Weak convergence of integrands and the young measure representation

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Abstract Validity of the Young measure representation is useful in the study of microstructure of ordered solids. Such a Young measure, generated by a minimizing sequence of gradients converging weakly in L^p, often needs to be evaluated on functions of p^{th} power polynomial growth. We give a sufficient condition for this in terms of the variational principle. The principal result concerns lower semicontinuity of functionals integrated over arbitrary sets, THEOREM 1.2. The question arose in the numerical analysis of configurations. Several applications are given. Of particular note, Young measure solutions of an evolution problem are found.

Contents

1 *Introduction*

For a lower semicontinuous functional of the form

$$\Phi(v) = \int_\Omega \varphi(\nabla v)\, dx, \quad v \in H^{1,p}(\Omega;\mathbb{R}^m),$$

the convergence property

$$\Phi(u^k) \rightarrow \Phi(u) \quad \text{and} \quad u^k \rightarrow u \text{ in } H^{1,p}(\Omega;\mathbb{R}^m) \text{ weakly}$$

for a particular sequence (u^k) does not by itself inform us of the behavior the sequence $(\varphi(\nabla u^k))$[1]. Here we show that if φ is nonnegative and has polynomial growth, then $(\varphi(\nabla u^k))$ is weakly convergent in $L^1(\Omega)$ to $\varphi(\nabla u)$. A consequence is that the Young measure generated by (∇u^k) represents the weak limit of a sequence $(\psi(\nabla u^k))$ when ψ is dominated by φ. Our interest in this question arose in the attempt to estimate convergence properties of numerical methods for functionals which are not lower semicontinuous, where φ plays the role of the relaxed density. Validity of the Young measure representation is useful knowledge in the study of the microstructure of ordered solids, cf. Ball and James [5,6], Chipot and Kinderlehrer [10], Ericksen [18-29], Fonseca [31-34], James [35], James and Kinderlehrer [36], Kinderlehrer [37], Kinderlehrer and Pedregal [38], Matos [41], and Pedregal [45,46]. We do not give any explicit applications to the numerical analysis in this paper except to say that our results confirm the validity of the Young measure representation for the limits of the approximations generated by finite element methods when the energy density has appropriate polynomial growth at infinity. We refer to [9,11,12,13,14] for discussions of these developments.

The proof of this and related facts is based on a method of Acerbi and Fusco [1] and subsequent application of the Dunford and Pettis criterion for weak convergence in L^1. Weak convergence of a sequence (f^k) in L^1 is sufficient but not necessary to give sense to the Young measure representation. Ball and Zhang [8] use the Chacon biting lemma to study this question under hypotheses weaker than ours.

The proofs of our results are in §1 - 3. Three applications are given in §4,5, and 6. The example of constraint management in §4 is a generalization of a result of S. Müller [44], cf. also K. Zhang [51]. In §5 a discussion of the Young measure representation when surface energies are present in the system, cf. [39]. Both of these use the convergence property above, or (1.3) below, without assuming that the functional is being driven to a minimum. An application to an evolution problem is given in §6, where it is shown how Young measure solutions may be found. This builds on some recent work of Slemrod [47]. Useful discussions of Young measures are given by Young [50] and Tartar [48,49], and more recently by Ball [3] and Evans [30]. One consequence of our considerations is that they lead to a notion of Young measures generated by

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functions whose gradients are in L^p for finite p, [45]. We begin with a description of our principal results.

THEOREM 1.1 *Let* φ *be continuous and quasiconvex and satisfy*

$$0 \leq \varphi(A) \leq C(1 + |A|^p), \quad A \in M, \tag{1.1}$$

where $1 \leq p \leq \infty$. *Suppose that*

$$u^k \to u \quad \text{in } H^{1,p}(\Omega) \text{ weakly and} \tag{1.2}$$

$$\int_\Omega \varphi(\nabla u)\, dx = \lim_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\, dx . \tag{1.3}$$

Then there is a subsequence (not relabled) of the (u^k) *such that*

$$\varphi(\nabla u^k) \to \varphi(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly.}$$

THEOREM 1.2 *Let* φ *be continuous and quasiconvex and satisfy*

$$0 \leq \varphi(A) \leq C(1 + |A|^p), \quad A \in \mathbb{M},$$

where $1 \leq p \leq \infty$. *If* $u^k \to u$ *in* $H^{1,p}(\Omega)$ *weakly , then*

$$\int_E \varphi(\nabla u)\, dx \leq \liminf_{k\to\infty} \int_E \varphi(\nabla u^k)\, dx \tag{1.4}$$

for every (measurable) $E \subset \Omega$.

We wish to discuss THEOREM 1.2 a little prior to giving the proof. First note that according to the generalizations of Morrey's Theorem [43], for example Acerbi and Fusco [1], (1.4) holds whenever E is a domain with Lipschitz boundary. This information is insufficient to deduce (1.4) for more general sets, which is the crux of the problem.

The case of THEOREM 2 with $p = \infty$ is automatic since $\{ \varphi(\nabla u^k) \}$ are uniformly bounded in this case. Indeed, choose M with the property

$$\| \varphi(\nabla u^k) \|_{L^\infty(\Omega)} \leq M \quad \text{for all } k.$$

Given E, let U be an open neighborhood of E with $|U - E| < \varepsilon$. Now U is the union of countably many cubes $\{ D_j \}$ with disjoint interiors and for each D_j, (1.4) holds. Hence

$$\begin{aligned}\int_U \varphi(\nabla u)\,dx &\leq \sum \int_{D_j} \varphi(\nabla u)\,dx \\ &\leq \sum \liminf \int_{D_j} \varphi(\nabla u^k)\,dx \\ &\leq \liminf \int_U \varphi(\nabla u^k)\,dx.\end{aligned}$$

Finally, we have that

$$\int_E \varphi(\nabla u)\,dx \leq \liminf \int_E \varphi(\nabla u^k)\,dx + 2M\varepsilon .$$

Thus, if $u^k \to u$ in $H^{1,\infty}(\Omega)$ weak*, then

$$\int_E \varphi(\nabla u)\,dx \leq \liminf \int_E \varphi(\nabla u^k)\,dx \tag{1.5}$$

for any measurable $E \subset \Omega$.

The case $p = 1$ for Theorems 1 and 2 is easy and will not be discussed.

To illustrate how the preceding results apply to the Young measure representation, let us introduce the Banach space, for $p > 1$ fixed,

$$E = \{ \psi \in C(\mathbb{M}) : \sup_{\mathbb{M}} \frac{|\psi(A)|}{|A|^p + 1} < \infty \} . \tag{1.6}$$

THEOREM 1.3 *Let* φ *be quasiconvex and satisfy, for some constants* $C \geq c > 0$,

$$\max\{ c|A|^p - 1, 0 \} \leq \varphi(A) \leq C(1 + |A|^p), \quad A \in \mathbb{M}, \tag{1.7}$$

where $1 \leq p \leq \infty$. *Suppose that*

$$u^k \to u \quad \textit{in } H^{1,p}(\Omega) \textit{ weakly and} \tag{1.8}$$

$$\int_\Omega \varphi(\nabla u)\, dx = \lim_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\, dx . \tag{1.9}$$

Let $\nu = (\nu_x)_{x\in\Omega}$ *be a Young measure generated by* (u^k). *Then for any* $\psi \in E$, *the sequence*

$$\psi(\nabla u^k) \to \overline{\psi} \quad \textit{in } \sigma(L^1(\Omega), L^\infty(\Omega)) \textit{ where}$$

$$\overline{\psi}(x) = \int_{\mathbb{M}} \psi(A)\, d\nu_x(A) \quad \textit{in } \Omega \textit{ a.e.} \tag{1.10}$$

Further, consider $W \in C(\mathbb{M})$ satisfying

$$W(A) \geq 0$$

and

$$c(|A|^p - 1) \leq W(A) \leq C(|A|^p + 1) \tag{1.11}$$

for some $p > 1$ and $0 < c \leq C$. Let

$$\mathcal{A}_\Omega(y_o) = \{ v \in H^{1,p}(\Omega) : v = y_o \text{ on } \partial\Omega \} \text{ where } y_o \in H^{1,p}(\Omega).$$

COROLLARY 1.4 *Let* W *satisfy* (1.11). *Suppose that* $(u^k) \subset \mathcal{A}_\Omega(y_o)$ *satisfies*

$$\lim_{k\to\infty} \int_\Omega W(\nabla u^k)\, dx = \inf_{\mathcal{A}_\Omega(y_o)} \int_\Omega W(\nabla v)\, dx . \tag{1.12}$$

and

$$u^k \rightarrow u \quad \textit{in } H^{1,p}(\Omega) \textit{ weakly.}$$

Let $\nu = (\nu_x)_{x \in \Omega}$ *be a Young measure generated by* (u^k). *Then for any* $\psi \in E$, *the sequence*

$$\psi(\nabla u^k) \rightarrow \bar{\psi} \quad \textit{in } \sigma(L^1(\Omega), L^\infty(\Omega)) \textit{ where}$$

$$\bar{\psi}(x) = \int_{\mathbb{M}} \psi(A)\, d\nu_x(A) \quad \textit{in } \Omega \textit{ a.e.} \tag{1.13}$$

In particular, the $(W(\nabla u^k))$ *converges to a limit energy density* $\bar{W}$ *in* $\sigma(L^1(\Omega), L^\infty(\Omega))$ *where*

$$\bar{W}(x) = \int_{\mathbb{M}} W(A)\, d\nu_x(A) \quad \textit{in } \Omega \textit{ a.e.} \tag{1.14}$$

A version of COROLLARY 1.4 has also been proved independently by Matos [42] who obtains an improved class E by combining Ekeland's Lemma with the reverse Hölder inequality, although the convergence is then restricted to $\sigma(L^1(\Omega'), L^\infty(\Omega'))$ for $\Omega' \subset\subset \Omega$.

Note that a particular consequence of THEOREM 1.3 is that the sequence $\{ | M \cdot \nabla u^k |^p \}$, for a constant matrix M, converges weakly in $L^1(\Omega)$, although not to $| M \cdot \nabla u |^p$. Another consequence concerns the relaxation of W, or its quasiconvexification, cf. [7,15,16] for example. Assume that $p > 1$. The integrand

$$W^\#(F) = \inf_{H_o^{1,\infty}(\Omega)} \frac{1}{|D|} \int_D W(F + \nabla\zeta)\, dx \tag{1.15}$$

is quasiconvex and relaxes the variational principle (1.8) in the sense that

$$\inf_{A_\Omega(y_o)} \int_\Omega W(\nabla v)\, dx = \inf_{A_\Omega(y_o)} \int_\Omega W^\#(\nabla v)\, dx .$$

Obviously a minimizing sequence for (1.8) is also a minimizing sequence for the functional with the integrand $W^\#$. For a given F, the infimum in (1.11) may or may not be realized, but given a minimizing sequence $u^k(x) = Fx + \zeta^k(x) \in H^{1,p}(\Omega;\mathbb{R}^m)$,

$$W^{\#}(F) = \lim_{k \to \infty} \int_{\Omega} W(\nabla u^k)\, dx \,.$$

Let $\mu = (\mu_x)_{x \in \Omega}$ be a Young measure generated by (u^k). We may assume that μ_x is independent of $x \in \Omega$, although we pass over the details of that here. Applying COROLLARY 1.4, we obtain in particular that

$$W^{\#}(F) = \int_{\mathbb{M}} W(A)\, d\mu(A) \,, \tag{1.16}$$

so the infimum is attained in a Young measure fashion. Moreover, the inequality $W^{\#} \leq W$ insures that

$$\operatorname{supp} \mu \subset \{A: W(A) = W^{\#}(A)\}.$$

Of course, if σ is any other Young measure generated by some sequence of the form $(v^k) \subset H^{1,p}(\Omega,\mathbb{R}^m)$ with $v^k = Fx$ on $\partial\Omega$, then

$$\int_{\mathbb{M}} W(A)\, d\mu(A) \leq \int_{\mathbb{M}} W(A)\, d\sigma(A) \,,$$

so μ satisfies an ersatz minimizing principle as well.

2 *Proof of* Theorem 1.2

Our aim is to give a proof of the second result. THEOREM 1.1 will be a corollary of it. For this we adopt a technique of Acerbi and Fusco which has an important ingredient from a paper of F.-C. Liu [40]. The technique uses these facts from Acerbi and Fusco:

LEMMA 2.1 *Let* $G \subset \mathbb{R}^n$ *have* $|G| < \infty$. *Assume that* $\{M_k\}$ *is a sequence of subsets of* G *such that for some* $\varepsilon > 0$

$$|M_k| > \varepsilon \qquad \textit{for all } k.$$

Then there is a subsequence k_j *for which*

$$\bigcap M_{k_j} \neq \emptyset .$$

LEMMA 2.2 *Let* $\{ f_k \}$ *be a sequence bounded in* $L^1(\Omega)$. *Then for each* $\varepsilon > 0$, *there is a triple* $(A_\varepsilon, \delta, S)$, *where* $A_\varepsilon \subset \Omega$ *with* $|A_\varepsilon| < \varepsilon$, $\delta > 0$, *and* S *is an infinite subset of the natural numbers, such that*

$$\int_D | f_k | \, dx \quad < \quad \varepsilon$$

whenever $D \cap A_\varepsilon = \emptyset$ *and* $|D| < \delta$.

For any $v \in C_0^\infty(\mathbb{R}^n)$, we set

$$M^*v(x) \; = \; M(|v(x)|) \; + \; M(|\nabla v(x)|)$$

where

$$Mf(x) \; = \; \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} | f(z) | \, dz$$

is the maximal funtion of f. It is well known that if $v \in C_0^\infty(\mathbb{R}^n)$, then $M^*v \in C(\mathbb{R}^n)$ and

$$\| M^*v \|_{L^p(\mathbb{R}^n)} \; \leq \; C(n,p) \, \| v \|_{H^{1,p}(\mathbb{R}^n)}, \; 1 < p \leq \infty, \tag{2.1}$$

and in particular, for any $\lambda > 0$,

$$|\{ M^*v \geq \lambda \}| \; \leq \; C(n,p) \, \lambda^{-p} \| v \|_{H^{1,p}(\mathbb{R}^n)}, \; 1 < p \leq \infty. \tag{2.2}$$

LEMMA 2.3 *Let* $v \in C_0^\infty(\mathbb{R}^n)$ *and* $\lambda > 0$. *Set* $H^\lambda = \{ M^*v < \lambda \}$. *Then*

$$\frac{|v(x) - v(y)|}{|x - y|} \; \leq \; C(n) \, \lambda, \; x, y \in H^\lambda , \tag{2.3}$$

where C(n) *depends only on* n.

We shall also make use of the well known fact that a Lipschitz function defined on a subset of $\mathbb{R}^n$ may be extended to all of $\mathbb{R}^n$ without increasing its Lipschitz constant.

PROOF OF THEOREM 1.2 We regard u^k and u as extended to functions in $H^{1,p}(\mathbb{R}^n)$ with norms controlled by their $H^{1,p}(\Omega)$ norms. Let $\varepsilon > 0$.

Step 1. Since the functional of (1.4) is continuous in $H^{1,p}(\mathbb{R}^n)$ in the norm topology, because of the upper bound on φ, we may find $z, z^k \in C_0^\infty(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} |\varphi(\nabla u) - \varphi(\nabla z)|\, dx < \varepsilon \tag{2.4}$$

$$\int_{\mathbb{R}^n} |\varphi(\nabla u^k) - \varphi(\nabla z + \nabla z^k)|\, dx < \varepsilon \tag{2.5}$$

and

$$\| u - u^k - z^k \|_{H^{1,p}(\mathbb{R}^n)} < \frac{1}{k}.$$

Thus $z^k \to 0$ in $H^{1,p}(\mathbb{R}^n)$ weakly and

$$\| z^k \|_{H^{1,p}(\mathbb{R}^n)} \leq M < \infty. \tag{2.6}$$

Set

$$H^\lambda = \{ M^* z < \lambda \} \quad \text{and} \quad H_k^\lambda = \{ M^* z^k < \lambda \}.$$

According to LEMMA 2.3, we may find $\zeta^k, \eta \in H^{1,\infty}(\mathbb{R}^n)$ such that $\zeta^k = z^k$ on H_k^λ and $\eta = z$ on H^λ with

$$\| \zeta^k \|_{L^\infty(\mathbb{R}^n)} \leq \| z^k \|_{L^\infty(H_k^\lambda)} \leq \lambda$$

and

$$\| \zeta^k \|_{H^{1,\infty}(\mathbb{R}^n)} \leq C(n)\,\lambda,$$

and the same for η. After extraction of a subsequence we may suppose that

$$\zeta^k \to \zeta \quad \text{in } H^{1,\infty}(\mathbb{R}^n) \text{ weak*}.$$

We apply LEMMA 2.2 to the sequence $\{ M^*(z^k)^p \}$. By (1.2) and (2.1) these functions are bounded in $L^1(\Omega)$. So given $\varepsilon' > 0$, there is a triple $(A_{\varepsilon'}, \delta, S)$ with $|A_{\varepsilon'}| < \varepsilon'$ and

$$\int_D M^*(z^k)^p \, dx < \varepsilon'$$

whenever $D \cap A_{\varepsilon'} = \emptyset$ and $k \in S$.

Now let $G = \{ \zeta \neq 0 \}$. Since the $z^k \to 0$ in $L^p(\mathbb{R}^n)$ in norm, we may assume that $z^k \to 0$ pointwise a.e. in Ω. Thus if we set $G_0 = G \cap \{ x \in \Omega: z^k(x) \to 0 \}$, then $|G_0| = |G|$. We write G_0 as a union,

$$G_0 = (G_0 \cap H_k^\lambda) \cup (G_0 \cap (\mathbb{R}^n - H_k^\lambda)) .$$

By (2.2),

$$|G_0 \cap (\mathbb{R}^n - H_k^\lambda)| \leq C\,\lambda^{-p} M \quad \text{for all } k. \tag{2.7}$$

This implies that

$$|G_0| = |G| \leq 2C\,\lambda^{-p} M . \tag{2.8}$$

Otherwise, namely if

$$|G_0| > 2C\,\lambda^{-p} M, \quad \text{then} \quad |G_0 \cap H_k^\lambda| > C\,\lambda^{-p} M,$$

by (2.7). Applying Lemma 2.1, there would be a subsequence k_j such that

$$G_o \cap \left(\bigcap H^{\lambda}_{k_j} \right) \neq \emptyset ,$$

and for x in this intersection,

$$\zeta(x) = \lim \zeta^k(x) = \lim z^k(x) = 0 ,$$

which contradicts the definition of the set G. Hence (2.8) holds.

Step 2 Since $\varphi(\nabla u) \in L^1(\Omega)$, we may find $\sigma, 0 < \sigma < \varepsilon$, and λ large enough that

$$\int_{A_\sigma \cup (\Omega - H^\lambda) \cup G} \varphi(\nabla u)\, dx < \varepsilon , \tag{2.9}$$

cf. (2.8) above. Let $E \subset \Omega$ be measurable and assume a subsequence of the u^k chosen (but not relabled) so that

$$\lim \int_E \varphi(\nabla u^k)\, dx = \liminf \int_E \varphi(\nabla u^k)\, dx .$$

Put

$$\alpha_k = \int_E \varphi(\nabla u^k)\, dx .$$

Since $\varphi \geq 0$, by (2.5)

$$\alpha_k \geq \int_{E \cap H^\lambda \cap H^{\lambda,k} \cap (\Omega - A_\sigma)} \varphi(\nabla u^k)\, dx$$

$$\geq -\varepsilon + \int_{E \cap H^\lambda \cap H^{\lambda,k} \cap (\Omega - A_\sigma)} \varphi(\nabla z + \nabla z^k)\, dx .$$

But $\nabla z = \nabla \eta$ and $\nabla z^k = \nabla \zeta^k$ in $H^\lambda \cap H^{\lambda,k}$ so that

$$\alpha_k \ \geq\ -\varepsilon \ + \int_{E\cap H^\lambda\cap H^{\lambda,k}\cap(\Omega-A_\sigma)} \varphi(\nabla\eta + \nabla\zeta^k)\, dx$$

$$= \ -\varepsilon \ + \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)} \varphi(\nabla\eta + \nabla\zeta^k)\, dx \ - \int_{E\cap H^\lambda\cap(\Omega-H^{\lambda,k})\cap(\Omega-A_\sigma)} \varphi(\nabla\eta + \nabla\zeta^k)\, dx$$

$$= \ -\varepsilon + \beta_k - \gamma_k .$$

Since $\nabla(\eta + \zeta^k)$ is uniformly bounded and φ is quasiconvex, by the remark (1.5), we have that for K sufficiently large

$$\beta_k + \varepsilon \ \geq \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)} \varphi(\nabla\eta + \nabla\zeta)\, dx \quad .$$

We now inspect γ_k. Using the bounds on $\nabla\eta$ and $\nabla\zeta^k$, and choosing λ large enough,

$$\begin{aligned}\gamma_k \ &\leq\ C(1+\lambda^p)\,|(\Omega - H_k^\lambda)\cap(\Omega - A_\sigma)| \\ &\leq\ C\,|\Omega - H_k^\lambda| \ + \int_{(\Omega-H^{\lambda,k})\cap(\Omega-A_\sigma)} CM^*(z^k)^p\, dx \\ &\leq\ C\varepsilon \ + \ C\sigma \ \leq\ 2C\varepsilon.\end{aligned}$$

Consequently, for k sufficiently large,

$$\alpha_k \ \geq\ -C\varepsilon \ + \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)} \varphi(\nabla\eta + \nabla\zeta)\, dx \quad . \tag{2.10}$$

Step 3. Again using the positivity of φ, from (2.10)

$$\alpha_k \ \geq\ -C\varepsilon \ + \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)\cap(\Omega-G)} \varphi(\nabla\eta + \nabla\zeta)\, dx \quad .$$

Since $\zeta = 0$ in $\Omega - G$, we have that $\nabla\zeta = 0$ in $\Omega - G$, so, since $\eta = z$ in H^λ, we deduce that

$$\alpha_k \geq -C\varepsilon + \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)\cap(\Omega-G)} \varphi(\nabla\eta)\,dx$$

$$\geq -C\varepsilon + \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)\cap(\Omega-G)} \varphi(\nabla z)\,dx \quad .$$

By (2.4) and (2.9),

$$\alpha_k \geq -(1+C)\varepsilon + \int_{E\cap H^\lambda\cap(\Omega-A_\sigma)\cap(\Omega-G)} \varphi(\nabla u)\,dx$$

$$\geq -(1+C)\varepsilon + \int_E \varphi(\nabla u)\,dx - \int_{E\cap[A_\sigma\cup(\Omega-H^\lambda)\cup G]} \varphi(\nabla u)\,dx$$

$$\geq -(2+C)\varepsilon + \int_E \varphi(\nabla u)\,dx \quad .$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved.

3 *Proofs of the other results*

PROOF OF THEOREM 1.1 This follows from the Dunford-Pettis criterion. Assume that the sequence $(\varphi(\nabla u^k))$ is not $\sigma(L^1,L^\infty)$ relatively compact. Then for some $\varepsilon > 0$ and every $\delta > 0$, there is an $A_\delta \subset \Omega$ and an integer k_δ such that $|A_\delta| < \delta$ and

$$\int_{A_\delta} \varphi(\nabla u^{k_\delta})\,dx > \varepsilon \ .$$

Since $\varphi(\nabla u) \in L^1(\Omega)$, there is a $\delta_o > 0$ such that if $|E| < \delta_o$, then

$$\int_E \varphi(\nabla u)\, dx \quad < \quad \varepsilon . \tag{3.1}$$

Let us choose in particular $\delta_j = 2^{-j}\,\delta_o$. Then there is a sequence A_j, $|A_j| < \delta_j$, and k_j such that

$$\int_{A_j} \varphi(\nabla u^{k_j})\, dx \quad > \quad \varepsilon \qquad \text{for all } j.$$

Let $E = \cup A_j$, so $|E| \le \delta_o$ and (3.1) holds. Thus

$$\varepsilon \;\le\; \int_E \varphi(\nabla u^{k_j})\, dx \;\le\; \int_\Omega \varphi(\nabla u^{k_j})\, dx \;-\; \int_{\Omega - E} \varphi(\nabla u^{k_j})\, dx \quad .$$

Letting $k_j \to \infty$, we have by THEOREM 1.2 and the hypothesis (1.3) that

$$\varepsilon \;\le\; \int_\Omega \varphi(\nabla u)\, dx) \;-\; \int_{\Omega - E} \varphi(\nabla u)\, dx$$

$$= \int_E \varphi(\nabla u)\, dx \quad < \quad \varepsilon ,$$

a contradiction.

PROOF OF THEOREM 1.3: This also follows by the Dunford-Pettis criterion, using THEOREM 1.1.

4 *Constraint management in a limit case*

Certain variational principles in elasticity constrain the admissible variations $v \in H^{1,s}(\Omega;\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$, to satisfy

$$\det \nabla v > 0 \quad \text{in } \Omega \text{ a.e.}$$

In the limit case $p = n$, $\det \nabla v \in L^1(\Omega)$ for $v \in H^{1,n}(\Omega;\mathbb{R}^n)$ but it is not necessarily integrable to any higher power. Thus it is not automatic that if $u^k \to u$ in $H^{1,n}(\Omega;\mathbb{R}^n)$ weakly, that $\det \nabla u^k \to \det \nabla u$ in $L^1(\Omega)$ weakly. In fact, without additional requirements, this condition does not hold. One may refer to the counterexamples in Ball and Murat [7]. However, much is known about this situation, as we summarize below.

First of all, the determinant is a null lagrangian, namely, if $u,v \in H^{1,n}(\Omega;\mathbb{R}^n)$ and $u\big|_{\partial\Omega} = v\big|_{\partial\Omega}$, then

$$\int_\Omega \det \nabla u \, dx = \int_\Omega \det \nabla v \, dx . \tag{4.1}$$

Assume that $u^k,u \in H^{1,n}(\Omega;\mathbb{R}^n)$ and

$$u^k \to u \text{ in } H^{1,n}(\Omega;\mathbb{R}^n) \text{ weakly.} \tag{4.2}$$

Then for a subsequence of the (u^k), not relabeled, cf. eg. [2] ,

$$\det \nabla u^k \to \det \nabla u \quad \text{in } \mathcal{D}'(\Omega). \tag{4.3}$$

Very recently, S. Müller [44] has shown that if (4.2) holds and $\det \nabla u^k \geq 0$, then

$$\det \nabla u^k \to \det \nabla u \quad \text{in } L^1_{loc}(\Omega) \text{ weakly.} \tag{4.4}$$

We give a slight generalization of Müller's result. With it, alternate proofs of some results in elasticity may be given, for example, some of those in Zhang [51].

THEOREM 4.1 *Let* $u^k,u \in H^{1,n}(\Omega;\mathbb{R}^n)$ *satisfy*

$$u^k \to u \quad \textit{in } H^{1,n}(\Omega;\mathbb{R}^n) \textit{ weakly,} \tag{4.5}$$

$$\det \nabla u^k \geq 0 \quad \textit{in } \Omega \textit{ a.e., and} \tag{4.6}$$

$$u^k \big|_{\partial\Omega} = u_0 \big|_{\partial\Omega}, \tag{4.7}$$

where $u_0 \in H^{1,n}(\Omega;\mathbb{R}^n)$ *is fixed. Then*

$$\det \nabla u^k \to \det \nabla u \quad \textit{in } L^1(\Omega) \textit{ weakly.} \tag{4.8}$$

PROOF First of all, $u = u_0$ on $\partial\Omega$. From Müller's result (4.4), we deduce that $\det \nabla u \geq 0$ in Ω a.e. By (4.1),

$$\int_\Omega \det \nabla u \, dx = \int_\Omega \det \nabla u^k \, dx = \int_\Omega \det \nabla u_0 \, dx \text{ , for all } k. \tag{4.9}$$

Now let

$$\varphi(A) = \max\{\det A, 0\}, \quad A \in \mathbb{M},$$

which is continuous, quasiconvex, and satisfies

$$0 \leq \varphi(A) \leq C(1 + |A|)^n, \quad A \in \mathbb{M}.$$

Then $\varphi(\nabla u^k) = \det \nabla u^k$ and $\varphi(\nabla u) = \det \nabla u$, so, trivially, by (4.9),

$$\int_\Omega \varphi(\nabla u) \, dx = \lim_{k\to\infty} \int_\Omega \varphi(\nabla u^k) \, dx .$$

Consequently, by THEOREM 1.1, possibly for a subsequence which we do not relable,

$$\det \nabla u^k \to \det \nabla u \quad \text{in } L^1(\Omega) \text{ weakly.} \qquad \text{QED}$$

We wish to remark that we used Müller's result to conclude that $\det \nabla u \geq 0$ in Ω a.e. We could also have used the biting convergence theorem of Zhang [51] for this purpose. The idea of THEOREM 4.1 is that the sequence (u^k) may arise as a minimizing sequence for some variational principle subject to (4.6). Additional information then follows from the theorem.

5 *Application to functionals with surface energies*

We consider a simple situation where cooperative bulk and surface energies are minimized. Let $\Omega \subset \mathbb{R}^n$ have smooth boundary Γ and set

$$E(v) = \int_\Omega W(\nabla v)\, dx + \int_\Gamma \tau(\nabla v, \nu)\, dS, \quad v \in C^1(\bar{\Omega};\mathbb{R}^m), \tag{5.1}$$

where ν denotes the exterior normal to Γ. The infimum of E over $C^1(\bar{\Omega};\mathbb{R}^m)$ is not necessarily the sum of the infima of its two summands, so we envision an application of our results when (1.3) will hold for each of the two terms but where these quantities will not be the unrestricted infima of their portions of the functional.

Assume that W is continuous and satisfies, for some $p > 1$ and $C \geq c > 0$,

$$\max\{ c|A|^p - 1, 0 \} \leq W(A) \leq C(1 + |A|^p), \quad A \in \mathbb{M}. \tag{5.2}$$

About τ we assume that it is continuous and, for some $s > 1$,

$$\begin{aligned} &0 \leq \tau(A,\nu) \quad \text{and} \\ &c(|A_{tan}|^s - 1) \leq \tau(A,\nu) \leq C(|A|^s + 1), \end{aligned} \qquad A \in \mathbb{M}, \tag{5.3}$$

where $A_{tan} = A(\mathbb{1} - \nu\otimes\nu)$ is the tangential part of A.

For a fixed $\nu \in \mathbb{S}^{n-1}$, let $D' \subset \{x \cdot \nu = 0\}$ be a domain and let dx' denote the $(n - 1)$-Lebesgue measure on D'. By $D' \times (-r,r)$, $r > 0$, we abbreviate the name of the set

$$\{x \in \mathbb{R}^n: x' = (\mathbb{1} - \nu \otimes \nu)x \in D' \text{ and } |x\cdot\nu| < r\}.$$

Let [E] denote the $n - 1$ dimensional Lebesgue measure of E. We define

$$\tau^\#(F,\nu) = \inf_{C'} \frac{1}{[D']} \int_{D'} \tau(F + \nabla\zeta, \nu)\, dx', \quad (F,\nu) \in \mathbb{M} \times \mathbb{S}^{n-1},$$

$$C' \;=\; C_0^1(D' \times (-r,r))\,. \tag{5.4}$$

We always suppose that $[\partial D'] = 0$. Clearly $\tau^\# \geq 0$ and is independent of r. The relaxation of the functional E is given by

$$E^\#(v) = \int_\Omega W^\#(\nabla v)\,dx + \int_\Gamma \tau^\#(\nabla v,\nu)\,dS\,, \qquad v \in C^1(\bar\Omega;\mathbb{R}^m), \tag{5.5}$$

where $W^\#(A)$ is the ordinary quasiconvexification of W and $\tau^\#$ is defined by (5.4). A special property of $\tau^\#$ is that

$$\tau^\#(A,\nu) \;=\; \tau^\#(A_{tan},\nu)\,, \qquad A \in \mathbb{M},$$

which implies that

$$c(|\,A_{tan}\,|^s - 1) \;\leq\; \tau^\#(A,\nu) \;\leq\; C(|\,A\,|^s + 1), \qquad A \in \mathbb{M}, \tag{5.6}$$

and that $\tau^\#$ is well defined on $H^{1,s}(\Gamma;\mathbb{R}^m)$. An easy generalization of [39] tells us that

$$\inf_{C^1(\bar\Omega)} E(v) \;=\; \inf_V E^\#(v)\,, \qquad V \;=\; H^{1,p}(\Omega;\mathbb{R}^m) \times H^{1,s}(\Gamma;\mathbb{R}^m)\,. \tag{5.7}$$

Let $(u^k) \subset V$ be a minimizing sequence for E. Then (u^k) is a minimizing sequence for $E^\#$, which is bounded in V. Suppose that $u \in V$ and $u^k \to u$ in V weakly. By lower semicontinuity,

$$E^\#(u) \;=\; \lim_{k\to\infty} E^\#(v) \;=\; \inf_{C^1(\bar\Omega)} E(v) \;=\; \inf_V E^\#(v) \quad \text{and}$$

$$\int_\Omega W^\#(\nabla u)\,dx \;=\; \lim_{k\to\infty} \int_\Omega W^\#(\nabla u^k)\,dx$$

$$\int_\Gamma \tau^\#(\nabla_{tan}u,\nu)\,dS \;=\; \lim_{k\to\infty} \int_\Gamma \tau^\#(\nabla u^k,\nu)\,dS\,. \tag{5.8}$$

We may apply THEOREM 1.3, or a slight generalization of it in the case of $(\tau^{\#}(\nabla u^k,\nu))$, to deduce that

$$W^{\#}(\nabla u^k) \to W^{\#}(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly and}$$
$$\tau^{\#}(\nabla u^k,\nu) \to \tau^{\#}(\nabla_{tan} u,\nu) \quad \text{in } L^1(\Gamma) \text{ weakly}.$$

If $\mu = (\mu_x)_{x\in\Omega}$ denotes a Young measure generated by (∇u^k), we have the limit energy representations

$$\overline{W}(x) = W^{\#}(\nabla u(x)) = \int_{\mathbb{M}} W(A)\, d\mu_x(A), \quad x \in \Omega,$$
$$\overline{\tau}(x) = \tau^{\#}(\nabla_{tan} u(x),\nu(x)) = \int_{\Gamma} \tau^{\#}(A,\nu(x))\, d\mu_x(A), \quad x \in \Gamma,$$

and

$$\int_{\Omega} \overline{W}(x)\, dx + \int_{\Gamma} \overline{\tau}(x)\, dS = \inf_{C^1(\overline{B})} E(v).$$

6 *Measure valued solutions of an evolution problem*

Some of our methods may be employed to study measure valued solutions of evolution problems. A more extensive treatment is given by Slemrod [47]; here we wish to explain merely how such solutions may come about. For further developments we refer to Demoulini [17]. To fix the ideas, we consider a scalar case. Suppose that $\varphi \in C^1(\mathbb{R}^n)$ satisfies

$$\begin{aligned} \max(c|a|^2 - 1, 0) \le \varphi(a) \le C(|a|^2+1) \\ |\nabla\varphi(a)| \le C|a| \end{aligned} \qquad a \in \mathbb{R}^n, \tag{6.1}$$

where $0 < c \le C$. Let $q(a) = \nabla\varphi(a)$. Our interest is in solutions, possibly Young measures, which in some sense satisfy

$$-\operatorname{div} \overline{q} + \frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{6.2}$$

$\mathbb{R}^+ = (0,\infty)$, subject to appropriate boundary conditions.

To render this more precise, let us agree that $\nu = (\nu_{x,t})_{(x,t)\in\Omega\times\mathbb{R}^+}$ is a Young measure solution of (6.2) provided that

ν is a family of probability measures and
$u \in L^\infty(\mathbb{R}^+; H^1_0(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(\Omega\times\mathbb{R}^+)$ which satisfy

$$-\operatorname{div}\bar{q} + \frac{\partial u}{\partial t} = 0 \quad \textit{in } H^1_0(\Omega\times\mathbb{R}^+), \tag{6.3}$$

$$u\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega}, \quad \textit{where} \tag{6.4}$$

$$\left.\begin{aligned} \bar{q}(x,t) &= \int_{\mathbb{R}^n} q(a)\, d\nu_{x,t}(a) \quad \textit{and} \\ \nabla u(x,t) &= \int_{\mathbb{R}^n} a\, d\nu_{x,t}(a) \end{aligned}\right\} \quad \textit{in } \Omega\times\mathbb{R}^+ \text{ a.e.} \tag{6.5}$$

Above, $u_0 \in H^1_0(\Omega)$ is given. Moreover, we shall impose the condition that

$$\nu \textit{ is generated by a sequence } (\nabla u^h),\ h > 0, \textit{ where } u^h \in L^\infty(\mathbb{R}^+; H^1_0(\Omega)). \tag{6.6}$$

The equation (6.5) means that

$$\int_0^\infty\int_\Omega \left(\bar{q}\cdot\nabla\zeta + \frac{\partial u}{\partial t}\zeta\right) dxdt = 0 \quad \text{for } \zeta \in H^1_0(\Omega\times\mathbb{R}^+). \tag{6.7}$$

We shall give an outline of the proof of

THEOREM 6.1 *Assume* (6.1) *about* φ. *Then there exists a Young measure solution* $\nu = (\nu_{x,t})_{(x,t)\in\Omega\times\mathbb{R}^+}$ *of*

$$-\operatorname{div}\bar{q} + \frac{\partial u}{\partial t} = 0 \quad \textit{in } \Omega\times\mathbb{R}^+,$$

satisfying (6.3) - (6.6). *In addition*

$$\operatorname{supp} \nu_{x,t} \subset \{ a \in \mathbb{R}^n : \varphi(a) = \varphi^{**}(a) \}, \quad \textit{in} \ \ \Omega \times \mathbb{R}^+ \text{ a.e.,} \tag{6.8}$$

where φ^{**} *is the convexification of* φ .

Recall that if $\varphi \in C^1(\mathbb{R}^n)$, then $\varphi^{**} \in C^1(\mathbb{R}^n)$, whence

$$q(a) = q^{**}(a) \quad \text{in} \quad \{ a \in \mathbb{R}^n : \varphi(a) = \varphi^{**}(a) \},$$

where $q^{**}(a) = \nabla\varphi^{**}(a)$. Note also that φ^{**} satisfies (6.1). Hence the

COROLLARY 6.2 *Assume* (6.1) *about* φ *and let* $\nu = (\nu_{x,t})_{(x,t) \in \Omega \times \mathbb{R}^+}$ *be a Young measure solution satisfying* (6.8). *Then* ν *is a solution of the relaxed problem*

$$-\operatorname{div} \overline{q^{**}} + \frac{\partial u}{\partial t} = 0 \quad \textit{in} \ \ \Omega \times \mathbb{R}^+. \tag{6.9}$$

The constructed solution has some additional properties which we shall describe in the sequel.

Step1 An equilibrium problem. Let $w \in H^1_0(\Omega)$ and $h > 0$ and consider

$$\Phi(v) = \Phi_h(v) = \int_\Omega (\varphi(\nabla v) + \frac{1}{2h} |v - w|^2)\, dx, \qquad v \in H^1_0(\Omega), \text{ and} \tag{6.10}$$

$$\Phi^{**}(v) = \int_\Omega (\varphi^{**}(\nabla v) + \frac{1}{2h} |v - w|^2)\, dx, \qquad v \in H^1_0(\Omega), \tag{6.11}$$

where φ^{**} is the convexification of φ. By a known relaxation theorem, cf. [16],

$$I = \inf_{H^1_0(\Omega)} \Phi(v) = \inf_{H^1_0(\Omega)} \Phi^{**}(v)\,. \tag{6.12}$$

Now let (v^k) be a minimizing sequence for $\Phi(v)$. We may assume there is a $u \in H^1_0(\Omega)$ such that

$$v^k \to u \quad \text{in } H^1_0(\Omega) \text{ weakly as } k \to \infty.$$

By lower semicontinuity,

$$\Phi(v^k) \to \Phi^{**}(u) \quad \text{as } k \to \infty,$$

and by the Rellich Theorem,

$$\int_\Omega | v^k - w |^2 \, dx \to \int_\Omega | u - w |^2 \, dx \quad \text{as } k \to \infty.$$

Hence

$$\int_\Omega \varphi^{**}(\nabla u) \, dx = \lim_{k \to \infty} \int_\Omega \varphi^{**}(\nabla v^k) \, dx = \lim_{k \to \infty} \int_\Omega \varphi(\nabla v^k) \, dx .$$

Hence by THEOREM 1.1,

$$\varphi^{**}(\nabla v^k) \to \varphi^{**}(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly and}$$
$$\varphi(\nabla v^k) \to \varphi^{**}(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly.}$$

Denoting by $\nu = (\nu_x)_{x \in \Omega}$ the Young measure generated by (∇v^k),

$$\operatorname{supp} \nu \subset \{ a \in \mathbb{R}^n : \varphi(a) = \varphi^{**}(a) \} ,$$

$$\varphi^{**}(\nabla u) = \overline{\varphi} = \overline{\varphi^{**}} \quad \text{and} \quad \overline{q} = \overline{q^{**}} \quad \text{in } \Omega \text{ a.e.,} \tag{6.13}$$

where

$$\overline{\psi}(x) = \int_{\mathbb{R}^n} \psi(a) \, d\nu_x(a) \quad \text{in } \Omega \text{ a.e.}$$

In fact, the Young measure representation holds for any $\psi \in E$, where

$$E = \{ \psi \in C(\mathbb{M}): \ \sup_{\mathbb{M}} \frac{|\psi(A)|}{|A|^2 + 1} < \infty \} .$$

We may now apply the technique developed in [10] to discuss stable Young measure minimizers of variational principles, cf. §5. As a consequence of this, we may write an equilibrium equation

$$\int_\Omega (\bar{q} \cdot \nabla\zeta + \frac{1}{h}(u - w)\, \zeta)\, dx = 0 \quad \text{for } \zeta \in H^1_0(\Omega). \tag{6.14}$$

Finally, the Young measure representation provides us with an elementary estimate for $\bar{q}$. Indeed, using the estimates of (6.1) and (6.13),

$$\begin{aligned} \int_\Omega |\bar{q}|^2 dx &\leq \int_\Omega \int_{\mathbb{R}^n} |q(a)|^2 d\nu_x(a)dx \\ &\leq C\int_\Omega \int_{\mathbb{R}^n} |a|^2 d\nu_x(a)dx \\ &\leq C\int_\Omega \int_{\mathbb{R}^n} (\varphi(a) + 1)\, d\nu_x(a)dx \\ &= C \int_\Omega (\varphi^{**}(\nabla u) + 1)\, dx \end{aligned} \tag{6.15}$$

Step 2 Approximate solution Let $u_o \in H^1_0(\Omega)$ be given and $h > 0$. We define a sequence of Young measure solutions $\nu^{h,j}$ and underlying functions $u^{h,j}$ by setting

$$\nu^{h,o} = \delta_{\nabla u_o} \quad \text{and} \quad u^{h,o} = u_o$$

and $\nu^{h,j+1}$ the solution of (6.12) with $w = u^{h,j}$ and $u^{h,j+1}$ its underlying function. We then are in possession of the energy densities

$$\varphi^{**}(\nabla u^{h,j}) = \langle \nu^{h,j}, \varphi \rangle = \langle \nu^{h,j}, \varphi^{**} \rangle \tag{6.16}$$

and the flux densities

$$\overline{q}^{h,j} = \langle \nu^{h,j}, q \rangle = \langle \nu^{h,j}, q^{**} \rangle . \tag{6.17}$$

Let $I^{h,j} = [hj, h(j+1))$, $\chi^{h,j} = \chi_{I^{h,j}}$, and

$$\lambda^{h,j}(t) = \begin{cases} 0 & t < hj \\ \frac{t}{hj} - 1 & hj \leq t < h(j+1) \\ 1 & h(j+1) \leq t \end{cases} .$$

Set

$$u^h(x,t) = \Sigma_j \{(1 - \lambda^{h,j}(t))\, u^{h,j}(x) + \lambda^{h,j}(t)\, u^{h,j+1}(x)\} \in L^\infty(\mathbb{R}^+; H^1_0(\Omega)) \tag{6.18}$$

and

$$\nu^h_{x,t} = \Sigma_j \chi^{h,j}(t)\, \nu^{h,j}_x \qquad \in E'. \tag{6.19}$$

Now from (6.18),

$$\frac{\partial u^h}{\partial t} = \frac{1}{h}(u^{h,j+1} - u^{h,j}) \quad \text{and} \quad \overline{q}^h = \langle \nu^h, q \rangle = \Sigma_j \overline{q}^{h,j} \chi^{h,j} \tag{6.20}$$

comprise a solution of

$$-\operatorname{div} \overline{q}^h + \frac{\partial u^h}{\partial t} = 0 \text{ in } H^{-1}(\Omega), \text{ for each } t,$$

from which it is elementary to check that

$$\int_0^\infty \int_\Omega \left(\overline{q}^h \cdot \nabla\zeta + \frac{\partial u^h}{\partial t} \zeta \right) dxdt = 0 \quad \text{for } \zeta \in H^1_0(\Omega \times \mathbb{R}^+). \tag{6.21}$$

Step 3 Estimates Uniform estimates are available for $u^h \in L^\infty(\mathbb{R}^+;H^1_o(\Omega))$ and $\frac{\partial u^h}{\partial t} \in L^2(\Omega\times \mathbb{R}^+)$. To begin, $u^{h,j}$ is admissible in the variational principle for $u^{h,j+1}$, so

$$\int_\Omega (\varphi^{**}(\nabla u^{h,j+1}) + \frac{1}{2h}\,|u^{h,j+1} - u^{h,j}|^2)\,dx \;\le\; \int_\Omega \varphi^{**}(\nabla u^{h,j})\,dx \,.$$

Hence

$$\int_\Omega \varphi^{**}(\nabla u^{h,j})\,dx \;\le\; \int_\Omega \varphi^{**}(\nabla u_o)\,dx \;=\; M^2 \tag{6.22}$$

and

$$\frac{1}{2h}\,\Sigma_j\,|u^{h,j+1} - u^{h,j}|^2 \;\le\; \int_\Omega \varphi^{**}(\nabla u_o)\,dx \;=\; M^2. \tag{6.23}$$

Since φ^{**} satisfies (6.1), the inequality (6.22) tells us that

$$\|\nabla u^{h,j}\|_{L^2(\Omega)} \;\le\; M\,. \tag{6.24}$$

By convexity of the L^2 norm and (6.24) we have that

$$\|u^h\|_{L^\infty(\mathbb{R}^+;H^1_o(\Omega))} \;\le\; M\,. \tag{6.25}$$

Rearranging a little in (6.23) and noting (6.20),

$$\int_0^\infty\int_\Omega |\frac{\partial u^h}{\partial t}|^2\,dxdt \;\le\; M^2\,. \tag{6.26}$$

Introduce the function

$$w^h(x,t) \;=\; \Sigma_j\; u^{h,j}(x)\,\chi^{h,j}(t) \;\in\; L^\infty(\mathbb{R}^+;H^1_o(\Omega))\,. \tag{6.27}$$

Then (6.24) implies that

$$\| w^h \|_{L^\infty(\mathbb{R}^+;H^1_0(\Omega))} \leq M . \qquad (6.28)$$

Finally, we wish to estimate $\bar{q}^h$ using (6.15), which provides the estimate

$$\| \bar{q}^h \|_{L^\infty(\mathbb{R}^+;L^2(\Omega))} \leq C \int_\Omega (\varphi^{**}(\nabla u^{h,j}) + 1)dx \leq C(M^2 + 1). \qquad (6.29)$$

Step 4 Passage to the limit We let $h \to 0$. From the estimates (6.25), (6.26), (6.28), and (6.29), we may extract a subsequence of h as $h \to 0$ and

$$\nu = (\nu_{x,t})_{(x,t) \in \Omega \times \mathbb{R}^+} \in E' \quad \text{with} \quad \text{supp}\, \nu \subset \{ \varphi(a) = \varphi^{**}(a) \}$$
and ν is a Young measure,

$$w \in L^\infty(\mathbb{R}^+;H^1_0(\Omega)) \quad \text{with} \quad \nabla w = \langle \nu, a \rangle ,$$

$$\bar{q} \in L^\infty(\mathbb{R}^+;L^2(\Omega)) \quad \text{with} \quad \bar{q} = \langle \nu, q \rangle = \langle \nu, q^{**} \rangle, \text{ and}$$
$$u \in L^\infty(\mathbb{R}^+;H^1_0(\Omega)) \quad \text{with} \quad \frac{\partial u}{\partial t} \in L^2(\Omega \times \mathbb{R}^+)$$

which satisfy

$$\int_0^\infty \int_\Omega (\bar{q} \cdot \nabla \zeta + \frac{\partial u}{\partial t} \zeta) \, dxdt = 0 \quad \text{for } \zeta \in H^1_0(\Omega \times \mathbb{R}^+). \qquad (6.30)$$

In fact, (6.30) above holds for $\zeta \in L^\infty(\mathbb{R}^+;H^1_0(\Omega))$. We remark that ν is a Young measure but it is not generated by the sequence (∇u^h) of (6.18), but rather by a diagonal subsequence of the functions which generate the (ν^h) of (6.19).

It remains to show that the Young measure ν and the limit function u are connected. We claim that $u = w$. In fact, we shall show that $\nabla u = \nabla w$ by means of an easy lemma.

LEMMA 6.3 *Let* $(f^{h,j}) \subset$ bounded set of $L^2(\Omega)$ *for* $h > 0$ *and* $j = 1,2,3,\ldots$, *and set*

$$f^h(x,t) = \sum_j f^{h,j}(x) \, \chi^{h,j}(t) \quad \textit{and}$$

$$g^h(x,t) = \Sigma_j \{(1 - \lambda^{h,j}(t))\, f^{h,j}(x) + \lambda^{h,j}(t)\, f^{h,j+1}(x)\},$$

where $\chi^{h,j}$ *is the characteristic function of* $[hj, h(j+1))$ *and*

$$\lambda^{h,j}(t) = \begin{cases} 0 & t < hj \\ \frac{t}{hj} - 1 & hj \le t < h(j+1) \\ 1 & h(j+1) \le t \end{cases} .$$

Suppose that

$$f^h \to f \quad \textit{and} \quad g^h \to g \quad \textit{in } L^2_{loc}(\Omega \times \mathbb{R}^+) \textit{ weakly.}$$

Then $f = g$.

PROOF It suffices to show that

$$\int_0^\infty \int_\Omega f\,\zeta\, dxdt = \int_0^\infty \int_\Omega g\,\zeta\, dxdt$$

for $\zeta \in C_0^\infty(\Omega)$ of the form $\zeta(x,t) = w(x)z(t)$. Let $z^{h,j} = z(hj)$ and

$$\zeta^h(x,t) = w(x) \Sigma_j\; z^{h,j} \chi^{h,j}(t) ,$$

$$\xi^h(x,t) = w(x) \Sigma_j \{(1 - \lambda^{h,j}(t))\, z^{h,j} + \lambda^{h,j}(t)\, z^{h,j-1}\} .$$

It is elementary to check that $\zeta^h \to \zeta$ and $\xi^h \to \zeta$ uniformly since z is smooth. Since

$$\int_0^\infty \int_\Omega f^h \xi^h\, dxdt = \int_0^\infty \int_\Omega g^h \zeta^h\, dxdt ,$$

the lemma follows. QED

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