WEAK GREEDY ALGORITHMS AND THE EQUIVALENCE BETWEEN SEMI-GREEDY AND ALMOST GREEDY MARKUSHEVICH BASES

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ABSTRACT. We introduce and study the notion of weak semi-greedy systems -which is inspired in the concepts of semi-greedy and branch semi-greedy systems and weak thresholding sets-, and prove that in infinite dimensional Banach spaces, the notions of semi-greedy, branch semi-greedy, weak semi-greedy, and almost greedy Markushevich bases are all equivalent. This completes and extends some results from [5], [9], and [13]. We also exhibit an example of a semi-greedy system that is neither almost greedy nor a Markushevich basis, showing that the Markushevich condition cannot be dropped from the equivalence result. In some cases, we obtain improved upper bounds for the corresponding constants of the systems.

1. Introduction.

Let X be a Banach space over the real or complex field \mathbb{K} , with dual space X'. A sequence $(x_i)_i$ in X is fundamental if $X = \overline{[x_i \colon i \in \mathbb{N}]}$, and it is minimal or a minimal system if there is a sequence of biorthogonal functionals $(x_i')_i \subseteq X'$ (i.e., $x_i'(x_j) = \delta_{ij}$ for every i, j); a sequence $(x_i')_i$ in X' is total if $x_i'(x) = 0$ for every $i \in \mathbb{N}$ implies that x = 0. A fundamental minimal system $(x_i)_i \subseteq X$ whose sequence of biorthogonal functionals is total is a Markushevich basis for X. From now on, unless otherwise stated $(x_i)_i \subseteq X$ denotes a fundamental minimal system for a Banach space X with (unique) biorthogonal functionals $(x_i')_i \subset X'$, and all of our Banach spaces are infinite dimensional. Given $x \in X$, supp(x) denotes the support of $x \in X$, that is the set $\{i \in \mathbb{N} \colon x_i'(x) \neq 0\}$. A decreasing ordering for x is an injective function $\varrho_x \colon \mathbb{N} \to \mathbb{N}$ such that $supp(x) \subseteq \varrho_x(\mathbb{N})$, and for all $i \leq j$

$$|x'_{\varrho_x(i)}(x)| \ge |x'_{\varrho_x(j)}(x)|.$$

The set of all decreasing orderings for a fixed $x \in X$ will be denoted by D(x). The greedy ordering for x is the decreasing ordering ϱ_x with the property that if i < j and $|x'_{\varrho_x(i)}(x)| = |x'_{\varrho_x(j)}(x)|$, then $\varrho_x(i) < \varrho_x(j)$.

Note that if $(x_i)_i \subseteq X$ is a fundamental minimal system and $(x_i')_i$ is a bounded sequence, then $(x_i)_i$ is bounded below (i.e., there is r > 0 such that $||x_i|| \ge r$ for each $i \in \mathbb{N}$) and $(x_i')_i$ is w^* -null, so the greedy ordering is well defined.

The Thresholding Greedy Algorithm (TGA) for a fundamental minimal system with bounded coordinates $(x_i)_i \subseteq X$ gives approximations to each $x \in X$ in terms of the greedy ordering. For $m \in \mathbb{N}$, the m-term greedy approximation to x is defined as follows:

$$\mathcal{G}_m(x) := \sum_{i=1}^m x'_{\rho(x,i)}(x) x_{\rho(x,i)},$$

where $\rho: X \times \mathbb{N} \to \mathbb{N}$ is the unique mapping such that for each $x \in X$, the function $\rho(x,\cdot)$ is the greedy ordering for x. In this paper, ρ will always denote this function.

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Using the convention that the sum over the empty set is zero, $\mathcal{G}_0(x) = 0$. As usual, given a finite subset A of \mathbb{N} , P_A denotes the projection with indices in A, that is $P_A(x) = \sum_{i \in A} x_i'(x)x_i$ and |A| denotes the cardinal of A.

The TGA was introduced by Temlyakov in [18], in the context of the trigonometric system, and extended by Konyagin and Temlyakov to general Banach spaces in [16], where the authors defined the concept of *greedy Schauder bases* (in the context of Schauder bases, "TGA" refers to any algorithm that gives approximations induced by a decreasing ordering; the greedy ordering is chosen for convenience to study general minimal systems; see [17]).

Definition 1.1. A Schauder basis $(x_i)_i \subseteq X$ is *greedy* if there is M > 0 such that for every $x \in X$ and each $m \in \mathbb{N}$,

$$||x - \mathcal{G}_m(x)|| \le M\sigma_m(x),$$

where $\sigma_m(x)$ is the best m-term approximation error given by

$$\sigma_m(x) = \inf\{\|x - y\| : |\operatorname{supp}(y)| \le m\}.$$

Notice that, by a density argument, $(x_i)_i \subseteq X$ is a greedy basis if and only if there is M > 0 such that for every $x \in X$ and each $m \in \mathbb{N}$,

$$||x - \sum_{i=1}^{m} x'_{\rho_x(i)}(x) x_{\rho_x(i)}|| \le M \sigma_m(x),$$

for some (or for all) $\rho_x \in D(x)$, which is the original definition in [16].

In [16], the authors also introduce the concept of quasi-greedy Schauder bases, developed independently for fundamental biorthogonal systems and quasi-Banach spaces (though with somewhat different terminology) by Wojtaszczyk in [20]. This concept was studied in several papers (see for example [1], [3], [9], [10], [13] and [17]). We follow the definition from [17].

Definition 1.2. A fundamental minimal system $(x_i)_i \subseteq X$ with bounded coordinates is *quasi-greedy* if there is M > 0 such that for all x and for each m,

$$||\mathcal{G}_m(x)|| \le M||x||.$$

It is clear from the definition that a fundamental minimal system is quasi-greedy if and only if there is a constant M > 0 such that for all x and for each m,

$$(2) ||x - \mathcal{G}_m(x)|| \le M||x||.$$

In the literature, the "quasi-greedy constant" of the system has been defined as the minimum M for which (1) holds (see, e.g., [9]), the minimum M for which (2) holds (see, e.g., [5]), or the minimum M for which both hold (see, e.g.,[1], [10], or [17]). The differences in notation in the literature have been discussed in [2], where the minimum M for which (2) holds is called the suppression quasi-greedy constant of the basis. We will refer to them as first quasi-greedy constant(1) and the second quasi-greedy constant (2) of the system, respectively.

A notion between being greedy and quasi-greedy is that of being almost greedy. This concept was introduced by Dilworth, Kalton, Kutzarova and Temlyakov for Schauder bases [10], and also studied in the context of Markushevich bases in several papers (see, among others, [2], [5], [11] and [13]).

Definition 1.3. A Markushevich basis $(x_i)_i \subseteq X$ with bounded coordinates is almost greedy if there is a constant M > 0 such that for each $x \in X$ and $m \in \mathbb{N}_0$,

(3)
$$||x - \mathcal{G}_m(x)|| \le M\widetilde{\sigma}_m(x),$$

where

$$\widetilde{\sigma}_m(x) = \inf\{\|x - P_A(x)\| \colon |A| \le m\}.$$

The minimum M for which the above inequality holds is called the *almost greedy* constant of the basis.

Another natural weakening of the greedy notion is the concept of semi-greedy Schauder bases that was introduced in [9]. This concept was later extended to Markushevich bases (see e.g., [5] and [12]) and can be considered for fundamental minimal systems in general. Before we give a definition, we set

$$\sigma_m(x) = \inf\{\|x - y\| \colon |\operatorname{supp}(y)| \le m \text{ and } y = P_{\operatorname{supp}(y)}(y)\},\$$

which extends the concept of the best-m-term approximation error to such systems.

Definition 1.4. A fundamental minimal system $(x_i)_i \subseteq X$ with bounded coordinates is *semi-greedy* if there exists M > 0 such that for every $x \in X$ and every $m \in \mathbb{N}$, there is $z \in [x_i : i \in \mathcal{GS}_m(x)]$ such that

$$||x - z|| \le M\sigma_m(x),$$

where $\mathcal{GS}_m(x) := \{\rho(x,i) : 1 \leq i \leq m\}$ is the greedy set of x of cardinality m. Under these conditions, z is called an m-term Chebyshev approximant to x, and the minimum M for which the inequality holds is called the semi-greedy constant of the system.

In semi-greedy systems, the *Chebyshev Greedy Algorithm* (CGA) is used instead of the TGA. The CGA gives generally better approximations than the TGA, since the approximations to x are not limited to the projections.

Weaker versions of the TGA and the CGA have been studied in several papers (see e.g. [7],[8], [11], [12], [13], [14] [17] and [19]). These algorithms give approximations in terms of *weak thresholding sets*, which are defined (according to [17]) as follows:

Definition 1.5. Let $(x_i)_i \subseteq X$ be a fundamental minimal system and let $0 < \tau \le 1$. Given $x \in X$ and $m \in \mathbb{N}$, a set $\mathcal{W}^{\tau}(x,m)$ of cardinality m is called an m-weak thresholding set for x with weakness parameter τ if

$$|x_i'(x)| \ge \tau |x_i'(x)|,$$

for all $i \in \mathcal{W}^{\tau}(x,m)$ and all $j \in \mathbb{N} \setminus \mathcal{W}^{\tau}(x,m)$. In this paper, $\mathcal{W}^{\tau}(x,m)$ always denotes one of these sets.

Weak thresholding greedy algorithms give approximations to $x \in X$ by projections $P_{\mathcal{W}^{\tau}(x,m)}$, whereas weak Chebyshev greedy algorithms give instead the best approximation in terms of the vectors in $[x_i : i \in \mathcal{W}^{\tau}(x,m)]$. In this paper, we focus on the following two properties:

Definition 1.6. A fundamental minimal system $(x_i)_i \subseteq X$ is weak almost greedy with weakness parameter $0 < \tau \le 1$ (WAG (τ)) and constant M if for every $x \in X$ and every $m \in \mathbb{N}$, there is a weak thresholding set $\mathcal{W}^{\tau}(x, m)$ such that

$$(4) ||x - P_{\mathcal{W}^{\tau}(x,m)}(x)|| \le M\widetilde{\sigma}_m(x).$$

Definition 1.7. Let $(x_i)_i \subseteq X$ be a fundamental minimal system and $0 < \tau \le 1$. The system is weak semi-greedy with weakness parameter τ (WSG(τ)) and constant M if for every $x \in X$ and every $m \in \mathbb{N}$, there is a weak thresholding set $\mathcal{W}^{\tau}(x, m)$ and $z \in [x_i : i \in \mathcal{W}^{\tau}(x, m)]$ such that

$$||x - z|| \le M\sigma_m(x).$$

In that case, z is called an m-term Chebyshev τ -greedy approximant for x.

Note that the WAG(τ) concept is an extension of the almost greedy concept (Definition 1.3) that correspons to $\tau=1$. Indeed, the greedy set $\mathcal{GS}_m(x)$ is a weak thresholding set with parameter 1, and $\mathcal{G}_m(x)=P_{\mathcal{GS}_m}(x)$, so any almost greedy system is WAG(1). Reciprocally, if $(x_i)_i$ is WAG(1), given $x\in X$, $m\in\mathbb{N}$, a set $\mathcal{W}^1(x,m)$ and $\epsilon>0$, one can choose $y\in X$ with the property that $|x_i'(y)|\neq |x_j'(y)|$ for all $i\neq j$ so that $||x-y||\leq \epsilon$ and $\mathcal{W}^1(x,m)=\mathcal{GS}_m(y)$ is the only m-weak thresholding set for y. It easily follows from this that (4) holds for every x, m, and every set $\mathcal{W}^1(x,m)$, so in particular (3) holds. Similarly, the notion of WSG(τ) systems is an extension of that of semi-greedy systems, which also corresponds to the case $\tau=1$.

We are now in a position to describe the goal of this paper, which is twofold. First, we study the relations between almost greedy and semi-greedy systems, and their relation with approximations involving weak thresholding sets. In particular, we prove that for Markushevich bases, the concepts of semi-greedy, almost greedy, $WSG(\tau)$ and $WAG(\tau)$ systems are all equivalent, extending results from [5] and [9]. Second, we focus our attention on some aspects that are significant on finite-dimensional spaces, in which the relations of their respective constants are of relevance. In particular, we answer a question from [13].

This paper is structured as follows: In Section 2 we study weak almost greedy systems. In Section 3, we define and study a separation property that will allow us to prove our main result, Theorem 4.2. Section 4 is devoted to the study of weak semi-greedy systems. Finally, in Section 5 we focus on finite-dimensional spaces and extend some results from [13].

2. Weak almost greedy systems.

In this section, we prove that the notions of WAG(τ) and almost greedy systems are equivalent. Note that one of the implications is immediate by definition. Indeed, any weak thresholding set $W^1(x,m)$ is also a weak thresholding set $W^{\tau}(x,m)$ for all $0 < \tau \le 1$ and almost greedy systems are WAG(1) systems, so they are WAG(τ) for all τ . Moreover, for almost greedy systems, it was proven in [17, Theorem 2.2] that (4) holds for every set $W^{\tau}(x,m)$, with M only depending on τ and the almost greedy constant of the basis (while [17, Theorem 2.2] is stated for Schauder bases, the proof does not use the Schauder property).

To prove that every WAG(τ) system is almost greedy, we will use the known equivalence between almost greediness and quasi-greediness plus democracy or superdemocracy - two concepts we define next. We also define the concept of hyperdemocracy, a natural extension of both democracy notions that has its roots in [9], [10] and [20].

Definition 2.1. A sequence $(x_i)_i \subseteq X$ is *superdemocratic* if there is K > 0 such that

(5)
$$\|\sum_{i \in A} a_i x_i\| \le K \|\sum_{j \in B} b_j x_j\|,$$

for any pair of finite sets $A, B \subseteq \mathbb{N}$ with $|A| \leq |B|$, and any scalars $(a_i)_{i \in A}, (b_j)_{j \in B}$ such that $|a_i| = |b_j|$ for every $i \in A$, $j \in B$. The minimum K for which the above inequality holds is called the *superdemocracy constant* of $(x_i)_i$. When (5) holds with $a_i = b_j = 1$ for all i, j, the sequence is *democratic*, and the corresponding constant is the *democracy constant* of $(x_i)_i$, whereas if (5) holds with $|a_i| \leq |b_j|$ for all i, j, we say that the sequence is *hyperdemocratic*, and that the corresponding minimum constant is the *hyperdemocracy constant* of $(x_i)_i$.

From [10, Theorem 3.3] and its proof, we extract the following result, which characterizes almost greedy Markushevich bases, valid for real and complex Banach spaces.

Theorem 2.2. Let $(x_i)_i \subseteq X$ be a Markushevich basis.

- (a) If $(x_i)_i$ is almost greedy with constant K_a , it is quasi-greedy with second quasi-greedy constant $K_{2q} \leq K_a$ and democratic with constant $K_d \leq K_a$.
- (b) If $(x_i)_i$ is quasi-greedy with first quasi-greedy constant K_{1q} and democratic with constant K_d , then it is almost greedy with constant $K_a \leq 32K_d(1 + K_{1q})^4$.

Almost greedy Markushevich bases can also be characterized as quasi-greedy and superdemocratic (see, e.g., [5], [9] or [7], which considers complex spaces and improves the order of the bound for K_a in (b) if democracy is replaced with superdemocracy) or quasi-greedy and hyperdemocratic. In fact, it follows at once from Theorem 2.2(b) that every quasi-greedy hyperdemocratic or superdemocratic basis is almost greedy, whereas a proof that an almost greedy system is hyperdemocratic can be obtained combining Theorem 2.2(a) with [20, Proposition 2] and [10, Lemma 2.2], with minor modifications for complex scalars. This implication is also established in Proposition 2.3 below, taking $\tau = 1$.

Also, note that in Theorem 2.2, the hypothesis that the minimal system $(x_i)_i$ is a Markushevich basis is not necessary, since an almost greedy system is clearly quasi-greedy, and a quasi-greedy system is a Markushevich basis. We give a simple proof of this fact that follows from [20, Theorem 1] (see also the proof of [3, Corollary 3.5]). If $(x_i)_i$ is quasi-greedy and $x_i'(x) = 0$ for all $i \in \mathbb{N}$, then $\mathcal{G}_m(y) = \mathcal{G}_m(y - x)$ for every $y \in X$ and every $m \in \mathbb{N}$. Thus,

$$||x|| \le ||y - x|| + ||y - \mathcal{G}_m(y)|| + ||\mathcal{G}_m(y - x)|| \le ||y - \mathcal{G}_m(y)|| + (1 + M)||y - x||.$$

Since the system is fundamental, for any $\epsilon > 0$ there is $m \in \mathbb{N}$ and $y \in [x_i : 1 \le i \le m]$ such that $||x - y|| \le \epsilon$. Given that $y = \mathcal{G}_m(y)$, it follows that $||x|| \le (1 + M)\epsilon$. As ϵ is arbitrary, this entails that x = 0.

Now we prove that every WAG(τ) is almost greedy. The proof is based on that of [13, Proposition 4.4], adapted for our purposes.

Proposition 2.3. Let $0 < \tau \le 1$, and let $(x_i)_i \subseteq X$ be a WAG (τ) system with constant M. Then, $(x_i)_i$ is a quasi greedy Markushevich basis with first quasigreedy constant $K_{1q} \le (1+M)(1+M^2\tau^{-4})$, and is hyperdemocratic with constant $K_{hd} \le M^2\tau^{-2}$. Hence, $(x_i)_i$ is almost greedy.

Proof. To prove the hyperdemocracy condition, fix nonempty finite sets $A, B \subseteq \mathbb{N}$ with $|A| \leq |B|$, and $(a_i)_{i \in A}, (b_j)_{j \in B}$ such that $|a_i| \leq |b_j|$ for every i, j, and choose a set $C \subseteq \mathbb{N}$ so that |C| = |B| and $C \cap (A \cup B) = \emptyset$. Assume without loss of generality that $a := \max_{i \in A} |a_i| > 0$. For every $0 < \epsilon < 1$ we have

$$(1 - \epsilon)a\tau \| \sum_{k \in C} x_k \| = \| \sum_{k \in C} (1 - \epsilon)a\tau x_k + \sum_{j \in B} b_j x_j - \sum_{j \in B} b_j x_j \|$$

$$\leq M\widetilde{\sigma}_{|C|} \left(\sum_{k \in C} (1 - \epsilon)a\tau x_k + \sum_{j \in B} b_j x_j \right)$$

$$\leq M \| \sum_{j \in B} b_j x_j \|,$$

$$(6)$$

the first inequality resulting from the fact that B is the only |C|-weak thresholding set for $\sum_{k \in C} (1 - \epsilon)a\tau x_k + \sum_{j \in B} b_j x_j$ with weakness parameter τ . Similarly, C is the

only |C|-weak thresholding set with weakness parameter τ for $\sum_{k \in C} (1+\epsilon)a\tau^{-1}x_k +$ $\sum_{i \in A} a_i x_i$, thus

$$\| \sum_{i \in A} a_i x_i \| = \| \sum_{i \in A} a_i x_i + \sum_{k \in C} (1 + \epsilon) a \tau^{-1} x_k - \sum_{k \in C} (1 + \epsilon) a \tau^{-1} x_k \|$$

$$\leq M \widetilde{\sigma}_{|C|} (\sum_{k \in C} (1 + \epsilon) a \tau^{-1} x_k + \sum_{i \in A} a_i x_i)$$

$$\leq (1 + \epsilon) a \tau^{-1} M \| \sum_{k \in C} x_k \|.$$
(7)

By letting $\epsilon \to 0$, it follows from (6) and (7) that

$$\|\sum_{i \in A} a_i x_i\| \le M^2 \tau^{-2} \|\sum_{j \in B} b_j x_j\|.$$

Therefore, $(x_i)_i$ is hyperdemocratic with $K_{hd} \leq M^2 \tau^{-2}$.

To prove that $(x_i)_i$ is quasi-greedy, first note that for every $x \in X$, there is a weak thresholding set $W^{\tau}(x,1)$, so $(|x_i'(x)|)_i$ is bounded. It follows by uniform boundedness that $(x_i')_i$ is bounded. Since $(x_i)_i$ is fundamental, this entails that $(x_i')_i$ is w^* -null. Now fix $x \in X$ and $m \in \mathbb{N}$. If $\mathcal{G}_m(x) = 0$, there is nothing to prove. Else, let

$$n := \max\{1 \le i \le m : x'_{o(x,i)}(x) \ne 0\}.$$

Given that $x_i'(x) \neq 0$ for all $i \in \mathcal{GS}_n(x)$ and $(x_i')_i$ is w^* -null, there is $j_0 \in \mathbb{N}$ such that for each $j \geq j_0$ and each $i \in \mathcal{GS}_n(x)$,

$$|x_i'(x)| < \tau |x_i'(x)|.$$

Thus, if $j \geq j_0$, every weak thresholding set $\mathcal{W}^{\tau}(x,j)$ contains $\mathcal{GS}_n(x)$. Hence, there is a minimum $m_1 \in \mathbb{N}$ for which there is a weak thresholding set $\mathcal{W}^{\tau}(x, m_1)$ containing $\mathcal{GS}_n(x)$ and such that (4) holds. Now if $\mathcal{W}^{\tau}(x, m_1) = \mathcal{GS}_n(x)$, then

$$\|\mathcal{G}_m(x)\| = \|\mathcal{G}_n(x)\| \le \|x\| + \|x - \mathcal{G}_n(x)\| = \|x\| + \|x - \sum_{i \in \mathcal{W}^{\tau}(x, m_1)} x_i'(x)x_i\|$$

$$\le (1+M)\|x\| \le (1+M)(1+M^2\tau^{-4})\|x\|.$$

On the other hand, if $\mathcal{GS}_n(x) \subsetneq \mathcal{W}^{\tau}(x, m_1)$, let $\mathcal{W}^{\tau}(x, m_1 - 1)$ be a weak thresholding set for which (4) holds. By the minimality of m_1 we get that $\mathcal{GS}_n(x) \not\subseteq$ $\mathcal{W}^{\tau}(x, m_1 - 1)$, so for every $i \in \mathcal{W}^{\tau}(x, m_1 - 1)$ we have

(8)
$$|x_i'(x)| \ge \tau |x_{\rho(x,n)}'(x)|.$$

Thus, if there is $i_0 \in \mathcal{W}^{\tau}(x, m_1 - 1) \setminus \mathcal{W}^{\tau}(x, m_1)$, it follows from (8) and Definition 1.5 that for all $j \in \mathcal{W}^{\tau}(x, m_1)$,

$$|x'_j(x)| \ge \tau |x'_{i_0}(x)| \ge \tau^2 |x'_{\rho(x,n)}(x)|.$$

On the other hand, if $W^{\tau}(x, m_1 - 1) \subseteq W^{\tau}(x, m_1)$, given that

$$\mathcal{GS}_n(x) \not\subseteq \mathcal{W}^{\tau}(x, m_1 - 1)$$
 and $\mathcal{GS}_n(x) \subseteq \mathcal{W}^{\tau}(x, m_1)$,

it follows that there is $1 \le i_1 \le n$ such that

$$\mathcal{W}^{\tau}(x, m_1) = \mathcal{W}^{\tau}(x, m_1 - 1) \cup \{\rho(x, i_1)\},\$$

which implies that (8) also holds for all $i \in \mathcal{W}^{\tau}(x, m_1)$. Therefore, in any case, we

$$|x_j'(x)| \ge \tau^2 |x_{\rho(x,n)}'(x)|$$

for all $j \in \mathcal{W}^{\tau}(x, m_1)$. In what follows, put $\mathcal{W} = \mathcal{W}^{\tau}(x, m_1)$. As for all $i \in \mathbb{N} \setminus \mathcal{GS}_n(x)$,

$$|x'_{\rho(x,n)}(x)| \ge |x'_i(x)|,$$

we obtain

$$\max_{i \in \mathcal{W} \setminus \mathcal{G}_n(x)} |x_i'(x)| \le \min_{j \in \mathcal{W}} \tau^{-2} |x_j'(x)|.$$

Hence, using that $\mathcal{G}_m(x) = \mathcal{G}_n(x)$ and applying the hyperdemocracy condition, we get

$$\|\mathcal{G}_{m}(x)\| \leq \|\sum_{i \in \mathcal{W}} x_{i}'(x)x_{i}\| + \|\sum_{i \in \mathcal{W}} x_{i}'(x)x_{i} - \mathcal{G}_{n}(x)\|$$

$$= \|\sum_{i \in \mathcal{W}} x_{i}'(x)x_{i}\| + \|\sum_{i \in \mathcal{W} \setminus \mathcal{GS}_{n}(x)} x_{i}'(x)x_{i}\|$$

$$\leq \|\sum_{i \in \mathcal{W}} x_{i}'(x)x_{i}\| + K_{hd}\|\sum_{i \in \mathcal{W}} \tau^{-2}x_{i}'(x)x_{i}\|$$

$$\leq (1 + M^{2}\tau^{-4})\|\sum_{i \in \mathcal{W}} x_{i}'(x)x_{i}\|$$

$$\leq (1 + M^{2}\tau^{-4})(\|x\| + \|x - \sum_{i \in \mathcal{W}} x_{i}'(x)x_{i}\|)$$

$$\leq (1 + M^{2}\tau^{-4})(\|x\| + M\tilde{\sigma}_{m_{1}}(x))$$

$$\leq (1 + M)(1 + M^{2}\tau^{-4})\|x\|.$$

This proves that $(x_i)_i \subseteq X$ is quasi-greedy (with K_{1q} as in the statement). Then $(x_i)_i$ is a Markushevich basis, and it is almost greedy by Theorem 2.2.

3. The finite dimensional separation property.

In this section, we introduce and study a separation property inspired by some of the proofs in [4]. We give upper bounds for a constant associated with this property. The constant plays a key role in some of our proofs involving Markushevich bases.

Definition 3.1. Let $(u_i)_i \subseteq X$ be a sequence. We say that $(u_i)_i$ has the *finite dimensional separation property* (FDSP) if there is a positive constant M such that for every separable subspace $Z \subset X$ and every $\epsilon > 0$, there is a basic subsequence $(u_{i_k})_k$ with basis constant no greater than $M + \epsilon$ satisfying the following: For every finite dimensional subspace $F \subset Z$ there is $j_F \in \mathbb{N}$ such that

(9)
$$||x|| \le (M + \epsilon)||x + z||,$$

for all $x \in F$ and all $z \in [u_{i_k} : k > j_F]$. We call any such subsequence a *finite dimensional separating subsequence* for (Z, M, ϵ) (and for $(u_i)_i$, leaving that implicit when clear), and we call the minimum M for which this property holds the *finite dimensional separation constant* M_{fs} of $(u_i)_i$.

Remark 3.2. Note that in order to check whether a subsequence is finite dimensional separating for (Z, M, ϵ) , it is enough to check that (9) holds for x with ||x|| = 1 and $z \in [u_{i_k} : k > j_F]$.

The following lemma gives a basic characterization for finite dimensional separating sequences. The proof is immediate and is left to the reader.

Lemma 3.3. Let $(u_i)_i \subseteq X$ be a sequence. For any sequence $(a_i)_i \subset \mathbb{K}$ with $a_i \neq 0$ for all i, and any bijection $\pi \colon \mathbb{N} \to \mathbb{N}$, $(u_i)_i$ has the finite dimensional separation property with constant M_{fs} if and only if, for any $l \in \mathbb{N}$, $(a_i u_{\pi(i)})_{i \geq l}$ does.

For our next result, we need the following technical lemmas; the second one is a variant of [4, Theorem 1.5.2].

Lemma 3.4. [4, Lemma 1.5.1] Let $S \subseteq X'$ be a subset such that S is bounded below and $0 \in \overline{S}^{w^*}$. Then, for every $\epsilon > 0$ and every finite dimensional subspace $F \subseteq X'$, there is $x' \in S$ such that for all $y' \in F$ and $b \in \mathbb{K}$,

$$||y'|| \le (1+\epsilon)||y'+bx'||.$$

Lemma 3.5. Let X be a Banach space and $(u_i')_i \subseteq X'$ a sequence such that $(u_i')_i$ is bounded below and $0 \in \overline{\{u_i'\}}_{i \in \mathbb{N}}^{w^*}$. Then, for any separable subspace $Z \subset X'$ and $\epsilon > 0$ there is a basic subsequence $(u_{i_n}')_n$ with basis constant no greater than $(1 + \epsilon)$ satisfying the following: For any finite dimensional subspace $F \subset Z$ and every $\xi > 0$, there is $j_{F,\xi} \in \mathbb{N}$ such that for all $y' \in F$ and $v' \in \overline{[u_{i_n}': n > j_{F,\xi}]}$,

$$||y'|| \le (1+\xi)||y'+v'||.$$

In particular, $(u_i)_i$ has the finite dimensional separation property with constant 1.

Proof. Choose a dense sequence $(v_i')_i$ in Z and a sequence of positive scalars $(\epsilon_i)_i$ so that $\prod_{i=1}^{\infty} (1+\epsilon_i) \leq (1+\epsilon)$. Since $0 \in \overline{\{u_i'\}}_{i \in \mathbb{N}}^{w^*}$ and $u_i' \neq 0$ for every $i \in \mathbb{N}$, it follows

that $0 \in \overline{\{u_i'\}}_{i \geq l}^{w^*}$ for every $l \in \mathbb{N}$. Thus, by Lemma 3.4, we can choose $i_2 > i_1 := 1$ so that for all $y' \in [v_{i_1}', u_{i_1}']$ and all $b \in \mathbb{K}$,

$$||y'|| \le (1 + \epsilon_1)||y' + bu'_{i_2}||.$$

By an inductive argument, we obtain a strictly increasing sequence of positive integers $\{i_n\}_{n\in\mathbb{N}}$ such that for all $y'\in[v'_s,u'_s\colon 1\leq s\leq i_n],\,b\in\mathbb{K}$ and $j\in\mathbb{N}$,

$$||y'|| \le (1 + \epsilon_n)||y' + bu'_{i_{n+1}}||.$$

Then, for any positive integers j < l, any $y' \in [v'_s, u'_s : 1 \le s \le i_j]$ and any scalars $(a_n)_{j < n \le l}$,

$$||y'|| \le \prod_{n=j}^{l-1} (1+\epsilon_n)||y' + \sum_{n=j+1}^{l} a_n u'_{i_n}|| \le \prod_{n=j}^{\infty} (1+\epsilon_n)||y' + \sum_{n=j+1}^{l} a_n u'_{i_n}||.$$

In particular, $(u'_{i_n})_n$ is basic with basis constant no greater than $\prod_{n=1}^{\infty} (1+\epsilon_n) \leq 1+\epsilon$, and the result holds for $F \subset [v'_i : 1 \leq i \leq n]$ for some $n \in \mathbb{N}$. Now, standard density arguments allow us to obtain the result for any finite dimensional subspace of $Z = [v'_i : i \in \mathbb{N}]$.

Remark 3.6. Suppose that there is a sequence $(u_i)_i \subseteq X$ such that $(u_i)_i$ is bounded below and $0 \in \overline{\{u_i\}_{i \in \mathbb{N}}}^w$. Via the canonical injection $X \hookrightarrow X''$, Lemma 3.5 remains valid for any separable subspace $Z \subseteq X$ where the basic sequence $(u_{i_n})_n$ is a subsequence of $(u_i)_i$.

Next, we consider the case in which 0 may not be a weak or a weak star accumulation point of the sequence. We use \widehat{x} to denote $x \in X$ as an element of the bidual space X'', and we use \widehat{X} to denote X as a subspace of X''. Also, for a bounded sequence $(u_i)_i \subseteq X$, we will consider $\beta((u_i)) := \overline{\{\widehat{u}_i\}_{i \in \mathbb{N}}^{w*}} \setminus \widehat{X}$.

Lemma 3.7. Let X be a Banach space and $(u_i)_i \subseteq X$ a bounded sequence such that $\overline{\{u_i\}}_{i\in\mathbb{N}}^w$ is not weakly compact. Then, $(u_i)_i$ has the finite dimensional separation property with constant $M_{fs} \leq M$, where

$$M := \left(2 + \inf_{x'' \in \beta((u_i))} \left\{ \frac{\|x''\|}{\operatorname{dist}(x'', \widehat{X})} \right\} \right)^2.$$

Proof. Since $\overline{\{u_i\}}_{i\in\mathbb{N}}^w$ is not weakly compact but $\overline{\{\widehat{u}_i\}}_{i\in\mathbb{N}}^{w*}$ is weak star compact, there is $x''\in\beta((u_i))=\overline{\{\widehat{u}_i\}}_{i\in\mathbb{N}}^{w*}\setminus\widehat{X}$, so M is well defined. Given $\epsilon>0$ and $Z\subset X$ a separable subspace, choose $0<\xi<1$ and $x_0''\in\overline{\{\widehat{u}_i\}}_{i\in\mathbb{N}}^{w*}\setminus\widehat{X}$ so that

(10)
$$M + \xi + \xi^2 + 2\xi(M + \xi) \le \frac{M + \epsilon}{1 + \xi},$$

and

(11)
$$(2 + \frac{\|x_0''\|}{\operatorname{dist}(x_0'', \widehat{X})})^2 \le M + \xi.$$

Let $Z_1 := \widehat{Z} + \overline{[x_0'', \widehat{u_i} : i \in \mathbb{N}]}$, and consider in Z_1 the seminormalized sequence $(\widehat{u}_i - x_0'')_i$. Then, there exists a basic subsequence $(\widehat{u}_{i_k} - x_0'')_k$ with basic constant no greater that $(1 + \xi)$ satisfying the conclusions of Lemma 3.5. Since $x_0'' \notin \widehat{X}$, there is a bounded linear functional x_1''' on $\widehat{X} \oplus [x_0'']$ such that for all $x \in X$ and all $b \in \mathbb{K}$,

$$x_1'''(\widehat{x} + bx_0'') = b.$$

Suppose that $\|\hat{x} + bx_0''\| = 1$ with $b \neq 0$. Then,

$$|x_1'''(\widehat{x} + bx_0'')| = |b| = |b| \frac{\|b^{-1}\widehat{x} + x_0''\|}{\|b^{-1}\widehat{x} + x_0''\|} = \frac{1}{\|b^{-1}\widehat{x} + x_0''\|} \le \frac{1}{\operatorname{dist}(x_0'', \widehat{X})}.$$

It follows that

(12)
$$||x_1'''|| \le \frac{1}{\operatorname{dist}(x_0'', \widehat{X})}.$$

By the Hahn–Banach Theorem, there is a norm-preserving extension of x_1''' to X'', which we also call x_1''' . Let $F_1:=[x_0'']$. By the choice of $(\widehat{u}_{i_k}-x_0'')_k$, there exists $j_{F_1,\xi}\in\mathbb{N}$ such that for all $z''\in\overline{[\widehat{u}_{i_k}-x_0''\colon k>j_{F_1,\xi}]}$ we have

(13)
$$||x_0''|| \le (1+\xi)||x_0'' + z''||.$$

In particular, given that $x_0'' \neq 0$ this implies that $x_0'' \notin [\widehat{u}_{i_k} - x_0'' : k > j_{F_1,\xi}]$. Thus, there is a bounded linear functional x_2''' on $[\widehat{u}_{i_k} - x_0'' : k > j_{F_1,\xi}] \oplus [x_0'']$ defined, for all $z'' \in [\widehat{u}_{i_k} - x_0'' : k > j_{F_1,\xi}]$ and all $b \in \mathbb{K}$, by

$$x_2'''(z'' + bx_0'') = b.$$

As before, for any $z'' \in [\widehat{u}_{i_k} - x_0'': k > j_{F_1,\xi}]$ and $b \neq 0$ such that $||z'' + bx_0''|| = 1$, we have

$$|x_2'''(z'' + bx_0'')| = |b| = \frac{1}{\|b^{-1}z'' + x_0''\|} \le \frac{1+\xi}{\|x_0''\|},$$
 by (13).

Thus.

$$||x_2'''|| \le \frac{1+\xi}{||x_0''||}.$$

Again, by the Hahn–Banach Theorem, we may consider x_2''' defined on X''. Now define, for $x'' \in X''$, the following bounded linear operators:

$$T(x'') := x'' + (x_2''' - x_1''')(x'')x_0'';$$

$$L(x'') := x'' - (x_2''' - x_1''')(x'')x_0''.$$

By (12) and (14) we get

$$||T(x'')|| \le ||x''|| + ||x_2'''|| ||x''|| ||x_0''|| + ||x_1'''|| ||x''|| ||x_0''|| \le (2 + \xi + \frac{||x_0''||}{\operatorname{dist}(x_0'', \widehat{X})}) ||x''||,$$

Hence, $||T|| \le 2 + \xi + \frac{||x_0''||}{\operatorname{dist}(x_0'', \widehat{X})}$ and the same bound holds for L.

From this, (10) and (11) we get

$$||T|||L|| \le \left(2 + \xi + \frac{||x_0''||}{\operatorname{dist}(x_0'', \widehat{X})}\right)^2 = \left(2 + \frac{||x_0''||}{\operatorname{dist}(x_0'', \widehat{X})}\right)^2 + \xi^2 + 2\xi\left(2 + \frac{||x_0''||}{\operatorname{dist}(x_0'', \widehat{X})}\right)^2 + \xi^2 + 2\xi\left(2 + \frac{||x_0''||}{\operatorname{dist}(x_0'', \widehat{X})}\right)^2$$

$$(15) \qquad \le M + \xi + \xi^2 + 2\xi(M + \xi) \le \frac{M + \epsilon}{1 + \xi}.$$

It is easy to check that T and L are inverses of each other. Then, since $T(\hat{u}_{i_k} - x_0'') =$ \widehat{u}_{i_k} for all $k > j_{F_1,\xi}$ and $(\widehat{u}_{i_k} - x_0'')_k$ is a basic sequence with basis constant no greater than $(1+\xi)$, it follows that $(u_{i_k})_{k>j_{F_1,\xi}}$ is a basic sequence with basis constant

$$K_b((u_{i_k})_{k>j_{F_1,\xi}}) \le ||T|||L||(1+\xi) \le M+\epsilon,$$

where the last inequality follows from (15). Now let $F \subset Z$ be a finite dimensional subspace, and let $j_F := \max\{j_{L(\widehat{F}),\xi}, j_{F_1,\xi}\}$. For any $x \in F$ and scalars $(a_k)_{j_F < k \le m}$, by the choice of $(\widehat{u}_{i_k} - x_0'')$ we have

$$\begin{split} \|x\| = &\|\widehat{x}\| \leq \|T\| \|L(\widehat{x})\| \leq \|T\| (1+\xi) \|L(\widehat{x}) + \sum_{j_F < k \leq m} a_k (\widehat{u}_{i_k} - x_0'') \| \\ = &\|T\| (1+\xi) \|L(\widehat{x} + \sum_{j_F < k \leq m} a_k \widehat{u}_{i_k})\| \leq \|T\| \|L\| (1+\xi) \|\widehat{x} + \sum_{j_F < k \leq m} a_k \widehat{u}_{i_k})\| \\ \leq &(M+\epsilon) \|x + \sum_{j_F < k \leq m} a_k u_{i_k}\|, \end{split}$$

where we apply again (15) to obtain the last inequality. By a density argument, it follows that $(u_{i_k})_{k>j_{F_1,\xi}}$ has the desired properties.

Remark 3.8. Note that the upper bound for M_{fs} given by Lemma 3.7 remains unchanged if one replaces $(u_i)_i$ with $(a_iu_i)_i$, for any seminormalized sequence $(a_i)_i$.

Corollary 3.9. Let $(u_i)_i$ be a seminormalized sequence. The following are equiva-

- (i) $(u_i)_i$ has a basic subsequence.
- (ii) Either 0 ∈ {u_i}_{i∈ℕ}^w, or {u_i}_{i∈ℕ}^w is not weakly compact.
 (iii) (u_i)_i has the finite dimensional separation property.

Proof. The equivalence (i) \iff (ii) was proven in [15] (see also [4, Theorem 1.5.6]). By Remark 3.6 and Lemma 3.7 it follows that (ii) \Longrightarrow (iii). Finally, (iii) \Longrightarrow (i) is clear.

Next we study the finite dimensional separation property of Markushevich bases, and give upper bounds for its constant in this context. Recall that for $0 < c \le 1$, a subspace $S \subset X'$ is said to be c-norming for X if

$$c\|x\| \le \sup_{\substack{x' \in S \\ \|x'\| = 1}} |x'(x)| \quad \forall x \in X.$$

We will use the following result.

Lemma 3.10. [4, Proposition 3.2.3] Let $(x_i)_i$ be a Schauder basis for X with basis constant K_b . Then $\overline{[x_i': i \in \mathbb{N}]} \subset X'$ is K_b^{-1} -norming for X.

Also recall that a sequence $(v_i)_i$ is a block basis of a Markushevich basis $(x_k)_k$ if there are sequences of positive integers $(n_i)_i$, $(m_i)_i$ with $n_i \leq m_i < n_{i+1}$ for all i and scalars $(b_k)_k$ such that

$$v_i = \sum_{k=n_i}^{m_i} b_k x_k,$$

with at least one nonzero b_k for each $i \in \mathbb{N}$. In particular, any subsequence of a Markushevich basis is a block basis of it.

Proposition 3.11. Let $(v_i)_i \subset X$ be a block basis of a Markushevich basis $(y_k)_k$ for $Y \subset X$ with biorthogonal functionals $(y'_k)_k$, and let $(a_i)_i$ be a scalar sequence such that $(z_i := a_i v_i)_i$ is seminormalized. The following hold:

- (a) Either $\overline{\{z_i\}}_{i\in\mathbb{N}}^w$ is not weakly compact, or $(z_i)_i$ is weakly null. Hence, (v_i) has the finite dimensional separation property, with the same constant as
- (b) If either $0 \in \overline{\{z_i\}}_{i \in \mathbb{N}}^w$ or X is a dual space and $0 \in \overline{\{z_i\}}_{i \in \mathbb{N}}^{w*}$, then $M_{fs} = 1$. (c) If $\overline{\{z_i\}}_{i \in \mathbb{N}}^w$ is not weakly compact, then

$$M_{fs} \le \left(2 + \inf\left\{\frac{\|x''\|}{\operatorname{dist}(x'', \widehat{X})} : x'' \in \overline{\{\widehat{z}_i\}}_{i \in \mathbb{N}}^{w*} \setminus \widehat{X}\right\}\right)^2.$$

- (d) If Y = X and $[y'_k : k \in \mathbb{N}]$ is c-norming, then $M_{fs} \leq c^{-1}$.
- (e) If Y = X and $(y_k)_k$ is a Schauder basis for X with constant K_b , then $M_{fs} \leq K_b$.

Proof. To prove (a), suppose that $\overline{\{z_i\}}_{i\in\mathbb{N}}^w$ is weakly compact. Then, since Y is weakly closed, given a subnet (z_{i_λ}) there is a further subnet $(z_{i_{\lambda_\theta}})$ and $v_0 \in Y$ such

$$z_{i_{\lambda_{\theta}}} \xrightarrow{w} v_0.$$

Since $(z_i)_i$ is a block basis of $(y_k)_k$, it follows that $y'_k(v_0) = 0$ for all $k \in \mathbb{N}$, so $v_0 = 0$. Thus, $(z_i)_i$ is weakly null. It follows by Corollary 3.9 that $(z_i)_i$ has the finite dimensional separation property, and by Lemma 3.3, so does $(v_i)_i$, with the same constant.

Lemma 3.5 and Remark 3.6 imply (b), and (c) follows by Lemma 3.7.

To prove (d), note that by (a), $(v_i)_i$ has a basic subsequence $(v_{i_l})_l$. Let $F \subset X$ be a finite dimensional subspace, and fix $0 < \epsilon < 1$. Choose $0 < \xi < 1$ so that

$$0 < \frac{c^{-1}(1-\xi)^{-1}}{1-c^{-1}(1-\xi)^{-1}\xi} \le c^{-1} + \epsilon$$

Take $(u_j)_{1 \leq j \leq m_1}$ unit vectors in F that form a ξ -net of the unit sphere of F.

As $\overline{[y'_k\colon k\in\mathbb{N}]}$ is c-norming so is $[y'_k\colon k\in\mathbb{N}]$. Hence, there is $m_2\in\mathbb{N}$ and unit vectors $(u'_j)_{1\leq j\leq m_1}\subset [y'_k\colon 1\leq k\leq m_2]$ such that $|u'_j(u_j)|\geq c(1-\xi)$ for all $1 \leq j \leq m_1$.

Now fix $x \in F$ with ||x|| = 1 and $v \in \overline{[v_{i_l}: l > m_2]}$, and choose $1 \le j \le m_1$ so that $||x - u_j|| \le \xi$. Note that $v \in \overline{[y_k: k > m_2]}$, so $y'_k(v) = 0$ for all $1 \le k \le m_2$. Hence,

$$1 \le c^{-1} (1 - \xi)^{-1} |u'_j(u_j)| = c^{-1} (1 - \xi)^{-1} |u'_j(u_j + v)| \le c^{-1} (1 - \xi)^{-1} ||u_j + v||$$

$$\le c^{-1} (1 - \xi)^{-1} ||x + v|| + c^{-1} (1 - \xi)^{-1} ||u_j - x||$$

$$\le c^{-1} (1 - \xi)^{-1} ||x + v|| + c^{-1} (1 - \xi)^{-1} \xi.$$

Thus,

$$||x|| = 1 \le \frac{c^{-1}(1-\xi)^{-1}}{1-c^{-1}(1-\xi)^{-1}\xi}||x+v|| \le (c^{-1}+\epsilon)||x+v||.$$

Finally, (e) follows by (d) and Lemma 3.10

4. Weak semi-greedy systems.

In this section, we prove our main results for weak semi-greedy minimal systems. It was proven in [9, Theorem 3.2] that every almost greedy Schauder basis is semigreedy with constant only depending on its democracy and quasi-greedy constants

(and thus, by Theorem 2.2, only on its almost greedy constant), with a proof valid also for Markushevich bases (see also [5, Theorem 1.10] and [12, Corollary 4.2]). Moreover, it is known that if $(x_i)_i$ is an almost greedy Markushevich basis, then for each $0 < \tau \le 1$ there is a constant M depending only on the first quasi-greedy and the democracy constants of the basis and τ such that the conditions of Definition 1.7 hold for all $x \in X$, $m \in \mathbb{N}$, and every weak thresholding set $\mathcal{W}^{\tau}(x,m)$. This fact was established in [13, Theorem 7.1] for finite dimensional Banach spaces, and the proof holds for the infinite dimensional case as well (see [8, Theorem 1.2]). On the other hand, results in the opposite direction are not yet complete. In [9, Theorem 3.6, it is proved that every semi-greedy Schauder basis for a Banach space with finite cotype is almost greedy. In [5, Theorem 1.10], the cotype condition is removed and it is proved that every semi-greedy Schauder basis is almost greedy with quasi-greedy and superdemocracy constants depending only on the basis constant and the semi-greedy constant, leaving the question of whether the implication from semi-greedy to almost greedy holds for general Markushevich bases ([5, Question 1]). Recently, Berná extended [5, Theorem 1.10] to a certain class of Markushevich bases (known as ρ -admisible) [6, Theorem 5.3]. To our knowledge the general case remained open until now. In this section, we complete the proof of the implication from semi-greedy to almost greedy Markushevich bases, and extend the result to $WSG(\tau)$ Markushevich bases. We also study the (weak) semi-greedy property for general minimal systems, without the Markushevich hypothesis. We begin with an auxiliary lemma.

Lemma 4.1. Let $(x_i)_i \subset X$ be a $WSG(\tau)$ system with constant K. Then both $(x_i)_i$ and $(x_i')_i$ are seminormalized.

Moreover,
$$\sup_i ||x_i|| \le 2K\tau^{-1}\inf_j (1 + ||x_j'|| ||x_j||) ||x_j||.$$

Proof. For every $x \in X$, there is a weak thresholding set $\mathcal{W}^{\tau}(x, 1)$, so $(|x'_i(x)|)_i$ is bounded. It follows by uniform boundedness that $(x'_i)_i$ is bounded. Since $x'_i(x_i) = 1$ for all $i \in \mathbb{N}$, $(x_i)_i$ is bounded below.

Given $i \neq j$, it follows from Definitions 1.5 and 1.7 that the only 1-weak thresholding set with weakness parameter τ for $x_i + 2\tau^{-1}x_j$ is

$$\mathcal{W}^{\tau}(x_i + 2\tau^{-1}x_j, 1) = \{j\}.$$

Let ax_j be a Chebyshev τ -greedy approximant for $x_i + 2\tau^{-1}x_j$. We have

$$||x_{i}|| \leq ||x_{i} + 2\tau^{-1}x_{j} - ax_{j}|| + ||2\tau^{-1}x_{j} - ax_{j}||$$

$$= ||x_{i} + 2\tau^{-1}x_{j} - ax_{j}|| + ||x'_{j}(x_{i} + 2\tau^{-1}x_{j} - ax_{j})x_{j}||$$

$$\leq (1 + ||x'_{j}|| ||x_{j}||) ||x_{i} + 2\tau^{-1}x_{j} - ax_{j}|| \leq (1 + ||x'_{j}|| ||x_{j}||) K\sigma_{1}(x_{i} + 2\tau^{-1}x_{j})$$

$$\leq 2(1 + ||x'_{j}|| ||x_{j}||)\tau^{-1}K||x_{j}||.$$

Thus, $(x_i)_i$ is bounded, which implies that $(x_i')_i$ is bounded below. Since the above inequality holds also for i = j, the bound in the statement follows by taking infimum over j and supremum over i.

Now we prove that $WSG(\tau)$ Markushevich bases are almost greedy. The proof combines arguments from the proofs of [9, Proposition 3.3] and [5, Theorem 1.10 b] - which we adapt to weak thresholding and weak Chebyshev greedy algorithms - with arguments based on the finite dimensional separation property - which allows us to work in the context of general Markushevich bases.

Theorem 4.2. Let $0 < \tau \le 1$, and let $(x_i)_i \subseteq X$ be a $WSG(\tau)$ system with constant K. The following are equivalent:

- (i) $(x_i)_i$ is almost greedy.
- (ii) $(x_i)_i$ is quasi-greedy.

- (iii) $(x_i)_i$ is a Markushevich basis.
- (iv) $(x_i)_i$ has the finite dimensional separation property.

If any (and thus all) of these conditions holds, $(x_i)_i$ has second quasi-greedy constant $K_{2q} \leq M_{fs}K + M_{fs}(M_{fs}+1)K^2\tau^{-2}$ and hyperdemocracy constant $K_{hd} \leq M_{fs}(M_{fs}+1)K^2\tau^{-2}$, where M_{fs} is the finite dimensional separation constant of $(x_i)_i$.

Proof. The implication (i) \Longrightarrow (ii) is immediate. The comments after Theorem 2.2 give that (ii) \Longrightarrow (iii) and Proposition 3.11 gives that (iii) \Longrightarrow (iv). To show that (iv) \Longrightarrow (i) fix $0 < \epsilon < 1$ and let $(x_{i_k})_k$ be a finite dimensional separation subsequence for (X, M_{fs}, ϵ) .

Fix $x \in X$ and $m \in \mathbb{N}$, assuming that $\mathcal{G}_m(x) \neq 0$ (otherwise, $||x - \mathcal{G}_m(x)|| = ||x||$ and there is nothing to prove). Let

$$n := \max \{1 \le j \le m : x'_{\rho(x,j)}(x) \ne 0\},\$$

and note that $\mathcal{G}_n(x) = \mathcal{G}_m(x)$. Since $x'_{\rho(x,n)}(x) \neq 0$ and $x'_i(x) \xrightarrow[i \to \infty]{} 0$, there is $j_0 \in \mathbb{N}$ such that for all $i \geq j_0$,

(16)
$$|x_i'(x)| \le \frac{\tau \epsilon}{2} |x_{\rho(x,n)}'(x)|.$$

Now take $F_0 := [x, x_i : 1 \le i \le j_0]$, let $W := \{i_k : j_{F_0} + 1 \le k \le j_{F_0} + n\}$ and set z as follows.

(17)
$$z := x - \mathcal{G}_n(x) + (1 + \epsilon)\tau^{-1}|x'_{\rho(x,n)}(x)| \sum_{j \in \mathcal{W}} x_j - \sum_{j \in \mathcal{W}} x'_j(x)x_j.$$

Since $i_{j_{F_0}+1} > j_0$, we deduce from (16) and the choice of z that for every $j \in \mathcal{W}$ and every $l \in \mathbb{N} \setminus \mathcal{W}$,

$$\tau |x_i'(z)| \ge (1+\epsilon)|x_{o(x,n)}'(x)| > |x_i'(x-\mathcal{G}_n(x))| = |x_i'(z)|.$$

It follows from this and Definition 1.5 that the only *n*-weak thresholding set for z with weakness parameter τ is \mathcal{W} . Let $u \in [x_j : j \in \mathcal{W}]$ be an *n*-term Chebyshev τ -greedy approximant for z. Notice that both x and $\mathcal{G}_n(x)$ belong to F_0 . Also, by Lemma 4.1, $(\|x_i\|)_i$ and $(\|x_i'\|)_i$ are bounded, say by N. As

$$\|\sum_{j \in \mathcal{W}} x_j'(x)x_j\| \le \sum_{j \in \mathcal{W}} \epsilon |x_{\rho(x,n)}'(x)| \|x_j\| \le \epsilon nN^2 \|x\|,$$

from (16), (17) and the choice of our subsequence we deduce that

$$||x - \mathcal{G}_{n}(x)|| \leq (M_{fs} + \epsilon)||z - u|| \leq (M_{fs} + \epsilon)K\sigma_{n}(z)$$

$$\leq (M_{fs} + \epsilon)K||x + (1 + \epsilon)\tau^{-1}|x'_{\rho(x,n)}(x)| \sum_{j \in \mathcal{W}} x_{j} - \sum_{j \in \mathcal{W}} x'_{j}(x)x_{j}||$$

$$\leq (M_{fs} + \epsilon)K(1 + \epsilon nN^{2})||x|| + (M_{fs} + \epsilon)K(1 + \epsilon)\tau^{-1}|x'_{\rho(x,n)}(x)|||\sum_{j \in \mathcal{W}} x_{j}||.$$
(18)

Now, in order to estimate $|x'_{\rho(x,n)}(x)| \|\sum_{j \in \mathcal{W}} x_j\|$ we set

$$w := (1 - \epsilon)\tau |x'_{\rho(x,n)}(x)| \sum_{j \in \mathcal{W}} x_j,$$

and let v be an n-term Chebyshev τ -greedy approximant for x + w. We claim that $v \in F_0$. To prove this, from (16) and the definition of F_0 we deduce that for all $i \in \mathbb{N}_{\geq i_0} \setminus \mathcal{W}$,

$$|x_i'(x+w)| = |x_i'(x)| \le \frac{\tau}{2} |x_{\rho(x,n)}'(x)| < \tau |x_{\rho(x,n)}'(x)|,$$

whereas for $i \in \mathcal{W}$,

$$|x_i'(x+w)| \le (1-\epsilon)\tau |x_{\rho(x,n)}'(x)| + \frac{\tau\epsilon}{2}|x_{\rho(x,n)}'(x)| < \tau |x_{\rho(x,n)}'(x)|.$$

Combining both inequalities above we get that

(19)
$$\{i \in \mathbb{N} \colon |x_i'(x+w)| \ge \tau |x_{\rho(x,n)}'(x)|\} \subseteq \{1,\ldots,j_0\}.$$

On the other hand, for all $1 \le i \le n$, we have $i_{j_{F_0}+1} > j_0 > \rho(x,i)$, so

$$|x'_{\rho(x,i)}(x+w)| = |x'_{\rho(x,i)}(x)| \ge |x'_{\rho(x,n)}(x)|.$$

From this and Definitions 1.5 and 1.7 we deduce that

$$|x_i'(x+w)| \ge \tau |x_{\rho(x,n)}'(x)|$$

for all $i \in \text{supp}(v)$ which, combined with (19), implies that $\text{supp}(v) \subseteq \{1, \ldots, j_0\}$. Since, by Definition 1.7, v is a linear combination of the x_i 's with i in its support, it follows that $v \in F_0$ (and so $x - v \in F_0$). Hence, applying the separation property of $(x_{i_k})_k$ we deduce that

$$(1 - \epsilon)\tau |x'_{\rho(x,n)}(x)| \| \sum_{j \in \mathcal{W}} x_j \| = \|w\| \le \|x + w - v\| + \|x - v\|$$

$$\le \|x + w - v\| + (M_{fs} + \epsilon)\|x + w - v\|$$

$$\le (1 + M_{fs} + \epsilon)K\sigma_n(x + w)$$

$$\le (1 + M_{fs} + \epsilon)K\|x\|.$$

Then

$$|x'_{\rho(x,n)}(x)| \|\sum_{j\in\mathcal{W}} x_j\| \le (1-\epsilon)^{-1} \tau^{-1} (1+M_{fs}+\epsilon) K \|x\|.$$

This result and (18) entail that

$$||x - \mathcal{G}_n(x)|| \le (M_{fs} + \epsilon)K(1 + \epsilon nN^2)||x|| + (M_{fs} + \epsilon)(1 + M_{fs} + \epsilon)\frac{1 + \epsilon}{1 - \epsilon}\tau^{-2}K^2||x||.$$

As $\mathcal{G}_n(x) = \mathcal{G}_m(x)$, letting $\epsilon \to 0$, we get

$$||x - \mathcal{G}_m(x)|| = ||x - \mathcal{G}_n(x)|| < (M_{fs}K + M_{fs}(M_{fs} + 1)K^2\tau^{-2})||x||.$$

Since x and m were chosen arbitrarily, this proves that $(x_i)_i$ is quasi-greedy with second quasi-greedy constant $K_{2q} \leq M_{fs}K + M_{fs}(M_{fs}+1)K^2\tau^{-2}$. Thus, it is a Markushevich basis.

Let us show the hyperdemocracy condition. Choose $\epsilon > 0$ and $(x_{i_k})_k$ as before, and take A and B finite subsets of $\mathbb N$ such that $|A| \leq |B|$, and $(a_i)_{i \in A}$, $(b_j)_{j \in B}$ scalars with $|a_i| \leq |b_j|$ for all $i \in A, j \in B$. Set

$$F_1 := [x_i \colon i \in A \cup B]$$
 and $a_0 := \max_{i \in A} |a_i|$,

assuming without loss of generality that $a_0 \neq 0$. Take $\mathcal{W} := \{i_k : j_{F_1} + 1 \leq k \leq j_{F_1} + |A|\}$, and define:

$$z_1 := \tau^{-1} a_0 \sum_{l \in \mathcal{W},} x_l, \qquad z_2 := \sum_{i \in A} a_i x_i \qquad \text{and} \qquad z_3 := \sum_{j \in B} b_j x_j.$$

Note that for all $i \in A$ and all $l \in \mathcal{W}$,

$$|x_i'(z_2 + (1+\epsilon)z_1)| = |x_i'(z_2)| = |a_i| < a_0(1+\epsilon) = \tau |x_i'(z_2 + (1+\epsilon)z_1)|.$$

By Definition 1.5, it follows that the only |A|-weak thresholding set for $z_2 + (1+\epsilon)z_1$ with parameter τ is \mathcal{W} . Let $u \in [x_i : i \in \mathcal{W}]$ be a |A|-term Chebyshev τ -greedy

approximant for $z_2 + (1 + \epsilon)z_1$. We have

(20)
$$\|\sum_{i \in A} a_i x_i\| = \|z_2\| \le (M_{fs} + \epsilon) \|z_2 + (1 + \epsilon)z_1 - u\|$$
$$\le (M_{fs} + \epsilon) K \sigma_{|A|} (z_2 + (1 + \epsilon)z_1)$$
$$\le (1 + \epsilon) (M_{fs} + \epsilon) K \|z_1\|.$$

Similarly, since for all $j \in B$, $(1 - \epsilon)\tau a_0 < \tau |b_j|$, the only |B|-weak thresholding set for $z_3 + (1 - \epsilon)\tau^2 z_1$ with parameter τ is B. Thus, by the WSG(τ) condition there is $v \in [x_i : i \in B]$ such that

$$||z_3 + (1 - \epsilon)\tau^2 z_1 - v|| \le K\sigma_{|B|}(z_3 + (1 - \epsilon)\tau^2 z_1) \le K||z_3|| = K||\sum_{j \in B} b_j x_j||.$$

Hence,

$$(1 - \epsilon)\tau^{2} ||z_{1}|| \leq ||z_{3} + (1 - \epsilon)\tau^{2}z_{1} - v|| + ||z_{3} - v||$$

$$\leq (1 + M_{fs} + \epsilon)||z_{3} + (1 - \epsilon)\tau^{2}z_{1} - v||$$

$$\leq (1 + M_{fs} + \epsilon)K||\sum_{j \in B} b_{j}x_{j}||.$$

From this and (20) we obtain

$$\|\sum_{i \in A} a_i x_i\| \le \frac{(1+\epsilon)(M_{fs}+\epsilon)(M_{fs}+1+\epsilon)}{(1-\epsilon)} K^2 \tau^{-2} \|\sum_{j \in B} b_j x_j\|.$$

We complete the proof of the hyperdemocracy property by letting $\epsilon \to 0$. Finally, an application of Theorem 2.2 gives that $(x_i)_i$ is almost greedy.

Corollary 4.3. Let $(x_i)_i$ be a Markushevich basis. The following are equivalent:

- (i) For every $0 < \tau \le 1$, $(x_i)_i$ is $WSG(\tau)$.
- (ii) There is $0 < \tau \le 1$ such that $(x_i)_i$ is $WSG(\tau)$.
- (iii) For every $0 < \tau \le 1$, $(x_i)_i$ is $WAG(\tau)$.
- (iv) There is $0 < \tau \le 1$ such that $(x_i)_i$ is $WAG(\tau)$.
- (v) $(x_i)_i$ is semi-greedy.
- (vi) $(x_i)_i$ is almost greedy.

Proof. The implications (i) \Longrightarrow (ii) and (iii) \Longrightarrow (iv) are immediate.

Also, $(v) \Longrightarrow (i)$ and $(vi) \Longrightarrow (iii)$ follow at once from the definitions.

That (ii) \Longrightarrow (vi) and (v) \Longrightarrow (vi) follow by Theorem 4.2.

By Proposition 2.3, we see that (iv) \Longrightarrow (vi).

Finally, (vi) \Longrightarrow (v) follows by [9, Theorem 3.2], as their proof holds for Markushevich bases.

When the conditions of Proposition 3.11(b) hold, we have $M_{fs}=1$ for any Markushevich basis. Thus, in such cases Theorem 4.2 gives upper bounds for the second quasi-greedy and the hyperdemocracy constant depending only on K and τ . On the other hand, and unlike the implication from (weak) almost greedy to semi-greedy, in general there is no upper bound for the almost greedy constant of a WSG(τ) Markushevich basis depending only on the WSG(τ) constant and τ . The following example illustrates that.

Example 4.4. Let $(e_i)_i$ and $(e'_i)_i$ be the unit vector basis of ℓ_1 and its sequence of coordinate functionals respectively. Given $\alpha > 0$, define

$$\begin{aligned} x_i &:= e_i + 2(\alpha + 1)(-1)^i e_1 & \text{for all } i \geq 2; \\ X &:= \overline{[x_i \colon i \geq 2]}; \\ x_i' &:= e_i' \big|_X & \text{for all } i \geq 2. \end{aligned}$$

The following statements hold:

- (a) $(x_i)_{i\geq 2}$ is a basic sequence equivalent to the unit vector basis of ℓ_1 with democracy constant $K_d > \alpha + 1$.
- (b) $(x_i)_{i\geq 2}$ is an almost greedy Markushevich basis for X with biorthogonal functionals $(x_i')_{i\geq 2}$. Its almost greedy constant and quasi-greedy constants are greater than α .
- (c) For every $0 < \tau < 1$, if $M(\tau)$ is a WAG (τ) constant for $(x_i)_{i \geq 2}$, then

$$M(\tau) > \tau \sqrt{\alpha + 1}$$
.

(d) $(x_i)_{i\geq 2}$ is a semi-greedy Markushevich basis for X with semi-greedy constant $K_s \leq 4$. Moreover, for every $x \in X$, $m \in \mathbb{N}$ and every set $\mathcal{W}^{\tau}(x,m)$, there are scalars $(b_i)_{i\in\mathcal{W}^{\tau}(x,m)}$ such that

$$||x - \sum_{i \in \mathcal{W}^{\tau}(x,m)} b_i x_i|| \le 4\tau^{-1} \sigma_m(x).$$

Proof. Let us show that (a) holds. For each $n \geq 2$, we have

$$\sum_{i=2}^{n} |a_i| \le \sum_{i=2}^{n} |a_i| + |\sum_{i=2}^{n} 2(\alpha+1)(-1)^i a_i| = \|\sum_{i=2}^{n} a_i x_i\| \le (3+2\alpha) \sum_{i=2}^{n} |a_i|.$$

We see that $(x_i)_{i\geq 2}$ is basic and equivalent to the unit vector basis of ℓ_1 , so in particular, it is democratic. Since $||x_2+x_3|| = ||e_2+2(\alpha+1)e_1+e_3-2(\alpha+1)e_1|| = 2$ and $||x_2|| = ||e_2+2(\alpha+1)e_1|| = 2\alpha+3$, it follows that

$$K_d \ge \frac{2\alpha + 3}{2} > \alpha + 1.$$

As $(x_i)_{i\geq 2}$ is equivalent to $(e_i)_{i\geq 2}$, it is an almost greedy Markushevich basis for X. Clearly, $(x_i')_{i\geq 2}$ is its biorthogonal sequence. To complete the proof of (b), let K_a, K_{1q} and K_{2q} be the almost greedy, first and second quasi-greedy constants of the system, respectively. By Theorem 2.2, we have $K_a \geq K_d > \alpha + 1$, and since $\mathcal{G}_1(x_2 + x_3) = x_2$, we get that $K_{1q} \geq \frac{2\alpha + 3}{2} > \alpha + 1$, so $K_{2q} > \alpha$. To prove (c), apply Proposition 2.3 to get

$$\alpha + 1 < K_d \le K_{hd} \le \tau^{-2} M^2(\tau),$$

from where the lower bound for $M(\tau)$ is obtained.

Finally, let us show that (d) holds. Fix $0 < \tau \le 1$, $x \in X$, $m \in \mathbb{N}$, and a set $\mathcal{W} = \mathcal{W}^{\tau}(x, m)$. Given $A \subseteq \mathbb{N}_{>1}$ with |A| = m and scalars $(a_i)_{i \in A}$, we proceed as follows: If $A = \mathcal{W}$, we choose $b_i := a_i$ for each i. Otherwise, fix π any bijection

$$\pi \colon \mathcal{W} \setminus A \to A \setminus \mathcal{W}.$$

For every $j \in \mathcal{W}$, we define

$$b_j := \begin{cases} a_j & \text{if } j \in A; \\ (-1)^{j+\pi(j)} a_{\pi(j)} & \text{otherwise.} \end{cases}$$

Let us estimate the ℓ_1 -norm $||x - \sum_{j \in \mathcal{W}} b_j x_j||$ in terms of $||x - \sum_{i \in A} a_i x_i||$. For the first coordinate we get

$$\begin{split} e_1'(x - \sum_{j \in \mathcal{W}} b_j x_j) = & e_1'(x) - \sum_{i \in \mathcal{W} \cap A} a_i e_1'(x_i) - \sum_{j \in \mathcal{W} \setminus A} a_{\pi(j)} (-1)^{j + \pi(j)} e_1'(x_j) \\ = & e_1'(x) - \sum_{i \in \mathcal{W} \cap A} a_i e_1'(x_i) - \sum_{j \in \mathcal{W} \setminus A} 2a_{\pi(j)} (\alpha + 1) (-1)^{j + \pi(j)} (-1)^j \\ = & e_1'(x) - \sum_{i \in \mathcal{W} \cap A} a_i e_1'(x_i) - \sum_{i \in A \setminus \mathcal{W}} 2a_i (\alpha + 1) (-1)^i \\ = & e_1'(x - \sum_{i \in A} a_i x_i). \end{split}$$

Now, suppose that l > 1. For $l \in \mathbb{N} \setminus A \cup \mathcal{W}$,

$$e'_{l}(x - \sum_{j \in \mathcal{W}} b_{j}x_{j}) = e'_{l}(x) = e'_{l}(x - \sum_{i \in A} a_{i}x_{i}).$$

Also, if $l \in A \cap \mathcal{W}$ then

$$e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j) = e'_l(x) - b_l = e'_l(x) - a_l = e'_l(x - \sum_{i \in A} a_i x_i).$$

On the other hand, if $l \in \mathcal{W} \setminus A$ we compute the l- and the $\pi(l)$ - coordinates components at the same time. From the fact that $\pi(l) \notin \mathcal{W}$ we deduce that $|e'_l(x)| = |x'_l(x)| \geq \tau |x'_{\pi(l)}(x)| = \tau |e'_{\pi(l)}(x)|$. Thus,

$$|e'_{l}(x - \sum_{j \in \mathcal{W}} b_{j}x_{j})| + |e'_{\pi(l)}(x - \sum_{j \in \mathcal{W}} b_{j}x_{j})| = |e'_{l}(x) - b_{l}| + |e'_{\pi(l)}(x)|$$

$$= |e'_{l}(x) - (-1)^{l+\pi(l)}a_{\pi(l)}| + |e'_{\pi(l)}(x)|$$

$$\leq 2\frac{|e'_{l}(x)|}{\tau} + |a_{\pi(l)}|$$

$$\leq 2\max\left\{2\frac{|e'_{l}(x)|}{\tau}, |a_{\pi(l)}|\right\}$$

$$\leq 2\max\left\{2\frac{|e'_{l}(x)|}{\tau}, 2|a_{\pi(l)}| - 2\frac{|e'_{l}(x)|}{\tau}\right\}$$

$$= \frac{4}{\tau}\tau\max\left\{\frac{|e'_{l}(x)|}{\tau}, |a_{\pi(l)}| - \frac{|e'_{l}(x)|}{\tau}\right\}.$$
(21)

Similarly,

$$|e'_{l}(x - \sum_{i \in A} a_{i}x_{i})| + |e'_{\pi(l)}(x - \sum_{i \in A} a_{i}x_{i})| = |e'_{l}(x)| + |e'_{\pi(l)}(x) - a_{\pi(l)}|$$

$$\geq \max \left\{ |e'_{l}(x)|, |a_{\pi(l)}| - |e'_{\pi(l)}(x)| \right\}$$

$$\geq \max \left\{ |e'_{l}(x)|, |a_{\pi(l)}| - \frac{|e'_{l}(x)|}{\tau} \right\}$$

$$\geq \tau \max \left\{ \frac{|e'_{l}(x)|}{\tau}, |a_{\pi(l)}| - \frac{|e'_{l}(x)|}{\tau} \right\}.$$

$$(22)$$

Comparing (21) and (22), we obtain

$$|e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j)| + |e'_{\pi(l)}(x - \sum_{j \in \mathcal{W}} b_j x_j)| \le \frac{4}{\tau} (|e'_l(x - \sum_{i \in A} a_i x_i)| + |e'_{\pi(l)}(x - \sum_{i \in A} a_i x_i)|).$$

Combining the above estimates we get

$$||x - \sum_{j \in \mathcal{W}} b_j x_j|| \le \frac{4}{\tau} ||x - \sum_{i \in A} a_i x_i||.$$

Now, the left-hand side of the inequality is greater than or equal to the infimum over all scalars $(b_i)_{i\in\mathcal{W}}$, which in fact is a minimum since \mathcal{W} is finite. Then, taking the infimum over all $A\subseteq\mathbb{N}$ with |A|=m and scalars $(a_i)_{i\in A}$ on the right-hand side, we conclude that

$$\min_{(b_j)_{j\in\mathcal{W}}\subseteq\mathbb{K}} \|x - \sum_{j\in\mathcal{W}} b_j x_j\| \le \frac{4}{\tau} \sigma_m(x).$$

Taking $\tau = 1$, we conclude that $(x_i)_{i \geq 2}$ is semi-greedy, and we get the bound for K_s .

A natural question in this context is whether the implication from $WSG(\tau)$ to almost greedy holds for all $WSG(\tau)$ systems, or - equivalently in light of Theorem 4.2 - whether every weak semi-greedy system is a Markushevich basis. The answer is negative. The following example shows a semi-greedy system that is neither quasi-greedy nor democratic.

Example 4.5. Let $(e_i)_i$ be the unit vector basis of \mathbf{c}_0 and let $(e'_i)_i$ be the sequence of biorthogonal functionals. Set

$$x_i := e_i + (-1)^i e_1$$
 for all $i \ge 2$;
 $x'_i := e'_i$ for all $i \ge 2$.

The following statements hold:

- (a) $(x_i)_{i\geq 2}$ is a fundamental minimal system for c_0 , but not a Markushevich basis. Thus, it is not quasi-greedy.
- (b) $(x_i)_{i\geq 2}$ is not democratic.
- (c) $(x_i)_{i\geq 2}$ is a semi-greedy system for \mathbf{c}_0 with semi-greedy constant no greater than 3. Moreover, for any $x\in X$, $m\in\mathbb{N}$ and every set $\mathcal{W}^{\tau}(x,m)$, there are scalars $(b_i)_{i\in\mathcal{W}^{\tau}(x,m)}$ such that

$$||x - \sum_{i \in \mathcal{W}^{\tau}(x,m)} b_i x_i|| \le 3\tau^{-1} \sigma_m(x).$$

Proof. To show that (a) holds, first note that for all $n \in \mathbb{N}$,

$$||e_1 - \sum_{i=1}^n \frac{x_{2i}}{n}|| = ||\sum_{i=1}^n \frac{e_{2i}}{n}|| = \frac{1}{n}.$$

This entails that $e_1 \in \overline{[x_i \colon i \geq 2]}$, so $(x_i)_{i\geq 2}$ is fundamental. Since $x_j'(e_1) = 0$ for every $j \geq 2$, $(x_i)_{i\geq 2}$ is not a Markushevich basis, and thus it is not quasi-greedy. To see that $(x_i)_{i\geq 2}$ is not democratic, notice that for all $n \in \mathbb{N}$,

$$\|\sum_{i=2}^{2n+1} x_i\| = \|\sum_{i=2}^{2n+1} e_i\| = 1,$$

but

$$\|\sum_{i=1}^{2n} x_{2i}\| = \|2ne_1 + \sum_{i=1}^{2n} e_i\| = 2n.$$

Hence, (b) holds. To prove (c), we proceed as in the proof of Example 4.4. Fix $0 < \tau \le 1$, $x \in X$, $m \in \mathbb{N}$ and a set $\mathcal{W} = \mathcal{W}^{\tau}(x,m)$. Take a set $A \subseteq \mathbb{N}_{>1}$ with |A| = m and $A \ne \mathcal{W}$, and scalars $(a_i)_{i \in A}$, and let

$$\pi \colon \mathcal{W} \setminus A \to A \setminus \mathcal{W}$$

be a bijection. For every $j \in \mathcal{W}$, define

$$b_j := \begin{cases} a_j & \text{if } j \in A; \\ (-1)^{j+\pi(j)} a_{\pi(j)} & \text{otherwise.} \end{cases}$$

Now, we estimate the supremum norm of $x - \sum_{j \in \mathcal{W}} b_j x_j$ in terms of that of $x - \sum_{i \in A} a_i x_i$.

First note that if l > 1 and $l \in \mathbb{N} \setminus A \cup \mathcal{W}$ or $l \in A \cap \mathcal{W}$, we have

$$e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j) = e'_l(x - \sum_{i \in A} a_i x_i).$$

This equality also holds for l = 1, indeed

$$e'_{1}(x - \sum_{j \in \mathcal{W}} b_{j}x_{j}) = e'_{1}(x) - \sum_{i \in \mathcal{W} \cap A} a_{i}e'_{1}(x_{i}) - \sum_{j \in \mathcal{W} \setminus A} a_{\pi(j)}(-1)^{\pi(j)}$$

$$= e'_{1}(x) - \sum_{i \in \mathcal{W} \cap A} a_{i}e'_{1}(x_{i}) - \sum_{j \in \mathcal{W} \setminus A} a_{\pi(j)}e'_{1}(x_{\pi(j)})$$

$$= e'_{1}(x) - \sum_{i \in \mathcal{W} \cap A} a_{i}e'_{1}(x_{i}) - \sum_{i \in A \setminus \mathcal{W}} a_{i}e'_{1}(x_{i})$$

$$= e'_{1}(x - \sum_{i \in A} a_{i}x_{i}).$$

For each $l \in \mathcal{W} \setminus A$, we have $|e'_l(x)| = |x'_l(x)| \ge \tau |x'_{\pi(l)}(x)| = \tau |e'_{\pi(l)}(x)|$. Hence, considering together the l- and the $\pi(l)$ -th coordinates we have

$$\max \left\{ |e'_l(x - \sum_{j \in \mathcal{W}} b_j x_j)|, |e'_{\pi(l)}(x - \sum_{j \in \mathcal{W}} b_j x_j)| \right\} = \max \left\{ |e'_l(x) - b_l|, |e'_{\pi(l)}(x)| \right\}$$

$$\leq \max \left\{ |e'_l(x) - \pm a_{\pi(l)}|, \frac{|e'_l(x)|}{\tau} \right\}$$

$$\leq \frac{|e'_l(x)|}{\tau} + |a_{\pi(l)}|$$

$$\leq 3 \max \left\{ \frac{|e'_l(x)|}{\tau}, |a_{\pi(l)}| - \frac{|e'_l(x)|}{\tau} \right\}.$$

Similarly, we obtain

$$\max \Big\{ |e_l'(x - \sum_{i \in A} a_i x_i)|, |e_{\pi(l)}'(x - \sum_{i \in A} a_i x_i)| \Big\} \ge \tau \max \Big\{ \frac{|e_l'(x)|}{\tau}, |a_{\pi(l)}| - \frac{|e_l'(x)|}{\tau} \Big\}.$$

From the inequalities given above,

$$||x - \sum_{j \in \mathcal{W}} b_j x_j|| \le \frac{3}{\tau} ||x - \sum_{i \in A} a_i x_i||.$$

The proof of (c) is completed by the same argument given in Example 4.4. \Box

Remark 4.6. The system of Example 4.5 can also be considered in ℓ_p for all $1 . With only minor adjustments to the calculations given above, we obtain that <math>(x_i)_{i\geq 2} \subseteq \ell_p$ is semi-greedy (with constant no greater than $3*2^{\frac{1}{p}}$), but neither democratic nor quasi-greedy.

Our next proposition shows that from any $WSG(\tau)$ system that is not a Markushevich basis, one can obtain an almost greedy Markushevich basis for the space, with superdemocracy and first quasi-greedy constants depending only on τ and the $WSG(\tau)$ constant of the system - and thus, by Theorem 2.2, with almost greedy constant also depending only on said constants. In order to prove our result, we need two technical lemmas. The notation used below is natural and according to the context.

Lemma 4.7. Let $\mathcal{B}_1 = (x_i)_{i \in \mathbb{N}}$ be a fundamental minimal system for Y, and suppose that $\mathcal{B}_2 := (x_0, x_i)_{i \in \mathbb{N}}$ is a fundamental minimal system for $X := \overline{[x_i : i \in \mathbb{N}_0]}$ with biorthogonal functionals $(x'_0, x'_i)_{i \in \mathbb{N}} \subseteq X'$ satisfying

$$||x_0|| ||x_0'|| = 1$$
 and $||x_0|| = \sup_{i \in \mathbb{N}} ||x_i||$.

The following hold:

(a) If \mathcal{B}_1 is quasi-greedy with first quasi-greedy constant $K_{1q}(\mathcal{B}_1)$, then \mathcal{B}_2 is quasi-greedy with first quasi-greedy constant

$$K_{1q}(\mathcal{B}_2) \le 2K_{1q}(\mathcal{B}_1) + 1.$$

(b) If \mathcal{B}_1 is superdemocratic with constant $K_{sd}(\mathcal{B}_1)$, then \mathcal{B}_2 is superdemocratic with constant

$$K_{sd}(\mathcal{B}_2) \leq 4K_{sd}(\mathcal{B}_1).$$

Proof. To prove (a), fix $x \in X$ and $m \in \mathbb{N}$. Then, we have

$$\mathcal{G}_{\mathcal{B}_2,m}(x) = \begin{cases} \mathcal{G}_{\mathcal{B}_1,m-1}(x - x_0'(x)x_0) + x_0'(x)x_0 & \text{if } 0 \in \mathcal{GS}_{\mathcal{B}_2,m}(x); \\ \mathcal{G}_{\mathcal{B}_1,m}(x - x_0'(x)x_0) & \text{otherwise.} \end{cases}$$

Thus,

$$\|\mathcal{G}_{\mathcal{B}_2,m}(x)\| \le K_{1q}(\mathcal{B}_1)\|x - x_0'(x)x_0\| + \|x_0'(x)x_0\| \le (2K_{1q}(\mathcal{B}_1) + 1)\|x\|.$$

It follows that \mathcal{B}_2 is quasi-greedy and $K_{1q}(\mathcal{B}_2) \leq 2K_{1q}(\mathcal{B}_1) + 1$.

To prove (b), suppose first that $D \subseteq \mathbb{N}_0$ is a finite nonempty set and take scalars $(a_k)_{k \in D}$ with $|a_k| = 1$ for each $k \in D$. If $0 \in D$, then

(23)
$$||x_0|| = ||x_0|| |x_0'(\sum_{k \in D} a_k x_k)| \le ||\sum_{k \in D} a_k x_k||.$$

On the other hand, if $0 \notin D$ we have

(24)
$$||x_0|| = \sup_{k \in \mathbb{N}} ||x_k|| \le K_{sd}(\mathcal{B}_1) ||\sum_{k \in D} a_k x_k||.$$

Now let $A, B \subseteq \mathbb{N}_0$ be finite nonempty sets with $|A| \leq |B|$, and take $(a_i)_{i \in A}$, $(b_j)_{j \in B}$ scalars such that $|a_i| = |b_j| = 1$ for all $i \in A, j \in B$. If $0 \notin A \cup B$, there is nothing to prove. If $0 \in A \setminus B$, by (24) we have

$$\| \sum_{i \in A} a_i x_i \| \le \|x_0\| + \| \sum_{i \in A \setminus \{0\}} a_i x_i \| \le \|x_0\| + K_{sd}(\mathcal{B}_1)\| \sum_{j \in B} b_j x_j \|$$

$$\le 2K_{sd}(\mathcal{B}_1)\| \sum_{i \in B} b_i x_j \|.$$

If $0 \in A \cap B$, by (23) and (24) we get

$$\| \sum_{i \in A} a_i x_i \| \le \|x_0\| + \| \sum_{i \in A \setminus \{0\}} a_i x_i \|$$

$$\le \| \sum_{j \in B} b_j x_j \| + K_{sd}(\mathcal{B}_1) \| \sum_{j \in B \setminus \{0\}} b_j x_j \|$$

$$\le (1 + K_{sd}(\mathcal{B}_1)) \| \sum_{j \in B} b_j x_j \| + K_{sd}(\mathcal{B}_1) \| x_0 \|$$

$$\le (1 + 2K_{sd}(\mathcal{B}_1)) \| \sum_{j \in B} b_j x_j \|.$$

If $0 \in B \setminus A$ and |B| > 1, we choose $i_0 \in A$ and using (23) we get that

$$\| \sum_{i \in A} a_i x_i \| \le \|x_{i_0}\| + \| \sum_{i \in A \setminus \{i_0\}} a_i x_i \| \le 2K_{sd}(\mathcal{B}_1) \| \sum_{j \in B \setminus \{0\}} b_j x_j \|$$

$$\le 2K_{sd}(\mathcal{B}_1) (\| \sum_{j \in B} b_j x_j \| + \|x_0\|)$$

$$\le 4K_{sd}(\mathcal{B}_1) \| \sum_{j \in B} b_j x_j \|.$$

The only case left is $A \neq B = \{0\}$. Then $A = \{i_0\}$ for some $i_0 \in \mathbb{N}$ and

$$\|\sum_{i \in A} a_i x_i\| = \|x_{i_0}\| \le \sup_{k \in \mathbb{N}} \|x_k\| = \|x_0\| = \|\sum_{j \in B} b_j x_j\|.$$

From the above estimations, \mathcal{B}_2 is superdemocratic and $K_{sd}(\mathcal{B}_2) \leq 4K_{sd}(\mathcal{B}_1)$.

The following result will allow us to handle the case $\sigma_m(x) = 0$ in the proof of Proposition 4.9.

Lemma 4.8. Let $(x_i)_i \subseteq X$ be a fundamental minimal system with both $(x_i)_i$ and $(x_i')_i$ bounded. If $x \in X$ is such that $\sigma_m(x) = 0$ for some $m \in \mathbb{N}$, then $|\operatorname{supp}(x)| \le m \text{ and } x = \mathcal{G}_m(x) = P_{\operatorname{supp}(x)}(x).$

Proof. Let B := supp(x). If |B| > m, there is $C \subset B$ with |C| = m + 1. Thus, if $A \subset \mathbb{N}$ and $|A| \leq m$, there is $j \in C \setminus A$. Then, for any scalars $(a_i)_{i \in A}$ it follows

$$||x - \sum_{i \in A} a_i x_i|| \ge \frac{|x_j'(x - \sum_{i \in A} a_i x_i)|}{||x_j'||} = \frac{|x_j'(x)|}{||x_j'||} \ge \frac{\min_{i \in C} |x_i'(x)|}{\max_{i \in C} ||x_i'||} > 0.$$

Taking infimum over such sets and scalars, we get a contradiction to the hypothesis that $\sigma_m(x) = 0$. Now let

$$M := \sup_{i} \{ \|x_i\|, \|x_i'\| \}.$$

Given that $\sigma_m(x) = 0$ and $|B| \leq m$, we have $\sigma_{2m}(x - P_B(x)) = 0$. Fix $\epsilon > 0$ and choose $A \subset X$ with |A| = 2m and scalars $(a_i)_{i \in A}$ so that

$$||x - P_B(x) - \sum_{i \in A} a_i x_i|| \le \epsilon.$$

For each $l \in A$, we have

$$|a_l| = |x_l'(x - P_B(x) - \sum_{i \in A} a_i x_i)| \le M||x - P_B(x) - \sum_{i \in A} a_i x_i|| \le M\epsilon.$$

Hence,

$$||x - P_B(x)|| \le \epsilon + ||\sum_{i \in A} a_i x_i|| \le \epsilon + \sum_{i \in A} M \epsilon ||x_i|| \le \epsilon + 2mM^2 \epsilon.$$

Since ϵ is arbitrary and m, M are fixed, we get $x = P_B(x)$, and thus $x = \mathcal{G}_m(x)$.

Now we can show that a weak semi-greedy system that is not a Markushevich basis can be slightly modified to obtain a Markushevich basis (and therefore, by Theorem 4.2, an almost greedy system).

Proposition 4.9. Let $0 < \tau \le 1$, and let $\mathcal{B} := (x_i)_i \subseteq X$ be a $WSG(\tau)$ system that is not a Markushevich basis, with constant $K_{ws}(\tau, \mathcal{B})$. There are $x_0 \in X$ and $x_0' \in X'$ such that

$$\overline{\{x_i\}}_{i\in\mathbb{N}}^w \subseteq \{x_i\}_{i\in\mathbb{N}} \cup [x_0],$$

and the system

$$\mathcal{B}_1 := (x_0, x_i - x_0'(x_i)x_0)_{i \in \mathbb{N}}$$

is an almost greedy Markushevich basis for X with biorthogonal functionals $(x'_0, x'_i)_i$. In addition, \mathcal{B}_1 has first quasi-greedy constant

$$K_{1a}(\mathcal{B}_1) < 3 + 4K_{ws}(\tau, \mathcal{B}) + 16K_{ws}(\tau, \mathcal{B})^2 \tau^{-2}$$

and superdemocracy constant

$$K_{sd}(\mathcal{B}_1) \le 32K_{ws}(\tau, \mathcal{B})^2 \tau^{-2}$$

Proof. By Theorem 4.2 and Corollary 3.9, the set $\overline{\{x_i\}}_{i\in\mathbb{N}}^w$ is weakly compact, and $0 \notin \overline{\{x_i\}}_{i\in\mathbb{N}}^w$. Then, there is a subnet $(x_{i_\lambda})_\lambda$ and $u_0 \in X \setminus \{0\}$ such that

$$x_i \xrightarrow{w} u_0$$
.

By the Hahn–Banach Theorem, there is $u'_0 \in X'$ such that

$$||u_0|| ||u_0'|| = 1$$
 and $u_0'(u_0) = 1$.

Now for $x \in X$ define the linear operator $P \colon X \to X$ by

$$P(x) := x - u_0'(x)u_0.$$

It is easy to check that P is a projection, $||P|| \leq 2$, $X = [u_0] \oplus P(X)$ and, as $x'_i(u_0) = 0$ for all $i \in \mathbb{N}$,

$$\mathcal{B}_2 := (x_i - u_0'(x_i)u_0)_{i \in \mathbb{N}}$$

is a fundamental minimal system for P(X) with biorthogonal functionals $(x_i'|_{P(X)})_i$. First, we show that \mathcal{B}_2 is a WSG (τ) system for P(X). Take $y \in P(X)$ and $m \in \mathbb{N}$, and fix $\epsilon > 0$. If $\sigma_{\mathcal{B}_2,m}(y) \neq 0$ we choose a set $A \subseteq \mathbb{N}$ with |A| = m and scalars $(a_i)_{i \in A}$ so that

$$||y - \sum_{i \in A} a_i(x_i - u_0'(x_i)u_0)|| \le (1 + \epsilon)\sigma_{\mathcal{B}_2,m}(y),$$

and define

$$z := y + \sum_{i \in A} a_i u_0'(x_i) u_0.$$

By hypothesis, there is a set $\mathcal{W}^{\tau}_{\mathcal{B}}(z,m)$ and scalars $(b_j)_{j\in\mathcal{W}^{\tau}_{\mathcal{B}}(z,m)}$ such that

$$||z - \sum_{j \in \mathcal{W}_{\mathcal{B}}^{\tau}(z,m)} b_j x_j|| \le K_{ws}(\tau, \mathcal{B}) \sigma_{\mathcal{B},m}(z).$$

Then, it follows that

$$||y - \sum_{j \in \mathcal{W}_{\mathcal{B}}^{\tau}(z,m)} b_{j}(x_{j} - u'_{0}(x_{j})u_{0})|| = ||P(z - \sum_{j \in \mathcal{W}_{\mathcal{B}}^{\tau}(z,m)} b_{j}x_{j})||$$

$$\leq 2||z - \sum_{j \in \mathcal{W}_{\mathcal{B}}^{\tau}(z,m)} b_{j}x_{j}||$$

$$\leq 2K_{ws}(\tau, \mathcal{B})||y - \sum_{i \in A} a_{i}(x_{i} - u'_{0}(x_{i})u_{0})||$$

$$\leq 2(1 + \epsilon)K_{ws}(\tau, \mathcal{B})\sigma_{\mathcal{B}_{2},m}(y).$$

Since $x_i'(y) = x_i'(z)$ for every $i \in \mathbb{N}$, the set $\mathcal{W}_{\mathcal{B}}^{\tau}(z, m)$ is also a weak thresholding set for y with respect to \mathcal{B}_2 , and we obtain the estimate

(25)
$$||y - \sum_{j \in \mathcal{W}_{\mathcal{B}_2}^{\tau}(y,m)} b_j(x_j - u_0'(x_j)u_0)|| \le 2(1 + \epsilon)K_{ws}(\tau, \mathcal{B})\sigma_{\mathcal{B}_2,m}(y).$$

Now suppose that $\sigma_{\mathcal{B}_2,m}(y) = 0$. By Lemma 4.1, both $(x_i)_i$ and $(x_i')_i$ are seminormalized. The former implies that $(x_i - u_0'(x_i)u_0)_i$ is bounded, so by Lemma 4.8, we have

$$y = \mathcal{G}_{\mathcal{B}_2,m}(y) = \sum_{j \in \mathcal{GS}_{\mathcal{B}_2,m}(y)} x'_j(y)(x_j - u'_0(x_j)u_0).$$

Hence, (25) holds also in this case taking $\mathcal{W}^{\tau}_{\mathcal{B}_2}(y,m) := \mathcal{GS}_{\mathcal{B}_2,m}(y)$ and $b_j := x_j'(y)$ for all $j \in \mathcal{W}^{\tau}_{\mathcal{B}_2}(y,m)$. Then we conclude that \mathcal{B}_2 is a WSG(τ) system for P(X) with constant

$$K_{ws}(\tau, \mathcal{B}_2) \le 2(1+\epsilon)K_{ws}(\tau, \mathcal{B}).$$

As $x_{i_{\lambda}} \xrightarrow{w} u_0$, we get that $x_{i_{\lambda}} - u'_0(x_{i_{\lambda}})u_0 \xrightarrow{w} 0$. Then, by Corollary 3.9, Theorem 4.2 and Proposition 3.11(b), it follows that \mathcal{B}_2 is an almost greedy Markushevich basis for P(X), with first quasi-greedy constant

$$(26) K_{1q}(\mathcal{B}_2) \le 1 + K_{ws}(\tau, \mathcal{B}_2) + 2K_{ws}(\tau, \mathcal{B}_2)^2 \tau^{-2}$$

$$\le 1 + 2(1 + \epsilon)K_{ws}(\tau, \mathcal{B}) + 8(1 + \epsilon)^2 K_{ws}(\tau, \mathcal{B})^2 \tau^{-2}$$

and with superdemocracy constant

(27)
$$K_{sd}(\mathcal{B}_2) \le 2K_{ws}(\tau, \mathcal{B}_2)^2 \tau^{-2} \le 8(1+\epsilon)^2 K_{ws}(\tau, \mathcal{B})^2 \tau^{-2}$$

Now set

$$a_0 := \frac{1}{\|u_0\|} \sup_{i \in \mathbb{N}} \|x_i - u_0'(x_i)u_0\|, \qquad x_0 := a_0 u_0, \qquad \text{and} \qquad x_0' := a_0^{-1} u_0'.$$

Since

$$||x_0|| ||x_0'|| = 1$$
, $||x_0|| = \sup_{i \in \mathbb{N}} ||x_i - x_0'(x_i)x_0||$ and $\mathcal{B}_2 = (x_i - x_0'(x_i)x_0)_i$,

we may apply Lemma 4.7. Letting $\epsilon \to 0$ in (26) and (27), an application of the lemma gives that \mathcal{B}_1 is a quasi-greedy Markushevich basis for X, with first quasi-greedy constant

$$K_{1q}(\mathcal{B}_1) \le 2K_{1q}(\mathcal{B}_2) + 1 \le 3 + 4K_{ws}(\tau, \mathcal{B}) + 16K_{ws}(\tau, \mathcal{B})^2\tau^{-2},$$

and it is superdemocratic with constant

$$K_{sd}(\mathcal{B}_1) \le 4K_{sd}(\mathcal{B}_2) \le 32K_{ws}(\tau, \mathcal{B})^2 \tau^{-2}.$$

To finish the proof, let $v \in X$ and suppose there is a subnet $(x_{i_{\gamma}})_{\gamma}$ such that

$$x_{i_{\gamma}} \xrightarrow{w} v$$
.

It is immediate that $x_j'(v) = 0$ for all $j \in \mathbb{N}$. Then, as \mathcal{B}_1 is a Markushevich basis for X we get that $v - x_0'(v)x_0 = 0$. This proves that $\overline{\{x_i\}}_{i \in \mathbb{N}}^w \subseteq \{x_i\}_{i \in \mathbb{N}} \cup [x_0]$. \square

5. Finite dimensional spaces and branch greedy algorithms.

In this section, we study the semi-greedy and almost greedy properties - and some weaker versions thereof - in finite dimensional Banach spaces, where each biorthogonal system is a greedy Schauder basis. Here, the questions concerning (weak) thresholding or Chebyshev greedy algorithms focus on the behavior and the relationships of their natural associated constants. We will consider branch semi-greedy and branch almost greedy bases, introduced and studied by Dilworth, Kutzarova, Schlumprecht and Wojtaszczyk in [13], and extend some of their results. Let us present the *branch* versions of the (weak) thresholding and Chebyshev greedy algorithms. For a fixed weakness parameter $0 < \tau < 1$ and a Markushevich basis $(x_i)_i$ with seminormalized coordinates, the algorithm is defined as follows. First, set

$$\mathcal{A}^{\tau}(x) := \{i \in \mathbb{N} \colon |e_i'(x)| \geq \tau \max_{j \in \mathbb{N}} |e_j'(x)|\},$$

and let $\mathcal{G}^{\tau} : X \setminus \{0\} \to \mathbb{N}$ be a function with the following properties:

(BG1) $\mathcal{G}^{\tau}(x) \in \mathcal{A}^{\tau}(x)$ for every $x \in X \setminus \{0\}$.

(BG2) $\mathcal{G}^{\tau}(\lambda x) = \mathcal{G}^{\tau}(x)$ for all $x \in X \setminus \{0\}$ and all $\lambda \in \mathbb{K} \setminus \{0\}$.

(BG3) If $\mathcal{A}^{\tau}(x) = \mathcal{A}^{\tau}(y)$ and $e'_i(x) = e'_i(y)$ for all $i \in \mathcal{A}^{\tau}(x)$, then $\mathcal{G}^{\tau}(x) = \mathcal{G}^{\tau}(y)$. For each $x \neq 0$, this defines a function $\rho_x^{\tau} \colon \{1, \dots, |\operatorname{supp}(x)|\} \to \mathbb{N}$ if $|\operatorname{supp}(x)| < \infty$ or $\rho_x^{\tau} \colon \mathbb{N} \to \mathbb{N}$ otherwise, given by $\rho_x^{\tau}(1) := \mathcal{G}^{\tau}(x)$, and for $2 \leq i \leq |\operatorname{supp}(x)|$,

$$\rho_x^{\tau}(i) := \mathcal{G}^{\tau}(x - \sum_{i=1}^{i-1} x'_{\varrho_x^{\tau}(j)}(x) x_{\varrho_x^{\tau}(j)}).$$

Similarly, for every $x \in X \setminus \{0\}$ and $m \in \mathbb{N}$, the *m-term branch greedy approximation* to x (with regard to a fixed branch greedy algorithm) is defined as

$$\mathcal{G}_m^{\tau}(x) := \sum_{i=1}^m x'_{\varrho_x^{\tau}(i)}(x) x_{\varrho_x^{\tau}(i)},$$

setting $x'_{\varrho_x^{\tau}(i)}(x)x_{\varrho_x^{\tau}(i)} := 0$ if $i > \max(\operatorname{supp}(x))$, and $\mathcal{G}_0^{\tau}(x) := 0$. The idea of choosing a branch associated to a weakness parameter τ is applied to different concepts (see [12]).

Definition 5.1. [13, Definition 6.1] Let $N \in \mathbb{N}$, and let E be a N-dimensional Banach space with a fundamental minimal system $(x_i')_{1 \leq i \leq N} \subseteq E$. The system is called *branch almost-greedy* with weakness parameter $0 < \tau \leq 1$ (BAG (τ)) and constant M if, for every $x \in E$ and every $0 \leq m \leq N$, we have

$$||x - \mathcal{G}_m^{\tau}(x)|| \le M\widetilde{\sigma}_m(x).$$

Definition 5.2. [13, Definition 7.3] Let $N \in \mathbb{N}$, and let E be a N-dimensional Banach space with fundamental minimal system $(x_i)_{1 \leq i \leq N} \subseteq E$. The system is called branch semi-greedy with weakness parameter $0 < \tau < 1$ (BSG (τ)) and constant M if, for every $x \in E$ and every $1 \leq m \leq N$, there are scalars $(a_i)_{1 \leq i \leq m}$ such that

$$||x - \sum_{i=1}^{m} a_i x_{\varrho_x^{\tau}(i)}|| \le M \sigma_m(x).$$

Remark 5.3. Note that if we consider the definition of WAG(τ) systems in the finite dimensional context, it is immediate that every BAG(τ) system with constant M is also WAG(τ) with constant no greater than M. The same relation exists between WSG(τ) and BSG(τ) systems.

Remark 5.4. Also, note that the greedy ordering provides a branch greedy algorithm with parameter τ for every $0 < \tau < 1$. We can simply define

$$\mathcal{G}^{\tau}(x) := \rho(x,1)$$

for all $x \in X$. It is easy to check that \mathcal{G}^{τ} satisfies (BG1), (BG2), and (BG3) for all $0 < \tau < 1$.

Every BAG(τ) system with constant M has quasi-greedy, democratic and almost greedy constants depending only on M and τ [13, Theorem 6.4, Corollary 6.5]. Also, for an almost greedy system, the conditions of Definition 1.7 hold for all $x \in X$, $m \in \mathbb{N}$, and every weak thresholding set $\mathcal{W}^{\tau}(x,m)$, with M depending only on the first quasi-greedy constant and τ ([13, Theorem 7.1]). This implies immediately that it is BSG(τ), and that every branch of the algorithm satisfies the BSG condition. Going in the opposite direction, that is from the BSG(τ) to the almost greedy (or, equivalently, the BAG(τ)) property, it was proved in [13, Theorem 7.7] that the almost greedy constant can be controlled by the BSG(τ) constant, τ , the basis constant, and the cotype constant of the space ([13, Theorem 7.7]). In the same

paper, the authors left open the question of whether the $BSG(\tau)$ property implies in general the BAG(τ) property, that is if the constant of the latter can be controlled by that of the former (see the question below [13, Definition 7.3]). Now we are in a position to answer that question and extend [13, Theorem 7.7].

First, note that in Example 4.4, we did not use that the space is infinite dimensional to prove any of the bounds for the constants of the system, except for the lower bound for the WAG(τ) constants, for which we used Proposition 2.3. The proofs of the rest of the bounds hold if we replace ℓ_1 with ℓ_1^n for any $n \geq 3$. The system in Example 4.4 has semi-greedy constant no greater than 4, and in fact, by (d), all branches of the algorithm satisfy the $BSG(\tau)$ condition with constant no greater than $4\tau^{-1}$. Hence, (a) and (b) show that there is no upper bound for the democracy, quasi-greedy or almost greedy constant that depends only on the semi-greedy or the BSG(τ) constant and τ . Thus, by [13, Theorem 6.4] and [13, Corollary 6.5], it follows that there is no such upper bound for the BAG(τ) constant, either.

Second, it is possible to remove the cotype condition from [13, Theorem 7.7], and also extend the result to any WSG(τ) system. To do so, next we provide a bound for the second quasi-greedy constant of such systems. For the proof we combine ideas from the proofs of [5, Theorem 1.10] and Theorem 4.2 with further arguments that allow us to handle the finite dimensional case.

Theorem 5.5. Let $N \in \mathbb{N}_{>1}$ and E be a N-dimensional Banach space with a $WSG(\tau)$ basis $(x_i')_{1 \leq i \leq N}$, $0 < \tau \leq 1$. If $(x_i)_{1 \leq i \leq N}$ has $WSG(\tau)$ constant $K_{ws}(\tau)$ and basis constant K_b , then $(x_i)_{1 \le i \le N}$ is quasi-greedy with second quasi-greedy constant

$$K_{2q} \le 5K_b^2 K_{ws}(\tau) + 6K_b^3 K_{ws}(\tau)^2 \tau^{-2}.$$

Proof. Let $N_1 := \lfloor \frac{N+1}{2} \rfloor$, and consider the finite sets $A_1 := \{j \in \mathbb{N} \colon 1 \leq j \leq N_1\}$ and $A_2 := \{j \in \mathbb{N} \colon N_1 < j \leq N\}$. Now, for all $x \in E$ and i = 1, 2 define the projection operators

$$P_i(x) := \sum_{j \in A_i} x_j'(x) x_j.$$

Fix $x \in E$ and $1 \le m \le N$, assuming without loss of generality that $\mathcal{G}_m(x) \ne x$ (else, there is nothing to prove). Set

$$m_1 := |A_1 \cap \mathcal{GS}_m(x)|$$
 and $m_2 := |A_2 \cap \mathcal{GS}_m(x)|$.

Note that

$$G_m(x) = G_{m_1}(P_1(x)) + G_{m_2}(P_2(x)).$$

Thus.

$$||x - \mathcal{G}_m(x)|| \le ||P_1(x) - \mathcal{G}_{m_1}(P_1(x))|| + ||P_2(x) - \mathcal{G}_{m_2}(P_2(x))||.$$

Let us consider first the case in which $m_1 \neq 0$ and $m_2 \neq 0$. Since $x \neq \mathcal{G}_m(x)$, it follows that $x'_{\rho(P_i(x),m_i)}(P_i(x)) \neq 0$ for $1 \leq i \leq 2$. Fix $0 < \xi < 1$, and let

$$y_1 := \tau^{-1} (1+\xi) |x'_{\rho(P_1(x),m_1)}(P_1(x))| \sum_{j=N_1+1}^{N_1+m_1} x_j;$$

$$y_2 := \tau^{-1} (1+\xi) |x'_{\rho(P_2(x), m_2)}(P_2(x))| \sum_{j=1}^{m_2} x_j.$$

Note that for any $N_1 < j \le (N_1 + m_1)$ and $1 \le i \le N_1$ or $(N_1 + m_1) < i \le N$, we have

$$\tau x_j'(y_1) = (1+\xi)|x_{\rho(P_1(x),m_1)}'(P_1(x))| > |x_i'(P_1(x) - \mathcal{G}_{m_1}(P_1(x)))|.$$

Hence, the only m_1 -weak thresholding set for

$$P_1(x) - \mathcal{G}_{m_1}(P_1(x)) + y_1$$

with weakness parameter τ is the set $\{j: N_1 < j \le N_1 + m_1\}$. Similarly, the only m_2 -weak thresholding set for

$$P_2(x) - \mathcal{G}_{m_2}(P_2(x)) + y_2$$

with weakness parameter τ is the set $\{j: 1 \leq j \leq m_2\}$. Let w_1 and w_2 be an m_1 -term and an m_2 -term Chebyshev τ -greedy approximant for $P_1(x) - \mathcal{G}_{m_1}(P_1(x)) + y_1$ and $P_2(x) - \mathcal{G}_{m_2}(P_2(x)) + y_2$, respectively. Considering that $||P_1|| \leq K_b$ and $||P_2|| \leq 1 + K_b$, we deduce that

$$||P_{1}(x) - \mathcal{G}_{m_{1}}(P_{1}(x))|| \leq K_{b}||P_{1}(x) - \mathcal{G}_{m_{1}}(P_{1}(x)) + y_{1} - w_{1}||$$

$$\leq K_{b}K_{ws}(\tau)\sigma_{m_{1}}(P_{1}(x) - \mathcal{G}_{m_{1}}(P_{1}(x)) + y_{1})$$

$$\leq K_{b}K_{ws}(\tau)||P_{1}(x)|| + K_{b}K_{ws}(\tau)||y_{1}||$$

$$\leq K_{b}^{2}K_{ws}(\tau)||x|| + K_{b}K_{ws}(\tau)||y_{1}||.$$

Analogously, we get that

$$||P_2(x) - \mathcal{G}_{m_2}(P_2(x))|| \le (1 + K_b)^2 K_{ws}(\tau) ||x|| + (1 + K_b) K_{ws}(\tau) ||y_2||.$$

Reasoning as before, we see that any weak thresholding set of cardinality m_1 for

$$P_1(x) + \tau^2(1-\xi)(1+\xi)^{-1}y_1$$

is contained in $\{1 \leq j \leq N_1\}$. So taking u_1 an m_1 -term Chebyshev τ -greedy approximant for $P_1(x) + \tau^2(1-\xi)(1+\xi)^{-1}y_1$, we deduce that

$$||y_{1}|| = \tau^{-2}(1-\xi)^{-1}(1+\xi)||\tau^{2}(1-\xi)(1+\xi)^{-1}y_{1}||$$

$$\leq \tau^{-2}(1-\xi)^{-1}(1+\xi)(1+K_{b})||P_{1}(x) - u_{1} + \tau^{2}(1-\xi)(1+\xi)^{-1}y_{1}||$$

$$\leq \tau^{-2}(1-\xi)^{-1}(1+\xi)(1+K_{b})K_{ws}(\tau)\sigma_{m_{1}}(P_{1}(x) + \tau^{2}(1-\xi)(1+\xi)^{-1}y_{1})$$

$$\leq \tau^{-2}(1-\xi)^{-1}(1+\xi)(1+K_{b})K_{ws}(\tau)||P_{1}(x)||$$

$$\leq \tau^{-2}(1-\xi)^{-1}(1+\xi)(1+K_{b})K_{b}K_{ws}(\tau)||x||.$$

Similarly, we obtain

$$||y_2|| \le \tau^{-2} (1-\xi)^{-1} (1+\xi) (1+K_b) K_b K_{ws}(\tau) ||x||.$$

From the above estimations, and letting $\xi \to 0$, we deduce that

(29)
$$||P_1(x) - \mathcal{G}_{m_1}(P_1(x))|| \le (K_b^2 K_{ws}(\tau) + (1 + K_b) K_b^2 K_{ws}^2(\tau) \tau^{-2}) ||x||;$$

$$(30) \quad \|P_2(x) - \mathcal{G}_{m_2}(P_2(x))\| \le ((1+K_b)^2 K_{ws}(\tau) + (1+K_b)^2 K_b K_{ws}^2(\tau)\tau^{-2})\|x\|.$$

Now suppose that $m_1 = 0$. Then $\mathcal{G}_{m_1}(P_1(x)) = 0$, so (29) is clear, and we can obtain (30) by the same argument as before because $m_2 \leq N_1$. Finally, assume $m_2 = 0$. Then, (30) is clear. Now, if $m_1 < N_1$, we apply the same argument as before to obtain (29). On the other hand, if $m_1 = N_1$, then $\mathcal{G}_{m_1}(P_1(x)) = P_1(x)$, so (29) is immediate.

To finish the proof, from (28), (29) and (30) we infer that

$$||x - \mathcal{G}_m(x)|| \le (5K_b^2 K_{ws}(\tau) + 6K_b^3 K_{ws}(\tau)^2 \tau^{-2})||x||,$$

from where the upper bound for K_{2q} is obtained.

Note that in [13, Theorem 7.4] it was proved that any system $(x_i)_{1 \leq i \leq N}$ that is $BSG(\tau)$ is also superdemocratic with constant depending only on the basis constant, τ , and the $BSG(\tau)$ constant. A careful look at the proof shows that it is also valid for $WSG(\tau)$ systems. Also, the bounds in Theorem 2.2 are extracted from the proof of [10, Theorem 3.3] (with minor modifications for complex scalars), which is valid for finite dimensional spaces. Combining these results with Theorem 5.5, we obtain the following extension of [13, Theorem 7.7].

Theorem 5.6. Let $N \in \mathbb{N}$ and let E be an N-dimensional Banach space. Let $(x_i)_{1 \leq i \leq N} \subset E$ be a $WSG(\tau)$ system for E, $0 < \tau \leq 1$, with constant K_{ws} and basis constant K_b . Then, $(x_i)_{1 \leq i \leq N} \subset E$ has almost greedy constant depending only on K_{ws} , τ and K_b .

Finally, we note that the branch thresholding algorithm can be and has been considered in infinite dimensional spaces as well. Indeed, in [13], the authors do so and prove that every weakly null semi-normalized branch quasi-greedy basic sequence has a quasi-greedy subsequence. If we extend the definitions of branch semi-greedy and branch almost greedy systems to the infinite dimensional context in the natural manner, it is immediate from the definitions that every semi-greedy system is branch semi-greedy, every branch semi-greedy system is weak semi-greedy, and the corresponding implications hold for the almost greedy case. Therefore, $BSG(\tau)$ and $BAG(\tau)$ Markushevich bases can be added to the equivalences in Corollary 4.3.

References

- F. Albiac, J. L. Ansorena, Characterization of 1-quasi-greedy bases. J. Approx. Theory 201 (2016), 7-12.
- [2] F. Albiac, J. L. Ansorena, Characterization of 1-almost-greedy bases. Rev. Mat. Complut. 30 (2017), 13-24.
- [3] F. Albiac, J. L. Ansorena, P. M. Berná, P. Wojtaszczyk, Greedy approximation for biorthogonal systems in quasi-Banach spaces. arXiv e-prints (2019), available at 1903.11651. To appear in press in Dissertationes Mathematicae.
- [4] F. Albiac, N. J. Kalton. Topics in Banach Space Theory, Second, Graduate Texts in Mathematics, vol. 233, Springer, [Cham], 2016. With a foreword by Gilles Godefory.
- [5] P. M. Berná. Equivalence between almost greedy and semi-greedy bases. J. Math. Anal. Appl. 470 (2019), 218–225.
- [6] P. M. Berná, Characterization of weight-semi-greedy bases. J. Fourier Anal. Appl. 26 (1) (2020) 1–21.
- [7] P. M. Berná, O. Blasco, G. Garrigós, Lebesgue inequalities for greedy algorithm in general bases, Rev. Mat. Complut. 30 (2017), 369–392.
- [8] P. M. Berná, Ó. Blasco, G. Garrigós, E. Hernández, T. Oikhberg, Lebesgue inequalities for Chebyshev thresholding greedy algorithms, Rev. Matem. Complut. 33 (3) (2020), 695-722.
- [9] S. J. Dilworth, N. J. Kalton, D. Kutzarova. On the existence of almost greedy bases in Banach spaces. Studia Math. 159 (2003), 67–101.
- [10] S. J. Dilworth, N. J. Kalton, D. Kutzarova, V. N. Temlyakov. The thresholding greedy algorithm, greedy bases, and duality. Constr. Approx. 19 (2003), 575–597.
- [11] S. J. Dilworth, D. Khurana. Characterizations of almost greedy and partially greedy bases. Jaen J. Approx. 11 (2019), 115–137.
- [12] S. J. Dilworth, D. Kutzarova, T. Oikhberg. Lebesgue constants for the weak greedy algorithm. Rev. Mat. Complut. 28 (2015), 393–409.
- [13] S. J. Dilworth, D. Kutzarova, Th. Schlumprecht, P. Wojtaszczyk. Weak thresholding greedy algorithms in Banach spaces. J. Funct. Anal. 263 (2012), 3900–3921.
- [14] S. Gogyan. On convergence of weak thresholding greedy algorithm in $L_1(0,1)$. J. Approx. Theory 161 (2009), 49–64.
- [15] M. Kadets, A Pełczyński. Basic sequences, bi-orthogonal systems and norming sets in Banach and Fréchet spaces. (Russian) Studia Math. 25 (1965), 297-323.
- [16] S. V. Konyagin, V. N. Temlyakov. A remark on greedy approximation in Banach spaces. East J. Approx. 3 (1999), 365–379.
- [17] S. V. Konyagin, V. N. Temlyakov. Greedy approximation with regard to bases and general minimal systems. Serdica Math. J. 28 (2002), 305–328.

- [18] V. N. Temlyakov, Greedy algorithm and m-term trigonometric approximation, Constr. Approx. 14 (1998), 569–587.
- [19] V. N. Temlyakov, The best m-term approximation and greedy algorithms. Advances in Comp. Math. 8 (1998), 249-265.
- [20] P. Wojtaszczyk, Greedy algorithm for general biorthogonal systems. J. Approx. Theory 107 (2000), 293–314.

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