WEAK CONVERGENCE OF HIGH-SPEED NETWORK TRAFFIC MODELS

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ABSTRACT. We consider a network traffic model consisting of an infinite number of sources linked to a server. Sources initiate transmissions to the server at Poisson time points. The duration of each transmission has a heavy tailed distribution. We show that suitable scalings of the traffic process converge to a totally skewed stable Lévy motion in Skorohod space, equipped with the Skorohod M_1 topology. This allows us to prove a heavy traffic theorem for a single server fluid model.

1. INTRODUCTION

In modern telecommunications traffic data, features such as long range dependence, self-similarity, and heavy tails are prominent. Furthermore, the characteristics of the data vary greatly over measurements at different sites and different points in time. This is especially true for internet data, as described in e.g. [32]. Black box time series modeling, which has been succesful in finite variance (low variability) settings, is inadequate for modeling teletraffic datasets. See e.g. [25, 20, 24, 23, 9, 21, 6, 7] and the discussions therein. Hence, we should attempt more structural modeling. The size of recent data sets of traffic network measurements, the dynamic nature of communication networks, and the complexity of the traffic they carry argue strongly in favor of parsimonious and structural network traffic models.

As a more structural traffic model, a superposition of a large number of ON/OFF type sources whose activity periods are heavy tailed, has received considerable attention. Such models are approximated by fluid models with $M/G/\infty$ inputs, sometimes referred to as infinite source Poisson models. (Cf. [13, 14, 16, 15, 18, 33, 34, 35, 28, 19, 5].) The $M/G/\infty$ input model described in [17] is of this type. Using a distributional limit theorem, Konstantopoulos and Lin ([17]) explain the suitability of an totally skewed stable Lévy motion as a macroscopic traffic model for a high-speed network switch. The limit process is self-similar, but somewhat surprisingly has independent increments. The marginal distribution of the Lévy stable motion matches those sometimes empirically obtained in practice ([11]). Also, results on the queue length process of queueing systems with Lévy inputs are reported in [17].

However, in [17] the convergence of the scaled traffic model to the limit Lévy motion is only shown to occur in the sense of finite dimensional distributions. To validate application of the limit model in telecom applications, it is important to establish convergence in a stronger sense of functional weak convergence in a suitable topology. Typically, when the limit process has almost surely continuous paths, one uses the Skorohod J_1 topology, or even the uniform topology. Konstantopoulos and Lin discuss why scaled infinite source Poisson models cannot converge in $(D[0, \infty), J_1)$ to a Lévy limit. A comprehensive discussion of these issues is in the recently released reports of Whitt [30, 31, 29].

The main result of the present paper shows that indeed, the scaled and centered cumulative traffic process converges to a skewed stable Lévy motion in $D[0, \infty)$ equipped with the Skorohod

M_1 topology. This result is then applied to obtain a heavy traffic limit theorem for a single server fluid queue.

The paper is organized as follows. In Section 2, the network traffic model considered is described in detail, and basic definitions and notation are given. Our assumptions, weaker than those used in [17], are explained. In Section 3, we give an alternative proof of finite dimensional convergence based on point process methods. This paves the way for proving weak M_1 convergence in $D[0, \infty)$, which is done in Section 4, using ideas from [2, 3]. Finally, in Section 5, the convergence results are applied to obtain a heavy traffic theorem for a single server fluid model with the given input traffic.

2. Infinite source Poisson model

We consider the M/G/∞ input model of incoming traffic to a communication network. Let $\{\Gamma_k, k \geq 1\}$ denote the points of a homogeneous Poisson process on $[0, \infty)$ with rate λ. Suppose at time Γ_k, a source starts a transmission, and continues to transmit for a period of length L_k, at possibly time-varying rate. Let $\zeta(t), t \in [0, \infty)$ be a monotone increasing, right continuous function describing the transmission schedule; the total amount of data transmitted by source k in time $t - \Gamma_k$ is given by $\zeta((t - \Gamma_k) \wedge L_k)$. Assume for convenience that $\zeta(s) = 0$ for $s < 0$. The total volume of traffic injected into the network between 0 and t is

$$(2.1) \qquad A(t) = \sum_{k=1}^{\infty} \zeta((t - \Gamma_k) \wedge L_k), \ t \geq 0.$$

We make the following assumptions on the distribution of L_k, and the schedule function ζ: Suppose

$$(2.2) \qquad P[L_k > x] = 1 - F(x) = x^{-\alpha} L(x), \ 1 < \alpha < 2, \ x > 0$$

for some slowly varying function $L(x)$, i.e. $L(tx)/L(t) \to 1$ as $t \to \infty$. Further, assume ζ is regularly varying with index β, that is:

$$(2.3) \qquad \lim_{t \to \infty} \frac{\zeta(xt)}{\zeta(t)} = x^{\beta}, \ \frac{\alpha}{2} < \beta < \alpha.$$

These assumptions subsume the case where sources transmit at unit rate; that is, $\zeta(t) = t$ and $\beta = 1$. Define

$$(2.4) \qquad b(t) := \inf\{x : P[\zeta(L_1) > x] \leq t^{-1}\}.$$

It is easy to see, that $\zeta(L_k)$ has regularly varying tail probabilities with index $-\alpha/\beta$ since $P[\zeta(L_1) > x] \sim 1 - F(\zeta^{\leftarrow}(x))$ is a composition of two regularly varying functions and is hence regularly varying ([8, 12, 4, 26]). It follows that $b(t)$ is regularly varying with index β/α.

Since $\alpha > 1$ and $\alpha/\beta > 1$,

$$(2.5) \qquad \mu := E(L_1) = \int_0^{\infty} x F(dx) < \infty, \quad \mu_\zeta := E(\zeta(L_1)) = \int_0^{\infty} \zeta(x) F(dx) < \infty.$$

3. Finite dimensional convergence: a point process approach

We now give a version of [17, Theorem 4] without some extraneous assumptions assumed in [17]. Our proof of this result, based on a point process approach, will help to show convergence in $D[0, \infty)$ endowed with Skorohod's M_1 topology.

Theorem 3.1. *Suppose $A(t)$ is defined by (2.1), $b(t)$ by (2.4) and μ_ζ by (2.5). Assume (2.2) and (2.3) hold. Then*

$$\frac{A(Tt) - \lambda Tt\mu_\zeta}{b(T)} \xrightarrow{f.d.d.} X_{\alpha/\beta}(t) \ as \ T \to \infty,$$

where $X_{\alpha/\beta}(t)$ is α/β stable Lévy motion with mean 0, skewness parameter 1, and scaling parameter $(\lambda C_{\alpha/\beta}^{-1})^{\beta/\alpha}$, where $C_{\alpha/\beta}$ is defined in (3.18). Here $\xrightarrow{f.d.d.}$ denotes convergence of the finite-dimensional distributions.

Proof. As a first step, we will show:

$$(3.1) \qquad \frac{A(T) - \lambda T\mu_\zeta}{b(T)} \Rightarrow X_{\alpha/\beta}(1).$$

The point process

$$(3.2) \qquad M = \sum_{k=1}^{\infty} \epsilon_{(\Gamma_k, L_k)} = PRM(\lambda dt \times F(dx)),$$

is a Poisson process on $[0, \infty) \times [0, \infty]$ with mean measure $\lambda dt \times F(dx)$ ([26]). Consider the cumulative input in $[0, T]$, denoted by $A(T)$. We can divide this input in two parts: the traffic from sources that have started and stopped transmitting before time T, and the traffic from sources that have started transmitting before T, and are still transmitting at T. Define (see Figure 1)

$$R_1^{(T)} = \{(x, y) \in [0, \infty) \times [0, \infty] : x + y \leq T\}, \quad R_2^{(T)} = \{(x, y) \in [0, \infty) \times [0, \infty] : x \leq T \leq x + y\}.$$

Then

$$A(T) = \sum_{k=1}^{\infty} \zeta(L_k) 1_{[(\Gamma_k, L_k) \in R_1^{(T)}]} + \sum_{k=1}^{\infty} \zeta(T - \Gamma_k) 1_{[(\Gamma_k, L_k) \in R_2^{(T)}]}$$

$$(3.3) \qquad = A_1(T) + A_2(T).$$

Since $A_1(T)$ and $A_2(T)$ are functions of M over the disjoint regions $R_1^{(T)}$ and $R_2^{(T)}$, $A_1(T)$ and $A_2(T)$ are independent.
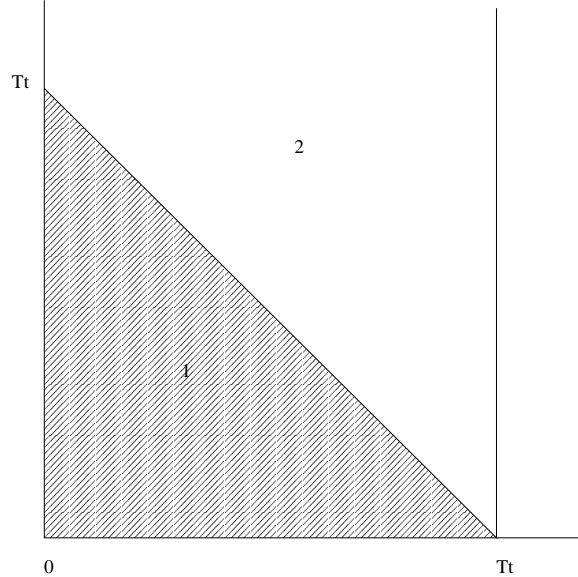
Observe that,

$$(3.4) \qquad E(M(R_1^{(T)})) = \int_{s=0}^{T} \lambda ds \int_{y=0}^{T-s} F(dy) = \lambda \hat{F}(T) \to \infty,$$

where $\hat{F}(T) := \int_0^T F(s)ds \sim TF(T) \sim T$, as $T \to \infty$, and

$$(3.5) \qquad E(M(R_2^{(T)})) = \int_0^T \lambda ds \int_{y=T-s}^{\infty} F(dy) = \lambda \int_0^T \bar{F}(T - s)ds = \lambda m(T) \xrightarrow{T \to \infty} \lambda \mu,$$

with $m(T) := \int_0^T \bar{F}(s)ds$. Also M restricted to $R_1^{(T)}$ is almost surely finite and hence has representation

$$(3.6) \qquad M|_{R_1^{(T)}} \stackrel{d}{=} \sum_{k=1}^{P_1(T)} \epsilon_{(\tau_k^{(T)}, J_k^{(T)})}$$

FIGURE 1. Traffic up to time T divided in two parts

where $P_1(T)$ is Poisson with parameter $\lambda \hat{F}(T)$, independent of the i.i.d. random pairs $\{(\tau_k^{(T)}, J_k^{(T)})\}$. The joint distribution of $(\tau_k^{(T)}, J_k^{(T)})$ is the normalized mean measure

$$\frac{\lambda \, ds \, F(dy)}{\lambda \hat{F}(T)}\Big|_{R_1^{(T)}} = \frac{ds \, F(dy)}{\hat{F}(T)}$$

and therefore the distribution of $J_k^{(T)}$ satisfies for $x < T$

$$(3.7) \qquad P[J_k^{(T)} \le x] = \frac{\int_0^{T-x} ds \, F(x) + \int_{T-x}^{T} ds \, F(T-s)}{\hat{F}(T)} = \frac{(T-x)F(x) + \hat{F}(x)}{\hat{F}(T)}$$

and for $x < T$

$$(3.8) \qquad P[J_k^{(T)} > x] = \frac{(T-x)\overline{F}(x) - \int_x^T \overline{F}(s) \, ds}{\hat{F}(T)}.$$

We can write

$$(3.9) \qquad A_1(T) \stackrel{d}{=} \sum_{k=1}^{P_1(T)} \zeta(J_k^{(T)}).$$

Next, we will analyze the tail behavior of $\zeta(J_k^{(T)})$ as $T \to \infty$. Note that $0 \le J_k^{(T)} \le T$, and hence $\zeta(0) \le \zeta(J_k^{(T)}) \le \zeta(T)$. We have

$$P(\zeta(J_k^{(T)}) > b(T)x) = P(J_k^{(T)} > \zeta^{\leftarrow}(b(T)x))$$

$$(3.10) \qquad = \frac{\{(T - \zeta^{\leftarrow}(b(T)x))\overline{F}(\zeta^{\leftarrow}(b(T)x)) - \int_{\zeta^{\leftarrow}(b(T)x)}^{T} \overline{F}(s) \, ds\}}{\hat{F}(T)}.$$

Now $\hat{F}(T) \sim T$ as $T \to \infty$. Since $\zeta^{\leftarrow}$ is regularly varying with index $1/\beta$, and $b(t)$ is regularly varying with index β/α, we have

$$\zeta^{\leftarrow}(b(T)x) \sim x^{1/\beta}\zeta^{\leftarrow}(b(T)) \sim x^{1/\beta}T^{1/\alpha}L^*(T)$$

for a slowly varying function L^*, and recalling $1 < \alpha < 2$ we see that

$$(3.11) \qquad \frac{T}{\zeta^{\leftarrow}(b(T)x)} \to \infty, \quad T \to \infty.$$

Putting things together, we get

$$TP(\zeta(J_k^{(T)}) > b(T)x) \sim (1 - \frac{\zeta^{\leftarrow}(b(T)x)}{T})T\bar{F}(\zeta^{\leftarrow}(b(T)x)) - \int_{\zeta^{\leftarrow}(b(T)x)}^{T} \bar{F}(s)ds$$

$$\sim (1 + o(1))x^{-\alpha/\beta} - \frac{\zeta^{\leftarrow}(b(T)x)}{T}\int_{1}^{T/\zeta^{\leftarrow}(b(T)x)} T\bar{F}(\zeta^{\leftarrow}(b(T)x)s)ds$$

$$(3.12) \qquad \to x^{-\alpha/\beta} - 0\int_{1}^{\infty} x^{-\alpha/\beta}s^{-\alpha}ds = x^{-\alpha/\beta}.$$

This result is equivalent to (see [26])

$$(3.13) \qquad TP(\frac{\zeta(J_k^{(T)})}{b(T)} \in \cdot) \xrightarrow{v} \eta(\cdot)$$

where η is the Lévy measure given by $\eta(dx) = \alpha/\beta x^{-\alpha/\beta-1}1_{(0,\infty]}(x)$. Using Karamata's theorem, we can verify that

$$(3.14) \qquad \limsup_{T\to\infty} T\mathrm{Var}(\frac{\zeta(J_1^{(T)})}{b(T)}1_{(|\frac{\zeta(J_1^{(T)})}{b(T)}|\leq M)}) < \infty$$

for any $M > 0$,

$$(3.15) \qquad \lim_{\epsilon\downarrow 0}\limsup_{T\to\infty} T\mathrm{Var}(\frac{\zeta(J_1^{(T)})}{b(T)}1_{(|\frac{\zeta(J_1^{(T)})}{b(T)}|\leq\epsilon)}) < \infty$$

and

$$(3.16) \qquad \lim_{M\to\infty}\limsup_{T\to\infty} TE(|\frac{\zeta(J_1^{(T)})}{b(T)}|1_{(|\frac{\zeta(J_1^{(T)})}{b(T)}|>M)}) = 0$$

Therefore, as in Section 2 of [22], it follows that in $(D[0,\infty), J_1)$,

$$(3.17) \qquad S_T(t) := (\sum_{k=1}^{[Tt]}\frac{\zeta(J_k^{(T)})}{b(T)} - [Tt]E(\frac{\zeta(J_k^{(T)})}{b(T)}1_{(\zeta(J_k^{(T)})\leq b(T))}) \Rightarrow S_{\alpha/\beta}(t)$$

where $S_{\alpha/\beta}(t)$ is skewed α/β stable Lévy motion. $S_{\alpha/\beta}(1)$ is a stable random variable with skewness parameter 1 and scaling parameter $C_{\alpha/\beta}^{-\beta/\alpha}$, where

$$(3.18) \qquad C_{\alpha/\beta} = \frac{1 - \frac{\alpha}{\beta}}{\Gamma(2 - \frac{\alpha}{\beta})\cos(\frac{\pi\alpha}{2\beta})}.$$

Furthermore, from the central limit theorem for Poisson random variables,

$$(3.19) \qquad B_T := \frac{(P_1(T) - \lambda \hat{F}(T))}{\sqrt{\lambda \hat{F}(T)}} \Rightarrow B(1)$$

where $B(1)$ has a $\mathcal{N}(0,1)$ distribution, and so $P_1(T)/T \overset{P}{\to} \lambda$. Consequently from (3.17) and (3.19)

$$\frac{A_1(T)}{b(T)} \overset{d}{=} \sum_{k=1}^{TP_1(T)/T} \frac{\zeta(J_k^{(T)})}{b(T)} = S_T\left(\frac{P_1(T)}{T}\right) + P_1(T)E\left(\frac{\zeta(J_k^{(T)})}{b(T)} 1_{(\zeta(J_k^{(T)}) \leq b(T))}\right)$$

$$= (S_{\alpha/\beta}(\lambda) + o_p(1)) + \frac{\sqrt{\lambda \hat{F}(T)} B_T + \lambda \hat{F}(T)}{b(T)} E(\zeta(J_k^{(T)}) 1_{(\zeta(J_k^{(T)}) \leq b(T))}),$$

so

$$(3.20) \qquad \frac{A_1(T)}{b(T)} - \frac{\lambda \hat{F}(T)}{b(T)} E(\zeta(J_k^{(T)}) 1_{(\zeta(J_k^{(T)}) \leq b(T))}) = S_{\alpha/\beta}(\lambda) + \frac{\sqrt{\lambda \hat{F}(T)}}{b(T)} B_T + o_p(1) \Rightarrow S_{\alpha/\beta}(\lambda)$$

as $T \to \infty$. Note that $b(T) \sim T^{\beta/\alpha} L_2(T)$, and $\beta > \alpha/2$, so that $\sqrt{\hat{F}(T)}/b(T) \sim T^{1/2}/b(T) \to 0$.

Also, observe that $S_{\alpha/\beta}(\lambda) \overset{d}{=} \lambda^{\beta/\alpha} S_{\alpha/\beta}(1)$, so $S_{\alpha/\beta}(\lambda)$ is a stable random variable with scaling parameter $(\lambda C_{\alpha/\beta}^{-1})^{\beta/\alpha}$. Next, we show:

Proposition 3.1. *With $A(t)$ defined as in (2.1), $b(t)$ as in (2.4), and $A_1(t)$ as in (3.3), we have*

$$\frac{A(T) - A_1(T)}{b(T)} \overset{P}{\to} 0,$$

as $T \to \infty$.

Proof. By (3.3) we need to show:

$$\frac{A_2(T)}{b(T)} \overset{P}{\to} 0.$$

We can write:

$$(3.21) \qquad A_2(T) \overset{d}{=} \sum_{k=1}^{P_2(T)} \zeta(T - \tau_k^{(T)}),$$

$$(3.22) \qquad M|_{R_2^{(T)}} \overset{d}{=} \sum_{k=1}^{P_2(T)} \epsilon_{(\tau_k^{(T)}, \eta_k^{(T)})}$$

where $P_2(T)$ is Poisson with parameter $\lambda m(T) \overset{T \to \infty}{\to} \lambda \mu$, and $P_2(T)$ is independent of the points $\{(\tau_k, \eta_k), k \geq 1\}$, which are in turn i.i.d. with distribution

$$\frac{\lambda ds F(dy)}{\lambda m(T)}\Big|_{R_2(T)} = \frac{ds F(dy)}{m(T)}\Big|_{R_2(T)}.$$

For $x < T$,

$$P(T - \tau_1^{(T)} \leq x) = P(\tau_1^{(T)} \geq T - x) = \frac{\int_{s=T-x}^{T} \lambda ds \bar{F}(T - s)}{\lambda m(T)} = \frac{m(x)}{m(T)}.$$

It suffices to show $EA_2(T)/b(T) \to 0$. We have

$$EA_2(T)/b(T) = \lambda m(T) E\zeta(T - \tau_1^{(T)})/b(T)$$
$$\sim \lambda \int_0^T \zeta(u) \bar{F}(u) \, du / b(T)$$
$$\sim (\text{const}) \, \zeta(T) T \bar{F}(T)/b(T).$$

(3.23)

We are done if we show

(3.24) $$\beta + 1 - \alpha < \beta/\alpha$$

or equivalently if $\beta\alpha + \alpha - \alpha^2 < \beta$. However

$$\beta\alpha - \alpha(\alpha - 1) < \beta\alpha - \beta(\alpha - 1) = \beta,$$

which proves the statement. $\square$

Combining Proposition 3.1 and (3.20), we have

(3.25) $$\frac{A(T)}{b(T)} - \lambda \frac{\hat{F}(T)}{b(T)} E(\zeta(J_k^{(T)}) 1_{[\zeta(J_k^{(T)}) \le b(T)]}) \Rightarrow S_{\alpha/\beta}(\lambda)$$

as $T \to \infty$.

In order to center $A(T)$ by $\lambda T \mu_\zeta$ we first subtract

$$\lambda \frac{\hat{F}(T)}{b(T)} \mu_\zeta - \lambda \frac{\hat{F}(T)}{b(T)} E(\zeta(J_k^{(T)}) 1_{[\zeta(J_k^{(T)}) \le b(T)]}) = \lambda \hat{F}(T) E\left(\frac{\zeta(J_k^{(T)})}{b(T)} 1_{[\zeta(J_k^{(T)}) > b(T)]}\right)$$
$$+ \lambda \frac{\hat{F}(T)}{b(T)} (\mu_\zeta - \mu_\zeta^{(T)}),$$

where $\mu_\zeta^{(T)} = E(\zeta(J_1^{(T)}))$. Now

$$\lambda \frac{\hat{F}(T)}{b(T)} (\mu_\zeta - \mu_\zeta^{(T)}) = \lambda \frac{\hat{F}(T)}{b(T)} \left[\iint_{\substack{0 \le s \le T \\ x < \infty}} \zeta(x) \frac{ds F(dx)}{T} - \iint_{\substack{0 \le s \le T \\ x \le T-s}} \zeta(x) \frac{ds F(dx)}{\hat{F}(T)} \right]$$

$$= \lambda \frac{\hat{F}(T)}{b(T)} \left[\iint_{\substack{0 \le s \le T \\ x < \infty}} \zeta(x) \frac{ds F(dx)}{\hat{F}(T)} \cdot \left(1 + \frac{\hat{F}(T)}{T} - 1\right) - \iint_{\substack{0 \le s \le T \\ x \le T-s}} \zeta(x) \frac{ds F(dx)}{\hat{F}(T)} \right]$$

$$= \lambda \frac{\hat{F}(T)}{b(T)} \left[\iint_{\substack{0 \le s \le T \\ x > T-s}} \zeta(x) \frac{ds F(dx)}{\hat{F}(T)} \right] + \frac{\lambda}{b(T)} T \mu_\zeta \left(\frac{\hat{F}(T)}{T} - 1 \right)$$

$$= \frac{\lambda}{b(T)} \iint_{\substack{0 \le s \le T \\ x > T-s}} \zeta(x) ds F(dx) + \frac{\lambda}{b(T)} \mu_\zeta (\hat{F}(T) - T)$$

$$= I + II.$$

We have

$$|II| = \frac{\lambda}{(T)} \mu_\zeta \int_0^T \bar{F}(u) \, du \le \frac{\lambda \mu_\zeta \mu}{b(T)} \to 0$$

and

$$I = \frac{\lambda}{b(T)} \int_{s=0}^{T} \Big(\int_{x=T-s}^{\infty} \zeta(x) F(dx) \Big) ds$$

$$\sim (const) \frac{\zeta(T) \bar{F}(T) T}{b(T)} \to 0$$

by regular variation of $\zeta(\cdot)$ and $b(\cdot)$, and (3.24). To summarize, for large T,

$$(3.26) \qquad \qquad \mu_\zeta - \mu_\zeta^{(T)} \le (const)\zeta(T)\bar{F}(T) + 2\lambda\mu\mu_\zeta/T.$$

We may conclude

$$(3.27) \qquad \lambda\frac{\hat{F}(T)}{b(T)}\mu_\zeta - \lambda\frac{\hat{F}(T)}{b(T)}E(\zeta(J_k^{(T)})1_{[\zeta(J_k^{(T)})\le b(T)]}) = \lambda\hat{F}(T)E(\frac{\zeta(J_k^{(T)})}{b(T)}1_{[\zeta(J_k^{(T)})>b(T)]}) + o(1).$$

Since $\hat{F}(T) \sim T$ and $T P(\zeta(J_k^{(T)}) > b(T)) \to 1$, we have from the Lemma on page 578 of [10], a variant of Karamata's theorem, that

$$(3.28) \qquad \qquad \lambda\hat{F}(T)E(\frac{\zeta(J_k^{(T)})}{b(T)}1_{[\zeta(J_k^{(T)})>b(T)]}) \to \lambda(\frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta} - 1}).$$

Furthermore,

$$\frac{\hat{F}(T)\mu_\zeta - \lambda T\mu_\zeta\lambda}{b(T)} = \frac{\Big(\int_0^T F(s)ds - T\Big)\mu_\zeta\lambda}{b(T)}$$

$$= \frac{-\lambda\mu_\zeta \int_0^T \bar{F}(s)ds}{b(T)} \sim \frac{-\lambda\mu_\zeta\mu}{b(T)}$$

$$\to 0,$$

as $T \to \infty$. We conclude:

$$(3.29) \qquad \qquad \frac{A(T) - \lambda T\mu_\zeta}{b(T)} \Rightarrow S_{\alpha/\beta}(\lambda) - \lambda(\frac{\alpha/\beta}{\alpha/\beta - 1}) =: X_{\alpha/\beta}(1),$$

where $X_{\alpha/\beta}(1)$ is an α/β stable random variable with mean 0, skewness 1, and scaling parameter $(\lambda C_{\alpha/\beta}^{-1})^{-\beta/\alpha}$.

Now, consider $A(Tt)$, $t > 0$ fixed. Recalling that $b(t)$ is regularly varying with index β/α, it follows that:
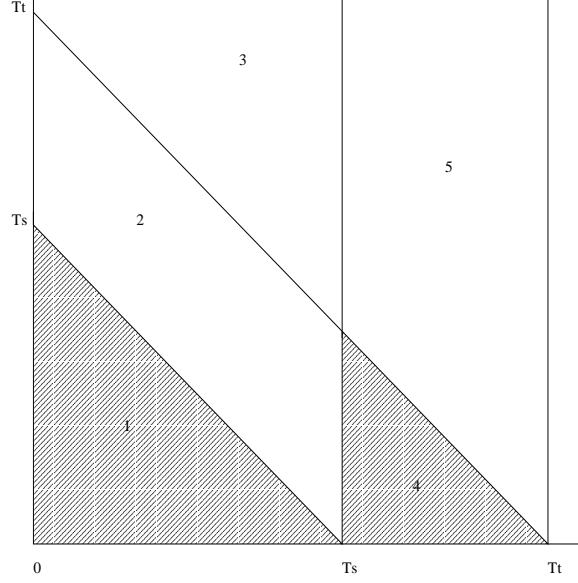
$$(3.30) \qquad \frac{A(Tt) - \lambda Tt\mu_\zeta}{b(T)} = \frac{b(Tt)}{b(T)}\frac{A(Tt) - \lambda Tt\mu_\zeta}{b(Tt)} \Rightarrow t^{\beta/\alpha}X_{\alpha/\beta}(1) \overset{d}{=} X_{\alpha/\beta}(t)$$

as $T \to \infty$.

Next, we show joint convergence. We content ourselves with considering two time points $0 \le s < t$. We divide the traffic between 0 and Tt into parts as illustrated in Figure 2.

From Proposition 3.1 we have

$$(3.31) \qquad \Big(\frac{A(Ts) - \lambda\mu_\zeta Ts}{b(T)}, \frac{A(Tt) - \lambda\mu_\zeta Tt}{b(T)} \Big) = \Big(\frac{A_1(Ts) - \lambda Ts}{b(T)}, \frac{A_1(Tt) - \lambda Tt}{b(T)} \Big) + \big(o_p(1), o_p(1)\big).$$

FIGURE 2. Traffic up to time Tt divided in five parts

Now

$$A_1(Tt) = A_1(Ts) + \iint_{\substack{Ts < u \leq Tt \\ 0 \leq u+y \leq Tt}} \zeta(y)\,M(du, dy) + \iint_{\substack{0 < u \leq Ts \\ Ts \leq u+y \leq Tt}} \zeta(y)\,M(du, dy)$$

$$=: A_1(Ts) + A_4' + A_2',$$

where the three summands are independent. Observe that by scaling

$$(3.32) \qquad\qquad A_4' \overset{d}{=} A_1(T(t-s)),$$

and A_2' is negligible since

$$\frac{E(A_2')}{b(T)} = \frac{1}{b(T)} E\left(\iint_{\substack{0 < u \leq Ts \\ Ts \leq u+y \leq Tt}} \zeta(y)\,M(du, dy) \right)$$

$$= \frac{1}{b(T)} \iint_{\substack{0 < u \leq Ts \\ Ts \leq u+y \leq Tt}} \zeta(y)\,\lambda\,du\,F(dy)$$

$$= \frac{\lambda}{b(T)} \int_{u=0}^{Ts} \int_{y=Ts-u}^{Tt-u} \zeta(y)\,du\,F(dy)$$

$$= \frac{\lambda T\zeta(T)\bar{F}(T)}{b(T)} \int_{u=0}^{s} \int_{y=s-u}^{t-u} \frac{\zeta(Ty)}{\zeta(T)} \frac{F(Tdy)}{\bar{F}(T)}\,du$$

$$= c_T(s,t) \frac{T\zeta(T)\bar{F}(T)}{b(T)}$$

$$\sim c(s,t) \frac{T\zeta(T)\bar{F}(T)}{b(T)},$$

where

$$(3.33) \qquad c(s,t) = \frac{\alpha}{(\alpha - \beta)(\beta - \alpha + 1)} \left[s^{\beta - \alpha + 1} - t^{\beta - \alpha + 1} + (t - s)^{\beta - \alpha + 1} \right].$$

This ratio goes to 0 due to (3.24) as required. The joint convergence then follows from (3.30), (3.31), (3.32). $\qquad\square$

Remark 3.1. For later purposes, note from the definitions of $c_T(s,t)$ and $c(s,t)$ that for $s < t$, these functions are increasing in t and decreasing in s.

Note that the tail behavior of the random variables $J_k^{(Tt)}$ as $T \to \infty$ does not depend on the value of t. To check this, note that $0 \le J_k^{(Tt)} \le Tt$, so $\zeta(0) \le \zeta(J_k^{(T)}) \le \zeta(Tt)$, and from (3.8),

$$
\begin{aligned}
P(\zeta(J_k^{(Tt)}) > b(T)x) &= P(J_k^{(Tt)} > \zeta^{\leftarrow}(b(T)x)) \\
&= \frac{\{(Tt - \zeta^{\leftarrow}(b(T)x))\bar{F}(\zeta^{\leftarrow}(b(T)x)) - \int_{\zeta^{\leftarrow}(b(T)x)}^{Tt} \bar{F}(s)\,ds\}}{\hat{F}(Tt)}.
\end{aligned}
$$

As $T \to \infty$, $\hat{F}(Tt) \sim Tt$. Now

$$
\begin{aligned}
TP(\zeta(J_k^{(Tt)}) > b(T)x) &\sim (1 - \frac{\zeta^{\leftarrow}(b(T)x)}{Tt})T\bar{F}(\zeta^{\leftarrow}(b(T)x)) - \frac{1}{t}\int_{\zeta^{\leftarrow}(b(T)x)}^{Tt} \bar{F}(s)\,ds \\
&\sim (1 + o(1))x^{-\alpha/\beta} - \frac{\zeta^{\leftarrow}(b(T)x)}{Tt}\int_{1}^{Tt/\zeta^{\leftarrow}(b(T)x)} T\bar{F}(\zeta^{\leftarrow}(B(T)x)s)\,ds \\
&\to x^{-\alpha/\beta} - 0\int_{1}^{\infty} x^{-\alpha/\beta}s^{-\alpha}\,ds = x^{-\alpha/\beta}
\end{aligned}
$$

since $T/\zeta^{\leftarrow}(b(T)x) \to \infty$ as $T \to \infty$, see (3.11). Hence, for every fixed t,

$$(3.34) \qquad TP(\zeta(J_k^{(Tt)}) > b(T)x) \to x^{-\alpha/\beta}$$

as $T \to \infty$.

4. Weak convergence in $D[0, \infty)$

In the previous section, we showed that the finite dimensional distributions of the scaled process $(A(Tt) - \lambda\mu_\zeta Tt)/b(T)$ converge as $T \to \infty$, to those of a skewed α/β stable Lévy motion $X_{\alpha/\beta}(t)$. For telecommunications applications such as the heavy traffic theorem discussed in the next section, we need convergence in the stronger functional sense in a suitable topology. When the limit process has almost surely continuous sample paths, one typically uses the Skorohod J_1-topology, or if possible the uniform topology. As remarked in [17, 30, 31, 29], it is not possible for a family of continuous processes $X_T \in C(0, \infty]$ to converge in the J_1 sense to a limit X that is discontinuous with positive probability.

Hence, we investigate convergence in a weaker topology than J_1 and show in this section that $(A(Tt) - \lambda\mu_\zeta Tt)))/b(T)$ converges to $X_{\alpha/\beta}(t)$ in $D[0, \infty)$ endowed with the M_1-topology.

The investigation of functional limit theorems for processes with paths in $D[0, 1]$ was started by Skorohod [27], who introduced four topologies on $D[0, 1]$, which he called J_1, J_2, M_1 and M_2. For defining the M_1 and J_1 topologies, we need the functions

$$(4.1) \qquad J(x_1, x_2, x_3) = \min\{|x_1 - x_2|, |x_3 - x_2|\}$$

and

(4.2) $M(x_1, x_2, x_3) =$ the distance from x_2 to the line segment joining x_1 and x_3

(4.3) $= \begin{cases} 0, & \text{if } x_2 \in [x_1 \wedge x_3, x_1 \vee x_3], \\ J(x_1, x_2, x_3), & \text{otherwise.} \end{cases}$

The function $M(x_1, x_2, x_3)$ is 0 if x_2 is in the line segment with endpoints x_1 and x_3 and is the minimum distance of x_2 to the endpoints of the segment otherwise.

Letting H stand for either J or M, we define the H-oscillation (see [27, 2, 3]) of a function $z(t)$ on $[0, 1]$ as

(4.4) $$\omega_\delta^H(Z) := \sup_{\substack{0 \le t_1 \le t \le t_2 \le 1 \\ 0 \le t_2 - t_1 \le \delta}} H(z(t_1), z(t_2), z(t_3)).$$

We need the following criterion from [27], Theorems 3.2.1 and 3.2.2, which can also be found as Proposition 2 in [3]:

Proposition 4.1. *Let $Z_n(t)$ be processes in $D[0, 1]$, whose finite dimensional distributions converge to those of a process $Z(t)$, which is almost surely continuous at $t = 0$ and $t = 1$. Let H stand for either J or M. Then weak H_1-convergence holds if and only if for every $\epsilon > 0$*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(\omega_\delta^H(Z_n) > \epsilon) = 0$$

Remark 4.1. *An analogous result holds for processes in $D[0, b]$, b arbitrary, provided the limit process $Z(t)$ is a.s. continuous at $t = 0$ and at $t = b$.*

As mentioned in [31], Section 8, p.20, convergence in $D[0, \infty)$ is equivalent to convergence in $D[0, b]$ for all $b > 0$. Let

(4.5) $$Z_T(t) = \frac{A(Tt) - \lambda \mu_\zeta Tt)}{b(T)}$$

By Proposition 4.1, we need to show

(4.6) $$\lim_{\delta \to 0} \limsup_{T \to \infty} P(\omega_\delta^M(Z_T) > \epsilon) = 0.$$

We will do this in the following Theorem.

Theorem 4.1. *For $0 \le t_1 \le t \le t_2 \le b$, there exists a T_1 depending on $(t_2 - t_1)$ such that for $T > T_1$, we have*

$$P(M(Z_T(t_1), Z_T(t), Z_T(t_2)) > \epsilon) \le \lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}(t_2 - t_1)^2.$$

Therefore

$$\limsup_{T \to \infty} P(\omega_\delta^M(Z_T) > \epsilon) \le K \lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}\delta$$

and Z_T converges in $(D[0, \infty), M_1)$.

Proof. If $t_1 = t = t_2$, there is nothing to prove. By (4.3):

$P(M(Z_T(t_1), Z_T(t), Z_T(t_2)) > \epsilon)$

$= P[Z_T(t) > Z_T(t_2) \vee Z_T(t_1) + \epsilon] + P[Z_T(t) < Z_T(t_2) \wedge Z_T(t_1) - \epsilon]$

$= P(Z_T(t) - Z_T(t_1) > \epsilon, Z_T(t_2) - Z_T(t) \le -\epsilon)$

(4.7) $+ P(Z_T(t) - Z_T(t_1) \le -\epsilon, Z_T(t_2) - Z_T(t) > \epsilon).$

Consider the first term on the right hand side of (4.7):

$$P(Z_T(t) - Z_T(t_1) > \epsilon, Z_T(t_2) - Z_T(t) \leq -\epsilon)$$

$$= P\left(\frac{A(Tt) - A(Tt_1) - \lambda\mu_\zeta T(t - t_1)}{b(T)} > \epsilon, \frac{A(Tt_2) - A(Tt) - \lambda\mu_\zeta T(t_2 - t)}{b(T)} \leq -\epsilon\right)$$

$$= P\left(\frac{A_1(Tt) - A_1(Tt_1) + A_2(Tt) - A_2(Tt_1) - \lambda\mu_\zeta T(t - t_1)}{b(T)} > \epsilon,\right.$$

$$\left.\frac{A_1(Tt_2) - A_1(Tt) + A_2(Tt_2) - A_2(Tt) - \lambda\mu_\zeta T(t_2 - t)}{b(T)} \leq -\epsilon\right).$$

Write

$$A_1(Tt) - A_1(Tt_1) = \iint_{\substack{0 \leq u \leq Tt_1 \\ Tt_1 \leq u+y \leq Tt}} \zeta(u) M(du\,dy) + \iint_{\substack{Tt_1 \leq u \leq Tt \\ 0 \leq u+y \leq Tt}} \zeta(u) M(du\,dy)$$

$$=: A_2' + A_1'(T(t - t_1)),$$

$$A_1(Tt_2) - A_1(Tt) = \iint_{\substack{0 \leq u \leq Tt \\ Tt \leq u+y \leq Tt_2}} \zeta(u) M(du\,dy) + \iint_{\substack{Tt \leq u \leq Tt_2 \\ 0 \leq u+y \leq Tt_2}} \zeta(u) M(du\,dy)$$

$$=: A_2'' + A_1''(T(t - t_1)).$$

The previous probability is

$$P\left[\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta(T(t - t_1))}{b(T)} + \frac{A_2' + A_2(Tt) - A_2(Tt_1)}{b(T)} > \epsilon,\right.$$

$$\left.\frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta(T(t_2 - t))}{b(T)} + \frac{A_2'' + A_2(Tt_2) - A_2(Tt)}{b(T)} < -\epsilon\right]$$

and using $(A \cup B) \cap (C \cup D) = AC \cup AD \cup BC \cup BD$ and subadditivity we get the bound

$$(4.8) \qquad P\left(\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)} > \epsilon/2, \frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta T(t_2 - t)}{b(T)} \leq -\epsilon/2\right)$$

$$+ P\left(\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)} > \epsilon/2, \frac{A_2'' + A_2(Tt_2) - A_2(Tt)}{b(T)} \leq -\epsilon/2\right)$$

$$+ P\left(\frac{A_2' + A_2(Tt) - A_2(Tt_1)}{b(T)} > \epsilon/2, \frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta T(t_2 - t)}{b(T)} \leq -\epsilon/2\right)$$

$$+ P\left(\frac{A_2' + A_2(Tt) - A_2(Tt_1)}{b(T)} > \epsilon/2, \frac{A_2'' + A_2(Tt_2) - A_2(Tt)}{b(T)} \leq -\epsilon/2\right).$$

Here A_1' and A_1'' are independent and identically distributed, as are A_2' and A_2''. Recalling we work on $[0, b]$, we may use Markov's inequality on the last three terms to get the bound

$$3\left(\frac{E(A_2') + E(A_2'') + 2E(A_2(Tb))}{b(T)\epsilon/2}\right) = \frac{6}{\epsilon}\left(\frac{E(A_2') + E(A_2'') + 2E(A_2(Tb))}{b(T)}\right)$$

$$(4.9) \qquad \sim (const)\frac{T\zeta(T)\bar{F}(T)}{b(T)} \to 0$$

as $T \to \infty$, by (3.24). The constant is of the form (cf. (3.33), (3.23))

$$\frac{6}{\epsilon}(c(t_1, t) + c(t, t_2) + k(b))$$

where $k(b)$ depends only on b.

For the first probability in (4.8), we can factor to get

$$P(\frac{A_1'(T(t-t_1)) - \lambda\mu_\zeta T(t-t_1)}{b(T)} > \epsilon/2, \frac{A_1''(T(t_2-t)) - \lambda\mu_\zeta T(t_2-t)}{b(T)} \le -\epsilon/2)$$

$$= P(\frac{A_1'(T(t-t_1)) - \lambda\mu_\zeta T(t-t_1)}{b(T)} > \epsilon/2)P(\frac{A_1''(T(t_2-t)) - \lambda\mu_\zeta T(t_2-t)}{b(T)} \le -\epsilon/2).$$

Consider the first factor

$$P(\frac{A_1'(T(t-t_1)) - \lambda\mu_\zeta T(t-t_1)}{b(T)} > \epsilon) = P(\frac{\sum_{k=1}^{P_1(T(t-t_1))} \zeta(J_k^{(T)}) - \lambda\mu_\zeta T(t-t_1)}{b(T)} > \epsilon)$$

$$= P(\frac{\sum_{k=1}^{P_1(T(t-t_1))}(\zeta(J_k^{(T)}) - \mu_\zeta^{(T)})}{b(T)} + \frac{P_1(T(t-t_1))\mu_\zeta^{(T)} - \lambda\mu_\zeta T(t-t_1)}{b(T)} > \epsilon)$$

$$\le P(\frac{\sum_{k=1}^{P_1(T(t-t_1))}(\zeta(J_k^{(T)}) - \mu_\zeta^{(T)})}{b(T)} > \epsilon/2) + P(\frac{P_1(T(t-t_1))\mu_\zeta^{(T)} - \lambda\mu_\zeta T(t-t_1)}{b(T)} > \epsilon/2)$$

(4.10) $= A + B.$

Now

$$B \le P(|(\frac{P_1(T(t-t_1)) - \lambda T(t-t_1)}{b(T)})\mu_\zeta| > \epsilon/4) + P(\frac{(\mu_\zeta - \mu_\zeta^{(T)})P_1(T(t-t_1))}{b(T)} > \epsilon/4)$$

$$= BI + BII.$$

Consider BI: From the central limit theorem for Poisson variables, we know that

$$B_{T(t-t_1)} := \frac{P_1(T(t-t_1)) - \lambda\hat{F}(T(t-t_1))}{\sqrt{\lambda\hat{F}(T(t-t_1))}} \Rightarrow B(1) \sim \mathcal{N}(0,1)$$

and

(4.11) $$E|B_{T(t-t_1)}| \le E(B_{T(t-t_1)}^2)^{1/2} = 1,$$

so

$$BI \le \frac{4}{\epsilon}E|\frac{P_1(T(t-t_1)) - \lambda T(t-t_1)}{b(T)}\mu_\zeta|$$

$$\le \frac{4}{\epsilon}\left(\mu_\zeta\frac{E|P_1(T(t-t_1)) - \lambda\hat{F}(T(t-t_1))|}{b(T)} + \mu_\zeta|\frac{\lambda(\hat{F}(T(t-t_1)) - T(t-t_1))}{b(T)}|\right)$$

$$\le \frac{4}{\epsilon}\left(\frac{\sqrt{\lambda\hat{F}(T(t-t_1))}}{b(T)}\mu_\zeta + |\frac{\lambda(\hat{F}(T(t-t_1)) - T(t-t_1))}{b(T)}\mu_\zeta|\right)$$

(4.12) $$\le \frac{4}{\epsilon}\left(\frac{\sqrt{\lambda\hat{F}(T(t-t_1)}}{b(T)}\mu_\zeta + \frac{\lambda\mu_\zeta\mu}{b(T)}\right) \to 0,$$

(see the argument between (3.28) and (3.29)) as $T \to \infty$, since $b(T)$ is regularly varying with index $\beta/\alpha > 1/2$.

To handle BII, we argue similarly:

$$\frac{(\mu_\zeta - \mu_\zeta^{(T)})P_1(T(t-t_1))}{b(T)} = \frac{(\mu_\zeta - \mu_\zeta^{(T)})\lambda\hat{F}(T(t-t_1))}{b(T)} + \frac{(\mu_\zeta - \mu_\zeta^{(T)})(P_1(T(t-t_1)) - \lambda\hat{F}(T(t-t_1)))}{b(T)}$$
$$= TII_1 + TII_2.$$

From (3.26),

$$TII_1 = (const)\,(t - t_1)\frac{\zeta(T)\bar{F}(T)T}{b(T)} + \lambda\frac{\mu_\zeta\mu}{b(T)}$$
$$\leq (const_1)(t_2 - t_1)\frac{\zeta(T)\bar{F}(T)T}{b(T)} \to 0$$

as $T \to \infty$ from (3.24). For TII_2 we have from (4.11)

$$P[TII_2 > \epsilon] \leq \frac{1}{\epsilon}E|B_{T(t-t_1)}|\mu_\zeta \cdot \frac{\sqrt{\lambda\hat{F}(T(t-t_1))}}{b(T)}$$

(4.13)
$$\leq \frac{1}{\epsilon}\mu_\zeta\frac{\sqrt{\lambda\hat{F}(T(t_2-t_1))}}{b(T)} \to 0,$$

as $T \to \infty$. This concludes the discussion of B.

Next, we consider A. We have

$$\sum_{k=1}^{P_1(T(t-t_1))}\left|\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}\right| \leq \sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]}\left|\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}\right| + \sum_{k=[\lambda\hat{F}(T(t-t_1))]+1}^{P_1(T(t-t_1))}\left|\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}\right|$$

and therefore

$$A \leq P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]}\left|\frac{(\zeta(J_k^{(T)}) - \mu_\zeta^{(T)})}{b(T)}\right| > \epsilon/2\right) + P\left(\sum_{k=1}^{|[\lambda\hat{F}(T(t-t_1))]-P_1(T(t-t_1))|}\left|\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}\right| > \epsilon/2\right)$$
$$=: AI + AII.$$

Now AII can be bounded using Markov's inequality, and the fact that $P_1(T(t-t_1))$ is a Poisson random variable:

$$AII \leq P\left(\left|\sum_{k=1}^{|[\lambda\hat{F}(T(t-t_1))]-P_1(T(t-t_1))|}\left|\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}\right|\right| > \epsilon/2\right)$$
$$\leq (2/\epsilon)E\left(\sum_{k=1}^{|[\lambda\hat{F}(T(t-t_1))]-P_1(T(t-t_1))|}\left|\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}\right|\right)$$
$$\leq (2/\epsilon)\mu_\zeta\frac{E|[\lambda\hat{F}(T(t-t_1))] - P_1(T(t-t_1))|}{b(T)}$$

(4.14)
$$\leq (2/\epsilon)\mu_\zeta\frac{\sqrt{\lambda\hat{F}(T(t_2-t_1))}}{b(T)} \to 0$$

as $T \to \infty$, as seen before.

Now we deal with AI. To bound this probability, we argue as in the proof of Lemma 2 of [2]: Pick $\eta > 0$ such that $\alpha/\beta - \eta > 1$. Let

$$(4.15) \qquad Y_{k,T} = \frac{(\zeta(J_k^{(T)}) - \mu_\zeta^{(T)})}{b(T)}$$

and define

$$(4.16) \qquad Y_{k,T}^{\leq} = Y_{k,T} 1_{(|Y_{k,T}| \leq 1)}, \qquad Y_{k,T}^{>} = Y_{k,T} 1_{(|Y_{k,T}| > 1)}$$

$$(4.17) \qquad \bar{Y}_{k,T}^{\leq} = Y_{k,T}^{\leq} - E(Y_{k,T}^{\leq}), \qquad \bar{Y}_{k,T}^{>} = Y_{k,T}^{>} - E(Y_{k,T}^{>})$$

Now

$$P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |Y_{k,T}| > \epsilon\right) \leq P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |\bar{Y}_{k,T}^{\leq}| > \epsilon/2\right) + P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |\bar{Y}_{k,T}^{>}| > \epsilon/2\right).$$

Using Chebychev's inequality with $p = \alpha/\beta - \eta$, followed by the von Bahr-Esseen inequality, which says that $E|\sum_{j=1}^m X_j|^p \leq 2\sum_{j=1}^m E|X_j|^p$ if $1 \leq p \leq 2$, and X_j a martingale difference sequence, we get

$$P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |\bar{Y}_{k,T}^{>}| > \epsilon/2\right) \leq (\epsilon/2)^{-(\alpha/\beta-\eta)} E\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |\bar{Y}_{k,T}^{>}|\right)^{[\alpha/\beta-\eta)}$$

$$\leq 2(\epsilon/2)^{-(\alpha/\beta-\eta)} [\lambda\hat{F}(T(t-t_1))] E|\bar{Y}_{k,T}^{>}|^{\alpha/\beta-\eta}.$$

Similarly, but using Chebychev's inequality with $p = \alpha/\beta + \eta$,

$$P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |\bar{Y}_{k,T}^{\leq}| > \epsilon/2\right) \leq 2(\epsilon/2)^{-(\alpha/\beta+\eta)} [\lambda\hat{F}(T(t-t_1))] E|\bar{Y}_{k,T}^{\leq}|^{\alpha/\beta-\eta}.$$

By Jensen's inequality,

$$E|Y_{k,T}^{\leq} - E(Y_{k,T}^{\leq})|^{\alpha/\beta+\eta} \leq 2^{\alpha/\beta+\eta-1}\{E|Y_{k,T}^{\leq}|^{\alpha/\beta+\eta} + |EY_{k,T}^{\leq}|^{\alpha/\beta+\eta}\} \leq 2^{\alpha/\beta+\eta} E|Y_{k,T}^{\leq}|^{\alpha/\beta+\eta}.$$

Similarly,

$$E|Y_{k,T}^{>} - EY_{k,T}^{>}|^{\alpha/\beta-\eta} \leq 2^{\alpha/\beta-\eta} E|Y_{k,T}^{>}|^{\alpha/\beta-\eta}.$$

Using Lemma 1 of [1], it follows that

$$\{\sup_T(TE|\bar{Y}_{k,T}^{\leq}|^{\alpha/\beta+\eta})\} \vee \{\sup_T(TE|\bar{Y}_{k,T}^{>}|^{\alpha/\beta-\eta}\} \leq C < \infty$$

and thus

$$P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |Y_{k,T}| > \epsilon\right) \leq C\epsilon^{-(\alpha/\beta+\eta)} \frac{[\lambda\hat{F}(T(t-t_1))]}{T}.$$

Now $\hat{F}(T(t-t_1)) \leq \hat{F}(T(t_2-t_1)) \sim T(t_2-t_1)$, so for T large enough (depending on $t_2 - t_1$),

$$(4.18) \qquad AI \leq P\left(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |Y_{k,T}| > \epsilon\right) \leq 2\lambda C\epsilon^{-(\alpha/\beta+\eta)}(t_2 - t_1).$$

Summarizing (4.10), (4.12), (4.13), (4.14), (4.18), we have for $t_2 - t_1 < \delta$ and T large enough, depending on $t_2 - t_1$ that

$$P(|\frac{A'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)}| > \epsilon)$$

$$\leq P(\sum_{k=1}^{[\lambda\hat{F}(T(t-t_1))]} |\frac{\zeta(J_k^{(T)}) - \mu_\zeta^{(T)}}{b(T)}| > \epsilon/2) + (\frac{16}{\epsilon})(\frac{\lambda\mu_\zeta\mu}{b(T)} + \mu_\zeta \frac{\sqrt{\lambda\hat{F}(T\delta)}}{b(T)})$$

$$\leq 2\lambda M\epsilon^{-(\alpha/\beta+\eta)}(t_2 - t_1) + (\frac{16}{\epsilon})(\frac{\lambda\mu_\zeta\mu}{b(T)} + \mu_\zeta \frac{\sqrt{\lambda\hat{F}(T\delta)}}{b(T)}).$$

Recall (4.7)

$$P(M(Z_T(t_1), Z_T(t), Z_T(t_2)) > \epsilon) = P(Z_T(t) - Z_T(t_1) > \epsilon, Z_T(t_2) - Z_T(t) \leq -\epsilon)$$
$$+ P(Z_T(t) - Z_T(t_1) \leq -\epsilon, Z_T(t_2) - Z_T(t) > \epsilon),$$

and from (4.8) and (4.9)

$$P(Z_T(t) - Z_T(t_1) > \epsilon, Z_T(t_2) - Z_T(t) \leq -\epsilon)$$
$$\leq P(\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)} > \epsilon, \frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta T(t_2 - t)}{b(T)} \leq -\epsilon) + C_1\frac{T\zeta(T)\bar{F}(T)}{b(T)}$$
$$= P(\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)} > \epsilon)P(\frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta T(t_2 - t)}{b(T)} \leq -\epsilon) + C_1\frac{T\zeta(T)\bar{F}(T)}{b(T)}$$
$$\leq P(|\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)}| > \epsilon)P(|\frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta T(t_2 - t)}{b(T)}| > \epsilon) + C_1\frac{T\zeta(T)\bar{F}(T)}{b(T)}.$$

Similarly,

$$P(Z_T(t) - Z_T(t_1) \leq -\epsilon, Z_T(t_2) - Z_T(t) > \epsilon)$$
$$\leq P(|\frac{A_1'(T(t - t_1)) - \lambda\mu_\zeta T(t - t_1)}{b(T)}| > \epsilon)P(|\frac{A_1''(T(t_2 - t)) - \lambda\mu_\zeta T(t_2 - t)}{b(T)}| > \epsilon) + C_1\frac{T\zeta(T)\bar{F}(T)}{b(T)}.$$

So since $A_1' \stackrel{d}{=} A_1''$,

$$P(M(Z_t(t_1), Z_T(t), Z_T(t_2)) > \epsilon)$$

$$\leq 2\left(2\lambda C\epsilon^{-(\alpha/\beta+\eta)}(t_2 - t_1) + (\frac{16}{\epsilon})(\frac{\lambda\mu_\zeta\mu}{b(T)} + \mu_\zeta \frac{\sqrt{\lambda\hat{F}(T\delta)}}{b(T)})\right)^2 + 2C_1\frac{T\zeta(T)\bar{F}(T)}{b(T)}$$

$$= 8\lambda^2 C^2\epsilon^{-2(\alpha/\beta+\eta)}(t_2 - t_1)^2 + 4(\frac{16}{\epsilon})(\frac{\lambda\mu_\zeta\mu}{b(T)} + \mu_\zeta \frac{\sqrt{\lambda\hat{F}(T\delta)}}{b(T)})2\lambda C\epsilon^{-(\alpha/\beta+\eta)}(t_2 - t_1)$$

$$+ 2((\frac{16}{\epsilon})^2(\frac{\lambda\mu_\zeta\mu}{b(T)} + \mu_\zeta \frac{\sqrt{\lambda\hat{F}(T\delta)}}{b(T)})^2 + 2C_1\frac{T\zeta(T)\bar{F}(T)}{b(T)}.$$

For $T > T_1$, with T_1 depending on $t_2 - t_1$, δ, $\alpha/\beta + \eta$, we have

(4.19) $$P(M(Z_T(t_1), Z_T(t), Z_T(t_2)) > \epsilon) \leq \lambda^2 C^2\epsilon^{-2(\alpha/\beta+\eta)}(t_2 - t_1)^2$$

Let $Z_{T,n}(t)$ be the step function built from the values of $Z_T(t)$ at $t = i/2^n, i = 1, \ldots, 2^n$. For $T > T_1$, also

$$P(M(Z_{T,n}(t_1), Z_{T,n}(t), Z_{T,n}(t_2)) > \epsilon) \leq \lambda^2 C^2 \epsilon^{-2(\alpha/\beta + \eta)}(t_2 - t_1)^2,$$

and thus, by Theorem 2 of [2], we have

$$P(\sup_{t_1 \leq c \leq d \leq e \leq t_2} M(Z_{T,n}(c), Z_{T,n}(d), Z_{T,n}(e)) > \epsilon) \leq K\lambda^2 C^2 \epsilon^{-2(\alpha/\beta + \eta)}(t_2 - t_1)^2.$$

Now, as in the proof of Lemma 1 of [2]: let $m = [\delta^{-1}]$ and partition $[0, b]$ with $t_i, i = 0, 1, \ldots, m+1$, where $t_0 = 0, t_{m+1} = b$, and $t_{i+1} - t_i = \delta, i = 0, 1, \ldots, m - 1$. The event $\{\omega_\delta^M(Z_{T,n}) \geq \epsilon\}$ implies $\{\max_{i=0,\ldots,m-1} \sup_{t_1 \leq c \leq d \leq e \leq t_2} M(Z_{T,n}(c), Z_{T,n}(d), Z_{T,n}(e)) \geq \epsilon\}$.

Indeed, if there exist points $c \leq d \leq e$ such that $e - c \leq \delta$, then these points belong to some interval $[t_i, t_{i+2}]$, so that $M(Z_{T,n}(c), Z_{T,n}(d), Z_{T,n}(e)) \geq \epsilon$ implies

$$\sup_{t_i \leq c \leq d \leq e \leq t_{i+2}} M(Z_{T,n}(c), Z_{T,n}(d), Z_{T,n}(e)) \geq \epsilon.$$

Therefore, for $T > T_1$, we have

$$\begin{aligned}
P(\omega_\delta^M(Z_{T,n}) \geq \epsilon) &\leq \sum_{i=0}^{m-1} P(\sup_{t_i \leq c \leq d \leq e \leq t_{i+2}} M(Z_{T,n}(c), Z_{T,n}(d), Z_{T,n}(e)) > \epsilon) \\
&\leq K\lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}(t_2 - t_1)^2 \\
&\leq [\delta^{-1}] K\lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}(2\delta)^2 \\
&\leq K'\lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}\delta.
\end{aligned}$$

Since almost all paths of Z_T are right continuous and $\omega_\delta^M(Z_{T,n}) \to \omega_\delta^M(Z_T)$ a.s., we have

$$P(\omega_\delta^M(Z_T) \geq \epsilon) = \lim_{n \to \infty} P(\omega_\delta^M(Z_{T,n}) \geq \epsilon),$$

and hence, for $T > T_1$

$$P(\omega_\delta^M(Z_T) \geq \epsilon) \leq K'\lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}\delta.$$

This implies

(4.20) $$\limsup_{T \to \infty} P(\omega_\delta^M(Z_T) > \epsilon) \leq K'\lambda^2 C^2 \epsilon^{2(\alpha/\beta + \eta)}\delta$$

and thus

(4.21) $$\lim_{\delta \to 0} \limsup_{T \to \infty} P(\omega_\delta^M(Z_T) \geq \epsilon) \leq \lim_{\delta \to 0} K'\lambda^2 C^2 \epsilon^{-2(\alpha/\beta + \eta)}\delta = 0$$

which proves weak M_1 convergence of $(A(Tt) - \lambda\mu_\zeta Tt)/b(T)$ to $X_{\alpha/\beta}(t)$. □

Remark 4.2. If, instead of the M function, we would have used the J-function, to attempt to prove J_1 convergence, the argument would be altered as follows. Since ζ is regularly varying with index β, $\zeta((1/2)^{1/\beta}L_k) \sim (1/2)\zeta(L_k)$ for L_k large. Heuristically, for each large job $\zeta(L_k)$, the halfway point $(1/2)\zeta(L_k)$ is also large. Take t_1 corresponding to the starting point of the large job, t as halfway point, and t_2 corresponding to the endpoint of the large job, we see that if there is a

large job arriving in the interval $[Tt_1, Tt_2]$, we can choose timepoints $Tt_1 \leq Tt_1' \leq Tt' \leq Tt_2' \leq Tt_2$ such that $J(Z_T(t_1'), Z_T(t_2'), Z_T(t_2')) > \epsilon$. Therefore,

$$P(J(Z_T(t_1), Z_T(t), Z_T(t_2)) > \epsilon) \cong P(\text{Large job arriving in } [Tt_1, Tt_2])$$

$$\cong P\left(\sum_{k=1}^{[\lambda T(t-t_1)]} \frac{\zeta(J_k^{(1)}) - E(\zeta(J_1^{(1)}))}{b(T)} > \epsilon \right)$$

$$+ P\left(\sum_{k=1}^{[\lambda T(t_2-t)]} \frac{\zeta(J_k^{(2)}) - E(\zeta(J_1^{(2)}))}{b(T)} \leq -\epsilon \right)$$

$$\cong O(t_2 - t_1)$$

which is not enough to make

$$P(J(Z_T(t_1), Z_T(t), Z_T(t_2)) > \epsilon) = O((t_2 - t_1)^{1+\beta}), \quad \beta > 0.$$

5. A SINGLE SERVER APPLICATION

Suppose traffic modeled by (2.1) enters a network, and queues at an intermediary switch. Let the cumulative output at the switch by time t be denoted by $R(t)$. Then the content of the fluid queue at time Tt is given by

$$Y(Tt) = A(Tt) - R(Tt) - \bigwedge_{s=0}^{Tt}(A(s) - R(s))$$

$$= A(Tt) - R(Tt) - \bigwedge_{s=0}^{t}(A(Ts) - R(Ts)).$$

Furthermore, assume

$$R(Tt) = R_T t$$

where

(5.1) $$\frac{R_T - \lambda\mu_\zeta T}{b(T)} \to c$$

for some constant c. Then since the process $A(T\cdot)$ has no fixed discontinuities,

$$\frac{A(Tt) - R(Tt)}{b(T)} = \frac{A(Tt) - \lambda\mu_\zeta Tt}{b(T)} - \left(\frac{R_T - \lambda\mu_\zeta T}{b(T)}\right)t \Rightarrow X_{\alpha/\beta}(t) - ct$$

as $T \to \infty$ in $(D[0,\infty), M_1)$, where $X_{\alpha/\beta}(t)$ is the totally skewed Lévy motion described in Theorem 3.1. (See [30], Corollary 7.1).

The reflection mapping $x_r(t) : x(t) \mapsto x(t) - \wedge_{s=0}^t x(s)$ is (Lipschitz) continuous in $(D[0,\infty), M_1)$ ([31], Theorem 3.2 and Remark 3.1; [30], Section 8). Therefore, we have by an application of the continuous mapping theorem:

(5.2) $$\frac{Y(Tt)}{b(T)} = \frac{A(Tt) - R(Tt) - \bigwedge_{s=0}^t(A(Ts) - R(Ts))}{b(T)} \Rightarrow X_{\alpha/\beta}(t) - ct - \bigwedge_{s=0}^t(X_{\alpha/\beta}(s) - cs).$$

In particular, suppose $c = 0$ in (5.1), which then becomes the 'heavy traffic' condition

(5.3) $$\frac{R_T - \lambda\mu_\zeta T}{b(T)} \to 0.$$

Then (5.2) yields

$$(5.4) \qquad \frac{Y(Tt)}{b(T)} \Rightarrow X_{\alpha/\beta}(t) - \bigwedge_{s=0}^{t} X_{\alpha/\beta}(s).$$

as $T \to \infty$, in $(D[0,\infty), M_1)$. To illustrate the heavy traffic interpretation of this result, we argue as follows:

Consider the model of section 2, but suppose we increase the arrival rate of source transmissions from λ to λT. Furthermore, we increase the speed of transmission. The new transmission schedule is given by $\zeta_T(t) := \zeta(Tt)$, and the new transmission durations $L_k^{(T)}$ have the same distribution as L_k/T:

$$P(L_k^{(T)} > \frac{x}{T}) = P(L_k > x) = 1 - F(x) = x^{-\alpha} L(x)$$

where $L(x)$ is as in (2.2). Then for the new system, the total traffic input by time t, $A_T(t)$, has the same distribution as $A(Tt)$ above. Now consider a fluid queue with input $A_T(t)$, and output $R_T(t) := R_T t$. Denote its content at time t by $Y_T(t)$. Under the heavy traffic condition (5.3), we see that in $(D[0,\infty), M_1)$:

$$(5.5) \qquad \frac{Y_T(t)}{b(T)} \Rightarrow X_{\alpha/\beta}(t) - \bigwedge_{s=0}^{t} X_{\alpha/\beta}(s) \text{ as } T \to \infty.$$

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