}\, d\theta_1 \int \prod_{i>1} |z_i - t_i|^{-2} f(\theta_1, \ldots, \theta_n)\, d\theta_2 \ldots d\theta_n \tag{7.1}$$

$$= \int |z_1 - t_1|^{-2}\, d\theta_1 F(\theta_1, z_2, \ldots, z_n),$$

say. Keeping $(z_2, \ldots, z_n)$ fixed, we see from this equation and Theorem 1 that $\delta^{-1}P\lambda(\cdot, z_2, \ldots, z_n)$ is in $\Lambda(U)$. The corresponding quasi-norm is dominated by $C\int F(\theta_1, z_2, \ldots, z_n)\, d\theta_1$, because of the remark in the beginning of §3. Putting $\int F\, d\theta_1 = G(z_2, \ldots, z_n)$, we thus have for $\alpha > 0$

$$m_2\left\{z_1 \in U: \delta^{-1}P\lambda(z_1, \ldots, z_n) > \alpha\right\} \leqslant C\alpha^{-1}G(z_2, \ldots, z_n), \tag{7.2}$$

where m_j is j-dimensional Lebesgue measure.

Clearly,

$$G(z_2, \ldots, z_n) = \int \prod_{i>1} |z_i - t_i|^{-2}\, d\theta_2 \ldots d\theta_n \int f(\theta_1, \ldots, \theta_n)\, d\theta_1. \tag{7.3}$$

But $\int f(\theta)\, d\theta_1$ can be considered as an integrable function on $\mathbf{T}^{n-1}$, with integral 1. Because of the induction hypothesis, (7.3) implies that G is in $\Lambda^{n-2}(U^{n-1})$. Since trivially $G \leqslant \Pi_{i>1}(1 - |z_i|)^{-2}$, we have $G \leqslant \alpha^{2(n-1)}$ on the set $B = \{(z_2, \ldots, z_n)\colon\ 1 - |z_i| > \alpha^{-1},\ i = 2, \ldots, n\}$. From (7.2) and Fubini's Theorem, we deduce

$$m_{2n}\{z \in U \times B: \delta^{-1}P\lambda(z) > \alpha\} \leq C\alpha^{-1}\int_B G\, dm_{2n-2}. \tag{7.4}$$

But (2.1) and the properties of G just stated imply

$$\begin{aligned}\int_B G\, dm_{2n-2} &\leq C\int_0^C \min\left(\alpha^{2(n-1)}, t^{-1}\left(\log(2+t^{-1})\right)^{n-2}\right) dt \\ &\leq C\left(\log(2+\alpha)\right)^{n-1}.\end{aligned} \tag{7.5}$$

Together with the trivial inequality $m_{2n}(U^n \setminus (U \times B)) \leq C\alpha^{-1}$, (7.4)–(7.5) imply

$$m_{2n}\{z \in U^n: \delta^{-1}P\lambda(z) > \alpha\} \leq C\alpha^{-1}\left(\log(2+\alpha)\right)^{n-1}.$$

This completes the induction for absolutely continuous measures λ. The general case then follows by an obvious limiting process. The necessity part of the theorem is thus proved.

Sufficiency. Suppose that u is n-harmonic in U^n, and that $\delta^{-1}u \in \Lambda^{n-1}(U^n)$. We follow closely the corresponding part of the proof of Theorem 2. The inequality from [3] used there is now applied to $|u|$ in suitable polydiscs, and yields

$$|u(z)| \leq C\delta(z)^{-1}\left(\log\left(2 + \delta(z)^{-1}\right)\right)^{n-1}.$$

For $0 < \eta < r_i < 1$, $i = 1, \ldots, n$, we have ($\theta_i = \arg t_i$)

$$\int_{\mathbf{T}^n} |u(\eta t)|\, d\theta \leq \int_{\mathbf{T}^n} |u(r_1 t_1, r_2 t_2, \ldots, r_n t_n)|\, d\theta,$$

since $|u|$ is n-subharmonic. If we integrate this inequality with respect to $(\Pi(1-r_i)^{-1})dr_1 \ldots dr_n$ over $\eta < r_i < 1 - \varepsilon$, we obtain a $2n$-dimensional integral on the right-hand side (cf. (4.8)). Essentially as in §4, we deduce that $\int_{\mathbf{T}^n}|u(\eta t)|\, d\theta$ is bounded as $\eta \to 0$, dividing by $(\log \varepsilon^{-1})^n$ this time. This means that u is the iterated Poisson integral of a measure, as required. (See [6, Theorem 2.1.3(e)].) To deal with the case when $\delta^{-1}|u|^p \in \Lambda^{n-1}(U^n)$, we carry out the same reasoning for $|u|^p$. This ends the proof of Theorem 5.

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