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WEAK FIELD LIMIT OF GENERAL RELATIVITY IN :RMS OF NEW VARIABLES: A HAMILTONIAN FRAMEWORK

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A lf-contained treatment of the linearization procedure for constrained Hamiltonian systen s first presented in a general setting. The procedure is then applied to general relati using triads and self-dual connections as the basic canonical variables. These result ave paved way to the quantization of weak gravitational waves in the connection and lc representations and to a study of the relation between these quanta and non-pertur ative canonical gravity. In the classical theory, they suggest a new approach to the treatn nt of gravitational perturbations and may be useful also to the theory underlying weak avity waves.

1. Introduction

In the mid-eighties, a new Hamiltonian framework for general relativity was introduced [1] which shifts emphasis from geometrodynamics to connection dynamics. The configuration variable of the theory is, in effect, a complex-valued SO(3) connection, A_a^i, on a spatial 3-slice Σ which can be interpreted as the pull-back to Σ of the self-dual part of the Lorentz spin connection on space-time. The conjugate momentum E_i^a is a vector density of weight 1 on Σ, which represents a "density weighted triad" on Σ. The resulting Hamiltonian framework has two key advantages over the well-known geometrodynamical framework (see, e.g., [2]). The first is that all field equations simplify considerably, being polynomial of low orders in the new canonical variables. Furthermore, dynamics of the theory has a natural geometrical interpretation: time evolution in space-time now corresponds to motion along a null geodesic in the space of connections. Such technical simplications have led to a number of results in classical relativity and differential geometry of self-dual metrics. (See, e.g., [3,4].) The second advantage is that, regarded as a theory of connections, general relativity is remarkably similar to Yang-Mills theory. Indeed, the phase spaces of the two theories are the same and the constraint surface of general relativity is naturally embedded in the constraint surface of Yang-Mills theory. This enables one to borrow into general relativity techniques from gauge theories [1,5]. The two features together have suggested a new approach to the construction of a non-perturbative quantum theory of gravity. (For recent reviews see [6].)

This paper has a dual purpose. First, we wish to present a systematic analysis of the weak field limit of this Hamiltonian description. Second, we will discuss in detail a general framework for linearizing constrained Hamiltonian systems. While this discussion is, in a sense, only a prelude to the linearization of general relativity, the results presented in this part are quite general and therefore of considerable interest in their own right.

In particular, they are applicable also to the investigations of cosmological perturbations, higher derivative theories of gravity, gauge theories, string theory, etc.

Linearization of the new Hamiltonian formulation of general relativity is motivated by three considerations. The first is conceptual. In Yang-Mills theory, quantization of the connection gives rise to a multiplet of massless spin-one particles. On the other hand, one expects the weak-field quanta of the gravitational field to be the states of a single massless spin two particle. The question therefore arises: How does the multiplet of spin-one quanta contained in the connection A_a^i reduce to a single spin-two quantum? The natural arena for analysing this issue is the weak field limit. One expects to resolve this issue essentially by counting of the number of true degrees of freedom and examining the form of the fields representing them, without having to actually quantize the system. We shall see that this expectation is correct. The second motivation comes from quantum gravity proper. In text-book treatments, graviton wavefunctions arise as positive frequency fields and multi-graviton wavefunctions are represented as holomorphic functionals of these positive frequency fields. In full quantum gravity, positive frequency fields are not available and the new Hamiltonian framework suggests that we use self-dual fields instead. Thus, the strategy envisaged is that quantum states be represented by (generalized) holomorphic functionals of self-dual connections A_a^i [1]. An important check on the viability of this idea is whether it can be applied successfully in the weak field limit, where the quantum theory is well understood. The first step in carrying out this program is the construction of the weak-field limit of the Hamiltonian framework based on self-dual connections. We carry out this step here. A subsequent paper will show that the quantization based on this framework does indeed lead to the familiar Fock space of gravitons even though an explicit decomposition into positive and negative frequency parts is not carried out. The last motivation comes from classical general relativity. Since the equations of the full theory simplify considerably in terms of the new canonical variables, one might expect the perturbation analysis to simplify in an analogous fashion. This may be the case especially when the backgrounds involved are conformally flat –as in FRW cosmologies– or have Weyl curvature of an especially simple form –as in the black hole spacetimes– since the connection A_a^i is a potential for the self-dual part of the Weyl tensor. An analysis of the simplest of these cases –perturbations off Minkowski space– is again the obvious first step.

In section 2, we present the general framework for linearizing a constrained Hamiltonian framework, emphasizing the subtle points such as linearization stability in the spatially compact context and linearization off time dependent backgrounds that one encounters in the cosmological contexts. Section 3 is devoted to the linearization of the new Hamiltonian framework for vacuum general relativity. In particular, we will see how the six "spin-1 modes" in the connection A_a^i reduce to the two "spin-2 modes" of the gravitational field in general relativity, discuss the dynamics of these true-degrees of freedom and analyse the linearization instability problems in the spatially compact cases.

Several of the results reported here were obtained a number of years ago and most of the material of section 3.2 has appeared in a monograph [7]. However, the results contained in sections 2.2, 2.3 and 3.3 are new and the older material is included both for completeness and because it is not easily accessible.

2. Linearization of constrained Hamiltonian systems

In this section, we will consider general Hamiltonian systems with first class constraints, where the constraints may or may not be related to the Hamiltonian. The notation will be as follows: The phase space will be denoted by Γ, the symplectic structure by $\Omega_{\alpha\beta}$, the constraint submanifold by $\bar{\Gamma}$, and the pull-back of the symplectic structure to $\bar{\Gamma}$ by $\bar{\Omega}_{\alpha\beta}$. On Γ, the inverse of the symplectic structure $\Omega^{\alpha\beta}$ is defined via: $\Omega^{\alpha\beta}\Omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. Given any function f, the Hamiltonian vectorfield X^{a}_{f} it defines on Γ will be defined as $X^{\alpha}_{f} = \Omega^{\alpha\beta}\nabla_{\beta}f$. Finally, the convention for the Posson brackets will be: $\{f,g\} = \Omega^{\alpha\beta}\nabla_{\alpha}f\nabla_{\beta}g \equiv -\mathcal{L}_{X_f}g$.

We will often work with constraint functions C_i; that is,the constraint sub-manifold $\bar{\Gamma}$ will be taken to be the one defined by the equations $C_i = 0$. Except in the second subsection, we will assume that C_i are "good coordinates", i.e., that their gradients $\nabla_a C_i$ do not vanish *anywhere* on $\bar{\Gamma}$. To clarify the structures involved, we will sometimes regard Γ as a 2n-dimensional manifold and $\bar{\Gamma}$ as a 2n-m-dimensional submanifold thereof; there will be m first class constraints and n-m true (configuration) degrees of freedom. However, our main discussion is not tied, in an essential way, to systems with only a finite number of degrees of freedom. To discuss field theories in detail, one has of course to specify the precise function spaces involved and handle the resulting analysis carefully. However, the basic results presented in this section are applicable to such systems as well.

Since the constraint sub-manifold $\bar{\Gamma}$ is of first class, by definition, it has the property that $\Omega^{\alpha\beta}n_{\beta}$ is tangential to $\bar{\Gamma}$ for every covariant normal n_{β} to $\bar{\Gamma}$ (for details, see. e.g., [8]). In terms of the constraint functions C_i, a complete set of linearly independent normals is given by $n^{(i)}_{\alpha} = \nabla_{\alpha}C_i$. Using the fact that $\bar{\Gamma}$ is a first-class constraint submanifold, it is easy to show that the constraint functions C_i on Γ satisfy $\{C_i, C_j\} = f^{k}_{ij}C_k$ for some functions f^{k}_{ij} on Γ. This last equation is in fact a necessary and sufficient condition that $\bar{\Gamma}$ is of first class. The Hamiltonian vector fields $X_i = \Omega^{\alpha\beta}\nabla_{\beta}C_i$ generated by the constraint functions represent infinitesimal gauge motions. Physical observables –such as the Hamiltonian– are to be invaraint under these motions. It therefore follows, in particular, that the Poisson brackets between the Hamiltonian H and the constraint functions C_i are of the form: $\{H, C_i\} = g^{j}_{i}C_j$. This condition in turn implies that the constraints are preserved by dynamical evolution.

This section is divided into three parts. In section 2.1, we discuss the issue of linearization about a time-independent background –i.e., a point in the phase space at which the Hamiltonian vector field vanishes– assuming that none of the gradients, $\nabla_{\alpha}C_i$, vanish there. The next two sub-sections extend the results of 2.1 in two different directions. In section 2.2, we allow for the possibility that some of the constraint functions may have a vanishing gradient at the background point. At generic points of $\bar{\Gamma}$, of course, $\nabla_{\alpha}C_i$ will not vanish since $\bar{\Gamma}$ is *defined* by the coordinate condition $C_i = 0$. However, sometimes the background point has special symmetries which can force the gradients to vanish there. When this happens, the system has "linearization instabilities." In section 2.3, we consider linearization about a time dependent background. Thus, now, the Hamiltonian vector field is not assumed to have zeros and linearization is carried out along a non-trivial integral curve. This discussion may be of interest, e.g., to the study of cosmological perturbations.

2.1 Linearization about a time-independent background

On the phase space, a time-independent solution is represented by a point, say p, on the constraint surface $\bar{\Gamma}$ at which the Hamiltonian vector field $X_H^\alpha \equiv \Omega^{\alpha\beta}\nabla_\beta H$ vanishes. Since $\Omega^{\alpha\beta}$ is non-degenerate, it follows that the gradient $\nabla_\alpha H$ of the Hamiltonian must also vanish at p. As explained above, in this section we assume that none of the gradients $\nabla_\alpha C_i$ of constraint functions C_i vanish at p. An illustrative application would be to the problem of linearization of Einstein's equations (possibly coupled to, say, the Yang-Mills-Higgs system) in the *asymptotically flat context.*

To obtain the Hamiltonian framework for the linearized theory, we begin by considering, within our phase space, infinitesimal deviations from p. These are represented by tangent vectors to Γ at p. Thus, the linearized phase space Γ_o is precisely the tangent space $T_p\Gamma$ to Σ. The symplectic tensor on Γ_o is just the restriction of $\Omega_{\alpha\beta}$ to p. We will denote it by $\Omega^o_{\alpha\beta}$ and its inverse by $\Omega_o^{\alpha\beta}$; thus $\Omega_o^{\alpha\beta}\Omega^o_{\beta\gamma} = \delta^\alpha_\gamma$. As expected, the linearized phase space, $(\Gamma_o, \Omega^o_{\alpha\beta})$, is a symplectic vector space. We will denote by ∂ the derivative operator on Γ_o which is compatible with its vector space structure.

The constraints of the full theory lead to linearized constraints on Γ_o. Consider a curve in the constraint surface $\bar{\Gamma}$ passing through p. The tangent vector to this curve at p represents an infinitesimal deviation from p satisfying constraints to first order. Hence, the constraint subspace $\bar{\Gamma}_o$ of the linearized phase space is the subspace of Γ_o which is tangential to the full constraint surface $\bar{\Gamma}$. Since the elements of $\bar{\Gamma}$ represent physically allowable states of the system, elements of $\bar{\Gamma}_o$ represent physically allowable pertubations. We can characterize these in terms of the constraint functions C_i as follows: Elements of $\bar{\Gamma}_o$ are those tangent vectors v at p which satisfy

$$C_i^L(v) := v^a(\nabla_a C_i)|_p = 0 \ , \tag{2.1}$$

for all $i = 1, 2, ..., m$. The functions $C_i^L(v)$ on Γ_o are thus the linearized constraints. Evaluating their Poisson brackets on Γ_o, one obtains:

$$\begin{aligned} \{C_i^L, C_j^L\}_o &= \Omega_o^{\alpha\beta}(\partial_\alpha C_i^L)(\partial_\beta C_j^L) \\ &= \Omega_o^{\alpha\beta}(\nabla_\alpha C_i)|_p(\nabla_\beta C_j)|_p \\ &= \{C_i, C_j\}|_p = 0 \ . \end{aligned} \tag{2.2}$$

Thus, not only are the linearized constraints of first class but their Poisson brackets are always *strongly* zero.

Let us next consider linearized gauge transformations. These are generated by the linearized constraints. Since $C_i^L(v)$ are all linear in their arguments, the Hamiltonian vector fields they generate are constant on the vector space Γ_o: ${}^oX_i^\alpha = \Omega_o^{\alpha\beta}\partial_\beta C_i$. They are naturally identified with the restriction to p of the constraint vector fields $X_i^\alpha := \Omega^{\alpha\beta}\nabla_\beta C_i^L$ of the full theory. Consequently, we have the following geometrical picture: $(\Gamma_o, \Omega^o_{\alpha\beta})$ is a 2n-dimensional symplectic vector space; $\bar{\Gamma}_o$ is a 2n-m-dimensional subspace of $\bar{\Gamma}_o$ and the linearized gauge directions span a m-dimensional subspace $\mathcal{G}^L$ of Γ_o. The true degrees of freedom of the linearized theory are represented by vectors in the 2(n-m)-dimensional vector space, $\hat{\Gamma}_o := \bar{\Gamma}_o/\mathcal{G}^L$. Since the linearized constaints are also of first class, the pull-back $\bar{\Omega}^o_{\alpha\beta}$ to $\bar{\Gamma}_o$ of the symplectic structure $\Omega^o_{\alpha\beta}$ projects down unambiguously to $\hat{\Gamma}_o$ and provides a natural symplectic structure $\hat{\Omega}^o_{\alpha\beta}$ thereon.

Let us now consider linearized dynamics. Recall first that the dynamics of the full theory is generated by the Hamiltonian vector field X_H^α. Since X_H^α vanishes at p, the 1-parameter family of diffeomorphisms it generates leaves p fixed and naturally induces the following linear transformation on the tanget space at $T_p\Gamma$: $v^\alpha \to T^\alpha{}_\beta v^\beta$ where $T^\alpha{}_\beta = (\nabla_\beta X_H^\alpha)|_p$. This provides us the linearized dynamics:

$$\dot{v}^\alpha = T^\alpha{}_\beta v^\beta \equiv -v^\beta \nabla_\beta X_H^\alpha|_p \ , \tag{2.3}$$

where, as before, v^α is a generic element of the linearized phase space Γ_o. (Note that $T^\alpha{}_\beta$ is independent of the choice of the torsion-free derivative operator used in its definition because X_H^α vanishes at p.) The right hand side of (2.3) defines the infinitesimal linearized dynamics. The first question is whether this evolution preserves the linearized symplectic structure $\Omega^o_{\alpha\beta}$. Now, on any linear phase space, $(\Gamma_o, \Omega^o_{\alpha\beta})$, the motion $v^\alpha \to T^\alpha{}_\beta v^\beta$ is an infinitesimal canonical transformation if and only if $T^\alpha{}_{[\beta}\, \Omega^o_{\gamma]\alpha} = 0$. For the $T^\alpha{}_\beta$ now under consideration, we have: $T^\alpha{}_\beta \Omega^o_{\gamma\alpha} = -(\nabla_\gamma \nabla_\beta H)|_p$. Hence the motion (2.3) does represent an infinitesimal canonical transformation. We can therefore ask for its generting function, i.e., for the Hamiltonian $H^L(v)$ of the linearized theory. Since the transformation under consideration is linear, the task of computing the Hamiltonian is straightforward. We have:

$$H^L(v) = \tfrac{1}{2} v^\alpha v^\beta (\nabla_\alpha \nabla_\beta H)|_p \tag{2.4}$$

Thus, the first derivative of the full Hamiltonian vanishes at p and its second derivative determines the Hamiltonian of the linearized theory.

We can now ask if the linearized Hamiltonian is guage invariant. A calculation analogous to (2.2) yields

$$\{H^L, C_i^L\}_o = 0 \ ,$$

whence the linearized Hamiltonian is indeed invariant under the linearized gauge transformations. Hence, it can be unambiguously projected down to the reduced phase space $\hat{\Gamma}_o$. Recall finally that there is freedom to add a multiple, say $h^i C_i$, of the constraints to the full Hamiltonian H since this addition does not affect dynamics of the true degrees of freedom of the full theory. However, under this addition, the Hamiltonian vector field X_H^α of the full theory does change; on the constraint surface Γ_o, we have: $X_H^\alpha \to X_H^\alpha + \Omega_o^{\alpha\beta} h^i \nabla_\beta C_i$. Therefore, unless the extra term, $\Omega_o^{\alpha\beta} h^i \nabla_\beta C_i$, vanishes at p, our assumption that the point p is left invariant by the full dynamical flow will not be satisfied. If the extra term does vanish, the framework will be applicable, although the linearized Hamiltonian flow will now acquire an additional term in the direction of gauge transformations and the linearized Hamiltonian will change correspondingly. On the physical phase space $\hat{\Gamma}_o$ on the other hand, the Hamiltonian will remain unaffected.

2.2 Linearization stability

We will now relax the assumption that none of the gradients $\nabla_\alpha C_i$ of the constraint functions C_i vanish at p. This situation often arises when the background point has a gauge symmetry. Consider, as an illustration, $SU(n)$ Yang-Mills theory in a spatially compact context, say on $T^3 \times R$. In this case, the phase space Γ is spanned by pairs

(A^i_a, E^a_i), where a is the tangent space index and i refers to the Lie algebra of $SU(n)$. The constraint surface $\bar{\Gamma}$ is now specified by the Gauss law:

$$C_\omega(A, E) := \int_{T^3} d^3x \; \omega_i \; D_a E^{ai} = 0 \qquad (2.5a)$$

for all Lie-algebra-valued functions $\omega^i(x)$ on T^3. The gradient of C_ω on Γ is given by:

$$\frac{\delta C_\omega}{\delta A^i_a} = -[\omega, E^a]^i, \quad \text{and} \quad \frac{\delta C_\omega}{\delta E^a_i} = -D_a\omega^i \; . \qquad (2.5b)$$

At a generic point on the phase space, the right sides of these two equations are non-zero for any given ω^i and one therefore generally *defines* $\bar{\Gamma}$ by (2.5a). However, at a point (A^i_a, E^a_i) which is invariant under gauge transformations generated by some ω^i_o, the right hand sides vanish for C_{ω_o}. If we happen to linearize about such a point, therefore, one of the key assumptions of section 2.1 is violated. (In particular, if we linearize about the "obvious" background ($A^i_a = 0, E^a_i = 0$), the gradients of C_ω vanish for all generators ω^i of *global* $SU(n)$ transformations.) As is well-known, this is the origin of the conical singularities in the configuration space of true degrees of freedom of the full Yang-Mills theory. In the context of the linearized theory, on the other hand, this "pathology" is responsible for linearization instabilities. We will return to this issue at the end of this subsection[1].

Let us return to the general context. As before, let there be m first class constraints $C_i = 0$ but let us suppose that the gradient of one of them, say C_1, vanishes at the background point p we have chosen. Then, C_1^L vanishes identically and does not give rise to a linearized constraint. Thus, it appears at first that $\bar{\Gamma}_o$ is 2n-m+1-dimensional; i.e. that the linearized theory has an additional degree of freedom relative to the full theory. However, as we will see, this conclusion is incorrect: C_1 does make its presence felt through a quadratic condition.

To see this, let us look at the situation from a geometric viewpoint. Consider a smooth parametrized curve $\gamma(s)$ passing through p and lying entirely in the constraint sub-manifold $\bar{\Gamma}$. Clearly, all the derivatives of constraints vanish along $\gamma(s)$. For constraints C_i, with $i = 2, 3, ...m$, the condition that their first derivative vanishes along $\gamma(s)$ constrains the permissible tangent vectors, forcing them to lie in a 2n-m+1-dimensional subspace of Γ_o. The condition that their second derivatives vanish along $\gamma(s)$ does not further restrict the tangent vectors; it is a constraint on the *rate of change* of the tangent vectors, or, on second-order variations. The situation is different for the constraint C_1 because its gradient vanishes at p. Now, the second derivative, $\dot{\gamma}^\alpha\nabla_\alpha(\dot{\gamma}^\beta\nabla_\beta C_1)|_p$ along the curve $\gamma(s)$ can be defined without the knowledge of the rate of change of the tangent vector:

$$[\dot{\gamma}^\alpha\nabla_\alpha(\dot{\gamma}^\beta\nabla_\beta C_1)]|_p = [\dot{\gamma}^\alpha\dot{\gamma}^\beta\nabla_\alpha\nabla_\beta C_1]|_p \; .$$

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[1] In the asymptotically flat context, to construct the phase space framework, one must specify appropriate boundary conditions. The constraint surface $\bar{\Gamma}$ can be defined by the equations: $C_\omega = 0$ for all Lie-algebra valued functions ω^i of *compact support.* Since none of the global gauge transformations are of compact support, in that case the gradients of C_ω do not vanish anywhere on $\bar{\Gamma}$ and the considerations of this sub-section are unnecessary.

Hence, the vanishing of the second derivative along $\gamma(s)$ *does* restrict the tangent vectors. Thus, in all, we do have all m constraints on Γ_o. However, one of them, $C_1^L(v) := \frac{1}{2}v^\alpha v^\beta[\nabla_\alpha\nabla_\beta C_1]|_p$, is quadratic in v. Thus, the linearized constraint surface $\bar{\Gamma}_o$ is 2n-m-dimensional as in the full theory. However, unlike in section 2.1, it is no longer a linear subspace of Γ_o. What would happen if we simply ignore the quadratic constraint C_1^L, and consider the resulting $2n-m+1$-dimensional linear space $\bar{\Gamma}'_o$ as the constraint surface of the linearized theory? As in section 2.1, these constraints strongly commute and we would obtain a first class system. Furthermore, the Hamiltonian would also strongly commute with these constraints. One would thus obtain a consistent Hamiltonian system which, however, would have one extra degree of freedom relative to the exact theory. Consequently, the tight relation between the linearized and the exact Hamiltonian descriptions of section 2.1 would be lost. More precisely, there will be solutions of this linearized system which would fail to give rise to a 1-parameter family of exact solutions. Thus, not all elements of the reduced phase space $\hat{\Gamma}'_o$ would represent allowable physical perturbations. This is the problem of linearized instability.

To investigate the structure of the correct linearized theory, let us now suppose that the gradients of q of the constraint functions, say $C_1, ..., C_q$, of the full theory vanish at p. Then, Γ_o inherits q quadratic constraints, $C_1^L, ..., C_q^L$ and $l = m - q$ linear constraints $C_{q+1}^L, ..., C_m^L$. As before, the l linear constraints strongly commute with one another. Furthermore, they strongly commute with the q quadratic constraints. For example, we have:

$$\begin{aligned}\{C_1^L, C_{q+1}^L\}_o &= \Omega_o^{\alpha\beta}\partial_\alpha C_1^L \partial_\beta C_{q+1}^L \\ &= [(\Omega^{\alpha\beta}\ v^\gamma\nabla_\gamma(\nabla_\alpha C_1)(\nabla_\beta C_{q+1})]|_p \\ &= [v^\gamma\nabla_\gamma\{C_1, C_{q+1}\}]|_p = 0\ .\end{aligned} \tag{2.6}$$

Finally, the q quadratic constraints are closed under the linearized Poisson bracket. For example, we have:

$$\begin{aligned}\{C_1^L, C_2^L\}_o &= \Omega_o^{\alpha\beta}\partial_\alpha C_1^L \partial_\beta C_2^L \\ &= \Omega^{\alpha\beta}\ v^\gamma\ v^\delta\ [(\nabla_\gamma\nabla_\alpha C_1)|_p \nabla_\delta(\nabla_\beta C_2)]|_p \\ &= v^\gamma v^\delta\ (\nabla_\gamma\nabla_\delta\{C_1, C_2\})|_p\ .\end{aligned} \tag{2.7}$$

The closure now follows from the fact that, since the gradients of C_1 and C_2 vanish at p, so does the gradient of the Poisson bracket $\{C_1, C_2\}$. A similar argument shows that the Poisson bracket between any quadratic constraint and the Hamiltonian is again a quadratic constraint.

To summarize, if there are m first class constraints such that the gradients of q of them happen to vanish at the background point p, the linearized theory acquires q quadratic constraints and $l = m - q$ linear constraints. The linear constraints strongly commute with all constraints as well as the Hamiltonian. The quadratic constraints are closed under the Poisson bracket. The linearized Hamiltonian commutes weakly with the quadratic constraints. Thus, on the constraint surface $\bar{\Gamma}_o$, the linearized Hamiltonian is gauge invariant. In a typical field theory context, there are only a finite number of quadratic constraints. These arise when the background point p has certain "global" gauge symmetries and they give rise to linearized instability.

2.3 Time dependent backgrounds

Let us now consider linearization around a time dependent background solution. The overall set-up is as in section 2.1. However, the background is now represented by a non-trivial integral curve, $\gamma(t)$, of X_H^α lying on the constraint submanifold $\bar{\Gamma}$ of the exact theory. For simplicity, we assume that none of the constraint functions C_i have vanishing gradients at any point along $\gamma(t)$. Furthermore, since this material is not needed in the next section –it is being presented only for completeness– the discussion will not be as detailed as in the last two sub-sections.

At each point of $\gamma(t)$ we have the tangent space $T_{\gamma(t)}\Gamma$, representing the possible infinitesimal deformations at that moment. We want to analyse the time evolution of these deformations in an appropriate Hamiltonian framework. We could choose any one point p along $\gamma(t)$ and take, as before, $(\Gamma_o = T_p\Gamma, \Omega^o_{\alpha\beta} = \Omega_{\alpha\beta}|_p)$ to be our phase space. However, to speak of evolution within Γ_o, we need an additional structure: a fiducial, kinematical isomorphism, say $I(t)$, from $T_{\gamma(t)}\Gamma$ to $T_P\Gamma$ for all t. The dynamical flow of the full theory is given by the integral curves of the Hamiltonian vector field X_H^α and, under the evolution by an amount τ, this maps the tangent vectors at $\gamma(t)$ to those at $\gamma(t+\tau)$. Therefore, if we have a fiducial isomorphism $I(t)$, we would then obtain a dynamical flow in the fixed phase space, $\Gamma_o = T_p\Gamma$, and study its properties. (Note that we can not use for our fiducial isomorphisms the ones provided by the Hamiltonian flow of the exact theory since this would lead us to trivial linearized dynamics; all perturbations would be trivially time independent.)

To obtain the required family of isomorphisms, let us fix, once and for all, a derivative operator ∇ on Γ which is compatible with the symplectic structure Ω, i.e., satisfies $\nabla_\alpha\Omega_{\beta\gamma} = 0$. We can now obtain the required isomorphisms $I(t)$ by parallel transporting tangent vectors from any point $\gamma(t)$ of the background solution to the fixed point p using this ∇. Since $\nabla_\alpha\Omega_{\beta\gamma} = 0$, the isomorphisms preserve the symplectic structure: $\Omega_{\alpha\beta}|_{\gamma(t)}v_1^\alpha v_2^\beta = \Omega_{\alpha\beta}|_p\ (I(t)v_1)^\alpha\ (I(t)v_2)^\beta$. In practice, the phase space Γ is often a symplectic *vector space* and one then chooses for ∇ the derivative operator which is compatible with its linear structure.

At each point $\gamma(t)$ of the background trajectory, the linearized constraint functions are defined, as in section 2.1, by

$$C_i^{L,\gamma(t)}(v) := v^\alpha(\nabla_\alpha C_i)|_{\gamma(t)}\ . \tag{2.8}$$

The physical perturbations $v \in T_{\gamma(t)}\Gamma$ satisfy $C_i^{L,\gamma(t)}(v) = 0$; they are tangential to the constraint sub-manifold $\bar{\Gamma}$ at the point $\gamma(t)$. Using the reasoning of section 2.1, it is straightforward to check that the Poisson bracket between these functions, computed using the symplectic tensor $\Omega_{\alpha\beta}|_{\gamma(t)}$, vanishes strongly. Now, through our isomorphisms $I(t)$, we can map these functions on $T_{\gamma(t)}$ to functions on the fixed linearized phase space Γ_o. The result is a 1-parameter family of constraint functions $C_i^{L,t}$ on Γ_o: $C_i^{L,t}(v) := C_i^{L,\gamma(t)}(I^{-1}(t)\cdot v)$ for all $v \in \Gamma_o$. The dependence on the parameter t merely reflects the fact that, for each t, linearization has been carried out at a different point, $\gamma(t)$, of $\bar{\Gamma}$. Since $I(t)$ preserve the symplectic structure, the Poisson brackets between $C_i^{L,t}$ also vanish strongly.

Let us next consider dynamics. Using the arguments of section 2.1, it again follows that, on $T_{\gamma(t)}\Gamma$, the infinitesimal time-evolution corresponds to the transformation:

$$\dot{v}^{\alpha}|_{\gamma(t)} = T^{\alpha}{}_{\beta}(t)v^{\beta} \equiv v^{\beta}(\nabla_{\beta}X^{\alpha}_{H})|_{\gamma(t)}$$

which is generated by:

$$H^{L,\gamma(t)}(v) := \tfrac{1}{2}v^{\alpha}v^{\beta}[\nabla_{\alpha}\nabla_{\beta}H]|_{\gamma(t)} \ . \tag{2.9}$$

We can push these functions to the fixed phase space Γ_o via our isomorphisms $I(t)$. The result is a Hamiltonian $H^{L,t}(v)$ with *explicit* time dependence. Finally, because of the explicit time-dependence, the Poisson brackets between the Hamiltonian and the constaints do not vanish. Rather, we have:

$$\begin{aligned}\{H^{L,t},\ C^{L,t}_i\}_o &= I(t)\cdot\ \{H^{L,\gamma(t)}, C^{L,\gamma(t)}_i\}_{\gamma(t)} \\ &= I(t)\cdot\ [v^{\alpha}\nabla_{\alpha}(X^{\beta}_{H}\nabla_{\beta}C_i) - X^{\beta}_{H}\nabla_{\beta}\ v^{\alpha}\nabla_{\alpha}C_i]|_{\gamma(t)} \ .\end{aligned} \tag{2.10}$$

When v^{α} satisfies the linearized constraint, the first term in the last step vanishes and the second can be interpreted as arising due to the explicit time dependence of the linearized constraints. Hence, on $\bar{\Gamma}_o$, (2.10) can be rewritten as:

$$\frac{d}{dt}C^{L,t}_i := \frac{\partial}{\partial t}C^{L,t}_i + \{H^{L,t}, C^{L,t}\} = 0 \ . \tag{2.11}$$

Thus, the Poisson bracket between the constraints and the Hamiltonian is non-zero but just compensates for the explicit time-dependence of the constraints.

3. Linearization of general relativity using triads and connections.

This section is divided into three parts. In the first, we briefly recall the Hamiltonian formulation of vacuum general relativity in terms of connections and triads. Using the framework of section 2.1, in the second subsection we carry out linearization of this Hamiltonian framework around Minkowskian initial data. In the last part, we use the results of section 2.2 to discuss the modifications required in the spatially compact context.

3.1 Hamiltonian general relativity

Fix a 3-manifold Σ which is to represent a Cauchy surface in the space-time picture. In section 3.2, we will focus on linearization off Minkowski space-time and Σ will then be assumed to be topologically R^3. In section 3.3, on the other hand, we will consider linearization off a flat space-time with spatial topology of a 3-torus, T^3. In this section, therefore, we will not commit ourselves to a specific topology.

For simplicity, in the discussion of the classical phase space structure, it will be convenient to first consider *complex* general relativity –i.e., complex 4-metrics g_{ab} satisfying vacuum Einstein's equations on a real 4-manifold $\Sigma \times R$– and then, at the end, restrict to the "real section" of the resulting phase space by imposing suitable reality conditions on the canonical variables. The phase space Γ of the complex theory can be taken to be the space of pairs (E^a_i, A^i_a), where E^a_i is a complex vector density of weight one, taking values in the Lie algebra of $SO(3)$ and A^i_a is a complex 1-form, also taking values in the Lie-algebra of $SO(3)$. Here, indices $a, b, c...$ refer to the tangent space of Σ while $i, j, k...$ refer to the Lie algebra of $SO(3)$. As the notation suggests, E^a_i serves as a (complex) triad while A^i_a serves as a connection. In terms of A^i_a, therefore, we can define a covariant derivative operator $\mathcal{D}$ which acts on $SO(3)$ "internal vectors" λ^i via: $\mathcal{D}_a\lambda^i = \partial_a\lambda^i + \epsilon^{ijk}A_{aj}\lambda_k$ where the $SO(3)$ indices are raised and lowered by the Killing-Cartan metric on the Lie algebra. (For simplicity, we have set Newton's constant G –which plays the same role here as Yang-Mills coupling constant– to one).

The variables (E^a_i, A^i_a) are canonically conjugate: the basic (non-vanishing) Poisson brackets of the full, non-linear theory are

$$\{E^a_i(x),\ A^j_b(y)\} = \delta^a_b\delta^j_i\delta^3(x,y) \tag{3.1}$$

In terms of these variables, the constraints of general relativity can be written as:

$$\mathcal{D}_aE^a_i = 0, \quad E^a_iF^i_{ab} = 0, \quad \text{and} \quad \epsilon^{ijk}E^a_iE^b_jF_{abk} = 0\ , \tag{3.2}$$

where $F_{ab}{}^i$ is the curvature of the connection A^i_a, defined by $F_{ab}{}^i := 2\partial_{[a}A^i_{b]} + \epsilon^{ijk}A_{ai}A_{bk}$. Note that the left hand side of the first constraint is well defined eventhough we have not introduced any derivative operator on spatial indices because E^a is a vector density of weight one. Thus, all equations are well-defined in terms of just the canonical variables without any background connections, derivative operators or metrics.

The relation of these phase space variables to those of "geometrodynamics" (see, e.g., [2]) is as follows. Taking E^a_i as orthonormal triads (of density weight one), we can define a 3-metric q_{ab} via:

$$q\ q^{ab} = \delta^{ij}E^a_iE^b_j\ ,$$

where $q :=$ det q_{ab}, and q^{ab} is the inverse metric. One can easily check that this relation does determine q_{ab} uniquely. Set $e^a_i = E^a_i/\sqrt{q}$; these serve as orthonormal triads (without any density weight) for the metric q_{ab}. They determine the Ricci rotation coefficients $\omega^{ij}_a \equiv \omega^k_a\epsilon_k{}^{ij}$ via $D_ae^b_i = \omega_{ai}{}^je^b_j$ where D_a is the unique torsion free derivative operator compatible with q_{ab}. Using these, we can now define the extrinsic curvature K_{ab} which essentially serves as the momentum canonically conjugate to the 3-metric q_{ab}. Set

$$K^i_a := i(A^i_a - \omega^i_a) \quad \text{and} \quad K_{ab} := K^i_ae_{bi}\ .$$

In terms of (E^a_i, K^i_a), the three constraint equations (3.2) can be interpreted as follows. The first equation –the Gauss constraint– implies that K_{ab} is symmetric, whence one can interpret it as extrinsic curvature on Σ in the 4-dimensional space-time. With this interpretation, the remaining two constraints of (3.2) can be shown [1] to be the well-known vector and scalar constraints of geometrodynamics [2]. Thus, the two formulations are equivalent. However, the formulation in terms of connections and triads has the advantage that all equations of the theory –the constraints, the Hamiltonians and the evolution equations– become simple, low order polynomials in terms of the canonical variables (E^a_i, A^i_a). (In addition, this description casts general relativity in the language of guage theories, thereby opening new avenues for quantum gravity. While this is the more important of the two advantages from a general, conceptual standpoint, it will not play a significant role in this paper.)

We can now summarize the situation with respect to dynamics. In this framework, the lapse field N turns out to be a scalar density of weight one while the shift field N^a is, as in geometrodynamics, a vector field on Σ. Given a pair, (N, N^a), the corresponding Hamiltonian is given by:

$$\begin{aligned} H_{N,\vec{N}}(E,A) &:= -\tfrac{1}{2}\int_\Sigma d^3x\ (NE_i^aE_j^bF_{abk}\epsilon^{ijk} + 2iN^aE_i^bF_{ab}{}^i) + \text{surface terms} \\ &= \int_\Sigma d^3x\ \mathcal{D}_a(NE_i^{[a}E_j^{b]})A_{bk}\epsilon^{ijk} + 2i\mathcal{D}_a(N^{[a}E_i^{b]})A_b^i\ . \end{aligned} \tag{3.3}$$

As expected, the volume terms in the first step are just linear combinations of constraints. The surface terms involve integrals over the boundary –if any– of Σ and, in the asymptotically flat context where the only boundary is at infinity, yield the ADM 4-momentum. The resulting evolution equations are:

$$\begin{aligned} \dot{E}_i^b &= -i\mathcal{D}_a(NE_j^{[a}E_k^{b]})\epsilon_i{}^{jk} + 2\mathcal{D}_a(N^{[a}E_i^{b]}) \\ \dot{A}_b^i &= iNE_j^aF_{abk}\epsilon^{ijk} + N^aF_{ab}{}^i \end{aligned} \tag{3.4}$$

So far, we have considered complex general relativity. This formulation is especially well-suited for discussing self-dual solutions of Einstein's equations which can be obtained simply by setting $A_a^i = 0$ in the constraint and the evolution equations. However, our interest in this paper lies in real general relativity. To recover this theory, one must restrict oneself to the appropriate "real section" of the complex phase space considered above. This can be done most easily by recalling that in the geometrodynamical formulation of real general relativity, the metrics and the extrinsic curvatures are both real. Thus, we are led to define the real section of the phase space as the subspace consisting of pairs (E_i^a, A_a^i) for which

$$q_{ab}^\star = q_{ab}, \quad \text{and} \quad K_{ab}^\star = K_{ab} \tag{3.5}$$

If these conditions are imposed initially, they are preserved by the time evolution (3.4). As stated, the conditions are not polynomial in our basic canonical variables (E_i^a, A_a^i). There *is* an alternate, polynomial formulation of these conditions (see [7], chapter 8). However, for the purpose of linearization, that formulation is less transparent and has no technical advantage over (3.5).

3.2 Linearization off Minkowski space-time

We will now apply the methods of section 2.1 to carry out a linearization of the above Hamiltonian formulation of general relativity off Minkowski space-time. Thus, in this section, the 3-manifold Σ will be assumed to be topologically R^3 and the phase-space Γ will consist of asymptotically flat [1] triads and connections. (We will not need the details of these boundary conditions here.)

Recall from section 2.1 that, to carry out linearization, we need to fix a background point p on the constraint surface $\bar{\Gamma}$. We will choose the obvious point that corresponds to Minkowski space-time: $p \equiv (E_i^a = {}^oE_i^a,\ A_a^i = 0)$, where ${}^oE_i^a$ is a flat triad on Σ. In the terminology of geometrodynamics, this corresponds to choosing the initial data consisting of a flat metric q_{ab}^o and vanishing extrinsic curvature K_{ab}. From the evolution equations (3.4) it is clear that this point is left invariant by the dynamical evolution corresponding to space-time translations which correspond to the following choices of lapse and shift: $N = \sqrt{q^o}$ and $N^a = N_o^a$, a translational Killing field of q_{ab}^o. That is, the Hamiltonian vector fields generated by $H_{N,\vec{N}}(E, A)$ with $N = \sqrt{q^o}$ and $N^a = N_o^a$ vanish at this background point p.

The tangent vectors v at p correspond to pairs of fields: $p = ((\delta E)_i^a,\ (\delta A)_a^i)$ satisfying appropriate boundary conditions. It is convenient to use the background (unweighted) triad $e_i^a := {}^oE_i^a/\sqrt{q^o}$ (and its inverse e_a^i) to convert all internal indices to spatial ones and deal exclusively with (complex) tensor fields. Let us therefore set:

$$h^{ab} = (\delta E)_i^a\ e^{bi}, \quad \text{and} \quad C_{ab} = (\delta A)_a^i\ e_{bi} \tag{3.6}$$

and allow ourselves to lower and raise tensor indices with q_{ab}^o and its inverse. The asymptotic conditions on the elements of the phase space Γ of the full theory [1] induce the following boundary conditions on linearized fields:

$$\begin{aligned} h^{ab} - \tfrac{1}{3}he^{ab} &= O(1/r^2), \quad h = O(1/r) \\ C_{ab} - \tfrac{1}{3}Ce_{ab} &= O(1/r^2), \quad C = O(1/r^3)\ . \end{aligned} \tag{3.7}$$

Thus, the linearized phase space Γ_o is spanned by pairs (h^{ab}, C_{ab}) of tensor fields on Σ, subject to the fall-off (3.7) at spatial infinity. The restriction of the symplectic structure Ω on Γ to this point p provides the symplectic structure Ω_o on Γ_o. The induced Poisson bracket relations are:

$$\{h^{ab}(x),\ C_{cd}(y)\}_o = -i\delta_c^a\delta_d^b\delta^3(x, y). \tag{3.8}$$

Having constructed the linearized phase space (Γ_o, Ω_o), our next task is to introduce linearlized constraint functionals. From (3.2) it follows that these are given by:

$$\partial_a h^{ap} - \eta^{amp}C_{am} = 0, \quad \partial_a C - \partial_b C_a{}^b = 0, \quad \text{and} \quad \partial_{[a}C_{bc]} = 0, \tag{3.9}$$

where η^{abc} is the Levi-Civita density on Σ. A straightforward calculation shows that, in accordance with the general result of section 2.1, the Poisson bracket between these linearized constraints vanishes strongly, i.e. everywhere on Γ_o. In particular, the constraints are of first class. Our next task is to analyse the canonical transformations they generate and construct the reduced phase space $\hat{\Gamma}_o$.

For this, let us smear the constraint by suitable test fields and obtain (complex-valued) functions on Γ_o. Set:

$$\begin{aligned} C_\omega^L(h, C) &:= i\int_\Sigma d^3x\ \omega_a(\partial_b h^{ba} - \eta^{bma}C_{bm})\ , \\ C_V^L(h, C) &:= i\int_\Sigma d^3x\ V^a(\partial_a C - \partial_b C_a{}^b) \\ C_\Lambda^L(h, C) &:= i\int_\Sigma d^3x\ \Lambda\ \eta^{abc}\ \partial_{[a}C_{bc]}, \end{aligned} \tag{3.10}$$

where the test fields ω_a, V^a and Λ vanish at infinity sufficiently rapidly. Then, the linearized constraint surface $\bar{\Gamma}_o$ can be specified by the equations:

$$C^L_\omega(h,C)=0, \quad C^L_V(h,C)=0, \quad \text{and} \quad C^L_\Lambda(h,C)=0\ , \tag{3.11}$$

for all test fields ω_a, V^a and Λ. The infinitesimal canonical transformations generated by these constraints are given by:

$$\begin{aligned}\triangle h^{ab} =& \eta^{abc}\omega_c + \partial^b V^a - (\partial_c V^c) q_o^{ab} - \eta^{abc}\partial_c \Lambda \\ \triangle C_{ab} =& \partial_a \omega_b .\end{aligned} \tag{3.12}$$

The right hand sides of (3.12) provide us with the linearized gauge directions. They span a subspace of $\bar{\Gamma}_o$ which we will denote, as before, by $\mathcal{G}^L$. The physical –or the reduced– phase space $\hat{\Gamma}_o$ is obtained by taking the quotient of $\bar{\Gamma}_o$ by $\mathcal{G}^L$. Alternatively, we can use a guage fixing procedure and select from each gauge orbit one and only one point, representing the true degrees of freedom in that gauge equivalence class. The physical phase space would then be displayed as a (gauge fixed) subspace of $\bar{\Gamma}_o$. This is the procedure we will now use.

Thus, the idea now is to simultaneously solve the linearized constraints (3.9) and eliminate the gauge freedom of (3.12). Since Σ is topologically R^3, the general solution to the last of constraint equations (3.9) is simply $C_{ab} = \partial_{[a} C_{b]}$ for some covector C_b on Σ. We can now use the gauge freedom contained in the second of equations (3.12) to make C_a vanish. The restricted gauge freedom is of the form $\omega_a = \partial_a \omega$ and can be exhausted by making C_{ab} trace-free. Thus, we have used the gauge freedom in the parameter $\omega_a(x)$ and the scalar constraint equation to make C_{ab} symmetric and trace-free. Next, consider the gauge freedom provided by the parameters $\Lambda(x)$ and $V^a(x)$. This freedom can be exhausted by making h^{ab} symmetric and traceless. There only remains the task of solving the two constraints; the first two of equations (3.9). They now imply that both h^{ab} and C_{ab} must be transverse (i.e., divergence-free). Thus, we can solve all the constraints and exhaust the gauge freedom by restricting ourselves to the subspace $\hat{\Gamma}_o$ consisting of pairs $(h^{ab}_{STT}, C^{STT}_{ab})$ of symmetric, transverse, traceless fields on Σ. These fields capture the two true degrees of freedom of the gravitational field per space point. They are related to the linearized geometrodynamical variables via:

$$\begin{aligned} h^{STT}_{ab} &= \tfrac{1}{2}\sqrt{q_o}(\delta q)^{ab}_{TT} \\ C^{STT}_{ab} &= -\eta_{bmn} \overset{o}{q}_{ap} \partial^m h^{np}_{TT} - i(\delta K)^{TT}_{ab} . \end{aligned} \tag{3.13}$$

We can now discuss linearized dynamics. For simplicity, let us consider a "pure time translation" and set the shift N^a to zero (and lapse N to $\sqrt{q^o}$). The linearized evolution equations can then be read-off from (3.4):

$$\begin{aligned} \dot{h}^{ab}_{STT} &= i\epsilon^{bm}{}_n \partial_m h^{an}_{STT} - i\sqrt{q^o}\ C^{ab}_{STT} \\ \dot{C}^{STT}_{ab} &= -i\epsilon_b{}^{mn}\partial_m C^{STT}_{an}\ , \end{aligned} \tag{3.14}$$

where ϵ_{abc} is the 3-form on Σ compatible with q^o_{ab}. As expected, the linearized Hamiltonian, generating this evolution, can be obtained by taking the second variation of the full Hamiltonian (3.3). We have:

$$H^L(h,C) = -\int_\Sigma d^3x\ (\sqrt{q^o}[{}^oq^{ac}\ {}^oq^{bd}C^{STT}_{ab}C^{STT}_{cd}] + 2C^{STT}_{am}\epsilon^{mb}{}_c\partial_b h^{ca}_{STT})\ . \qquad (3.15)$$

So far, we have considered *complex* linearized fields. To recover the real theory, we have to impose the linearized analog of the reality conditions (3.5). From (3.13), it follows that these are given by:

$$(h^{ab}_{STT})^\star = (h^{ab}_{STT}) \quad \text{and} \quad (C^{STT}_{ab} + \eta_{bmn}q^o_{ap}\partial^m h^{np}_{STT})^\star = -(C^{STT}_{ab} + \eta_{bmn}q^o_{ap}\partial^m h^{np}_{STT})\ . \qquad (3.16)$$

Using these conditions, we can simplify the form of the linearized Hamiltonian. On the real section of the linearized, reduced phase space, it has a particularly simple and convenient form:

$$H^L(h,C) = \int_\Sigma d^3x\ \sqrt{q^o}\ C_{ab}(x)(C^{ab}(x))^\star\ , \qquad (3.17)$$

which resmbles the Hamiltonian of an harmonic oscillator, $H = ZZ^\star$, expressed in terms of Bargmann (i.e. creation-annihilation type) variables. Thus, (3.17) exhibits the dynamics of linearized gravity in terms of that of an assembly of harmonic oscillators. Not only does this form bring out the fact that the Hamiltonian is positive definite, but it also facilitates the transition to quantum theory.

3.3 Linearization off flat $T^3 \times R$

In this section, we will summarize the modifications in the linearization procedure needed for the case when the underlying spatial topology is compact. Since we are linearizing off flat space, the simplest such topology is that of a 3-torus, T^3. In terms of geometrodynamical variables, the corresponding problem of linearization was discussed in detail in [9]. While our procedure is technically different, our final conclusions are the equivalent to those of that reference.

Since the spatial manifold Σ is compact, in the full theory, there are no boundary conditions to impose and surface terms are absent. The constraints and the evolution equations are the same as in section 3.1. We can choose the same background point p on $\bar\Gamma$ and construct the linearized phase space exactly as in section 3.2 (except that we do not have to worry about the boundary conditions (3.7)). We again have the basic Poisson brackets (3.8) and linearized constraints (3.9)–(3.11). Thus, the extraction of the constraint surface $\bar\Gamma_o$ is unaffected. What changes is the next step: isolation of the true degrees of freedom.

Let us discuss this point in some detail.

Again, the last of the constraint equations (3.9) implies that $C_{[ab]}$ is closed. However, since the 3-torus is topologically non-trivial, we can not conclude that $C_{[ab]}$ is exact. Rather, now we have to use the Hodge theorem which ensures that every closed form can be decomposed into an exact form and an harmonic form and set $C_{[ab]} = \partial_{[a}C_{b]} + C^{\rm har}_{[ab]}$. As before, using the gauge freedom provided by $\omega_a(x)$, we can eliminate the exact part

$\partial_{[a}C_{b]}$ and this makes $C_{[ab]}$ harmonic. This partially fixes the gauge freedom in $\omega_a(x)$. The remaining gauge freedom lies in choosing a $\omega_a(x)$ which is closed. Using the Hodge decomposition, let us set $\omega_a = \partial_a\omega + \omega_a^{\rm har}$. As before, we can use the freeom in the scalar $\omega(x)$ to restrict the trace C of C_{ab}. From (3.12), it follows that, under the gauge transformation parametrized by ω, C transforms as $C \to C + \partial^a\omega_a \equiv C + \partial^a\partial_a\omega$, (where the harmonic part does not contribute since it is divergence-free.) Using this freedom, we can eliminate the part of $C(x)$ which is orthogonal (in the L^2-inner product) to the kernel of the Laplacian $\partial^a\partial_a$. That is, we can exhaust the freedom in $\omega(x)$ by making $C(x)$ a constant, C_o. Thus, we have solved the scalar constraint and used most of the freedom in $\omega_a(x)$ to make $C_{[ab]} = C^{\rm har}_{[ab]}$. The remaning gauge freedom is that in letting $\omega_a(x)$ itself be harmonic.

We now turn to h^{ab}. We can use the restricted gauge freedom in $\omega_a(x)$ and the gauge freedom available through parameters $V^a(x)$ and $\Lambda(x)$ to restrict the form of h^{ab}. From (3.12), we have:

$$\Delta h^{[ab]} = \eta^{abc}\omega_c^{\rm har} + \partial^{[b}V^{a]} - \eta^{abc}\partial_c\Lambda \ .$$

Therefore, using the Hodge decomposition of $h^{[ab]}$, we can make $h^{[ab]}$ vanish. This exhausts the freedom available in $\omega_a^{\rm har}(x)$ and $\Lambda(x)$ but leaves the freedom of choosing a closed [2] $V^a(x)$. Let us again use the Hodge decomposition of V^a. As with $C(x)$ above, one can eliminate the freedom of using an exact V^a by making the trace of h^{ab} a constant multiple of $\sqrt{q^o}$: $h(x) = h_o\sqrt{q^o}(x)$. Finally, we use the first two of the constraint equations (3.9). The second equation tells us that C_{ab} is divergence-free. As for the first, it is straightforward to check that, with $h^{ab} = h^{(ab)}$ of constant trace and $C_{[ab]} = C^{\rm har}_{[ab]}$, the solutions to it are necessarily of the form $C_{[ab]} = 0$ and $\partial_a h^{ab} = 0$. To summarize, we have solved the linearized constraints (3.9) and eliminated the gauge freedom to conclude that h^{ab} and C_{ab} are symmetric, transverse and of constant trace. (The freedom in choosing V^a harmonic leaves these fields unchanged and is therefore spurious.)

This may seems surprising at first because, in addition to the two (configuration) degrees of freedom per space point contained in the transverse, traceless parts of C_{ab}, we have an additional, global degree of fredom contained in the trace of C_o. Recall, however, from section 2.2 that, due to the potential linearization instability problems, there may be additional constraints. This is indeed the case.

First, since we are in the spatially compact context, we can let the smearing field Λ in (3.10) to be a constant, Λ_o. For this choice, $C^L_{\Lambda_o}(h, C)$ vanishes identically *on the entire linearized phase space* Γ_o, whence the gradient of the constraint C_{Λ_o} vanishes at p. Therefore, as discussed in section 2.2, we must consider a second order variation of the scalar constraint (3.2) (smeared with Λ_o) to recover all the constraints in the linearized theory. This is easy to do. The result is the following quadratic constraint on the linearized fields:

$$C^L_{\Lambda_o}(h,C) \equiv \tfrac{2}{3}C_o^2 V_o - \int_\Sigma d^3x\ (\sqrt{q^o}\ q_o^{ac}q_o^{bd}C^{STT}_{ab}C^{STT}_{cd} - 2C^{STT}_{am}\ \epsilon^{mb}{}_c\ \partial_b h^{ca}_{STT}) = 0\ , \quad (3.18)$$

[2] For simplicity of notation, we will not distinguish between forms and vector densities here; the background metric q^o_{ab} enables one to set isomorphisms between the two spaces.)

where V_o is the volume of the 3-torus with respect to the background metric q^o_{ab}. Thus, C_o is not free but is determined by the symmetric, transverse, traceless degrees of freedom.

Inded, the situation is just the opposite of what one might have initially expected: in the case of the 3-torus, there are in fact *less* degrees of freedom than in the Minkowskian case of section 3.2! This comes about because there are three *additional* quadratic constraints. To see this, consider the second linearized constraint functional C^L_V of (3.10). If $V^a = \sqrt{q^o} V_o^a$, where V_o^a is a constant vector field on (T^3, q^o_{ab}), $C^L_V(h, C)$ vanishes *on the entire linearized phase space* Γ_o. Thus, there are three additional quadratic constraints. (V_o^a are global Killing fields on the background space (T^3, q^o_{ab}); as expected, we are encountering the linearized instability problem.) It is easy to work these out from the vector constraint of the full theory. We have:

$$C^L_{V_o}(h, C) \equiv \int_\Sigma d^3x \; (h^{ab} \mathcal{L}_{\vec{V}_o} C_{ab}) = 0. \tag{3.19}$$

Thus, not all $(h^{ab}_{STT},\ C^{STT}_{ab})$ can serve as true degrees of freedom of the linearized theory. To do so, they must satisfy (3.19). Only then can one be sure that the pair $(h^{ab}_{STT},\ C^{STT}_{ab})$ represents a tangent vector to a curve in the constraint surface $\bar{\Gamma}$ of the full theory and thus represents a genuine physical perturbation. Note, however, that these are just three *global* constriaints, whence the "total number of degrees of freedom" is essentially unaffected.

4. Discussion

In this paper, we first developed a general framework for linearizing Hamiltonian systems with first class constraints and then applied it to general relativity, using connections and triads as the basic canonical variables. In particular, to illustrate the problems associated with linearization instability, we considered both the spatially open and the spatially compact cases. We will now conclude with a few remarks.

1. The discussion of section 2.2 can be used to see how the single spin-2 mode of the gravitational field arises in the connection framework. If we take only the Gauss constraint into account and ignore, for the moment, the vector and the scalar constraints of general relativity, the situation is exactly as in the Yang-Mills theory. Then, the connections A^i_a would describe three spin-1 particles, each having two helicity states. It is the vector and the scalar constraints that together remove four of these six degrees of freedom, distilling the two degrees corresponding to a single spin-2 quantum.

2. In section 3.3, we saw that the fact that the background metric q^o_{ab} has three "translational" Killing vectors has an important consequence: contrary to one's first expectation, not all symmetric, transverse, traceless fields $(h^{ab}_{STT}, C^{STT}_{ab})$ represent physical perturbations of the gravitational field. To qualify as physical modes, they must satisfy three (global) constraints, (3.19). Thus, there is a key difference between considering gravitational radiation in Minkowski space and that in a box, with periodic identification. This is quite striking since we are, in particular, used to approximating electromagnetic ratiation in Minkowski space by "putting the system in a box." Indeed, this is the strategy often adopted in the textbook treatments of quantization of the Maxwell field and in the rigorous treatments that aim at avoiding infra-red problems. The presence of the constraints (3.19)

shows that considerable care is needed in using a similar strategy in the gravitational case. This result is in complete agreement with the main results of [9, 10] which were obtained in the geometrodynamical framework. However, the contrast between the Maxwell theory and linearized gravity is perhaps more striking in the connection dynamics framework.

3. It is also striking that while the scalar and the vector constraints lead to quadratic constraints (3.18) and (3.19) on linearized fields, the Gauss constraint does not. In particular, it is clear from (3.10) that the linearized Gauss constraint $C^L_\omega(h, C)$ continues to be non-trivial even if the smearing field ω_a is assumed to be constant on T^3. This might seem surprising at first since we noted in the beginning of section 2.2 that the Yang-Mills theory also faces the problem of linearization instability off trivial initial data: ($A^i_a = 0,\ E^a_i = 0$). Note however, that in the case of linearized gravity, we are effectively linearizing about the point ($A^i_a = 0,\ E^a_i = \sqrt{q^o}\delta^a_i$) of the Yang-Mills phase space and this point is *not* left invariant by any infinitesimal gauge transformation ω^i. Finally, one might wonder why it is that only the translations –and not the rotations– that lead to quadratic constraints. The answer is simply that rotations fail to be globally defined on T^3 and can not therefore be used as smearing fields $V^a(x)$ in (3.10).

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