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WEAK COMPACTNESS IS EQUIVALENT TO THE FIXED POINT PROPERTY IN c_0

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Abstract. A nonempty, closed, bounded, convex subset of c_0 has the fixed point property if and only if it is weakly compact.

1. Introduction

In 1981, B. Maurey [9], E. Odell and Y. Sternfeld [11], and R. Haydon, E. Odell and Y. Sternfeld [6] published results on the existence of fixed points of nonexpansive maps on subsets of c_0. The most general result of these was due to Maurey who used ultrapower techniques to prove that nonempty, weakly compact, convex subsets of c_0 have the fixed point property. That is, every nonexpansive mapping of a nonempty, weakly compact, convex subset of c_0 into itself has a fixed point.

Recently there have been several articles investigating the converse of Maurey's theorem. In 1998, E. Llorens-Fuster and B. Sims [8] showed that the closed, bounded, convex subsets of c_0 with nonempty interior fail to have the fixed point property, and that there exist nonempty convex subsets of c_0 that are compact in a topology slightly weaker than the weak topology that also fail to have the fixed point property. (Recently, M. Japón Pineda [7] extended this second result to Banach spaces containing c_0.) Llorens-Fuster and Sims' investigations led them to conjecture that closed, bounded, convex subsets of c_0 with the fixed point property are weakly compact.

Partial results along these lines have been obtained in Domínguez Benavides, Japón Pineda and Prus [3] and Dowling, Lennard and Turett [4]. In both of these articles, the authors provide characterizations of the weakly compact convex subsets of c_0 in terms of the fixed point property for certain classes of mappings. For example, in [4], it is shown that a closed, bounded, convex subset of c_0 is weakly compact if and only if all of its nonempty, closed, convex subsets have the fixed point property for nonexpansive mappings. However, this result is not strong enough to prove the conjecture of Llorens-Fuster and Sims. The result only guarantees that a closed, bounded, convex subset K of c_0 that is not weakly compact contains a further closed, bounded, convex subset K_0 that fails the fixed point property. It

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does not guarantee that the set K itself fails the fixed point property. Thus, the following theorem both improves and clarifies the situation.

Theorem 1. *Let K be a nonempty, closed, bounded, convex subset of $(c_0, ||\cdot||_\infty)$. Then K is weakly compact if and only if K has the fixed point property ; i.e., every nonexpansive mapping $U : K \longrightarrow K$ has a fixed point.*

Moreover, if K is non-weakly compact, there exists a contractive *mapping $T : K \longrightarrow K$ (i.e., $\|T(u) - T(v)\|_\infty < \|u - v\|_\infty$ for all $u, v \in K$ with $u \neq v$), such that T is fixed point free.*

Thus, as conjectured by Llorens-Fuster and Sims, the converse of Maurey's c_0 theorem is true. We remark that our result holds when the underlying scalar field is $\mathbb{R}$ and also when it is $\mathbb{C}$.

The proof of the theorem depends on the notions of asymptotically isometric c_0-basic sequences and asymptotically isometric c_0-summing basic sequences. Recall from [4] that a sequence $(y_n)_{n\in\mathbb{N}}$ in a Banach space X is an asymptotically isometric c_0-summing basic sequence if there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ such that

$$\sup_{n\in\mathbb{N}} \left(\frac{1}{1+\varepsilon_n}\right) \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{n=1}^{\infty} t_n\, y_n\right\| \leq \sup_{n\in\mathbb{N}} (1+\varepsilon_n) \left|\sum_{j=n}^{\infty} t_j\right|,$$

for all $(t_n)_{n\in\mathbb{N}} \in c_{00}$, the space of finitely nonzero sequences. Recall also that if a sequence $(y_n)_{n\in\mathbb{N}}$ is an asymptotically isometric c_0-summing basic sequence and if the sequence $(w_n)_{n\in\mathbb{N}}$ is defined by $w_n := y_n - y_{n-1}$, where $y_0 := 0$, then for all $(t_n)_{n\in\mathbb{N}} \in c_0$,

$$(*) \qquad \sup_{n\in\mathbb{N}} \left(\frac{1}{1+\varepsilon_n}\right) |t_n| \leq \left\|\sum_{n=1}^{\infty} t_n\, w_n\right\| \leq \sup_{n\in\mathbb{N}} (1+\varepsilon_n)\, |t_n|.$$

As in [4], a sequence $(w_n)_{n\in\mathbb{N}}$ satisfying $(*)$ is called an asymptotically isometric c_0-basic sequence. For information concerning asymptotically isometric c_0-basic sequences and asymptotically isometric c_0-summing basic sequences, see [4] and the references therein. The following result (Theorem 4 in [4]) as well as certain technical details in its proof will prove crucial.

Theorem 2. [4] *Let K be a closed, bounded, convex subset of $(c_0, \|\cdot\|_\infty)$ that is not weakly compact. Then K contains a nonzero multiple of an asymptotically isometric c_0-summing basic sequence.*

Thus, if K is a closed, bounded, convex subset of c_0 that is not weakly compact, then there exists $L > 0$ and an asymptotically isometric c_0-summing basic sequence $(y_n)_{n\in\mathbb{N}}$ in c_0 such that the sequence $(L\, y_n)_{n\in\mathbb{N}}$ is in K. With $K_0 := \overline{\text{co}}\{y_n\}$, it is easy to see that there exists a nonexpansive map $T : LK_0 \to LK_0$ without a fixed point if and only if there exists a nonexpansive map $\widetilde{T} : K_0 \to K_0$ without a fixed point. Merely define $\widetilde{T}$ as the composition of the maps: multiplication by L, T, and multiplication by $1/L$. Thus, in utilizing the theorem to prove Theorem 1, it suffices to assume that $L = 1$; i.e., that K contains an asymptotically isometric c_0-summing basic sequence.

The technical detail from the proof of the above theorem that will be used in the proof of Theorem 1 is a specific property of the asymptotically isometric c_0-basic sequence $(w_n)_{n\in\mathbb{N}}$ constructed in [4]. In particular, there exists a strictly increasing sequence $(N_{M(n)})_{n\geq 0}$ in $\mathbb{N}\cup\{0\}$ such that $N_{M(0)} := 0$ and the elements $w_n = (w_i^n) = (y_i^n - y_i^{n-1})$ satisfy

$$(\spadesuit) \qquad \max_{N_{M(n-1)}<i\leq N_{M(n)}} \left| \sum_{k=1}^{\infty} t_k\, w_i^k \right| \geq |t_n| - \left(\frac{\delta}{4^{n-2}}\right) \|t\|_\infty$$

for all $n \in \mathbb{N}$ and all $(t_k)_{k\in\mathbb{N}} \in c_0$. Here $\delta \in (0, 4^{-7})$ is a constant.

We remark that this condition ($\spadesuit$) is crucial in [4] for establishing the left inequality in $(*)$ above, so that ($\spadesuit$) and the right inequality in $(*)$ provide us with a sharpening of Theorem 2.

2. Proof of Theorem 1

Proof. Since one direction is just a restatement of Maurey's result, it is only necessary to show that, if K is not weakly compact, then there exists a nonexpansive self-map T of K without a fixed point. Indeed, we will show that there exists a *contractive* such T, i.e., $\|T(u) - T(v)\|_\infty < \|u - v\|_\infty$ for all $u, v \in K$ with $u \neq v$.

Fix a closed, bounded, convex subset K of c_0 that is not weakly compact. By our comments above, we may assume that K contains an asymptotically isometric c_0-summing basic sequence $(y_n)_{n\in\mathbb{N}}$ where there is no loss in generality in assuming that $\varepsilon_n < 2^{-1}\cdot 4^{-n}$, for all $n \geq 2$. Then $(w_n)_{n\in\mathbb{N}}$ defined by $w_n := y_n - y_{n-1}$, where $y_0 := 0$, is an asymptotically isometric c_0-basic sequence. Defining the closed, convex subset $K_0 := \overline{\text{co}}\{y_n\}$ of K and using $y_n = w_1 + \cdots + w_n$ for $n \in \mathbb{N}$ yields:

$$K_0 = \overline{\text{co}}\{y_n\} = \left\{ \sum_{n=1}^{\infty} t_n w_n : (t_n)_{n\in\mathbb{N}} \in c_0,\ 1 = t_1 \geq t_2 \geq \cdots \geq 0 \right\}.$$

We begin by defining a nonexpansive map S from c_0 into K_0 as a composition of four mappings: firstly $R : c_0 \longrightarrow c_0^{\downarrow}$, followed by $J : c_0^{\downarrow} \longrightarrow A$, then $V : A \longrightarrow K_0$, and finally $M : K_0 \longrightarrow K_0$. Here,

$$c_0^{\downarrow} := \{s = (s_n)_{n\in\mathbb{N}} \in c_0 : s_1 \geq s_2 \geq s_3 \geq \cdots \geq 0\} \quad \text{and}$$
$$A := \{t = (t_n)_{n\in\mathbb{N}} \in c_0 : 1 = t_1 \geq t_2 \geq \cdots \geq 0\} = \{t \in c_0^{\downarrow} : t_1 = 1\}.$$

(Note that the sets A and K_0 coincide when each $w_n = e_n$; i.e., $(y_n)_{n\in\mathbb{N}}$ is the usual summing basis of c_0.) We will successively define and discuss each mapping below.

For all $u = (u_1, u_2, \ldots) \in c_0$, let $R(u) := u^* = (u_1^*, u_2^*, \ldots) \in c_0$ be the decreasing (i.e., non-increasing) rearrangement of u; i.e., $u^* = (|u_{\rho(1)}|, |u_{\rho(2)}|, \ldots)$ for some one-to-one mapping $\rho : \mathbb{N} \to \mathbb{N}$ such that $u_1^* \geq u_2^* \geq u_3^* \geq \cdots$. Note that $u^* \in c_0^{\downarrow}$.

Although the basic properties of R are well known, we shall derive those that we use herein, for the sake of completeness. Fix $u = (u_n)_{n\in\mathbb{N}} \in c_0$. Define $u_1^* := \max_{n\in\mathbb{N}} |u_n|$, which equals $|u_{n_1}|$ for some $n_1 \in \mathbb{N}$. We may assume that n_1 is minimal with this property. Next, $u_2^* := \max\{|u_n| : n \in \mathbb{N}\backslash\{n_1\}\} = |u_{n_2}|$ for some minimal $n_2 \in \mathbb{N}\backslash\{n_1\}$. Generally, for each $k \in \mathbb{N}$, we inductively define

$$u_{k+1}^* := \max\{|u_n| : n \in \mathbb{N}\backslash\{n_1, \ldots, n_k\}\},$$

which equals $|u_{n_{k+1}}|$ for some minimal $n_{k+1} \in \mathbb{N}\backslash\{n_1, \ldots, n_k\}$. It is easy to see from this definition that $u^* = (u_n^*)_{n\in\mathbb{N}}$ is decreasing and belongs to c_0. (Moreover, the map $\rho : \mathbb{N} \to \mathbb{N}$ mentioned above is given by $\rho(k) := n_k$, for all $k \in \mathbb{N}$.) By the definition, it is straightforward to check the following fact: for all $u \in c_0$, for each $k \in \mathbb{N}$,

$$(\clubsuit) \qquad u_k^* = \min_{F \subseteq \mathbb{N}:\, \#(F)=k-1} \ \max_{n \in \mathbb{N}\setminus F} |u_n|.$$

As a direct consequence of this min-max characterization, it follows that for all $u, v \in c_0$,

$$|u_k^* - v_k^*| \le \|u - v\|_\infty \ , \ \forall\, k \in \mathbb{N}.$$

Therefore, for all $u, v \in c_0$, $||u^* - v^*||_\infty \le ||u - v||_\infty$. Hence, rearrangement is a *nonexpansive* mapping on c_0.

Let us now consider the second mapping J. Using an idea of Llorens-Fuster and Sims [8, The proof of Proposition 1], we define $J : c_0^{\downarrow} \longrightarrow A$ by

$$J(s) := (1, s_1 \wedge 1, s_2 \wedge 1, s_3 \wedge 1, \dots), \text{ for all } s = (s_n)_{n\in\mathbb{N}} \in c_0^{\downarrow}.$$

The following fact is well known (see, for example, [10, Theorem 1.1.1 (x)]): for all $r, s, t \in \mathbb{R}$,

$$|s - t| = |s \vee r - t \vee r| + |s \wedge r - t \wedge r|.$$

Consequently, it is easy to see that J is nonexpansive on $c_0^{\downarrow}$.

The third mapping $V : A \longrightarrow K_0$ is formed by taking each $t = (t_n)_{n\in\mathbb{N}} \in A$ to $\sum_{k=1}^{\infty} t_k\, w_k \in K_0$. Lastly, $M : K_0 \longrightarrow K_0$ is the identity averaged together with the iterates of a right shift operator on K_0. Indeed, define $Q : K_0 \longrightarrow K_0$ by

$$Q(\sigma) := w_1 + t_1\, w_2 + t_2\, w_3 + \cdots + t_n\, w_{n+1} + \cdots \ , \text{ for all } \sigma = \sum_{k=1}^{\infty} t_k\, w_k \in K_0.$$

Also, let I be the identity operator on K_0 and $Q^2 := Q \circ Q$, $Q^3 := Q \circ Q \circ Q$, and so on. Next, we define $M : K_0 \longrightarrow K_0$ by

$$M := \frac{1}{2}\, I + \frac{1}{2^2}\, Q + \frac{1}{2^3}\, Q^2 + \frac{1}{2^4}\, Q^3 + \cdots \ .$$

Now, putting everything together, define $S : c_0 \longrightarrow K_0$ by $S := M \circ V \circ J \circ R = MVJR$. Furthermore, let us introduce some useful notation. Put $\widetilde{u} := u^* \wedge 1 = (u_n^* \wedge 1)_{n\in\mathbb{N}}$, for all $u \in c_0$. Note that for each $u \in c_0$, $JR(u) = (1, \widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \dots)$. Then for all $u \in c_0$,

$$\begin{aligned} S(u) &= MV(JR(u)) = M(V(1, \widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \dots)) \\ &= M(w_1 + \widetilde{u}_1\, w_2 + \widetilde{u}_2\, w_3 + \cdots + \widetilde{u}_n\, w_{n+1} + \cdots) \\ &= \frac{1}{2}\, (w_1 + \widetilde{u}_1\, w_2 + \widetilde{u}_2\, w_3 + \widetilde{u}_3\, w_4 + \widetilde{u}_4\, w_5 + \cdots) \\ &\quad + \frac{1}{2^2}\, (w_1 + w_2 + \widetilde{u}_1\, w_3 + \widetilde{u}_2\, w_4 + \widetilde{u}_3\, w_5 + \cdots) \\ &\quad + \frac{1}{2^3}\, (w_1 + w_2 + w_3 + \widetilde{u}_1\, w_4 + \widetilde{u}_2\, w_5 + \widetilde{u}_3\, w_6 + \cdots) + \cdots ; \text{ and so} \end{aligned}$$

$$\begin{aligned} S(u) = w_1 \quad &+ \quad \left(\frac{1}{2} + \frac{1}{2}\, \widetilde{u}_1\right) w_2 + \left(\frac{1}{4} + \frac{1}{4}\, \widetilde{u}_1 + \frac{1}{2}\, \widetilde{u}_2\right) w_3 + \cdots \\ &+ \quad \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}}\, \widetilde{u}_1 + \frac{1}{2^{n-2}}\, \widetilde{u}_2 + \frac{1}{2^{n-3}}\, \widetilde{u}_3 + \cdots + \frac{1}{2}\, \widetilde{u}_{n-1}\right) w_n + \cdots \ . \end{aligned}$$

We will show below that S is contractive on c_0, using the fact that $||\widetilde{u}-\widetilde{v}||_\infty \le ||u-v||_\infty$, for all $u, v \in c_0$. Fix $u, v \in c_0$ with $u \neq v$. Define $\alpha_j := \widetilde{u}_j - \widetilde{v}_j$, for all $j \in \mathbb{N}$. From above,

$$\begin{aligned} S(u) - S(v) &= \frac{1}{2}\,\alpha_1\, w_2 + \left(\frac{1}{4}\,\alpha_1 + \frac{1}{2}\,\alpha_2\right) w_3 + \cdots \\ &+ \left(\frac{1}{2^{n-1}}\,\alpha_1 + \frac{1}{2^{n-2}}\,\alpha_2 + \cdots + \frac{1}{2}\,\alpha_{n-1}\right) w_n + \cdots\ ; \end{aligned}$$

and using the rightmost inequality in $(*)$ above, together with the fact that $\varepsilon_n < 2^{-1}\,4^{-n}$ for all $n \ge 2$, it follows that

$$\begin{aligned} ||S(u)-S(v)||_\infty &\le \sup_{n\ge 2}\,(1+\varepsilon_n)\left|\frac{1}{2^{n-1}}\,\alpha_1 + \frac{1}{2^{n-2}}\,\alpha_2 + \cdots + \frac{1}{2}\,\alpha_{n-1}\right| \\ &\le \sup_{n\ge 2}(1+\varepsilon_n)\left(\frac{1+2\,\varepsilon_2}{2^{n-1}}\,\frac{1}{1+2\,\varepsilon_2}\,|\alpha_1| + \frac{1+2\,\varepsilon_3}{2^{n-2}}\,\frac{1}{1+2\,\varepsilon_3}\,|\alpha_2| + \dots \right.\\ &\qquad\left. + \frac{1+2\,\varepsilon_n}{2^1}\,\frac{1}{1+2\,\varepsilon_n}\,|\alpha_{n-1}|\right) \\ &\le \max_{m\ge 1}\frac{1}{1+2\,\varepsilon_{m+1}}\,|\alpha_m|\cdot B, \text{ where} \end{aligned}$$

$$\begin{aligned} B &:= \sup_{n\ge 2}(1+\varepsilon_n)\left(\frac{1+2\,\varepsilon_2}{2^{n-1}} + \frac{1+2\,\varepsilon_3}{2^{n-2}} + \cdots + \frac{1+2\,\varepsilon_n}{2^1}\right) \\ &\le \sup_{n\ge 2}\left(1+\frac{1}{2\,(4^n)}\right)\left(\left[\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^1}\right]\right.\\ &\qquad\left. + \left[\frac{1}{4^2\,2^{n-1}} + \frac{1}{4^3\,2^{n-2}} + \cdots + \frac{1}{4^n\,2^1}\right]\right) \\ &= \sup_{n\ge 2}\left(1+\frac{1}{2\,(4^n)}\right)\left(1 - \frac{1}{2^{n-1}} + \left[\frac{1}{2^{n+3}} + \frac{1}{2^{n+4}} + \cdots + \frac{1}{2^{2n+1}}\right]\right) \\ &\le \sup_{n\ge 2}\left(1+\frac{1}{4^n}\right)\left(1-\frac{1}{2^{n-1}}+\frac{1}{2^{n+2}}\right) \le \sup_{n\ge 2}\left(1+\frac{1}{4^n}-\frac{7}{2^{n+2}}\right) = 1. \end{aligned}$$

So,

$$\|S(u)-S(v)\|_\infty \le \max_{m\ge 1}\frac{|\widetilde{u}_m - \widetilde{v}_m|}{1+2\,\varepsilon_{m+1}}.$$

Recall that $u \neq v$. If $\widetilde{u} = \widetilde{v}$, then $\|S(u)-S(v)\|_\infty = 0 < \|u-v\|_\infty$. Otherwise, $\widetilde{u} \neq \widetilde{v}$, which implies

$$\|S(u)-S(v)\|_\infty \le \max_{m\ge 1}\frac{|\widetilde{u}_m - \widetilde{v}_m|}{1+2\,\varepsilon_{m+1}} < \max_{m\ge 1}|\widetilde{u}_m - \widetilde{v}_m| = ||\widetilde{u}-\widetilde{v}||_\infty \le ||u-v||_\infty,$$

and therefore $\|S(u)-S(v)\|_\infty < \|u-v\|_\infty$, for all $u, v \in c_0$ with $u \neq v$. Thus, S is contractive.

Finally, define $T : K \to K_0 \subseteq K$ to be the restriction of S to K. The mapping T is contractive.

We will next show that T fails to have a fixed point in K. Observe first that from the definition above, one can readily show that for all $u \in c_0$, for each $k \in \mathbb{N}$,

$$(\diamondsuit) \qquad u_k^* = \max_{G\subseteq\mathbb{N}:\,\#(G)=k}\ \min_{n\in G}|u_n|.$$

Now assume, to get a contradiction, that $u \in K$ is a fixed point of T and recall that $0 < \delta < 4^{-7}$. Then

$$\begin{aligned} u &= T(u) = w_1 + \left(\frac{1+\widetilde{u}_1}{2}\right) w_2 + \left(\frac{1}{4} + \frac{1}{4}\,\widetilde{u}_1 + \frac{1}{2}\,\widetilde{u}_2\right) w_3 + \cdots \\ &+ \left(\frac{1}{2^{k-1}}\,\widetilde{u}_1 + \frac{1}{2^{k-2}}\,\widetilde{u}_2 + \frac{1}{2^{k-3}}\,\widetilde{u}_3 + \cdots + \frac{1}{2}\,\widetilde{u}_{k-1}\right) w_k + \cdots \\ &=: \sum_{k=1}^{\infty} t_k\, w_k. \end{aligned}$$

Applying (♠) with $n = 1$ yields that there exists $i_1 \in \{1, \ldots, N_{M(1)}\}$ such that

$$\begin{aligned} |u_{i_1}| &= |(T(u))_{i_1}| = \left|\sum_{k=1}^{\infty} t_k\, w_{i_1}^k\right| \\ &\geq |t_1| - \frac{\delta}{4^{1-2}}\,\|t\|_\infty = 1 - 4\,\delta. \end{aligned}$$

So, $u_1^* \geq |u_{i_1}| \geq 1 - 4\,\delta$; and thus $\widetilde{u}_1 = u_1^* \wedge 1 \geq 1 - 4\,\delta$.

Next, applying (♠) with $n = 2$, we see that there exists $i_2 \in \{N_{M(1)} + 1, \ldots, N_{M(2)}\}$ such that

$$\begin{aligned} |u_{i_2}| &= |(T(u))_{i_2}| \geq |t_2| - \frac{\delta}{4^{2-2}}\,\|t\|_\infty = \frac{1}{2} + \frac{1}{2}\,\widetilde{u}_1 - \delta \\ &\geq \frac{1}{2} + \frac{1}{2}\,(1 - 4\,\delta) - \delta = 1 - 2\,\delta - \delta \geq 1 - 4\,\delta. \end{aligned}$$

Hence, using (♢) above and the fact that $i_2 > i_1$, it follows that $u_2^* \geq \min\{|u_{i_1}|, |u_{i_2}|\} \geq 1 - 4\,\delta$; and consequently $\widetilde{u}_2 = u_2^* \wedge 1 \geq 1 - 4\,\delta$.

Furthermore, applying (♠) with $n = 3$ gives that there exists $i_3 \in \{N_{M(2)}+1, \ldots, N_{M(3)}\}$ such that

$$\begin{aligned} |u_{i_3}| &= |(T(u))_{i_3}| \geq |t_3| - \frac{\delta}{4^{3-2}}\,\|t\|_\infty = \frac{1}{4} + \frac{1}{4}\,\widetilde{u}_1 + \frac{1}{2}\,\widetilde{u}_2 - \frac{\delta}{4} \\ &\geq \frac{1}{4} + \frac{1}{4}\,(1 - 4\,\delta) + \frac{1}{2}\,(1 - 4\,\delta) - \frac{\delta}{4} \geq 1 - 2\,\delta - \delta - \frac{\delta}{2} \geq 1 - 4\,\delta. \end{aligned}$$

Since $i_3 > i_2 > i_1$, we may use (♢) to see that $u_3^* \geq \min\{|u_{i_1}|, |u_{i_2}|, |u_{i_3}|\} \geq 1 - 4\,\delta$; and therefore $\widetilde{u}_3 = u_3^* \wedge 1 \geq 1 - 4\,\delta$.

Continuing inductively, it follows that there exists a strictly increasing sequence of positive integers $(i_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$|u_{i_n}| \geq 1 - 2\,\delta - \delta - \frac{\delta}{2} - \cdots - \frac{\delta}{2^{n-2}} \geq 1 - 4\,\delta.$$

It follows from (♢) that for all $n \in \mathbb{N}$, $\widetilde{u}_n \geq 1 - 4\,\delta > 0$, which contradicts the fact that $\widetilde{u} \in c_0$. Thus, T is fixed point free on K. This completes the proof of Theorem 1. □

3. Some remarks and a further result

The conclusions of Theorem 1 do not hold in every Banach space. Goebel and Kuczumow [5] and Soardi [12] have given examples of closed, bounded, convex, non-weakly compact sets in ℓ^1 and $L^\infty[0,1]$ that have the fixed point property, and Dale Alspach [1] has given an example of a weakly compact, convex subset of $L^1[0,1]$ failing to have the fixed point property. There are spaces, other than c_0, for which the conclusions of Theorem 1 hold. However, it is unclear if any of these spaces are significantly different from c_0.

Corollary 3. *Let K be a nonempty, closed, bounded, convex subset of $(c_0(\Gamma), \|\cdot\|_\infty)$ where Γ is uncountable. Then K is weakly compact if and only K has the fixed point property.*

Proof. Since it is known [2] that weakly compact convex sets in $c_0(\Gamma)$ have the fixed point property for nonexpansive maps, assume that K is not weakly compact. Then K contains a sequence $(x_n)_{n\in\mathbb{N}}$ with no weakly convergent subsequence. Let Γ_1 denote the support of the sequence $(x_n)_{n\in\mathbb{N}}$; that is, $\Gamma_1 = \{\gamma \in \Gamma : x_n(\gamma) \neq 0 \text{ for some } n \in \mathbb{N}\}$ where $x_n = (x_n(\gamma))_{\gamma\in\Gamma}$. Then $\overline{\text{co}}\{x_n\}$ lies in the subspace $c_0(\Gamma_1) \times \prod_{\gamma\in\Gamma\setminus\Gamma_1}\{0\}$ of $c_0(\Gamma)$ and, since Γ_1 is countably infinite, $c_0(\Gamma_1) \times \prod_{\gamma\in\Gamma\setminus\Gamma_1}\{0\}$ is linearly isometric to c_0. Thus the previous results can be applied. To be specific, let P_1 denote the canonical projection from $c_0(\Gamma)$ onto $c_0(\Gamma_1)$. The closed, bounded, convex, non-weakly compact set $P_1(K)$ contains the closed, bounded, convex, non-weakly compact set $\overline{\text{co}}\{P_1(x_n)\}$; and, since this set lies in a closed subspace isometric to c_0, $\overline{\text{co}}\{P_1(x_n)\}$ contains (without loss of generality) an asymptotically isometric c_0-summing basic sequence $(y_n)_{n\in\mathbb{N}}$. Letting $(w_n)_{n\in\mathbb{N}}$ denote the asymptotically isometric c_0-basic sequence corresponding to $(y_n)_{n\in\mathbb{N}}$ and defining the closed, convex subset $K_0 := \overline{\text{co}}\{y_n\}$ implies, as before, that

$$K_0 = \overline{\text{co}}\{y_n\} = \left\{ \sum_{n=1}^{\infty} t_n w_n : (t_n)_{n\in\mathbb{N}} \in c_0,\ 1 = t_1 \geq t_2 \geq \cdots \geq 0 \right\}.$$

If the mapping $S : c_0 \longrightarrow K_0$ is defined as in the proof of Theorem 1 (with $c_0(\Gamma_1)$ identified with c_0) and if T is defined as the restriction of S to $P_1(K)$, then T is a nonexpansive self-map of $P_1(K)$ without a fixed point. Next, with $\overrightarrow{0}$ denoting the zero element in $c_0(\Gamma \setminus \Gamma_1)$, note that $y_n \times \overrightarrow{0}$ lies in $\overline{\text{co}}\{x_n\}$ and thus is an element of K. Therefore, since the range of T is actually a subset of K_0, the map $U : K \to K$ defined by $U(x) = (T \circ P_1(x)) \times \overrightarrow{0}$ is a nonexpansive self-map of K without a fixed point and the proof is complete. □

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