Bernoulli **10**(5), 2004, 847–860

Weak convergence of empirical copula processes

JEAN-DAVID FERMANIAN[1], DRAGAN RADULOVIĆ[2] and MARTEN WEGKAMP[3]

[1]*CDC IXIS Capital Markets, 47 Quai d'Austerlitz, 75648 Paris Cedex 13, France*
[2]*Department of Mathematics, Florida Atlantic University, Boca Raton FL 33431-0991, USA*
[3]*Department of Statistics, Florida State University, Tallahassee FL 32306–4330, USA.*
E-mail: wegkamp@stat.fsu.edu

Weak convergence of the empirical copula process has been established by Deheuvels in the case of independent marginal distributions. Van der Vaart and Wellner utilize the functional delta method to show convergence in $\ell^\infty([a, b]^2)$ for some $0 < a < b < 1$, under restrictions on the distribution functions. We extend their results by proving the weak convergence of this process in $\ell^\infty([0, 1]^2)$ under minimal conditions on the copula function, which coincides with the result obtained by Gaenssler and Stute. It is argued that the condition on the copula function is necessary. The proof uses the functional delta method and, as a consequence, the convergence of the bootstrap counterpart of the empirical copula process follows immediately. In addition, weak convergence of the smoothed empirical copula process is established.

Keywords: empirical copula process; smoothed empirical copula processes; weak convergence

1. Introduction

Every multivariate cumulative distribution function (cdf) H on $\mathbb{R}^p$ can be put in the form

$$H(x_1, \ldots, x_p) = C(F_1(x_1), \ldots, F_p(x_p)) \tag{1}$$

for some function $C: [0, 1]^p \to [0, 1]$, where $F_1, \ldots, F_p$ denote the marginal cdfs (see, for example, Nelsen 1999). The function C is called the *copula* or *dependence function* associated with H, and in itself is a distribution function on $[0, 1]^p$ with uniform marginals. The representation in (1) is unique on the range of $(F_1, \ldots, F_p)$, a result due to Sklar (1959). For some historical notes, we refer to Schweizer (1991) and the recent surveys by Joe (1997) and Nelsen (1999).

Copulas capture the dependence structure among the components X_j of the vector $(X_1, \ldots, X_p)$, irrespective of their marginal distributions F_j. In fact, Lemma 1 below asserts that we may assume without loss of generality the X_j to be uniformly distributed on [0,1]. In other words, copulas allow marginal distributions and dependence structure to be modelled separately; this ability has recently led to a revival of their use, for example, in studying the joint probability of default of several borrowers in finance and actuarial sciences, or, more generally, of a set of correlated extreme events. See, for example, Schweizer and Sklar (1974), Genest and McKay (1986a, 1986b), Genest and Rivest (1993), Genest *et al.* (1995), Capéraà *et al.* (1997), Bouyé *et al.* (2000), Schönbucher and Schubert (2001), and Embrechts *et al.* (2002).

In order to simplify our notation and exposition, we will consider only two-dimensional copulas in this paper ($p = 2$). The general case, however, can easily be deduced from our results. Let (X, Y) be a bivariate random vector with joint cdf $H(x, y)$ and *continuous* marginal cdfs $F(x)$ and $G(y)$. Its associated copula C is defined, for all real numbers x and y, by

$$H(x, y) = C(F(x), G(y)). \tag{2}$$

Since F and G are continuous, the copula C defined in (2) is unique, and we may write

$$C(u, v) = H(F^-(u), G^-(v)), \qquad 0 \leqslant u, v \leqslant 1,$$

where F^- and G^- are the *generalized* quantile functions of F and G, respectively. Recall that the generalized inverse of a cdf F is defined as

$$F^-(u) = \inf\{t \in \mathbb{R} \mid F(t) \geqslant u\}, \qquad 0 \leqslant u \leqslant 1.$$

Based on independent copies $(X_1, Y_1), \ldots, (X_n, Y_n)$, we construct the empirical distribution function

$$\mathbb{H}_n(x, y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{X_i \leqslant x, Y_i \leqslant y\}}, \qquad -\infty < x, y < +\infty,$$

and let $\mathbb{F}_n(x)$ and $\mathbb{G}_n(y)$ be its associated marginal distributions, that is,

$$\mathbb{F}_n(x) = \mathbb{H}_n(x, +\infty) \quad \text{and} \quad \mathbb{G}_n(y) = \mathbb{H}_n(+\infty, y), \qquad -\infty < x, y < +\infty.$$

We define the *empirical* copula function $\mathbb{C}_n$ by

$$\mathbb{C}_n(u, v) = \mathbb{H}_n\big(\mathbb{F}_n^-(u), \mathbb{G}_n^-(v)\big), \qquad 0 \leqslant u, v \leqslant 1,$$

and the (ordinary) *empirical copula process*

$$\mathbb{Z}_n(u, v) \equiv \sqrt{n}(\mathbb{C}_n - C)(u, v), \qquad 0 \leqslant u, v \leqslant 1.$$

The function $\mathbb{C}_n$ was briefly discussed by Ruymgaart (1973, pp. 6–13) in the introduction of his doctoral thesis. Deheuvels (1979) investigated the consistency of $\mathbb{C}_n$ and Deheuvels (1981a; 1981b) obtained the exact law and the limiting process of $\mathbb{Z}_n$ when the two margins are independent. Rüschendorf (1976, Theorem 3.3) established weak convergence of the related empirical multivariate rank-order process in the Skorokhod space endowed with stronger metrics than the Skorokhod metric. As a corollary, we may conclude weak convergence of $\mathbb{Z}_n$, but his technical conditions are not optimal. Gaenssler and Stute (1987) proved weak convergence of the empirical copula process $\mathbb{Z}_n$ in the Skorokhod space $D([0, 1]^2)$. We will show weak convergence in the space $\ell^\infty([0, 1]^2)$ using the functional delta method and argue that the required regularity on C, to wit, that C has continuous partial derivatives, cannot be dispensed with. The weak convergence of the bootstrap counterpart follows almost immediately by the functional delta method for the bootstrap.

Statistical applications in hypothesis testing for independence, asymptotic normality of rank statistics, and the bootstrap are provided.

Section 3 deals with the smoothed empirical copula process that is obtained by taking kernel estimates $\hat{\mathbb{H}}_n$, $\hat{\mathbb{F}}_n$ and $\hat{\mathbb{G}}_n$ in lieu of the ordinary empirical cdfs considered in Section 2. The resulting estimate $\hat{\mathbb{C}}_n = \hat{\mathbb{H}}_n(\hat{\mathbb{F}}_n^-, \hat{\mathbb{G}}_n^-)$ and smoothed empirical copula process

$$\hat{\mathbb{Z}}_n(x, y) = \sqrt{n}(\hat{\mathbb{C}}_n - C)(x, y), \qquad 0 \leqslant x, y \leqslant 1,$$

have not been studied before, and we give conditions under which $\hat{\mathbb{Z}}_n - \mathbb{Z}_n$ converges to zero.

2. Weak convergence of empirical copula processes

By an elegant application of the functional delta-method, van der Vaart and Wellner (1996, p. 389) proved the weak convergence of the (ordinary) empirical copula process $\mathbb{Z}_n$ to a Gaussian process in $\ell^\infty([a, b]^2)$ when $a > 0$ and $b < 1$. Theorem 3 below shows that actually $\{\mathbb{Z}_n(x, y), 0 \leqslant x, y \leqslant 1\}$ converges weakly to a Gaussian process in $\ell^\infty([0, 1]^2)$, provided $C(x, y)$ has continuous partial derivatives only. Gaenssler and Stute (1987, Chapter 5) proved weak convergence in $D([0, 1]^2)$ using standard empirical process techniques. Lemmas 1 and 2 below allow us to obtain weak convergence in $\ell^\infty([0, 1]^2)$ using the delta method, as well as the bootstrap counterpart.

We first introduce some more notation. Define the pseudo-variables

$$(X^*, Y^*) = (F(X), G(Y)),$$

with distribution function

$$H^*(x, y) = \mathbb{P}\{X_1^* \leqslant x, Y_1^* \leqslant y\} = H(F^-(x), G^-(y)),$$

and marginal cdfs $F^*(x) = H^*(x, +\infty)$ and $G^*(y) = H^*(+\infty, y)$. Notice that $F^*(x)$ and $G^*(y)$ are both uniform distributions on [0, 1]. The copula function associated with $H^*(x, y)$ is denoted by $C^*(u, v) = H^*(F^{*-}u, G^{*-}v)$ for $0 \leqslant u, v \leqslant 1$. Finally, let $\mathbb{H}_n^*(x, y)$ be the empirical distribution function based on $(X_1^*, Y_1^*), \ldots, (X_n^*, Y_n^*)$ with marginal distributions $\mathbb{F}_n^*(x) = \mathbb{H}_n^*(x, +\infty)$ and $\mathbb{G}_n^*(y) = \mathbb{H}_n^*(+\infty, y)$, and let $\mathbb{C}_n^*(x, y)$ be its associated empirical copula function. The following lemma states that we can assume with impunity that the marginal distributions are uniform.

Lemma 1. *Let F, G be continuous distribution functions. We have*

$$C(x, y) = C^*(x, y) = H^*(x, y) \qquad \textit{for all } x, y \in [0, 1].$$

Moreover,

$$\mathbb{C}_n\left(\frac{i}{n}, \frac{j}{n}\right) = \mathbb{C}_n^*\left(\frac{i}{n}, \frac{j}{n}\right) \qquad \textit{for } i, j = 0, 1, \ldots, n. \tag{3}$$

The first assertion is well known. The fact that $\mathbb{C}_n$ and $\mathbb{C}_n^*$ agree on the grid points

$(i/n, j/n)$ is more surprising and is stated, but not proved, in Gaenssler and Stute (1987, p. 51). In fact, since both $\mathbb{C}_n$ and $\mathbb{C}_n^*$ are constant on the pavements $(i/n, (i+1)/n] \times (j/n, (j+1)/n]$, we have $\mathbb{C}_n(u, v) = \mathbb{C}_n^*(u, v)$ for all $0 \leqslant u, v \leqslant 1$.

Proof. The first claim of the lemma follows easily from the continuity of F and G, the definitions of C, C^* and H^*, and the fact that F^* and G^* are uniform. For the proof of (3), let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the order statistics of the sample $X_1, \ldots, X_n$. Similarly, $Y_{(1)} < \ldots < Y_{(n)}$ are the order statistics of the sample $Y_1, \ldots, Y_n$. Define $X_{(0)} = Y_{(0)} = -\infty$ and $X_{(n+1)} = Y_{(n+1)} = +\infty$, and set $i_n = i/n$ and $j_n = j/n$. Observe that, with probability one, $X_{(i)} \geqslant F^- F(X_{(i)}) > X_{(j)}$ and likewise $Y_{(i)} \geqslant G^- G(Y_{(i)}) > Y_{(j)}$ for all $i > j$, so that $\mathbb{H}_n(X_{(i)}, Y_{(j)}) = \mathbb{H}_n(F^- F(X_{(i)}), G^- G(Y_{(j)}))$. Hence, with probability one,

$$
\begin{aligned}
\mathbb{C}_n(i_n, j_n) &= \mathbb{H}_n(X_{(i)}, Y_{(j)}) \qquad \text{as } \mathbb{F}_n(X_{(i)}) = i_n \text{ and } \mathbb{G}_n(Y_{(j)}) = j_n \\
&= \mathbb{H}_n(F^- F(X_{(i)}), G^- G(Y_{(j)})) \\
&= \mathbb{H}_n^*(F(X_{(i)}), G(Y_{(j)})) \qquad \text{since } \mathbb{H}_n^*(x, y) = \mathbb{H}_n(F^- x, G^- y) \\
&= \mathbb{H}_n^*(X_{(i)}^*, Y_{(j)}^*) \qquad \text{where } X_{(i)}^* = F(X_{(i)}) \text{ and } Y_{(j)}^* = G(Y_{(j)}) \\
&= \mathbb{C}_n^*(i_n, j_n) \qquad \text{since } \mathbb{F}_n^*(X_{(i)}^*) = i_n \text{ and } \mathbb{G}_n^*(Y_{(j)}^*) = j_n.
\end{aligned}
$$

This conclude the proof of the lemma. □

The next result shows that the map

$$
\phi(H)(u, v) = H(F^-(u), G^-(v)), \qquad 0 \leqslant u, v \leqslant 1, \tag{4}
$$

is Hadamard differentiable at H^*.

Lemma 2. *Let $H(x, y)$ have compact support $[0, 1]^2$, and marginal distributions $F(x)$ and $G(y)$ that are continuously differentiable on its support with strictly positive densities $f(x)$ and $g(y)$, respectively. Furthermore, assume that $H(x, y)$ is continuously differentiable on $[0, 1]^2$. Then the map $\phi: D([0, 1]^2) \to \ell^\infty([0, 1]^2)$ defined in (4), which transforms the cdf H into its copula function C_H, is Hadamard differentiable tangentially to $C([0, 1]^2)$.*

Proof. As in van der Vaart and Wellner (1996, p. 389), we observe that mapping H into its copula function can be decomposed as

$$
H \mapsto (H, F, G) \mapsto (H, F^-, G^-) \mapsto H \circ (F^-, G^-).
$$

The first map and the third map are Hadamard differentiable, as pointed out in the proof of Lemma 3.9.28 in van der Vaart and Wellner (1996). The second map is Hadamard differentiable as a consequence of Lemma 3.9.23 in van der Vaart and Wellner (1996, p. 386), which states that the inverse mapping $F \mapsto F^-$ as a mapping $D_2 \subset D[0, 1] \mapsto \ell^\infty[0, 1]$ is Hadamard differentiable at F tangentially to $C[0, 1]$. Here D_2 is the collection of distribution functions of measures that concentrate on [0, 1]. Apply the chain rule to show that $H \mapsto H \circ (F^-, G^-)$ is Hadamard differentiable. □

The combination of Lemmas 1 and 2 and the functional delta method immediately yields the following result.

Theorem 3. *Suppose that H has continuous marginal distribution functions and that the copula function $C(x, y)$ has continuous partial derivatives. Then the empirical copula process $\{\mathbb{Z}_n(x, y), 0 \leqslant x, y \leqslant 1\}$ converges weakly to the Gaussian process $\{\mathbb{G}_C(x, y), 0 \leqslant x, y \leqslant 1\}$ in $\ell^\infty([0, 1]^2)$.*

Proof. First notice that for all $x, y \in [0, 1]$, there exist i_n, j_n such that $\mathbb{C}_n(x, y) = \mathbb{C}_n(i_n, j_n)$, which, coupled with Lemma 1, yields $\sqrt{n}(\mathbb{C}_n - C)(x, y) = \sqrt{n}(\mathbb{C}_n^* - C^*)(x, y)$. Since $H^*(x, y) = C(x, y)$ satisfies the conditions of Lemma 2, invoke the functional delta method, Theorem 3.9.4 in van der Vaart and Wellner (1996), to conclude the proof. □

The limiting Gaussian process can be written as

$$\mathbb{G}_C(u, v) = \mathbb{B}_C(u, v) - \partial_1 C(u, v)\mathbb{B}_C(u, 1) - \partial_2 C(u, v)\mathbb{B}_C(1, v),$$

where $\mathbb{B}_C$ is a Brownian bridge on $[0, 1]^2$ with covariance function

$$\mathbb{E}[\mathbb{B}_C(u, v) \cdot \mathbb{B}_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$$

for each $0 \leqslant u, u', v, v' \leqslant 1$.

Regarding the assumption on C, we note that every copula C is Lipschitz and its partial derivatives exist for almost all points in $[0, 1]^2$ (see, for example, Nelsen 1999). Careful inspection of the proof of Theorem 3 reveals that we require smoothness of the partial derivatives only in order to apply the functional delta method (cf. Lemma 2). This observation suggests that one might be able to relax this assumption. That would be useful, since there are many statistically relevant cases where the desired copula function does not have continuous partial derivatives. For example, $C(s, t) = \max(0, s + t - 1)$ (X is symmetric and $Y = -X$) does not have continuous partial derivatives. Unfortunately, the following result, which in a sense is the converse of Theorem 3, indicates that there is actually very little we can do.

Theorem 4. *Let F and G be continuous distribution functions. Assume that the inverses F^{-1} and G^{-1} exist and that there exists at least one point $(s^*, t^*) \in (0, 1)^2$ for which the four quantities*

$$A_1 \equiv \lim_{h \nearrow 0} \frac{C(s^* + h, t^*) - C(s^*, t^*)}{h}, \qquad A_2 \equiv \lim_{h \searrow 0} \frac{C(s^* + h, t^*) - C(s^*, t^*)}{h},$$

$$\bar{A}_1 \equiv \lim_{h \nearrow 0} \frac{C(s^*, t^* + h) - C(s^*, t^*)}{h}, \qquad \bar{A}_2 \equiv \lim_{h \searrow 0} \frac{C(s^*, t^* + h) - C(s^*, t^*)}{h}$$

are not all equal. Then $\{\mathbb{Z}_n(x, y), 0 \leqslant x, y \leqslant 1\}$ *does not converge to a tight Gaussian process.*

Proof. We have

$$\begin{aligned}\mathbb{Z}_n(s, t) &= \sqrt{n}\left[\mathbb{H}_n(\mathbb{F}_n^-(s), \mathbb{G}_n^-(t)) - H(F^{-1}(s), G^{-1}(t))\right] \\ &= \sqrt{n}\left[(\mathbb{H}_n - H)(\mathbb{F}_n^-(s), \mathbb{G}_n^-(t)) - (\mathbb{H}_n - H)(F^{-1}(s), G^{-1}(t))\right] \\ &\quad + \sqrt{n}[H(\mathbb{F}_n^-(s), \mathbb{G}_n^-(t)) - H(F^{-1}(s), G^{-1}(t))] + \sqrt{n}(\mathbb{H}_n - H)(F^{-1}(s), G^{-1}(t)).\end{aligned} \tag{5}$$

The first term on the right of (5) is $o_p(1)$ since the process $\sqrt{n}(\mathbb{H}_n - H)$ is stochastically equicontinuous. The third term converges to a normal random variable for every fixed (s, t). Evaluated at the point (s^*, t^*), the second term is equal to

$$\begin{aligned}&\sqrt{n}[C(F \circ \mathbb{F}_n^-(s^*), G \circ \mathbb{G}_n^-(t^*)) - C(s^*, t^*)] = A_1\sqrt{n}(F \circ \mathbb{F}_n^-(s^*) - s^*)\mathbb{I}_{\{\mathbb{F}_n^-(s^*) < F^{-1}(s^*)} \\ &\quad + A_2\sqrt{n}(F \circ \mathbb{F}_n^-(s^*) - s^*)\mathbb{I}_{\{\mathbb{F}_n^-(s^*) > F^{-1}(s^*)\}} + \bar{A}_1\sqrt{n}(G \circ \mathbb{G}_n^-(t^*) - t^*)\mathbb{I}_{\{\mathbb{G}_n^-(t^*) < G^{-1}(t^*)\}} \\ &\quad + \bar{A}_2\sqrt{n}(G \circ \mathbb{G}_n^-(t^*) - t^*)\mathbb{I}_{\{\mathbb{G}_n^-(t^*) > G^{-1}(t^*)\}} + o_p(1).\end{aligned} \tag{6}$$

Thus, the asymptotic behaviour of (6) depends on the values of the left- and right-hand limits. If one of the four constants A_1, A_2, $\bar{A}_1$ or $\bar{A}_2$ differs from the others, then (6) does not converge for $n \to \infty$. Notice that the events in the indicator functions occur with a non-zero probability, and as a result the limiting process (if it exists) is not Gaussian. Moreover, a very similar argument reveals that

$$\begin{aligned}&\mathbb{Z}_n(s^* + \delta, t^*) - \mathbb{Z}_n(s^* - \delta, t^*) \\ &\quad = A_1\xi_1^\delta\mathbb{I}_{\left\{\mathbb{F}_n^-(s^*+\delta) < F^{-1}(s^*+\delta)\right\}} + A_2\xi_1^\delta\mathbb{I}_{\left\{\mathbb{F}_n^-(s^*+\delta) > F^{-1}(s^*+\delta)\right\}} \\ &\qquad - A_1\xi_2^\delta\mathbb{I}_{\left\{\mathbb{F}_n^-(s^*-\delta) < F^{-1}(s^*-\delta)\right\}} - A_2\xi_2^\delta\mathbb{I}_{\left\{\mathbb{F}_n^-(s^*-\delta) > F^{-1}(s^*-\delta)\right\}} + o_p(1),\end{aligned} \tag{7}$$

where

$$\xi_1^\delta = \lim_{n\to\infty} \sqrt{n}(\mathbb{F}_n(s^* + \delta) - F(s^* + \delta)), \qquad \xi_2^\delta = \lim_{n\to\infty} \sqrt{n}(\mathbb{F}_n(s^* - \delta) - F(s^* - \delta)).$$

Assume that $A_1 \neq A_2$. Then the right-hand side of (7) does not converge to 0 as $\delta \to 0$ and $n \to \infty$ because

$$\lim_{\delta \searrow 0} \liminf_{n\to\infty} \mathbb{P}\{\mathbb{F}_n^-(s^* + \delta) < F^{-1}(s^* + \delta); \mathbb{F}_n^-(s^* - \delta) > F^{-1}(s^* - \delta)\} > 0.$$

This implies that the process $\mathbb{Z}_n(s, t)$ is not stochastically equicontinuous, which establishes the claim. □

The covariance structure of $\mathbb{Z}_n$ might be complicated to estimate, and the bootstrap methodology provides an attractive alternative to estimate the finite-sample distribution of

$\mathbb{Z}_n$. We will show that the bootstrap 'works', but first we need some additional notation. Let $(X_{1,B}, Y_{1,B}), \ldots, (X_{n,B}, Y_{n,B})$ be the bootstrap sample obtained by sampling with replacement from the original observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. We write $\mathbb{H}_{n,B}$ for the empirical cdf based on the bootstrap sample, and denote its associated empirical copula function by $\mathbb{C}_{n,B}$.

Theorem 5. *Let F, G be continuous distribution functions. The conditional distribution of $\{\sqrt{n}(\mathbb{C}_{n,B} - \mathbb{C}_n)(x, y), 0 \leqslant x, y \leqslant 1\}$ converges to the same limiting Gaussian process as $\{\sqrt{n}(\mathbb{C}_n - C)(x, y), 0 \leqslant x, y \leqslant 1\}$ in $\ell^\infty([0, 1]^2)$ in probability.*

Proof. We can invoke the same uniform transformation trick as in Lemma 1. We already know that $\mathbb{C}_n(i_n, j_n) = \mathbb{C}_n^*(i_n, j_n)$, and it is easily verified that $\mathbb{C}_{n,B}(i_n, j_n) = \mathbb{C}_{n,B}^*(i_n, j_n)$ as well, where $\mathbb{C}_{n,B}^*$ is the empirical copula function based on $(F(X_{i,B}), G(Y_{i,B}))$. Hence

$$\sqrt{n}(\mathbb{C}_{n,B} - \mathbb{C}_n)(i_n, j_n) = \sqrt{n}(\mathbb{C}_{n,B}^* - \mathbb{C}_n^*)(i_n, j_n).$$

The conclusion follows by observing that the map $\phi: H^* \mapsto C^* = C_{H^*}$ is Hadamard differentiable (cf. Lemma 2), and hence $\sqrt{n}(\phi(\mathbb{H}_{n,B}) - \phi(\mathbb{H}_n))$ converges weakly if and only if $\sqrt{n}(\phi(\mathbb{H}_n) - \phi(H))$ is weakly convergent by Theorem 3.9.11 in van der Vaart and Wellner (1996, p. 378). □

We mention some consequences of the convergence results. Deheuvels (1981a) proposed among other related procedures the Kolmogorov–Smirnov type statistic

$$T \equiv \sup_{0 \leqslant s, t \leqslant 1} |\sqrt{n}(\mathbb{C}_n - C)(s, t)|$$

for testing the independence hypothesis $H_0: C(s, t) = s \cdot t$. He calculated the limiting distribution of T under this null hypothesis. The results established here are useful for computing the asymptotic power of this test under various alternatives.

In the early 1970s there was considerable interest in multivariate rank-order statistics; see, for example, Ruymgaart *et al.* (1972), Ruymgaart (1974) and Rüschendorf (1976). Such statistics are of the form

$$R_n = \frac{1}{n}\sum_{i=1}^{n} J(\mathbb{F}_n(X_i), \mathbb{G}_n(Y_i)),$$

and asymptotic normality of R_n has been established under regularity assumptions on $J: [0, 1]^2 \to \mathbb{R}$. However, by simply observing that

$$\frac{1}{n}\sum_{i=1}^{n} J(\mathbb{F}_n(X_i), \mathbb{G}_n(Y_i)) = \int_{[0,1]^2} J(u, v)\, \mathrm{d}\overline{\mathbb{C}}_n(u, v),$$

where $\overline{\mathbb{C}}_n$ is the cadlag version of the empirical copula function defined in the proof of Theorem 6 below, and

$$\mathbb{E}J(F(X), G(Y)) = \int_{[0,1]^2} J(u, v)\, \mathrm{d}C(u, v),$$

we can invoke the weak convergence of the empirical copula process to deduce asymptotic normality of R_n. To our knowledge this method establishes normality of R_n under the weakest set of assumptions in the literature.

Theorem 6. *Let H have continuous marginals and let C have continuous partial derivatives. Assume that J is of bounded variation, continuous from above and with discontinuities of the first kind (Neuhaus, 1971). Then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{J(\mathbb{F}_n(X_i), \mathbb{G}_n(Y_i)) - \mathbb{E}J(F(X_i), G(Y_i))\} \xrightarrow{D} \int_{[0,1]^2} \mathbb{G}_C(u, v)\,\mathrm{d}J(u, v).$$

In particular, the limiting distribution is Gaussian.

Proof. Let

$$\overline{\mathbb{C}}_n(u, v) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{\{\mathbb{F}_n(X_i)\leqslant u, \mathbb{G}_n(Y_i)\leqslant v\}}, \qquad u, v \in [0, 1].$$

It is easily seen that $\mathbb{C}_n$ and $\overline{\mathbb{C}}_n$ coincide on the grid $\{(i/n, j/n), 1 \leqslant i, j \leqslant n\}$. The subtle difference lies in the fact that $\mathbb{C}_n$ is left-continuous with right-hand limits, whereas $\overline{\mathbb{C}}_n$ on the other hand is right-continuous with left-hand limits. The difference between $\mathbb{C}_n$ and $\overline{\mathbb{C}}_n$, however, is small:

$$\sup_{0\leqslant u,v\leqslant 1} |\mathbb{C}_n(u, v) - \overline{\mathbb{C}}_n(u, v)| \leqslant \max_{1\leqslant i,j\leqslant n}\left|\mathbb{C}_n\left(\frac{i}{n}, \frac{j}{n}\right) - \mathbb{C}_n\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\right| \leqslant \frac{2}{n}.$$

As a consequence, Theorem 3 also implies that the related process $\overline{\mathbb{Z}}_n \equiv \sqrt{n}(\overline{\mathbb{C}}_n - C)$ converges weakly to $\mathbb{G}_C$, provided C has continuous partial derivatives. By the continuous mapping theorem and using integration by parts, we obtain

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{J(\mathbb{F}_n(X_i), \mathbb{G}_n(Y_i)) - \mathbb{E}J(F(X_i), G(Y_i))\} &= \sqrt{n}\int_{[0,1]^2} J(u, v)\,\mathrm{d}(\overline{\mathbb{C}}_n - C)(u, v) \\
&= \sqrt{n}\int_{[0,1]^2}\left[(\overline{\mathbb{C}}_n - C)(u, v) - (\overline{\mathbb{C}}_n - C)(u, 1) - (\overline{\mathbb{C}}_n - C)(1, v)\right]\mathrm{d}J(u, v) \\
&\quad - \int_{[0,1]}\sqrt{n}(\overline{\mathbb{C}}_n(u, 1) - u)\,\mathrm{d}J(u, 0) - \int_{[0,1]}\sqrt{n}(\overline{\mathbb{C}}_n(1, v) - v)\,\mathrm{d}J(0, v) \\
&= \int_{[0,1]^2}\sqrt{n}(\overline{\mathbb{C}}_n - C)(u, v)\,\mathrm{d}J(u, v) + O(n^{-1/2}) \xrightarrow{D} \int_{[0,1]^2}\mathbb{G}_C(u, v)\,\mathrm{d}J(u, v).
\end{aligned}$$

The integration by parts formula is specified in Proposition 3.2.1 of Fermanian (1996). Since a continuous, linear transformation of a tight Gaussian process is normally distributed, the limit has a Gaussian distribution. □

3. Weak convergence of smoothed empirical copula processes

The smoothed empirical distribution function $\hat{\mathbb{H}}_n(x, y)$ is defined by

$$\hat{\mathbb{H}}_n(x, y) = \frac{1}{n}\sum_{i=1}^{n} K_n(x - X_i, y - Y_i).$$

Here $K_n(x, y) = K(a_n^{-1}x, a_n^{-1}y)$, and

$$K(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} k(u, v)\,\mathrm{d}u\,\mathrm{d}v,$$

for some bivariate kernel function $k: \mathbb{R}^2 \mapsto \mathbb{R}$, with $\int k(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1$, and a sequence of bandwidths $a_n \downarrow 0$ as $n \to \infty$. For notational convenience, we have chosen the same bandwidth sequence for each margin. This assumption can easily be dropped. For small enough bandwidths a_n, the empirical cdf $\mathbb{H}_n$ and the smoothed empirical cdf $\hat{\mathbb{H}}_n$ are almost indistinguishable:

Lemma 7. *Assume that F and G are Lipschitz, $a_n \to 0$,*

$$\int_{\mathbb{R}^2}(|x| + |y|)\,\mathrm{d}K(x, y) < \infty \quad \textit{and} \quad \sup_{x,y}\sqrt{n}|\mathbb{E}\hat{\mathbb{H}}_n(x, y) - H(x, y)| \to 0.$$

Then

$$\sqrt{n}\sup_{x,y}|\hat{\mathbb{H}}_n(x, y) - \mathbb{H}_n(x, y)| \xrightarrow{P} 0,$$

and, in particular, the smoothed empirical process $\{\sqrt{n}(\hat{\mathbb{H}}_n - H)(x, y), x, y \in \mathbb{R}\}$ converges weakly to a tight Brownian bridge in $D(\mathbb{R}^2)$.

Proof. According to van der Vaart (1994), we only have to check that

$$\sup_{s,t}\int\left[\int\left(\mathbb{I}_{\{x+\varepsilon\leqslant s, y+\eta\leqslant t\}} - \mathbb{I}_{\{x\leqslant s, y\leqslant t\}}\right)\mathrm{d}K_n(\varepsilon, \eta)\right]^2 \mathrm{d}H(x, y) \to 0. \tag{8}$$

After applications of Jensen's inequality and Fubini's theorem, we can bound the term on the left in (8) by

$$\sup_{s,t}\int([F(s) - F(s - a_n x)] + [G(t) - G(t - a_n y)])\,\mathrm{d}K(x, y)$$

which tends to zero as $a_n \to 0$ by our assumptions on F, G and K. □

The assumption on the bias term in the statement of Lemma 7 can be handled by means of some smoothness assumptions on H and regularity of K and a_n:

Lemma 8. *Assume that H has a bounded pth derivative,* $\lim_{n\to\infty}\sqrt{n}a_n^p = 0$,

$$\int_{\mathbb{R}^2} x^k y^l k(x, y) \mathrm{d}x \, \mathrm{d}y = 0, \qquad 1 \leqslant k + l < p,$$

and $\int |x|^k |y|^l |k(x, y)| \, \mathrm{d}x \, \mathrm{d}y < \infty$, $1 \leqslant k + l \leqslant p$. *Then we have*

$$\sup_{x,y} \sqrt{n} |\mathbb{E}\hat{\mathbb{H}}_n(x, y) - H(x, y)| = \sqrt{n} a_n^p.$$

Proof. The result follows readily after a simple Taylor expansion. □

We study the weak convergence of the smoothed empirical copula process

$$\hat{\mathbb{Z}}_n(x, y) = \sqrt{n}(\hat{\mathbb{C}}_n - C)(x, y), \qquad 0 \leqslant x, y \leqslant 1,$$

based on the smoothed empirical copula function

$$\hat{\mathbb{C}}_n(x, y) = \hat{\mathbb{H}}_n(\hat{\mathbb{F}}_n^-(x), \hat{\mathbb{G}}_n^-(y)).$$

The following lemma establishes asymptotic tightness of the smoothed empirical process. The proof is given at the end of the section.

Lemma 9. *Let* $C(x, y)$ *have continuous partial derivatives and assume that the assumptions of Lemma 7 hold. Then the process* $\{\hat{\mathbb{Z}}_n(x, y) : (x, y) \in [0, 1]^2\}$ *is stochastically equicontinuous, that is, for all* $\eta > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}\left\{ \sup_{|x - x'| \leqslant \delta, |y - y'| \leqslant \delta} |\hat{\mathbb{Z}}_n(x, y) - \hat{\mathbb{Z}}_n(x', y')| > \eta \right\} = 0.$$

We have obtained the following result:

Theorem 10. *Under the assumptions of Lemma 7 and provided C has continuous partial derivatives, the smoothed empirical copula process* $\{\hat{\mathbb{Z}}_n(u, v), 0 \leqslant u, v \leqslant 1\}$ *converges weakly to the Gaussian process* $\{\mathbb{G}_C(u, v), 0 \leqslant u, v \leqslant 1\}$ *in* $\ell^\infty([0, 1]^2)$.

Proof. In view of Lemma 9, we only have to show the finite-dimensional convergence of the process $\{\hat{\mathbb{Z}}_n(u, v), 0 \leqslant u, v \leqslant 1\}$. Take $x, y \in \mathbb{R}$ arbitrarily, and set $u = F(x)$, $v = G(y)$ and $\hat{u} = \hat{\mathbb{F}}_n(x)$ and $\hat{v} = \hat{\mathbb{G}}_n(y)$. Note that $\hat{u} \xrightarrow{P} u$ and $\hat{v} \xrightarrow{P} v$ by Lemma 7, and argue that, since $\hat{\mathbb{Z}}_n$ is stochastically equicontinuous,

$$\hat{\mathbb{Z}}_n(u, v) = \hat{\mathbb{Z}}_n(\hat{u}, \hat{v}) + o_p(1)$$

$$= \sqrt{n}(\hat{\mathbb{H}}_n - H)(x, y) + \sqrt{n}[C(u, v) - C(\hat{u}, \hat{v})] + o_p(1)$$

$$= \sqrt{n}(\hat{\mathbb{H}}_n - H)(x, y) + \sqrt{n}\Big[(F - \hat{\mathbb{F}}_n)(x)\partial_1 C(u, v) + (G - \hat{\mathbb{G}}_n)(y)\partial_2 C(u, v)\Big] + o_p(1) \tag{9}$$

$$= \sqrt{n}(\mathbb{H}_n - H)(x, y) + \sqrt{n}[(F - \mathbb{F}_n)(x)\partial_1 C(u, v) + (G - \mathbb{G}_n)(y)\partial_2 C(u, v)] + o_p(1) \tag{10}$$

$$= \mathbb{Z}_n(u, v) + o_p(1);$$

(9) holds since C has continuous partial derivatives, and (10) by Lemma 7. The required finite-dimensional convergence of the process follows from Theorem 3. □

Note that we could not prove the last result in the same way as for the empirical copula process $\mathbb{Z}_n$. Indeed, the transformation of Lemma 1 no longer works for smoothed empirical cdfs. In contrast, we can repeat the same arguments leading to Theorem 10 to prove Theorem 3.

Proof of Lemma 9. Observe that

$$\mathbb{P}\left\{\sup_{|u-u'|\leqslant\delta, |v-v'|\leqslant\delta} |\hat{\mathbb{Z}}_n(u, v) - \hat{\mathbb{Z}}_n(u', v')| > \eta\right\} \leqslant I + II,$$

with

$$I = \mathbb{P}\left\{\sup_{|u-u'|\leqslant\delta, |v-v'|\leqslant\delta}\left|\sqrt{n}\Big(\hat{\mathbb{C}}_n(\hat{\mathbb{F}}_n(\hat{\mathbb{F}}_n^- u), \hat{\mathbb{G}}_n(\hat{\mathbb{G}}_n^- v)) - C(F(\hat{\mathbb{F}}_n^- u), G(\hat{\mathbb{G}}_n^- v))\Big)\right.\right.$$
$$\left.\left. -\sqrt{n}\Big(\hat{\mathbb{C}}_n(\hat{\mathbb{F}}_n(\hat{\mathbb{F}}_n^- u'), \hat{\mathbb{G}}_n(\hat{\mathbb{G}}_n^- v')) - C(F(\hat{\mathbb{F}}_n^- u'), G(\hat{\mathbb{G}}_n^- v'))\Big)\right| > \frac{\eta}{2}\right\}$$

$$II = \mathbb{P}\left\{\sup_{|u-u'|\leqslant\delta, |v-v'|\leqslant\delta}\left|\sqrt{n}\Big(C(\hat{\mathbb{F}}_n(\hat{\mathbb{F}}_n^- u), \hat{\mathbb{G}}_n(\hat{\mathbb{G}}_n^- v)) - C(F(\hat{\mathbb{F}}_n^- u), G(\hat{\mathbb{G}}_n^- u))\Big)\right.\right.$$
$$\left.\left. - \sqrt{n}\Big(C(\hat{\mathbb{F}}_n(\hat{\mathbb{F}}_n^- u'), \hat{\mathbb{G}}_n(\hat{\mathbb{G}}_n^- v')) - C(F(\hat{\mathbb{F}}_n^- u'), G(\hat{\mathbb{G}}_n^- v'))\Big)\right| > \frac{\eta}{2}\right\}.$$

We will deal with the two terms I and II separately. The first term can be handled by noticing that, by Lemma 7,

$$\hat{\mathbb{C}}_n(x,\, y) = \mathbb{H}_n(\hat{\mathbb{F}}_n^-(x),\, \hat{\mathbb{G}}_n^-(y)) + o_p(n^{-1/2}) = \mathbb{H}_n^*\Big(F(\hat{\mathbb{F}}_n^-(x)),\, G(\hat{\mathbb{G}}_n^-(y))\Big) + o_p(n^{-1/2}),$$

where $\mathbb{H}_n^*(u, v)$ is the empirical cdf based on $(F(X_1), G(Y_1)), \ldots, (F(X_n), G(Y_n))$. Therefore, denoting $\sqrt{n}\big(\mathbb{H}_n^* - H^*\big)$ by $\mathbb{W}_n^*$, the first probability can be bounded by

$$I \leqslant \mathbb{P}\left\{ 2\sup_{\substack{(x,x'):|\hat{\mathbb{F}}_n(x)-\hat{\mathbb{F}}_n(x')|\leqslant\delta,\\ (y,y'):|\hat{\mathbb{G}}_n(y)-\hat{\mathbb{G}}_n(y')|\leqslant\delta}} |\mathbb{W}_n^*(F(x),\, G(y)). \right.$$

$$\left. -\sqrt{n}\big(\mathbb{H}_n^* - H^*\big)(F(x'),\, G(y'))| > \frac{\eta}{4} \right\} + o(1)$$

$$\leqslant \mathbb{P}\left\{ \sup_{\substack{(u,u'):|u-u'|\leqslant 3\delta,\\ (v,v'):|v-v'|\leqslant 3\delta}} |\mathbb{W}_n^*(u,\, v) - \mathbb{W}_n^*(u',\, v')| > \frac{\eta}{4} \right\}$$

$$+ \mathbb{P}\left\{ \sup_x |\hat{\mathbb{F}}_n(x) - F(x)| > \delta \right\} + \mathbb{P}\left\{ \sup_y |\hat{\mathbb{G}}_n(y) - G(y)| > \delta \right\} + o(1),$$

which tends to 0 as $n \to \infty$ and $\delta \downarrow 0$ from the weak convergence of the process $\mathbb{W}_n^*$.

The second term II can be made arbitrarily small by invoking the fact that C has continuous partial derivatives so that

$$\Big[C(u,\, v) - C\Big(F\hat{\mathbb{F}}_n^- u,\, G\hat{\mathbb{G}}_n^- v\Big)\Big] - \Big[C(u',\, v') - C\Big(F\hat{\mathbb{F}}_n^- u',\, G\hat{\mathbb{G}}_n^- v'\Big)\Big]$$

$$= -C'(u,\, v)\,.\, \Big((\hat{\mathbb{F}}_n - F)\hat{\mathbb{F}}_n^- u' - (\hat{\mathbb{F}}_n - F)\hat{\mathbb{F}}_n^- u,\, (\hat{\mathbb{G}}_n - G)\hat{\mathbb{G}}_n^- v' - (\hat{\mathbb{G}}_n - G)\hat{\mathbb{G}}_n^- v\Big)$$

$$+ o(\|\hat{\mathbb{F}}_n - F\|_\infty + \|\hat{\mathbb{G}}_n - G\|_\infty)$$

for $u \to u'$, $v \to v'$. Next, observe that

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{|u-u'|\leqslant\delta}\sqrt{n}|(\hat{\mathbb{F}}_n - F)\hat{\mathbb{F}}_n^- u' - (\hat{\mathbb{F}}_n - F)\hat{\mathbb{F}}_n^- u| > \eta\right\}\\
&= \mathbb{P}\left\{\sup_{x,x':|\hat{\mathbb{F}}_n x - \hat{\mathbb{F}}_n x'|\leqslant\delta}\sqrt{n}|(\hat{\mathbb{F}}_n - F)x' - (\hat{\mathbb{F}}_n - F)x| > \eta\right\}\\
&\leqslant \mathbb{P}\left\{\sup_{x,x':|Fx - Fx'|\leqslant 3\delta}\sqrt{n}|(\mathbb{F}_n - F)x' - (\mathbb{F}_n - F)x| > \frac{\eta}{2}\right\}\\
&\quad + \mathbb{P}\left\{\|F - \hat{\mathbb{F}}_n\|_\infty > \delta\right\} + \mathbb{P}\left\{\sqrt{n}\|\hat{\mathbb{F}}_n - \mathbb{F}_n\|_\infty > \frac{\eta}{4}\right\}\\
&= \mathbb{P}\left\{\sup_{|u-u'|\leqslant 3\delta}\sqrt{n}|(\mathbb{F}_n^* - F^*)u' - (\mathbb{F}_n^* - F^*)u| > \frac{\eta}{2}\right\}\\
&\quad + \mathbb{P}\left\{\|F - \hat{\mathbb{F}}_n\|_\infty > \delta\right\} + \mathbb{P}\left\{\sqrt{n}\|\hat{\mathbb{F}}_n - \mathbb{F}_n\|_\infty > \frac{\eta}{4}\right\}\\
&\to 0, \qquad n \to \infty, \delta \downarrow 0,
\end{aligned}
$$

by the weak convergence of the uniform empirical process and Lemma 7. Similarly, we can show that

$$
\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left\{\sup_{|v-v'|\leqslant\delta}\left|(\hat{\mathbb{G}}_n - G)\hat{\mathbb{G}}_n^- v' - (\hat{\mathbb{G}}_n - G)\hat{\mathbb{G}}_n^- v\right| > \eta\right\} \to 0.
$$

Hence, *II* is asymptotically negligible as well, and the proof is complete. □

Acknowledgements

The authors thank David Mason for providing the relevant Gaenssler and Stute (1987) reference.

References

Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G. and Roncalli, T. (2000) *Copulas for Finance. A Reading Guide and Some Applications.* Paris: Groupe de Recherche Opérationnelle, Crédit Lyonnais.

Capéraà, P., Fougères, A.-L. and Genest, C. (1997) A nonparametric estimation procedure for bivariate extreme value copulas. *Biometrika*, **84**, 567–577.

Deheuvels, P. (1979) La fonction de dépendance empirique et ses propriétés. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, **65**, 274–292.

Deheuvels, P. (1981a) A Kolmogorov–Smirnov type test for independence and multivariate samples. *Rev. Roumaine Math. Pures Appl.*, **26**(2) 213–226.

Deheuvels, P. (1981b) A nonparametric test for independence. *Publ. Inst. Statist. Univ. Paris.*, **26**(2), 29–50.

Embrechts, P., McNeil, A. and Straumann, D. (2002) Correlation and dependence in risk management: properties and pitfalls. In M.A.H. Dempster (ed.), *Risk Management: Value at Risk and Beyond*, pp. 176–223. Cambridge: Cambridge University Press.

Fermanian, J-D. (1996) Multivariate hazard rates under random censorship. CREST Working Paper 9603.

Gaenssler, P. and Stute, W. (1987) *Seminar on Empirical Processes*, DMV Sem. 9. Basel: Birkhäuser.

Genest, C. and MacKay, R.J. (1986a) Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. *Rev. Canad. Statist.*, **14**, 145–159.

Genest, C. and MacKay, R.J. (1986b) The joy of copulas: Bivariate distributions with uniform marginals. *Amer. Statist.*, **40**, 280–283.

Genest, C. and Rivest, L.-P. (1993) Statistical inference procedures for bivariate Archimedean copulas. *J. Amer. Statist. Assoc.*, **88**, 1034–1043.

Genest, C., Ghoudi, K. and Rivest, L.-P. (1995) A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, **82**, 543–552.

Joe, H. (1997) *Multivariate Models and Dependence Concepts*. London: Chapman & Hall.

Nelsen, R.B. (1999) *An Introduction to Copulas*, Lecture Notes in Statist. 139. New York: Springer-Verlag.

Neuhaus, G. (1971) On the weak convergence of stochastic processes with multidimensional time parameter. *Ann. Math. Statist.*, **42**, 1285–1295.

Rüschendorf, L. (1976) Asymptotic distributions of multivariate rank order statistics. *Ann. Statist.*, **4**, 912–923.

Ruymgaart, F.H. (1973) *Asymptotic Theory for Rank Tests for Independence*, MC Tract 43. Amsterdam: Mathematisch Instituut.

Ruymgaart, F.H. (1974) Asymptotic normality of nonparametric tests for independence. *Ann. Statist.*, **2**, 892–910.

Ruymgaart, F.H., Shorack, G.R. and van Zwet, W.R. (1972) Asymptotic normality of nonparametric tests for independence. *Ann. Math. Statist.*, **43**, 1122–1135.

Schönbucher, P. and Schubert, D. (2000) Copula dependent default risk in intensity models. Preprint, Department of Statistics, Bonn University.

Schweizer, B. (1991) Thirty years of copulas. In G. Dall'Aglio, S. Kotz and G. Salinetti (eds), *Advances in Probability Distributions with Given Marginals*. Dordrecht: Kluwer.

Schweizer, B. and Sklar, A. (1974) Operations on distribution functions not derivable from operations on random variables. *Studia Math.*, **52**, 43–52.

Sklar, A. (1959) Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, **8**, 229–231.

van der Vaart, A.W. (1994) Weak convergence of smoothed empirical processes. *Scand. J. Statist.*, **21**, 501–504.

van der Vaart, A.W. and Wellner, J.A. (1996) *Weak Convergence and Empirical Processes*. New York: Springer-Verlag.

Received April 2002 and revised March 2004