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WEAK CONVERGENCE OF WEIGHTED EMPIRICAL CUMULATIVES BASED ON RANKS

By Hira Lal Koul and Robert G. Staudte, Jr.

Michigan State University

The weak convergence of weighted empirical cumulatives based on the ranks of independent, not necessarily identically distributed, observations to a continuous Gaussian process is proved. The results contain a shorter proof of a central limit theorem by Dupač and Hájek (1969) *Ann. Math. Statist*. Analogous results are proved for signed rank processes.

0. Summary. In this paper we consider the asymptotic behavior under general alternatives of the processes $S_n^*(v) = \sum_{i=1}^n c_{ni} I\{R_{ni} \leqq (n+1)v\}$, $0 \leqq v \leqq 1$, and their signed rank counterparts.

Theorem 1.1 of Section 1 gives a simpler proof of the Theorem 1 of [2], namely the asymptotic normality of $S_n^*(v)$ for a fixed point v. Theorem 2.1 of Section 2 proves the analogous result for the signed rank statistics.

Theorems 3.1 and 3.2 give sufficient conditions for weak convergence of the suitably normalized processes $\{S_n^*(v) : 0 \leqq v \leqq 1\}$ and their signed rank counterparts to appropriate continuous Gaussian processes. The results include as special cases the asymptotic distribution of generalized Kolmogorov–Smirnov statistics as studied in Theorems V. 3.5.1 and VI. 3.2.1 of [3].

The proof in all three sections uses results of [4] pertaining to the weighted empirical cumulatives based on $\{X_{ni}\}$.

1. Linear rank statistics. For each $n \geqq 1$ let $R_{n1}, \cdots, R_{nn}$, denote the ranks of independent observations $X_{n1}, \cdots, X_{nn}$ having respective continuous distribution functions $F_{n1}, \cdots, F_{nn}$. Let $c_{n1}, \cdots, c_{nn}$ be given real numbers. We define the linear rank statistics

$$S_n^* = \sum_{i=1}^n c_{ni} I\{R_{ni} \leqq (n+1)v\}$$

where I is the set indicator function and v is a fixed point in the unit interval. *The point v will remain fixed throughout Sections 1 and 2.*

For notational convenience we shall hereafter suppress the subscript n on the above given entities and on all functions of them except for the empirical distribution function. Also, we shall write $\sum_i$ and $\max_i$ for $\sum_{i=1}^n$ and $\max_{1\leqq i\leqq n}$, respectively. For any distribution function G, $G^{-1}(t)$ will denote the usual inverse $\inf\{x : G(x) \geqq t\}$.

We define

$$H_n(x) = n^{-1} \sum_i I\{X_i \leqq x\}, \qquad -\infty < x < \infty$$
$$H(x) = n^{-1} \sum_i F_i(x), \qquad -\infty < x < \infty$$
$$L_i(t) = F_i(H^{-1}(t)), \qquad 0 \leqq t \leqq 1, \quad 1 \leqq i \leqq n.$$

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We are here concerned only with the asymptotic behavior of S^*, so we may consider the simpler (for our purposes) and asymptotically equivalent statistic

$$S = \sum_i c_i I\{X_i \leqq H_n^{-1}(v)\} . \tag{1.1}$$

The main result concerning S follows.

THEOREM 1.1. *Let S be given by* (1.1) *and assume that*

$$\lim_{n\to\infty} \sigma_c^{-1} \max_i |c_i| = 0 \qquad \textit{where} \quad \sigma_c^2 = \sum_i c_i^2 , \tag{1.2}$$

and

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \max_i |L_i(v+\delta) - L_i(v-\delta)| = 0 . \tag{1.3}$$

Furthermore assume that there exist real numbers $l_i(v)$, $1 \leqq i \leqq n$, such that for every $K > 0$

$$\lim_{n\to\infty} \max_i \max_{|t-v| \leqq Kn^{-\frac{1}{2}}} n^{\frac{1}{2}} |L_i(t) - L_i(v) - (t-v)l_i(v)| = 0 . \tag{1.4}$$

Denoting $n^{-1} \sum_i c_i l_i(v)$ by $\bar{c}$, we also require

$$\limsup_{n\to\infty} n^{\frac{1}{2}} \sigma_c^{-1} |\bar{c}| < +\infty . \tag{1.5}$$

Then S is asymptotically normal with parameters (μ, σ^2), where

$$\mu = \sum_i c_i L_i(v) \qquad \textit{and}$$

$$\sigma^2 = \sum_i (c_i - \bar{c})^2 L_i(v)(1 - L_i(v)) , \qquad \textit{provided that} \tag{1.6}$$

$$\liminf_{n\to\infty} \sigma^2/\sigma_c^2 > 0 . \tag{1.7}$$

REMARKS. Since the above stated theorem bears strong resemblance to Theorem DH1 of [1], (which is the complemented version of Theorem 1 of [2]), some comments are in order. First, assumptions (1.2), (1.4), and (1.7) correspond to conditions (2.2), (2.13), and (2.22) of [2]. Condition (1.3) above is not quite comparable to condition (2.12) of [2], but it appears to be less restrictive. In any case (2.12) and (2.13) of [2] together imply the boundedness of the l_i's, and hence the condition (1.5) above. Taken together, then, the assumptions of Theorem 1.1 above are somewhat weaker than those of DH1. On the other hand, the conclusions of DH1 of [1] are stronger than those of the above theorem in that it asserts not only $\mathscr{L}(\sigma^{-1}(S^* - \mu)) \to N(0, 1)$ but also $E(\sigma^{-1}(S^* - \mu)) \to 0$ and $E[\sigma^{-1}(S^* - \mu)]^2 \to 1$ as $n \to \infty$. The main advantages of the proof of the above Theorem 1.1 are that it is shorter and the role played by the conditions (1.3) and (1.4) is clear.

PROOF OF THEOREM 1.1. The proof will be based on three lemmas that follow. We define the weighted empirical cumulative functions

$$V_c(t) = \sigma_c^{-1} \sum_i c_i [I\{X_i \leqq H^{-1}(t)\} - L_i(t)] , \qquad 0 \leqq t \leqq 1 . \tag{1.8}$$

In the special case $c_i = 1$, $1 \leqq i \leqq n$, we write $V_1(t)$ for $V_c(t)$. Note that

$$V_1(t) = n^{\frac{1}{2}}[H_n(H^{-1}(t)) - t] , \qquad 0 \leqq t \leqq 1 . \tag{1.9}$$

Letting $T = \sigma_c^{-1}[S - \mu]$, one may easily show that

(1.10) $$T = V_c(H(H_n^{-1}(v))) + \sigma_c^{-1} \sum_i c_i[L_i(H(H_n^{-1}(v))) - L_i(v)] .$$

This relationship between T and V_c is basic to the proof of the theorem.

In the sequel we shall write $Y_n = o_p(1)$ for "Y_n converges to zero in probability" and $Y_n = O_p(1)$ for "Y_n is bounded in probability"; all limits are taken with $n \to +\infty$, unless otherwise specified.

LEMMA 1.1. $$\sup_{0<t<1} |H(H_n^{-1}(t)) - t| = o_p(1) .$$

PROOF.

$$\begin{aligned}\sup_{0<t<1} &|H(H_n^{-1}(t)) - t| \\ &\leqq \sup_{0<t<1} |H(H_n^{-1}(t)) - H_n(H_n^{-1}(t))| + \sup_{0<t<1} |H_n(H_n^{-1}(t)) - t| \\ &\leqq \sup_{-\infty<x<\infty} |H(x) - H_n(x)| + \frac{1}{n} = \sup_{0\leqq t\leqq 1} |H_n(H^{-1}(t) - t| + \frac{1}{n} .\end{aligned}$$

But the first term on the right converges in probability to zero by (1.4) of [4], and the lemma follows at once.

LEMMA 1.2. *If condition* (1.3) *is satisfied then for any* $\varepsilon > 0$ *and* v *fixed*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \Pr\{\sup_{|t-v|<\delta} |V_c(t) - V_c(v)| > \varepsilon\} = 0 .$$

PROOF. Let $G(t) = \sigma_c^{-2} \sum_i c_i^2 L_i(t)$, $0 \leqq t \leqq 1$. Then by Lemmas 2.1 and 2.2 of [4], there is a K independent of δ and v such that for all n

$$\begin{aligned}\Pr &\{\sup_{|t-v|<\delta} |V_c(t) - V_c(v)| > \varepsilon\} \\ &\leqq \frac{K}{\varepsilon^2} [G(v + \delta) - G(v - \delta)]^2 + 2 \Pr\left\{|V_c(v + \delta) - V_c(v - \delta)| \geqq \frac{\varepsilon}{4}\right\} ,\end{aligned}$$

which is

$$\leqq \frac{K + 32}{\varepsilon^2} [G(v + \delta) - G(v - \delta)]^2$$

by Chebyshev's Inequality. The lemma now follows from (1.3) and the fact that

$$G(v + \delta) - G(v - \delta) \leqq \max_i (L_i(v + \delta) - L_i(v - \delta)) .$$

LEMMA 1.3. *If condition* (1.3) *is satisfied, then for any* $\varepsilon > 0$

$$\lim_{n\to\infty} \Pr\{|V_c(H(H_n^{-1}(v)) - V_c(v)| > \varepsilon\} = 0 .$$

PROOF. This statement is immediate from Lemmas 1.1 and 1.2 above.

PROOF OF THEOREM 1.1 (continued). In view of (1.10) and the definition of T it suffices to prove that T is asymptotically normal with mean 0 and variance σ^2/σ_c^2.

By Lemmas 1.1 and 1.3

(1.11) $$T = V_c(v) + o_p(1) + Y$$

where $$Y = \sigma_c^{-1} \sum_i c_i[L_i(H(H_n^{-1}(v))) - L_i(v)] .$$

We also have

$$
\begin{aligned}
n^{\frac{1}{2}}[H(H_n^{-1}(v)) - v] &= n^{\frac{1}{2}}[H(H_n^{-1}(v)) - H_n(H_n^{-1}(v))] + O(n^{-\frac{1}{2}}) \\
&= -V_1(H(H_n^{-1}(v)) + O(n^{-\frac{1}{2}}) \qquad \text{by (1.9)} \\
&= -V_1(v) + o_p(1) \qquad \text{by Lemma 1.3.}
\end{aligned} \tag{1.12}
$$

Since the sequence $\{V_1(v)\}$ has an asymptotically normal distribution, it follows that for any given $\varepsilon > 0$ there exist K_ε and $N_{1\varepsilon}$ such that $n > N_{1\varepsilon}$ implies

$$
P[|H(H_n^{-1}(v)) - v| \leqq K_\varepsilon n^{-\frac{1}{2}}] \geqq 1 - \varepsilon\,. \tag{1.13}
$$

Moreover by assumption (1.4) there exists numbers $\{l_i(v)\}$ such that for any given $\varepsilon > 0$ there exists $N_{2\varepsilon}$ such that $n > N_{2\varepsilon}$ implies

$$
\max_i \sup_{|t-v|<n^{-\frac{1}{2}}K_\varepsilon} n^{\frac{1}{2}}|L_i(t) - L_i(v) - (t - v)l_i(v)| < \varepsilon\,. \tag{1.14}
$$

We define

$$
A_{n,\varepsilon} = \{|H(H_n^{-1}(v)) - v| \leqq K_\varepsilon n^{-\frac{1}{2}}\} \qquad \text{and}
$$

$$
Z_{n,i,\varepsilon} = \{L_i(H(H_n^{-1}(v))) - L_i(v) - [H(H_n^{-1}(v)) - v]l_i(v)\}I(A_{n,\varepsilon})\,, \qquad 1 \leqq i \leqq n\,. \tag{1.15}
$$

Now observe that in view of (1.14) and (1.13) we have

$$
P(\max_i n^{\frac{1}{2}}|Z_{n,i,\varepsilon}| > \varepsilon) < \varepsilon \qquad \text{for} \quad n > N_\varepsilon \tag{1.16}
$$

where $N_\varepsilon = \max(N_{1\varepsilon}, N_{2\varepsilon})$.

Furthermore we have

$$
\begin{aligned}
Y &= YI(A_{n,\varepsilon}) + YI(A_{n,\varepsilon}^c) \\
&= \sigma^{-1}n^{\frac{1}{2}}\bar{c}[H(H_n^{-1}(v)) - v] + Z_{n,0,\varepsilon} + \sigma_c^{-1}\textstyle\sum_i c_i Z_{n,i,\varepsilon}
\end{aligned} \tag{1.17}
$$

where

$$
Z_{n,0,\varepsilon} = \{Y - \sigma_c^{-1}\bar{c}n^{\frac{1}{2}}[H(H_n^{-1}(v)) - v]\}I(A_{n,\varepsilon}^c)\,.
$$

Note that

$$
P(Z_{n,0,\varepsilon} \neq 0) \leqq P(A_{n,\varepsilon}^c) < \varepsilon \qquad \text{for} \quad n > N_\varepsilon\,. \tag{1.18}
$$

By the Cauchy inequality and (1.16)

$$
P[\sigma_c^{-1}|\textstyle\sum_i c_i Z_{n,i,\varepsilon}| > \varepsilon] < \varepsilon \qquad \text{for} \quad n > N_\varepsilon\,. \tag{1.19}
$$

Hence in view of (1.19), (1.18) and (1.17) we have

$$
Y = \sigma_c^{-1}n^{\frac{1}{2}}\bar{c}[H(H_n^{-1}(v)) - v] + o_p(1)\,. \tag{1.20}
$$

Consequently, by (1.10), (1.11), (1.12) and (1.13) we have in view of (1.5) that

$$
T - V_c(v) + \sigma_c^{-1}n^{\frac{1}{2}}\bar{c}V_1(v) = o_p(1)\,. \tag{1.21}
$$

But $V_c(v) - \sigma_c^{-1}n^{\frac{1}{2}}\bar{c}V_1(v)$ equals

$$
\sigma_c^{-1}\textstyle\sum_i (c_i - \bar{c})[I\{X_i \leqq H^{-1}(v)\} - L_i(v)] \tag{1.22}
$$

which under (1.2) and (1.7) has an asymptotically normal distribution by the Lindeberg–Feller Central Limit Theorem. The proof is complete upon observation that (1.22) has asymptotic mean 0 and variance $\sigma_c^{-2}\sigma^2$.

2. Signed rank statistics. In this section we prove the analogue of Theorem 1.1 for signed rank statistics. This result has been previously obtained by the authors in [5] using the projection method and other techniques of Dupač and Hájek in [2]. The proof presented here, however, is considerably shorter.

Given the model of Section 1, define

$$H_{n+}(x) = n^{-1} \sum_i I\{|X_i| \leqq x\}\,, \qquad 0 \leqq x < \infty$$
$$H_{+}(x) = n^{-1} \sum_i [F_i(x) - F_i(-x)]\,, \qquad 0 \leqq x < \infty\,.$$

Furthermore letting $p_i = F_i(0)$, $q_i = 1 - p_i$, $1 \leqq i \leqq n$ define, for $0 \leqq t \leqq 1$,

$$\begin{aligned} L_{i1}^{+}(t) &= [F_i(H_+^{-1}(t)) - p_i]/q_i && \text{if } q_i > 0 \\ &= 0 && \text{if } q_i = 0 \\ L_{i2}^{+}(t) &= [p_i - F_i(-H_+^{-1}(t))]/p_i && \text{if } p_i > 0 \\ &= 0 && \text{if } p_i = 0 \\ L_i^{+}(t) &= q_i L_{i1}^{+}(t) + p_i L_{i2}^{+}(t) \\ \mu_i^{+}(t) &= q_i L_{i1}^{+}(t) - p_i L_{i2}^{+}(t)\,. \end{aligned}$$

For a fixed point v in the unit interval, define the signed rank statistic

$$S^{+} = \sum_i c_i I\{|X_i| \leqq H_{n+}^{-1}(v)\} s(X_i) \tag{2.1}$$

where $s(x) = I\{x \geqq 0\} - I\{x \leqq 0\}$.

THEOREM 2.1. *Assume* (1.2) *holds. Also, assume*

(2.2) (i) $\lim_{\delta\to 0} \limsup_{n\to\infty} |L_{ij}^{+}(v+\delta) - L_{ij}^{+}(v-\delta)| = 0\,,$
$j = 1, 2,\ 1 \leqq i \leqq n$ *and*
(ii) *that there exist numbers* $\{l_{i1}^{+}(v)\}$ *and* $\{l_{i2}^{+}(v)\}$ *such that for all* $K > 0$

$$\lim_{n\to\infty} \max_i \max_{|t-v| \leqq Kn^{-\frac{1}{2}}} n^{\frac{1}{2}} |L_{ij}^{+}(t) - L_{ij}^{+}(v) - (t-v) l_{ij}^{+}(v)| = 0\,, \qquad j = 1, 2\,.$$

Defining $\tilde{c}_{+} = n^{-1} \sum c_i \{q_i l_{i1}^{+}(v) - p_i l_{i2}^{+}(v)\}$, *we further assume*

$$\limsup_{n\to\infty} n^{\frac{1}{2}} \sigma_c^{-1} |\tilde{c}_{+}| < \infty\,. \tag{2.3}$$

Then S^{+} *given by* (2.1) *is asymptotically normal with parameters* (μ_+, σ_+^2), *where* $\mu_+ = \sum c_i \mu_i^{+}(v)$ *and*

$$\begin{aligned} \sigma_+^2 = \sum \{ & c_i^2 [L_i^{+}(v) - (\mu_i^{+}(v))^2] + \tilde{c}_+^2 L_i^{+}(v)(1 - L_i^{+}(v)) \\ & - 2 c_i \tilde{c}_+ [\mu_i^{+}(v)(1 - L_i^{+}(v))]\} \end{aligned}$$

provided that

$$\liminf_{n\to\infty} \sigma_+^2/\sigma_c^2 > 0\,. \tag{2.4}$$

REMARK. Note that if the X_i's are symmetric random variables $\tilde{c}_+ = 0 = \mu_+$, and $\sigma_+^2 = \sum_i c_i^2 L_i^{+}(v)$.

PROOF OF THEOREM 2.1. The method of proof is similar to that of Theorem 1.1, so we shall be brief. We define, for $0 \leqq t \leqq 1$,

$$V_c^{+}(t) = \sigma_c^{-1} \sum_i c_i [I\{|X_i| \leqq H_+^{-1}(t)\} s(X_i) - \mu_i^{+}(t)]\,. \tag{2.5}$$

Then denoting $\sigma_c^{-1}[S^+ - \mu^+]$ by T^+ we obtain

(2.6) $$T^+ = V_c^+(H_+(H_{n+}^{-1}(v))) + \sigma_c^{-1} \sum_i c_i[\mu_i^+(H_+(H_{n+}^{-1}(v))) - \mu_i^+(v)] .$$

In view of the proof of Theorem 1.1 and the decomposition it should be clear that we need results analogous to Lemmas 1.1, 1.2 and 1.3.

LEMMA 2.1. $$\sup_{0<t<1} |H_+(H_{n+}^{-1}(t)) - t| = o_p(1) .$$

PROOF. This statement follows from Lemma 1.1 applied to the random variables $|X_1|, |X_2|, \cdots, |X_n|$.

LEMMA 2.2. *Assume that* v *is fixed in* [0, 1] *and that* (2.2) (i) *is satisfied. Then for any* $\varepsilon > 0$

(2.7) $$\lim_{\delta\to 0} \limsup_{n\to\infty} \Pr\{\sup_{|t-v|<\delta} |V_c^+(t) - V_c^+(v)| > \varepsilon\} = 0 .$$

PROOF. Using (1.8) and (2.5) we may write

(2.8) $$V_c^+(t) = W_{c1}(t) + W_{c2}(t)$$

where

$$W_{c1}(t) = V_c(H(H_+^{-1}(t))) - V_c(H(0))$$

and

$$W_{c2}(t) = V_c(H(-H_+^{-1}(t))) - V_c(H(0)) .$$

It is clear that the lemma will follow if we verify condition (2.7) with V_c^+ replaced by W_{c1} and then by W_{c2}. Now

$$W_{c1}(t) = \sigma_c^{-1} \sum_i c_i[I\{0 < H_+(X_i) \leqq t\} - q_i L_{i1}^+(t)]$$

is a weighted empirical process defined in terms of random variables with improper but finite and continuous distribution functions $q_i L_{i1}^+$, $1 \leqq i \leqq n$. Lemmas 2.1 and 2.2 of [4] also hold for such processes; and if $G_1(t) = \sigma_c^{-2} \sum_i c_i^2 q_i L_{i1}^+(t)$ they imply for $\varepsilon > 0$,

$$\Pr\{\sup_{|t-v|<\delta} |W_{c1}(t) - W_{c1}(v)| > \varepsilon\} \leqq K[G_1(v+\delta) - G_1(v-\delta)]/\varepsilon^2 ,$$

Since $G_1(v+\delta) - G_1(v-\delta) \leqq \max_i |L_{i1}(v+\delta) - L_{i1}(v-\delta)|$, and since the W_{c2} process may be treated in the same way, the lemma follows from (2.2) (i) above.

An immediate consequence of Lemmas 2.1 and 2.2 is

LEMMA 2.3. *If* (2.2) (i) *is satisfied, then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{|V_c^+(H_+(H_{n+}^{-1}(v))) - V_c^+(v)| > \varepsilon\} = 0 .$$

PROOF OF THEOREM 2.1 (continued). Using (2.6) and Lemma 2.3 we obtain

(2.9) $$T^+ = V_c^+(v) + o_p(1) + Y^+ ,$$ where

(2.10) $$Y^+ = \sigma_c^{-1} \sum_i c_i[\mu_i^+(H_+(H_{n+}^{-1}(v))) - \mu_i^+(v)] .$$

Then, denoting W_{cj} by W_{1j} when $c_i = 1$, $1 \leqq i \leqq n$, and writing $W(v)$ for $W_{11}(v) - W_{12}(v)$, we have

(2.11) $$n^{\frac{1}{2}}[H_{n+}(H_+^{-1}(v)) - v] = W(v) ,$$

so that we have an analogue of (1.12), namely

$$n^{\frac{1}{2}}[H_+(H_{n+}^{-1}(v)) - v] = -W(v) + o_p(1). \tag{2.12}$$

Next, using (2.2) (ii) and (2.12) above and the fact that the sequence $\{W(v)\}$ is bounded in probability one may show, with details similar to the derivation of (1.21), that

$$T^+ - V_c^+(v) + n^{\frac{1}{2}}\bar{c}_+\sigma_c^{-1}W(v) = o_p(1). \tag{2.13}$$

Now using the definitions of V_c^+ and W we have

$$\begin{aligned} V_c^+(v) - n^{\frac{1}{2}}\bar{c}_+\sigma_c^{-1}W(v) &= \sigma_c^{-1}\textstyle\sum_i \{c_i[I(|X_i| \leqq H_+^{-1}(v))s(X_i) - \mu_i^+(v)] \\ &\qquad - \bar{c}_+[I\{|X_i| \leqq H_+^{-1}(v)\} - L_i^+(v)]\} \\ &= \sigma_c^{-1}Z_+, \qquad \text{say,} \end{aligned} \tag{2.14}$$

where Z_+ is asymptotically normal (with parameters $(0, \sigma_+^2)$ by the Lindeberg Central Limit Theorem). The proof is complete.

3. Weak convergence.

In Sections 1 and 2 we determined the asympotic distribution of the statistics T_n and T_n^+ at an arbitrary but fixed point v in the unit interval. Here we shall give sufficient conditions for convergence of the processes $\{T_n(v): 0 \leqq v \leqq 1\}_n$ and $\{T_n^+(v): 0 \leqq v \leqq 1\}_n$ to the appropriate continuous Gaussian processes. As before the dependence on n of X_{ni}, F_{ni}, c_{ni}, and functions of them will be notationally suppressed, with the exception of T_n, T_n^+, and the processes K_n, K_n^+ defined below. We consider the linear rank case first.

THEOREM 3.1. *Assume* (1.2) *of Theorem* 1.1 *and*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \max_i \max_{|t-s|<\delta} |L_i(t) - L_i(s)| = 0, \tag{3.1}$$

and that

(3.2) *there exist functions* l_i, $1 \leqq i \leqq n$, *on* $[0, 1]$ *such that for all* $K > 0$

$$\lim_{n\to\infty} \max_i \max_{0\leqq s\leqq 1} \max_{|t-s|\leqq Kn^{-\frac{1}{2}}} n^{\frac{1}{2}}|L_i(t) - L_i(s) - (t-s)l_i(s)| = 0.$$

Furthermore, defining $\bar{c}(t) = n^{-1}\sum_i c_i l_i(t)$ *and* $\sigma^2(t) = \sum_i (c_i - \bar{c}(t))^2 L_i(t)(1 - L_i(t))$, $0 \leqq t \leqq 1$ *we assume*

$$\limsup_{n\to\infty} \sup_{0\leqq t\leqq 1} n^{\frac{1}{2}}|\bar{c}(t)|\sigma_c^{-1} < +\infty, \tag{3.3}$$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{|t-s|\leqq\delta} n^{\frac{1}{2}}|\bar{c}(t) - \bar{c}(s)|\sigma_c^{-1} = 0 \qquad \text{and} \tag{3.4}$$

$$\liminf_{n\to\infty} \frac{\sigma^2(t)}{\sigma_c^2} > 0 \qquad \text{for} \quad 0 < t < 1. \tag{3.5}$$

Finally, defining

$$K_n(t) = V_c(t) - n^{\frac{1}{2}}\bar{c}(t)\sigma_c^{-1}V_1(t), \qquad 0 \leqq t \leqq 1, \tag{3.6}$$

we assume

$$\begin{aligned} C(t, s) &= \lim_{n\to\infty} \operatorname{Cov}(K_n(t), K_n(s)) \\ &= \lim_{n\to\infty} \sigma_c^{-2} \textstyle\sum_i (c_i - \bar{c}(s))(c_i - \bar{c}(t))L_i(s)[1 - L_i(t)] \end{aligned} \tag{3.7}$$

exists for all $0 \leqq s \leqq t \leqq 1$.

Then $\{T_n\}$ converges weakly to a Gaussian process T having continuous sample paths and the following properties:

$$T(0) = 0 = T(1), \qquad ET = 0, \qquad \textit{and} \tag{3.8}$$
$$\operatorname{Cov}(T(s), T(t)) = C(s, t), \qquad 0 \leqq s, \quad t \leqq 1.$$

REMARK 1. Since $S_n^*(v) = S_n((n+1)v/n)$ it is clear that the above theorem holds for $T_n^*(v) = T_n((n+1)v/n)$, $0 \leqq v \leqq 1$.

REMARK 2. In (3.2) we may assume without loss of generality that the l_i's are measurable functions and $n^{-1}\sum_{i=1}^n l_i(s) = 1$, $0 \leqq s \leqq 1$. For if (3.2) is satisfied for $\{l_i : 1 \leqq i \leqq n\}$ then it is satisfied for $\{l_i^* : 1 \leqq i \leqq n\}$, where $l_i^*(s) = n^{\frac{1}{2}}[L_i(s + n^{-\frac{1}{2}}) - L_i(s)]$, $0 \leqq s \leqq 1$.

REMARK 3. Conditions (1.2) and (3.3) may together be replaced by the boundedness condition $\limsup n^{\frac{1}{2}} \max_i |c_i|\sigma_c^{-1} < \infty$, for in view of the last remark

$$\sup_{0 \leqq t \leqq 1} n^{\frac{1}{2}}|\bar{c}(t)|\sigma_c^{-1} \leqq n^{\frac{1}{2}} \max_i |c_i|\sigma_c^{-1}.$$

REMARK 4. Condition (3.4) may be replaced by the stronger condition

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \max_i \max_{|t-s| \leqq \delta} |l_i(t) - l_i(s)| = 0$$

since

$$n^{\frac{1}{2}}|\bar{c}(t) - \bar{c}(s)|\sigma_c^{-1} \leqq \max_i |l_i(t) - l_i(s)|.$$

REMARK 5. In the special case $F_i = F$, $i = 1, \cdots, n$ we have $L_i(t) = t$ and $l_i(t) = 1$ for $0 \leqq t \leqq 1$, so conditions (3.1) through (3.4) are trivially satisfied. Moreover, $\operatorname{Cov}(K_n(t), K_n(s)) = s(1-t)$, $0 \leqq s \leqq t \leqq 1$ so (3.7) is satisfied. Thus Theorem 3.1 includes Theorem V. 3.5.1. of [3].

PROOF OF THEOREM 3.1. We first observe that conditions (3.1), (3.2), (3.3) and Lemma 2.3 of [4] enable us to strengthen (1.21) of Theorem 1.1 to

$$\sup_{0 \leqq t \leqq 1} |T_n(t) - K_n(t)| = o_p(1). \tag{3.9}$$

Moreover, (3.1), (3.2), (3.3), and (3.5) imply (1.3), (1.4), (1.5), and (1.7) for each fixed point t in [0, 1], so that the distributions of $\{T_n(t_1), \cdots, T_n(t_r)\}$ converge to that of Gaussian distribution with $r \times r$ covariance matrix $[C(t_i, t_j)]$, where C is given by (3.7).

Finally, in view of (3.9) we need only to show that for any $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \Pr\{\sup_{|s-t| < \delta} |K_n(s) - K_n(t)| > \varepsilon\} = 0. \tag{3.10}$$

But $|K_n(t) - K_n(s)|$ is bounded above by

$$|V_c(t) - V_c(s)| + n^{\frac{1}{2}}\sigma_c^{-1}|V_1(t)|\,|\bar{c}(t) - \bar{c}(s)| + n^{\frac{1}{2}}\sigma_c^{-1}|\bar{c}(s)|\,|V_1(t) - V_1(s)|$$

and hence (3.10) follows from conditions (3.3), (3.4) and Lemma 2.3 of [4]. The proof is complete.

We now state the weak convergence result for the signed rank processes $\{T_n^+(v) : 0 \leqq v \leqq 1\}$ defined in Section 2.

THEOREM 3.2. *Assume that conditions* (3.1) *and* (3.2) *of Theorem* 3.1 *are satisfied for each of the sets* $\{L_{i1}^+: 1 \leqq i \leqq n\}$ *and* $\{L_{i2}^+: 1 \leqq i \leqq n\}$, *and denote the respective sets of approximating functions by* $\{l_{i1}^+: 1 \leqq i \leqq n\}$ *and* $\{l_{i2}^+: 1 \leqq i \leqq n\}$. *Then, defining* $\tilde{c}_+(t)$ *and* $\sigma_+^2(t)$ *as in Theorem* 2.1 *for* $0 \leqq t \leqq 1$, *we assume conditions* (3.3), (3.4) *and* (3.5) *are satisfied when* $\tilde{c}_+$, σ_+^2 *replace* $\tilde{c}$, σ^2.

Define $K_n^+(t) = V_c^+(t) - n^{\frac{1}{2}}\tilde{c}_+(t)\sigma_c^{-1}W(t)$ *and assume that*

$$C_+(t, s) = \lim_{n\to\infty} \operatorname{Cov}(K_n^+(t), K_n^+(s)) \tag{3.11}$$

exists for $0 \leqq s,\ t \leqq 1$.

Then $\{T_n^+\}$ *converges weakly to a continuous Gaussian process* W^+ *which satisfies*

$$W^+(0) = 0\,, \qquad EW^+ = 0\,, \qquad \textit{and}$$
$$\operatorname{Cov}(W^+(s), W^+(t)) = C^+(s, t)\,, \qquad 0 \leqq s\,, \quad t \leqq 1\,.$$

PROOF OF THEOREM 3.2. The proof of Theorem 3.1 can be modified to prove this result in the same way that the proof of Theorem 1.1 was modified to prove Theorem 2.1.

REMARK 1. Remarks 1 through 3 following Theorem 3.1 apply to Theorem 3.2 with obvious modifications.

REMARK 2. Even in the special case $F_i = F$, $1 \leqq i \leqq n$, Thoerem 3.2 is to our knowledge an unpublished result. It is the analogue of Theorem V. 3.5.1 of [3] for the signed rank statistics (2.1). Furthermore, if $c_i = 1$ for all i, we observe that

$$\begin{aligned}\sup_{0\leqq v\leqq 1} |T_n^+(v)| = \sup_{0<x<\infty} n^{\frac{1}{2}} |\{H_n(x) - H_n(0)\} - \{H_n(0) - H_n(x)\} \\ - \{F(x) - F(0)\} + \{F(0) - F(-x)\}|\end{aligned}$$

which is precisely the statistic τ_n^* proposed by Smirnov [6] in 1946 to test the symmetry of F. Smirnov considered only the null distribution but Theorem 3.2 allows us to compute the asymptotic distribution of τ_n^* under a general class of alternatives.

If the c_i's are allowed to be arbitrary (subject to (1.2)) we obtain a class of generalized Smirnov staistics $\sup_{0\leqq t\leqq 1} |T_n^+(t)|$ for testing symmetry which are analogous to the generalized Kolmogorov-Smirnov statistics studied in Chapter V. 3 of [3].

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