Weak logarithmic Sobolev inequalities and entropic convergence

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Abstract

In this paper we introduce and study a weakened form of logarithmic Sobolev inequalities in connection with various others functional inequalities (weak Poincaré inequalities, general Beckner inequalities...). We also discuss the quantitative behaviour of relative entropy along a symmetric diffusion semi-group. In particular, we exhibit an example where Poincaré inequality can not be used for deriving entropic convergence whence weak logarithmic Sobolev inequality ensures the result.

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1 Introduction

Since the beginning of the nineties, functional inequalities (Poincaré, logarithmic (or F-) Sobolev, Beckner's like, transportation) turned to be a powerful tool for studying various problems in Probability theory and in Statistics: uniform ergodic theory, concentration of measure, empirical processes, statistical mechanics, particle systems for non linear p.d.e.'s, stochastic analysis on path spaces, rate of convergence of p.d.e....
Among such functional inequalities, Poincaré inequality and its generalizations (weak and super Poincaré) deserved particular interest, as they are the most efficient tool for the study of isoperimetry, concentration of measure and $\mathbb{L}^2$ long time behaviour (see e.g. [RW01, Wan00, Wan05, BCR05a, BCR05c]). However (except the usual Poincaré inequality) they are not easily tensorizable nor perturbation stable. That is why super-Poincaré inequalities have to be compared with (generalized) Beckner's inequalities or with additive φ-Sobolev inequalities (see [Wan05, BCR05c, BCR05b]).
But for some aspects, generalized Poincaré inequalities are insufficient. Indeed $\mathbb{L}^2$ controls are not well suited in various situations (statistical mechanics, non linear p.d.e), where entropic controls are more natural. It is thus interesting to look at generalizations of Gross logarithmic Sobolev inequality. In this paper we shall investigate weak logarithmic Sobolev inequalities (the "super" logarithmic Sobolev inequalities have already been investigated by Davies and Simon, or Röckner and Wang).
In order to better understand the previous introduction and what can be expected, let us introduce some definitions and recall some known facts. In all the paper μ denotes an absolutely continuous probability measure on a given riemannian manifold M. We also assume that μ is symmetric for a "nice" diffusion semi-group P_t (that is, associated to a non explosive diffusion process).
Let $\mathcal{C}_b^1(M)$ the set of bounded and derivable functions on M.

Definition 1.1 *We say that the measure μ satisfies a weak Poincaré inequality,* **WPI***, if there exists a non-increasing function $\beta_{WP} : (0, +\infty) \to \mathbb{R}^+$, such that for all $s > 0$ and any $f \in \mathcal{C}_b^1(M)$,*

$$\mathbf{Var}_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \beta_{WP}(s) \int |\nabla f|^2 d\mu + s\,\mathbf{Osc}^2(f), \tag{\textbf{WPI}}$$

where $\mathbf{Osc}(f) = \sup f - \inf f$.

Weak Poincaré inequalities have been introduced by Röckner and Wang in [RW01]. If β_{WP} is bounded, we recover the (classical) Poincaré inequality, while if $\beta_{WP}(s) \to \infty$ as $s \to 0$ we obtain a weaker inequality.
Actually, as shown in [RW01] any Boltzman measure ($d\mu = e^{-V}\,dx$) on $\mathbb{R}^n$ with a locally bounded potential V satisfies some **WPI** (the result extends to any manifold with Ricci curvature bounded from below by a possibly negative constant, according to Theorem 3.1 in [RW01] and the local Poincaré inequality shown by Buser [Bus82] in this framework). **WPI** furnishes an isoperimetric inequality, hence (sub-exponential) concentration of measure (see [RW01, BCR05a]). It also allows to describe non exponential decay of the $\mathbb{L}^2$ norm of the semi group, i.e. **WPI** is linked to inequalities like

$$\forall t \geqslant 0, \quad \mathbf{Var}_\mu(P_t f) \leq \xi(t)\,\mathbf{Osc}^2(f),$$

for some adapting function ξ (relations between β_{WP} and ξ will be recalled later). Recall that an uniform decay of the Variance, is equivalent to its exponential decay which is equivalent to the usual Poincaré inequality.
If we replace the variance by the entropy the latter argument is still true. Indeed (at least for bounded below curvature) an uniform decay of $\mathbf{Ent}_\mu(P_t h)$ is equivalent to its exponential decay which is equivalent to the logarithmic Sobolev inequality. In order to describe non exponential decays, it is thus natural to introduce the following definition:

Definition 1.2 *We say that the measure μ satisfies a weak logarithmic Sobolev inequality,* **WLSI***, if there exists a non-increasing function $\beta_{WL} : (0, +\infty) \to \mathbb{R}^+$, such that for all $s > 0$ and any $f \in \mathcal{C}_b^1(M)$,*

$$\mathbf{Ent}_\mu(f^2) := \int f^2 \log\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq \beta_{WL}(s) \int |\nabla f|^2 d\mu + s\,\mathbf{Osc}^2(f)\,. \tag{\textbf{WLSI}}$$

Remark that **WPI** is translation invariant. Hence it is enough to check it for non negative functions f and for such functions we get $\mathbf{Var}_\mu(f) \leq \mathbf{Ent}_\mu(f^2)$. Hence **WLSI** is stronger than **WPI** (we shall prove a more interesting result), and we can expect that **WLSI** (with a non bounded β_{WL}) allows to describe all the sub-gaussian measures, in particular all super-exponential (and sub-gaussian measures) for which a strong form of Poincaré inequality holds.

Remark 1.3 *We may always choose $\beta_{WP}(s) = 0$ for $s \geqslant 1$. It is not immediate that a similar property holds for β_{WL}. However recall Rothaus inequality*

$$\mathbf{Ent}_\mu(f^2) \leq \mathbf{Ent}_\mu\left((\bar{f})^2\right) + 2\mathbf{Var}_\mu(f)\,,$$

where $\bar{f} = f - \int f d\mu$. If $1 = \mathbf{Osc}(f)$ then one has $\|\bar{f}\|_\infty \leq 1$ and we obtain $\mathbf{Ent}_\mu\left((\bar{f})^2\right) \leq 1/e$. Thus, by homogeneity,

$$\mathbf{Ent}_\mu(f^2) \leq (2 + 1/e)\,\mathbf{Osc}^2(f)\,.$$

Hence for **WPI** *and* **WLSI** *what is important is the behaviour of β near 0 (we may always choose β as a constant for $s \geqslant s_a > 0$ with $0 < s_a \leq 2 + 1/e^2$).*

In order to understand the picture and to compare all these inequalities we shall call upon another class of inequalities, namely measure-capacity inequalities introduced by Barthe and Roberto [BR03] and then extensively used in [Che05, BCR05b, BCR05a, BCR05c]. Recall that, given measurable sets $A \subset \Omega$, the capacity $Cap_\mu(A, \Omega)$, is defined as

$$Cap_\mu(A,\Omega) := \inf \left\{ \int |\nabla f|^2 d\mu;\; \mathbb{I}_A \le f \le \mathbb{I}_\Omega \right\},$$

where the infimum is taken over all locally Lipschitz functions on M. If now A satisfies $\mu(A) \le 1/2$ we note

$$Cap_\mu(A) := \inf \left\{ Cap_\mu(A,\Omega);\; A \subset \Omega,\; \mu(\Omega) \le 1/2 \right\}. \tag{1}$$

A measure-capacity inequality is an inequality of the form

$$\frac{\mu(A)}{\gamma(\mu(A))} \le Cap_\mu(A), \tag{2}$$

for some function γ. They are in a sense universal, since they only involve the energy (Dirichlet form) and the measure. Furthermore, a remarkable feature is that most of known inequalities involving various functionals (variance, p-variance, F functions of F-Sobolev inequalities, entropy etc...) can be compared (in a non sharp form) with some measure-capacity inequalities.
We shall thus start by characterizing **WLSI** via measure-capacity inequalities. Then we will study the one dimensional case, in the spirit of Muckenhoupt or Bobkov-Götze criteria for Poincaré or logarithmic Sobolev inequalities (see e.g. [ABC$^+$00] chapter 6). We shall then discuss in details the relationship between **WLSI** and the generalized Poincaré inequalities. Finally we shall discuss various properties and consequences of **WLSI**. In the final sections, we study in details the decay of entropy for large time. In particular we show that for a μ reversible gradient diffusion process, very mild conditions on the initial law are sufficient to ensure an entropic decay of type e^{-t^β} when μ satisfies interpolating inequalities between Poincaré and Gross introduced by Latala and Oleszkiewicz [LO00], those conditions preventing estimation via Poincaré inequalities. We also give the elements to compute this decay under general **WLSI**. The particular case of the double sided exponential measure is detailed.

Let us finally remark that the limitation to finite dimensional space is only instrumental and the main results would be readily extendable to infinite dimensional space with capacity defined to suitable Dirichlet forms (assuming for example the existence of a *carré du champ* operator).

2 Weak logarithmic Sobolev inequalities

2.1 Characterization via capacity-measure condition

We start this section by characterizing **WLSI** in terms of measure-capacity inequalities.

Theorem 2.1 *Assume that the measure μ satisfies a* **WLSI** *with function β_{WL}, then for every $A \subset M$ such that $\mu(A) \le 1/2$,*

$$\forall s > 0, \qquad \frac{\mu(A)\log\left(1 + \frac{1}{2\mu(A)}\right) - s}{\beta_{WL}(s)} \le Cap_\mu(A).$$

Proof
$\lhd$ Let $A \subset \Omega$ with $\mu(\Omega) \le 1/2$ and let f be a locally Lipschitz function satisfying $\mathbb{I}_A \le f \le \mathbb{I}_\Omega$. The variational definition of the entropy implies

$$\mathbf{Ent}_\mu(f^2) \geqslant \int f^2 g d\mu,$$

for all g such that $\int e^g d\mu \leq 1$. Apply this inequality with

$$g = \begin{cases} \log\left(1 + \frac{1}{2\mu(A)}\right) & \text{on } A \\ 0 & \text{on } \Omega\backslash A \\ -\infty & \text{on } \Omega^c \end{cases}$$

which satisfies $\int e^g d\mu \leq 1$. It yields $\mathbf{Ent}_\mu(f^2) \geqslant \mu(A)\log\left(1 + \frac{1}{2\mu(A)}\right)$.
Therefore by the weak logarithmic Sobolev inequality and the definition of the capacity we obtain

$$\mu(A)\log\left(1 + \frac{1}{2\mu(A)}\right) \leq \beta_{WL}(s) Cap_\mu(A, \Omega) + s.$$

Taking the infimum over sets Ω with measure at most $1/2$ and containing A we obtain

$$\forall s > 0, \quad \frac{\mu(A)\log\left(1 + \frac{1}{2\mu(A)}\right) - s}{\beta_{WL}(s)} \leq Cap_\mu(A).$$

$\triangleright$

Theorem 2.2 *Let $\beta : (0, +\infty) \to \mathbb{R}^+$ be non-increasing function such that for every $A \subset M$ with $\mu(A) \leq 1/2$ one has*

$$\forall s > 0, \quad \frac{\mu(A)\log\left(1 + \frac{e^2}{\mu(A)}\right) - s}{\beta(s)} \leq Cap_\mu(A). \tag{3}$$

Then the measure μ satisfies a **WLSI** *with the function $\beta_{WL}(s) = 16\beta(3s/14)$, for $s > 0$.*

Proof
$\triangleleft$ Let $f \in \mathcal{C}_b^1(M)$ we will prove that

$$\forall s > 0, \quad \mathbf{Ent}_\mu(f^2) \leq 16\beta(s) \int |\nabla f|^2 d\mu + 14s/3\, \mathbf{Osc}^2(f). \tag{4}$$

Let m be a median of f under μ and let $\Omega_+ = \{f > m\}$, $\Omega_- = \{f < m\}$. Then, using the argument of Lemma 5 in [BR03], we obtain

$$\mathbf{Ent}_\mu(f^2) \leq \sup\left\{\int F_+^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu \leq e^2 + 1\right\}$$
$$+ \sup\left\{\int F_-^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu \leq e^2 + 1\right\}, \tag{5}$$

where $F_+ = (f - m)\mathbb{1}_{\Omega_+}$ and $F_- = (f - m)\mathbb{1}_{\Omega_-}$.
We will study the first term in the right hand side, the second one will be treated by the same method.
There are two cases depending on the value of s. Let $s_1 := \frac{1}{2}\log\left(1 + 2e^2\right)$, and assume that $s \in (0, s_1)$.
The function $x \mapsto x\log(1 + e^2/x)$ is increasing on $(0, \infty)$, and realize a bijection between $(0, 1/2]$ and $(0, s_1]$. We get that the function

$$c \mapsto \mu(\Omega_0)\log\left(1 + \frac{e^2}{\mu(\Omega_0)}\right),$$

where $\Omega_0 = \left\{F_+^2 \geqslant c\right\}$, is non-decreasing on $(0, \infty)$ and realize a surjection on $(0, s_1)$. Then for any $s \in (0, s_1)$ there exists at least $c > 0$ such that

$$\mu(\Omega_0)\log\left(1 + \frac{e^2}{\mu(\Omega_0)}\right) = s. \tag{6}$$

Pick some $\rho \in (0,1)$ and introduce for any $k \in \mathbb{N}$, $\Omega_k = \left\{F_+^2 \geqslant c\rho^k\right\}$. The sequence $(\Omega_k)_k$ is increasing so that, for every function $h \geqslant 0$,

$$\int F_+^2 h d\mu = \int_{\Omega_0} F_+^2 h d\mu + \sum_{k>0} \int_{\Omega_k \setminus \Omega_{k-1}} F_+^2 h d\mu.$$

For the first term we get

$$\int_{\Omega_0} F_+^2 h d\mu \leq \mathbf{Osc}^2(f) \int_{\Omega_0} h \, d\mu,$$

then Lemma 6 of [BR03] implies that

$$\sup\left\{\int_{\Omega_0} h d\mu;\ h \geqslant 0,\ \int e^h d\mu \leq e^2+1\right\} = \mu(\Omega_0)\log\left(1+\frac{e^2}{\mu(\Omega_0)}\right).$$

So that, using the definition of c (equality (6)) we get

$$\sup\left\{\int_{\Omega_0} F_+^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu \leq e^2+1\right\} \leq s\,\mathbf{Osc}^2(f).$$

For the second term we have for all $k>0$, due to the fact that $c\rho^k \leq F_+^2 \leq c\rho^{k-1}$ on $\Omega_k \setminus \Omega_{k-1}$,

$$\int_{\Omega_k \setminus \Omega_{k-1}} F_+^2 h d\mu \leq c\rho^{k-1} \int_{\Omega_k \setminus \Omega_{k-1}} h d\mu.$$

Then we obtain using again Lemma 6 of [BR03], for any $k>0$,

$$\sup\left\{\int_{\Omega_k \setminus \Omega_{k-1}} F_+^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu \leq e^2+1\right\} \leq c\rho^{k-1}\mu(\Omega_k \setminus \Omega_{k-1})\log\left(1+\frac{e^2}{\mu(\Omega_k \setminus \Omega_{k-1})}\right).$$

Using now inequality (3) we get

$$c\rho^{k-1}\left(\mu(\Omega_k \setminus \Omega_{k-1})\log\left(1+\frac{e^2}{\mu(\Omega_k \setminus \Omega_{k-1})}\right)\right) \leq c\rho^{k-1}\beta(s)Cap_\mu(\Omega_k \setminus \Omega_{k-1}) + sc\rho^{k-1}.$$

Let set for any $k>0$,

$$g_k = \min\left\{1, \left(\frac{F_+ - \sqrt{c\rho^{k+1}}}{\sqrt{c\rho^k} - \sqrt{c\rho^{k+1}}}\right)_+\right\},$$

so that we have $\mathbb{1}_{\Omega_k} \leq g_k \leq \mathbb{1}_{\Omega_+}$ recall that $\mu(\Omega_+) = 1/2$. This implies, using the definition of $Cap_\mu(\Omega_k \setminus \Omega_{k-1})$ (inequality (1)),

$$c\rho^{k-1}Cap_\mu(\Omega_k \setminus \Omega_{k-1}) \leq \frac{1}{\rho(1-\sqrt{\rho})^2}\int_{\Omega_{k+1} \setminus \Omega_k} |\nabla F_+|^2 d\mu.$$

Note that the constant c satisfies $c \leq \|F_+\|_\infty^2 \leq \mathbf{Osc}^2(f)$. We can now finish the proof in the case $s \in (0, s_1)$,

$$\begin{aligned}
\sup\left\{\int F_+^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu \leq e^2+1\right\} \leq\ & \sup\left\{\int_{\Omega_0} F_+^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu\right\} + \\
& \sum_{k>0} \sup\left\{\int_{\Omega_{k+1} \setminus \Omega_k} F_+^2 h d\mu;\ h \geqslant 0,\ \int e^h d\mu\right\} \\
\leq\ & s\,\mathbf{Osc}^2(f) + sc\rho^{k-1} + \\
& \beta(s)\sum_{k>0}\frac{1}{\rho(1-\sqrt{\rho})^2}\int_{\Omega_{k+1} \setminus \Omega_k} |\nabla F_+|^2 d\mu \\
\leq\ & \frac{\beta(s)}{\rho(1-\sqrt{\rho})^2}\int |\nabla F_+|^2 d\mu + s\frac{2-\rho}{1-\rho}\mathbf{Osc}^2(f).
\end{aligned}$$

Using inequality (5) and the previous inequality for F_- we get

$$\forall s \in (0, s_1), \quad \mathbf{Ent}_\mu(f^2) \le \frac{\beta(s)}{\rho(1-\sqrt{\rho})^2} \int |\nabla f|^2 d\mu + 2s\frac{2-\rho}{1-\rho}\mathbf{Osc}^2(f), \tag{7}$$

for all $\rho \in (0,1)$. Choosing $\rho = 1/4$ furnishes inequality (4) for any $s \in (0, s_1)$.
Assume now that $s \geqslant s_1$, then take $c = 0$ and we get

$$\mu(\Omega_0) \log\left(1 + \frac{e^2}{\mu(\Omega_0)}\right) \le s,$$

and the same argument used for $s \in (0, s_1)$ implies

$$\forall s \geqslant s_1, \quad \mathbf{Ent}_\mu(f^2) \le 2s\mathbf{Osc}^2(f). \tag{8}$$

Then inequality (8) and the previous result implies inequality (4) for any $s > 0$.
Note that we do not obtain the optimal function $\beta_{WL}(s)$ for s large, but, as explained in Remark 1.3, this is not important for the **WLSI**. $\rhd$

Remark 2.3 *The following two inequalities hold*

$$\frac{\frac{\mu(A)}{2}\log\left(1+\frac{1}{2\mu(A)}\right)}{\beta_{WL}\left(\frac{\mu(A)}{2}\log\left(1+\frac{1}{2\mu(A)}\right)\right)} \le \sup_{s>0}\left\{\frac{\mu(A)\log\left(1+\frac{1}{2\mu(A)}\right)-s}{\beta_{WL}(s)}\right\} \le \frac{\mu(A)\log\left(1+\frac{1}{2\mu(A)}\right)}{\beta_{WL}\left(\mu(A)\log\left(1+\frac{1}{2\mu(A)}\right)\right)}$$

and

$$\frac{\frac{\mu(A)}{2}\log\left(1+\frac{e^2}{\mu(A)}\right)}{\beta_{WL}\left(\frac{\mu(A)}{2}\log\left(1+\frac{e^2}{\mu(A)}\right)\right)} \le$$

$$\sup_{s>0}\left\{\frac{\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)-s}{\beta_{WL}(s)}\right\} \le \frac{\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)}{\beta_{WL}\left(\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)\right)}. \tag{9}$$

The proof of these inequalities is the same as in [BCR05a, Theorem 1]. The lower bounds of these inequalities correspond to a specific choice, $s = \frac{\mu(A)}{2}\log\left(1+\frac{1}{2\mu(A)}\right)$ *for the first one and* $s = \frac{\mu(A)}{2}\log\left(1+\frac{e^2}{\mu(A)}\right)$ *for the second one. For the upper bound of the first inequality we use the fact that*

$$\sup_{s>0}\left\{\frac{\mu(A)\log\left(1+\frac{1}{2\mu(A)}\right)-s}{\beta_{WL}(s)}\right\} \le \sup_{0<s<\mu(A)\log\left(1+\frac{1}{2\mu(A)}\right)}\left\{\frac{\mu(A)\log\left(1+\frac{1}{2\mu(A)}\right)}{\beta_{WL}(s)}\right\},$$

and the non-increasing property of β gives the result. The method holds for the second inequality.

2.2 An Hardy like criterion on $\mathbb{R}$

Proposition 2.4 *Let μ be a probability measure on $\mathbb{R}$. Assume that μ is absolutely continuous with respect to the Lebesgue measure and denote by ρ_μ its density. Let m be a median of μ and $\beta_{WL} : (0,\infty) \to \mathbb{R}^+$ be non-increasing. Let C be the optimal constant such that for all $f \in \mathcal{C}_b^1(\mathbb{R})$,*

$$\forall s > 0, \quad \mathbf{Ent}_\mu(f^2) \le C\beta_{WL}(s)\int |\nabla f|^2 d\mu + s\,\mathbf{Osc}^2(f).$$

Then $\max(b_-,b_+)\le C\le\max(B_-,B_+)$, *where*

$$b_+ := \sup_{x>m}\frac{\frac{\mu([x,+\infty))}{2}\log\left(1+\frac{1}{2\mu([x,+\infty))}\right)}{\beta_{WL}\left(\frac{\mu([x,+\infty))}{2}\log\left(1+\frac{1}{2\mu([x,+\infty))}\right)\right)}\int_m^x\frac{1}{\rho_\mu},$$

$$b_- := \sup_{x<m}\frac{\frac{\mu((-\infty,x])}{2}\log\left(1+\frac{1}{2\mu((-\infty,x])}\right)}{\beta_{WL}\left(\frac{\mu((-\infty,x])}{2}\log\left(1+\frac{1}{2\mu((-\infty,x])}\right)\right)}\int_x^m\frac{1}{\rho_\mu},$$

$$B_+ := \sup_{x>m}\frac{16\mu([x,+\infty))\log\left(1+\frac{e^2}{\mu([x,+\infty))}\right)}{\beta_{WL}\left(\frac{14}{3}\mu([x,+\infty))\log\left(1+\frac{e^2}{\mu([x,+\infty))}\right)\right)}\int_m^x\frac{1}{\rho_\mu}, \tag{10}$$

$$B_- := \sup_{x<m}\frac{16\mu((-\infty,x])\log\left(1+\frac{e^2}{\mu((-\infty,x])}\right)}{\beta_{WL}\left(\frac{14}{3}\mu((-\infty,x])\log\left(1+\frac{e^2}{\mu((-\infty,x])}\right)\right)}\int_x^m\frac{1}{\rho_\mu}.$$

Proof

$\lhd$ The proof of the lower bound on C is exactly the same as in [BCR05a, Theorem 3] using Theorem 2.1 and Remark 2.3.

For the upper bound denote $F_+=(f-f(m))\mathbb{1}_{[m,+\infty)}$ and $F_-=(f-f(m))\mathbb{1}_{(-\infty,m]}$. Then

$$\mathbf{Ent}_\mu\left(f^2\right)\le\mathbf{Ent}_\mu\left(F_+^2\right)+\mathbf{Ent}_\mu\left(F_-^2\right).$$

We work separately with the two terms and we explain the arguments for $\mathbf{Ent}_\mu\left(F_+^2\right)$ only. We follow the method of proof in [BCR05a, Theorem 3].

Using inequality (10) we get

$$\forall x>m,\quad \frac{16\mu([x,+\infty))\log\left(1+\frac{e^2}{\mu([x,+\infty))}\right)}{\beta_{WL}\left(\frac{14}{3}\mu([x,+\infty))\log\left(1+\frac{e^2}{\mu([x,+\infty))}\right)\right)}\int_m^x\frac{1}{\rho_\mu}\le B_+.$$

This means that

$$\forall x>m,\quad \frac{16\mu([x,+\infty))\log\left(1+\frac{e^2}{\mu([x,+\infty))}\right)}{B_+\beta_{WL}\left(\frac{14}{3}\mu([x,+\infty))\log\left(1+\frac{e^2}{\mu([x,+\infty))}\right)\right)}\le Cap_\mu([x,+\infty),[m,+\infty)),$$

recall that $Cap_\mu([x,+\infty),[m,+\infty))=1/(\int_m^x\frac{1}{\rho_\mu})$.

If $A\subset[m,+\infty)$ then $Cap_\mu(A,[m,+\infty))=Cap_\mu([\inf A,+\infty),[m,+\infty))$, the function

$$t\mapsto\frac{16t\log\left(1+\frac{e^2}{t}\right)}{\beta_{WL}\left(\frac{14}{3}t\log\left(1+\frac{e^2}{t}\right)\right)}$$

is increasing on $(0,\infty)$, we get

$$\forall A\subset[m,+\infty),\quad \frac{16\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)}{B_+\,\beta_{WL}\left(\frac{14}{3}\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)\right)}\le Cap_\mu(A,[m,+\infty)).$$

Using now inequality (9) one has for all $A\subset[m,+\infty)$,

$$\sup_{s>0}\left\{16\frac{\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)-s}{B_+\,\beta_{WL}(\frac{14}{3}s)}\right\}\le Cap_\mu(A,[m,+\infty)),$$

and then by the same argument as in Theorem 2.2 one has

$$\mathbf{Ent}_\mu(F_+^2) \le B_+\beta_{WL}(s)\int |\nabla F_+|^2 d\mu + s\mathbf{Osc}(f)^2.$$

It follows that $C \le B_+$. The same argument gives also $C \le B_-$ and the proposition is proved. $\triangleright$

Corollary 2.5 ([BCR05a]) *Let Φ be a function on $\mathbb{R}$ such that $d\mu_\Phi(x) := e^{-\Phi(x)}dx$, $x \in \mathbb{R}$ is a probability measure and let $\varepsilon \in (0,1)$.*
Assume that there exists an interval $I = (x_0, x_1)$ containing a median m of μ such that $|\Phi|$ is bounded on I, and Φ is twice differentiable outside I with for any $x \notin I$,

$$\Phi'(x) \neq 0, \frac{|\Phi''(x)|}{\Phi'(x)^2} \le 1-\varepsilon \text{ and}$$
$$A'\Phi(x) \le \Phi(x) + \log\left|\Phi'(x)\right| \le A\Phi(x), \tag{11}$$

for some constants $A, A' > 0$.
Let β be a non-increasing function on $(0,\infty)$. Assume that there exists $c > 0$ such that for all $x \notin I$ it holds

$$\frac{\Phi(x)}{\Phi'(x)^2} \le c\beta\left(\frac{Ae^{-\Phi(x)}\Phi(x)}{|\Phi'(x)|}\right).$$

Then μ_Φ satisfies a **WLSI** *with function $C\beta$ for some constant $C > 0$.*

Proof
$\triangleleft$ Corollary 2.4 of [BCR05a] gives for $x \ge x_1$,

$$\mu([x,+\infty)) \le \frac{e^{-\Phi(x)}}{\varepsilon\Phi'(x)} \le \frac{2-\varepsilon}{\varepsilon}\mu([x,+\infty)).$$

Then using Proposition 2.4 and inequality (11) we get the result. $\triangleright$

Let us give two examples:

- For $\alpha > 0$, the measure $dm_\alpha(t) = \alpha(1+|t|)^{-1-\alpha}dt/2$, $t \in \mathbb{R}$ satisfies the **WLSI** with the function

  $$\forall s > 0, \quad \beta_{WL}(s) = C\frac{(\log 1/s)^{1+2/\alpha}}{s^{2/\alpha}},$$

  for some constant $C > 0$.

- Let $\alpha \in (0,2)$ and defined the probability measure $d\mu_\alpha(t) = Z_\alpha e^{-|t|^\alpha}dt$, $t \in \mathbb{R}$, (Z_α is a normalization constant). Then μ_α satisfies the **WLSI** with the function

  $$\forall s > 0, \quad \beta_{WL}(s) = C(\log 1/s)^{(2-\alpha)/\alpha},$$

  for some $C > 0$.

  Contrary to the **WPI**, one can study the case $\alpha \in [1,2]$. In particular for $\alpha = 2$ we get that β_{WL} is bounded, i.e. we recover (with a non sharp constant) the classical logarithmic Sobolev inequality for the gaussian measure.

3 Weak Logarithmic Sobolev inequalities and generalized Poincaré inequalities

3.1 Link with weak Poincaré inequalities and classical Poincaré inequality

Barthe, Cattiaux and Roberto investigated in [BCR05a] the measure-capacity criterion for **WPI**. Their results read as follows: **WPI** with a function β_{WP} implies a measure-capacity inequality with $\gamma(u) = 4\beta_{WP}(u/4)$ (see inequality (2)) while a measure-capacity inequality with non-increasing function γ implies **WPI** with $\beta_{WP} = 12\gamma$ (we may assume that $\gamma(u) = \gamma(1/2)$ for $u \geqslant 1/2$). Comparing with Theorem 2.1 and Theorem 2.2, we can state

Proposition 3.1 *Assume that a probability measure μ satisfies a* **WLSI** *with function β_{WL} then μ satisfies a* **WPI** *with function β_{WP} defined by*

$$\forall s > 0, \quad \beta_{WP}(s) = \frac{24\beta_{WL}\left(\frac{s}{2}\log\left(1+\frac{1}{2s}\right)\right)}{\log\left(1+\frac{1}{2s}\right)}. \tag{12}$$

Conversely, a **WPI** *with function β_{WP} implies a* **WLSI** *with function β_{WL}, defined by,*

$$\begin{cases} \forall s \in (0, s_0), \quad \beta_{WL}(s) = c'\beta_{WP}\left(c\dfrac{s}{\log(1/s)}\right)\log(1/s), \\ \forall s \geqslant s_0, \quad \beta_{WL}(s) = c'\beta_{WP}\left(c\dfrac{s_0}{\log(1/s_0)}\right)\log(1/s_0), \end{cases} \tag{13}$$

for some universal constants $c, c', s_0 > 0$.
Finally assume that μ satisfies a **WLSI** *with function β_{WL}, then it verifies a classical Poincaré inequality if and only if there exist $c_1, c_2 > 0$ such that for s small enough,*

$$\beta_{WL}(s) \leq c_1 \log(c_2/s).$$

Proof
◁ For the first statement, first note that β_{WP} is non-increasing. Then Theorem 2.1 and Remark 2.3 imply that for all A such that $\mu(A) \leq 1/2$,

$$\frac{\frac{\mu(A)}{2}\log\left(1+\frac{1}{2\mu(A)}\right)}{\beta_{WL}\left(\frac{\mu(A)}{2}\log\left(1+\frac{1}{2\mu(A)}\right)\right)} \leq Cap_\mu(A).$$

This means that for all A such that $\mu(A) \leq 1/2$,

$$\frac{12\mu(A)}{\beta_{WP}(\mu(A))} \leq Cap_\mu(A),$$

where β_{WP} is defined by (12), the result holds using Theorem 2.2 of [BCR05a].
To prove the second statement we use the same argument (replacing Theorem 2.1 by Theorem 2.2) and the fact that there exist constants $A, A', s_0 > 0$ such that

$$\forall s \in (0, s_0), \quad A'\frac{s}{\log(1/s)} \leq \varphi^{-1}(s) \leq A\frac{s}{\log(1/s)}, \tag{14}$$

where $\varphi(s) = s\,\log(1+e^2/s)$. Then μ satisfies a **WLSI** with function β_{WL} defined by (13). Note that β_{WL} is non-increasing.
Finally, the last two results prove that $\beta_{WL}(s) \leq c_1\log(c_2/s)$ for s enough is equivalent to the classical Poincaré inequality. ▷

Remark 3.2 • *It is interesting to remark that when considering the usual derivation "Logarithmic Sobolev inequality implies Poincaré inequality" by means of test function* $1 + \epsilon g$ *and* $\epsilon \to 0$*, we get a worse result: a weak logarithmic Sobolev inequality with function* β *implies a weak Poincaré inequality with the same function* β*, whereas the result of Proposition 3.1 gives a better result.*

- *As a byproduct, we get that any Boltzman's measure (with a locally bounded potential) satisfies some* **WLSI** *if Ricci(M) is bounded from below (see [RW01]).*

- *Finally the above proof shows that we obtain the best function (up to multiplicative constants) for* **WPI** *or* **WLSI** *as soon as we have the best function for the other. In particular we recover the good functions for the examples 2.2.*

3.2 Link with super Poincaré inequalities

Let us recall the definition of the super Poincaré inequality introduced by Wang in [Wan00].

Definition 3.3 *We say that the measure* μ *satisfies a super Poincaré inequality,* **SPI***, if there exists a non-increasing function* $\beta_{SP} : [1, +\infty) \to \mathbb{R}^+$*, such that for all* $s \geqslant 1$ *and any smooth functions* f*,*

$$\int f^2 d\mu \leq \beta_{SP}(s) \int |\nabla f|^2 d\mu + s \left(\int |f| d\mu \right)^2. \tag{SPI}$$

Note that as for **WLSI** in Remark 1.3, for the **SPI** what is important is the behaviour of β near ∞ (we may always choose $\beta_{SP}(s) = \beta_{SP}(1)$ for $1 \leq s \leq s_b$ where s_b is a constant). As for Proposition 3.1 we can now relate **WLSI** and **SPI**.

Proposition 3.4 *Suppose that* μ *satisfies a* **WLSI** *with function* β_{WL}*. Assume that* β_{WL} *verifies that* $x \mapsto \beta_{WL}\left(\frac{\log(x/2)}{2x}\right)/\log(x/2)$ *is non-increasing on* $(2, \infty)$*.*
Then μ *satisfies a* **SPI** *with function* β_{SP} *given by*

$$\forall t \geqslant 2e, \qquad \beta_{SP}(t) = 2 \frac{\beta_{WL}\left(\frac{\log(t/2)}{2t}\right)}{\log(t/2)}, \tag{15}$$

β_{SP} *being constant on* $[1, 2e)$*.*

Proof
$\lhd$ If μ satisfes a **WLSI** then one obtains by Theorem 2.1 and Remark 2.3:

$$\frac{\frac{\mu(A)}{2} \log\left(1 + \frac{1}{2\mu(A)}\right)}{\beta_{WL}\left(\frac{\mu(A)}{2} \log\left(1 + \frac{1}{2\mu(A)}\right)\right)} \leq Cap_\mu(A), \tag{16}$$

for any $A \subset M$, with $\mu(A) \leq 1/2$. Finally the function $t \mapsto t\,\beta_{WL}\left(\frac{\log(t/2)}{2t}\right)/\log(t/2)$ is clearly non decreasing for $t \geqslant 2e$, then Corollary 6 of [BCR05c] gives the result. $\rhd$

Note that the last proposition is not entirely satisfactory. We hope that **WLSI** is equivalent to **SPI** via a measure-capacity measure criterion.

3.3 Link with general Beckner inequalities

Definition 3.5 *Let $T : [0,1] \to \mathbb{R}^+$, be a non-decreasing function, satisfying in addition $x \mapsto T(x)/x$ is non-increasing on $(0,1]$.*
We say that a measure μ satisfies a general Beckner inequality, **GBI**, *with function T if for all smooth function f,*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu - \left(\int |f|^p d\mu\right)^{\frac{2}{p}}}{T(2-p)} \le \int |\nabla f|^2 d\mu. \qquad (\mathbf{GBI})$$

Note that our hypotheses imply that

$$\forall x \in [0,1], \qquad T(1)x \le T(x) \le T(1).$$

The two extremal cases correspond respectively to the Poincaré inequality (T is constant, $T(x) = T(1)$) and the logarithmic Sobolev inequality ($T(x) = T(1)x$). The intermediate cases $T(x) = x^a$ for $0 \le a \le 1$, have been introduced and studied in [LO00], while a study of general T is partly done in [BCR05b]. Also note that (up to multiplicative constants) the interesting part of T is its behaviour near 0, that is we can always define T near the origin and then take it equal to a large enough constant.
In [BCR05b] Theorem 10 and Lemma 9, it is shown that (up to a multiplicative constant) **GBI** is equivalent to a measure-capacity, inequality (2), with the function

$$\gamma(u) = T\left(\frac{1}{\log\left(1+\frac{1}{u}\right)}\right), \qquad (17)$$

for $u > 0$ small enough. More precisely a **GBI** implies a measure-capacity inequality with the function 6γ defined on (17). Conversely a measure-capacity inequality with the function γ implies a **GBI** with the function $20T$. We thus obtain:

Proposition 3.6 *Assume that μ satisfies a* **WLSI** *with function β_{WL}. Let*

$$\forall t \in (0,1], \qquad T(t) = t\beta_{WL}\left(\frac{1}{4te^{1/t}}\right). \qquad (18)$$

Assume that T is non-decreasing on $(0, t_a]$ for some $t_a \in (0,1]$. Then the measure μ satisfies a **GBI** *with function $20T$.*
Conversely assume that μ satisfies a **GBI** *with function T, then μ satisfies a* **WLSI** *with function β_{WL} given by*

$$\beta_{WL}(s) = C\,T\left(C'\frac{1}{\log(1/s)}\right)\log(1/s), \qquad (19)$$

for $s > 0$ small enough and some constants C, C'.

Proof
$\lhd$ Assume that μ satisfies a **WLSI** with function β_{WL}. Using Theorem 2.1 and Remark 2.3 one has inequality (16). Using the fact that

$$\forall x \in (0,1], \qquad \log\left(1+\frac{1}{2x}\right) \geqslant \frac{1}{2}\log\left(1+\frac{1}{x}\right),$$

one obtains that inequality (16) implies that the function T defined on (18) satisfies a measure-capacity inequality. The function $x \mapsto T(x)/x$ is non-increasing and due to the fact that T is non-decreasing by hypothesis, then Theorem 10 and Lemma 9 of [BCR05b] prove that μ satisfies a **GBI** of function T.

To prove the second statement we need also Theorem 10 and Lemma 9 of [BCR05b], Theorem 2.2 and inequality (14). $\triangleright$

Note that if the function T defined on (18) is non-decreasing near 0 then one can prove that $\beta_{WL}(s) \leq c_1 \log(c_2/s)$ for s small enough and some constants $c_1, c_2 > 0$. Then by Proposition 3.1, μ satisfies a Poincaré inequality. The last proposition can be applied only for measures satisfying a Poincaré inequality.

3.4 Link with another weak logarithmic Sobolev inequality

The next inequality is useful to control the decay in entropy of the semigroup. It will by used in Theorem 4.2 in the next section.

Theorem 3.7 *If μ satisfies a* **WLSI** *with function β_{WL}, then μ satisfies for any smooth function f any $u > 0$ small enough,*

$$\mathbf{Ent}_\mu(f^2) \leq \beta_{SWL}(u) \int |\nabla f|^2 d\mu + \sqrt{3}u \, \left(\mathbf{Var}_\mu(f^2)\right)^{\frac{1}{2}} , \tag{20}$$

with

$$\beta_{SWL}(u) = 16 \beta_{WL} \left(\frac{\kappa \, u^3}{\log^6(1/u)} \right)$$

for some universal constant $\kappa > 0$ and $u > 0$ small enough.

Proof

$\triangleleft$ According to Theorem 2.1 and Remark 2.3 we know that for every $A \subset M$ such that $\mu(A) \leq 1/2$,

$$Cap_\mu(A) \geqslant \frac{\frac{\mu(A)}{2} \log\left(1 + \frac{1}{2\mu(A)}\right)}{\beta_{WL}\left(\frac{\mu(A)}{2} \log\left(1 + \frac{1}{2\mu(A)}\right)\right)} \geqslant \frac{\frac{\mu(A)}{2k} \log\left(1 + \frac{e^2}{\mu(A)}\right)}{\beta_{WL}\left(\mu(A) \log\left(1 + \frac{e^2}{\mu(A)}\right)\right)}$$

for $k = \log(1 + 2e^2)/\log(2)$ using $k \log(1 + y/2) \geqslant \log(1 + e^2 y)$ for $y \geqslant 2$ and that β_{WL} is non-increasing. Hence we are in the situation of Theorem 2.2 with $\beta = 2k \, \beta_{WL}$.
Note that we may assume that f is non-negative.

We shall use the notations in the proof of Theorem 2.2, in particular $\Omega_0 = \{F_+^2 \geqslant c\}$ for some positive c, but we will choose another c than in the referred proof.
Indeed the first quantity we have to control is $\int_{\Omega_0} F_+^2 h d\mu$ which is less than

$$\left(\int_{\Omega_0} h^2 d\mu \right)^{\frac{1}{2}} \left(\int F_+^4 d\mu \right)^{\frac{1}{2}} .$$

We thus have to bound

$$\begin{aligned} X_0 &:= \sup\{ \int_{\Omega_0} h^2 d\mu; \, h \geqslant 0, \int e^h d\mu \leq 1 + e^2 \} \\ &= \sup\{ \int_{\Omega_0} h^2 d\mu; \, h \geqslant 0, \int_{\Omega_0} e^h d\mu \leq e^2 + \mu(\Omega_0) \} , \end{aligned}$$

(see [BR03] Lemma 6 for the latter equality). But $\varphi(x) = (1+\log^2(x))\, \mathbb{I}_{x\geqslant e} + \frac{2}{e}x\, \mathbb{I}_{x<e}$ is concave on $\mathbb{R}_+$. It follows that

$$\begin{aligned}
\varphi\left(\frac{e^2+\mu(\Omega_0)}{\mu(\Omega_0)}\right) &\geqslant \int_{\Omega_0} \varphi(e^h)\,\frac{d\mu}{\mu(\Omega_0)} \\
&\geqslant \int_{\Omega_0} \left((1+h^2)\,\mathbb{I}_{h\geqslant 1}\right)\frac{d\mu}{\mu(\Omega_0)} \\
&\geqslant \int_{\Omega_0} h^2\,\frac{d\mu}{\mu(\Omega_0)} - \int_{\Omega_0} h^2\,\mathbb{I}_{h<1}\,\frac{d\mu}{\mu(\Omega_0)} \\
&\geqslant \int_{\Omega_0} h^2\,\frac{d\mu}{\mu(\Omega_0)} - 1\,,
\end{aligned}$$

so that

$$X_0 \leq \mu(\Omega_0)\left(2+\log^2\left(1+\frac{e^2}{\mu(\Omega_0)}\right)\right) := \psi(\mu(\Omega_0))\,.$$

For $s \leq 1$ we thus choose c such that

$$\psi(\mu(\Omega_0)) = s^a\,,$$

for some $a>0$, this choice being possible since ψ is increasing on $[0,1/2[$, the maximal possible s being greater than 1.
We can mimic now the proof of Theorem 2.2 and obtain

$$\begin{aligned}
\sup\left\{\int F_+^2 h d\mu;\ h\geqslant 0,\ \int e^h d\mu \leq e^2+1\right\} \ &\leq\ \sqrt{s^a}\left(\int F_+^4 d\mu\right)^{\frac{1}{2}} + s\,\frac{c}{1-\rho} \\
&+\ \frac{\beta_{WL}(s)}{\rho(1-\sqrt{\rho})^2}\int |\nabla F_+|^2 d\mu\,.
\end{aligned} \tag{21}$$

It remains to estimate c. Note that there exists an universal constant θ such that $\psi^{-1}(x) \geqslant \theta\, x/\log^2(1+\frac{e^2}{x})$. It follows

$$\theta\,\frac{s^a}{\log^2(1+\frac{e^2}{s^a})} \leq \mu(\Omega_0) \leq \frac{\int F_+^2 d\mu}{c} \leq \frac{\left(\int F_+^4 d\mu\right)^{\frac{1}{2}}}{c}\,,$$

so that choosing $a = 2/3$ and $\rho = 1/4$ we finally obtain

$$\begin{aligned}
\sup\left\{\int F_+^2 h d\mu;\ h\geqslant 0,\ \int e^h d\mu \leq e^2+1\right\} \ &\leq\ s^{\frac{1}{3}}\left(1+\frac{4}{3\theta}\log^2(1+\frac{e^2}{s^{2/3}})\right)\left(\int F_+^4 d\mu\right)^{\frac{1}{2}} \\
&+\ 16\beta_{WL}(s)\int |\nabla F_+|^2 d\mu\,.
\end{aligned} \tag{22}$$

The same inequality for F_- and the elementary $\sqrt{a}+\sqrt{b} \leq \sqrt{2}\,\sqrt{a+b}$ yield, since there exists an universal constant θ' such that the inverse function of $s \mapsto \sqrt{2}s^{\frac{1}{3}}\left(1+\frac{4}{3\theta}\log^2(1+\frac{e^2}{s^{2/3}})\right)$ is greater than $u \mapsto \theta'\, u^3/(\log^6(1/u))$ for $u>0$ small enough,

$$\mathbf{Ent}_\mu(f^2) \leq 16\beta_{WL}\left(\frac{\theta'\,u^3}{1+\log^6(1+\frac{e^2}{u^2})}\right)\int |\nabla f|^2 d\mu + u\left(\int (f-m)^4 d\mu\right)^{\frac{1}{2}}\,. \tag{23}$$

Since we have assumed that f is non-negative, a median of f^2 is m^2, and $(f-m)^4 \leq (f^2-m^2)^2$. Finally, if M denotes the mean of f^2,

$$\int \left((f^2-M)-(m^2-M)\right)^2 d\mu = \mathbf{Var}_\mu(f^2) + (m^2-M)^2$$

and since $m^2 - M$ is a median of $f^2 - M$, provided $m^2 - M \geqslant 0$

$$\mathbf{Var}_\mu(f^2) \geqslant \int (f^2 - M)^2 \, \mathbb{1}_{f^2 - M \geqslant m^2 - M} \, d\mu \geqslant \frac{1}{2} \, (m^2 - M)^2$$

while if $m^2 - M \leq 0$

$$\mathbf{Var}_\mu(f^2) \geqslant \int (f^2 - M)^2 \, \mathbb{1}_{f^2 - M \leq m^2 - M} \, d\mu \geqslant \frac{1}{2} \, (m^2 - M)^2 .$$

We thus finally obtain

$$\int (f - m)^4 d\mu \leq 3 \, \mathbf{Var}_\mu(f^2)$$

and the proof is completed. $\triangleright$

One may of course derive other weak logarithmic Sobolev inequalities by this method, such inequalities as well as further applications will be treated elsewhere. We will apply this theorem in Section 4 for studying the decay to the equilibrium of the semigroup.

4 Convergence of the associated semigroup

In this section we shall study entropic convergence for the semi-group. Let h be a bounded density of function with respect to the measure μ. The two results of this section connect the decay of the entropy with the infinite norm of h. More precisely, using the **WLSI** we will compute the function $C(t, \|h\|_\infty)$ such that for all $t > 0$,

$$\mathbf{Ent}_\mu(\mathbf{P_t} h) \leq C(t, \|h\|_\infty).$$

Note that we have $C(t, \|h\|_\infty) \to 0$ when t goes to ∞.

The first result connects the decay of the entropy with the oscillation of h:

Proposition 4.1 *If μ satisfies a* **WLSI** *with function β_{WL}, then for any $h \geqslant 0$ with $\int h d\mu = 1$, for t large enough,*

$$\mathbf{Ent}_\mu(\mathbf{P_t} h) \leq (2 + e^{-1} + \varepsilon) \, \xi_\varepsilon(t) \, \mathbf{Osc}^2(\sqrt{h}) \tag{24}$$

where $\xi_\varepsilon(t)$ is given by, for r small enough,

$$\xi_\varepsilon^{-1}(r) = -\frac{1}{2} \beta_{WL}(r) \log\left(\frac{r}{\varepsilon}\right).$$

Conversely, if there exists ξ decreasing such that, for any $h \geqslant 0$ with $\int h d\mu = 1$ we get

$$\forall t > 0, \qquad \mathbf{Ent}_\mu(\mathbf{P_t} h) \leq \xi(t) \, \mathbf{Osc}^2(\sqrt{h}),$$

then μ satisfies a **WLSI** *with function $\beta_{WL}(t) = \psi^{-1}(t)$ where $\psi(t) = 2\sqrt{2\,\xi(t)}$. In particular if $\xi(t) \leq c e^{-\alpha t}$, μ satisfies a Poincaré inequality.*

Proof

$\triangleleft$ Denote $I(t) = \mathbf{Ent}_\mu(\mathbf{P_t} h)$. $I'(t) = -\frac{1}{2} \int \frac{|\nabla \mathbf{P_t} h|^2}{\mathbf{P_t} h} d\mu$, thus the weak logarithmic Sobolev inequality yields

$$I'(t) \quad \leq \quad -\frac{2}{\beta_{WL}(r)} I(t) + \frac{2r}{\beta_{WL}(r)} \mathbf{Osc}^2(\sqrt{\mathbf{P_t} h}).$$

Using Gronwall's lemma yields

$$\mathbf{Ent}_\mu(\mathbf{P_t} h) \leq \inf_{r>0} \left\{ r \sup_{s \in [0,t]} \mathbf{Osc}^2(\sqrt{\mathbf{P_s} h}) + e^{-2t/\beta_{WL}(r)} \mathbf{Ent}_\mu(h) \right\}.$$

Use now $\mathbf{Osc}^2(\sqrt{\mathbf{P_t}h}) \leq \mathbf{Osc}^2(\sqrt{h})$ and $\mathbf{Ent}_\mu(h) \leq (2+1/e)\,\mathbf{Osc}^2(\sqrt{h})$ we proved in Remark 1.3. Then we choose r such that $r = \varepsilon\, e^{-2t/\beta_{WL}(r)}$.

Let us prove the second statement. Denote $f = \sqrt{h}$. According to [Cat04] (2.5) with $\alpha_1 = -1$ and $\alpha_2 = 2$ it holds

$$\mathbf{Ent}_\mu(h) \leq t \int |\nabla f|^2 d\mu + 2\,\log \int f\,\mathbf{P_t}h\,d\mu\,. \tag{25}$$

But

$$\begin{aligned}\int f\,\mathbf{P_t}h\,d\mu &= \int f\;(1 + (\mathbf{P_t}h - 1))\;d\mu \\ &\leq 1 + \int (f - \int f d\mu)\,(\mathbf{P_t}h - 1)\,d\mu \\ &\leq 1 + \mathbf{Osc}(f) \int |\mathbf{P_t}h - 1|\,d\mu \\ &\leq 1 + \mathbf{Osc}(f)\sqrt{2\,\mathbf{Ent}_\mu(\mathbf{P_t}h)} \\ &\leq 1 + \sqrt{2\xi(t)}\,\mathbf{Osc}^2(f)\,,\end{aligned}$$

where we used successively $\int f d\mu \leq 1$, Pinsker inequality and the hypothesis. It remains to use $\log(1+a) \leq a$ to get the first result. The particular case follows from Proposition 3.1. $\triangleright$

The previous result is the exact analogue of Theorem 2.1 in [RW01] for **WPI**. The converse statement (Theorem 2.3 in [RW01]) is remarkable in the following sense: it implies in particular that any exponential decay ($\mathbf{Var}_\mu(\mathbf{P_t}f) \leq ce^{-\alpha t}\Psi(f - \int f d\mu)$) for any Ψ such that $\Psi(af) = a^2\Psi(f)$ (in particular $\Psi(f) = \mathbf{Osc}^2(f)$) implies a (true) Poincaré inequality. This result is of course very much stronger than the usual one involving a $\mathbb{L}^2$ bound. Its proof lies on the fact that $t \mapsto \log(\int (\mathbf{P_t}f)^2 d\mu)$ is convex. This convexity property (even without the log) fails in general for the relative entropy (Bakry-Emery renowned criterion was introduced for ensuring such a property). Actually a similar statement for the entropy is false.

Not that the previous result is only partly satisfactory for the convergence of the entropy. Indeed recall that for a density of probability h, the following holds

$$\mathbf{Var}_\mu(\sqrt{h}) \leq \mathbf{Ent}_\mu(h) \leq \mathbf{Var}_\mu(h)$$

so that a weak Poincaré inequality implies for $t > 0$

$$\mathbf{Ent}_\mu(\mathbf{P_t}h) \leq \xi_\varepsilon^{WP}(t)\,(1+\varepsilon)\mathbf{Osc}(h)\,,$$

where $(\xi_\varepsilon^{WP})^{-1}(r) = -\beta_{WP}(r)\log(r/\varepsilon)$, whereas our WLSI implies

$$\mathbf{Ent}_\mu(\mathbf{P_t}h) \leq \xi_\varepsilon^{WLS}(t)\,(2 + e^{-1} + \varepsilon)\mathbf{Osc}^2(\sqrt{h})$$

so that even for very small time, the WLSI can be of no use for particular bounded density h (namely if $(2 + e^{-1} + \varepsilon)\mathbf{Osc}^2(\sqrt{h}) > (1+\varepsilon)\mathbf{Osc}(h)$). This fact is a little bit disappointing as when a true logarithmic Sobolv holds it is well known that for small time the LSI always furnishes lower bounds than Poincaré inequality (and justifies the use of LSI for this kind of evaluation).

In order to correct this unsatisfactory point, at least when a Poincaré inequality holds, and always for bounded density h, we will make use of the other weak logarithmic Sobolev inequality stated in Theorem 3.7. Indeed, another way to control entropy decay was introduced in [CG05, Theorem 1.13]. It was proved there that a Poincaré inequality (with constant C_P) is equivalent to a restricted logarithmic Sobolev inequality

$$\mathbf{Ent}_\mu(h) \leq C\,(1 + \log(\parallel h \parallel_\infty)) \int \frac{|\nabla h|^2}{h}\,d\mu$$

for all bounded density of probability h, where C only depends on C_P. It follows that

$$\mathbf{Ent}_\mu(\mathbf{P_t}h) \le e^{-\frac{t}{C(1+\log(\|h\|_\infty))}}\,\mathbf{Ent}_\mu(h)$$

for such an h.

We shall describe below one result in this direction for **WLSI**, using Theorem 3.7 and Poincaré inequality.

Proposition 4.2 *Let μ be a probability measure satisfying a* **WLSI** *with function β_{WL} and a Poincaré inequality with constant C_P. Let β_{SWL} be the function defined in Theorem 3.7. Then for all $f \in \mathcal{C}_b^1(M)$,*

$$\mathbf{Ent}_\mu\big(f^2\big) \le A(C_P, \| f \|_\infty) \int |\nabla f|^2 d\mu$$

where

$$A(C_P, \|f\|_\infty) = \inf_{u\in(0,s_0]} \Big\{\beta_{SWL}(u) + u\sqrt{3C_P}\|f\|_\infty^2\Big\},$$

and $(0, s_0]$ is the set where β_{SWL} is defined. As a consequence, for all $t \geqslant 0$,

$$\mathbf{Ent}_\mu(\mathbf{P_t}h) \le e^{-t\,/\,A(C_P,\|h\|_\infty^{\frac{1}{2}})}\,\mathbf{Ent}_\mu(h)$$

for any bounded density of probability h.

Proof

$\lhd$ Due to homogeneity we may assume that $\int |\nabla f|^2 d\mu = 1$ (if it is 0 the result is obvious). But since μ satisfies a Poincaré inequality

$$\mathbf{Var}_\mu(f^2) \le 4C_P \int f^2\,|\nabla f|^2 d\mu \le 4C_P \parallel f^2 \parallel_\infty,$$

so that $\mathbf{Ent}_\mu\big(\mathbf{P_t} f^2\big) \le \beta_{SWL}(u) + 2u \parallel f \parallel_\infty^2 \sqrt{3C_P}$, by Theorem 3.7. $\rhd$

Note now that the previous entropic decay is always better for small time. Indeed if

$$t \le \frac{C_P A(C_P, \parallel h \parallel_\infty^{\frac{1}{2}})}{A(C_P, \parallel h \parallel_\infty^{\frac{1}{2}}) - C_P} \log\left(\frac{\mathbf{Var}_\mu(h)}{\mathbf{Ent}_\mu(h)}\right)$$

then the entropic decay obtained by Proposition 4.2 is better than the estimate obtained with Poincaré inequality.

Let $\alpha \in [1,2]$ and $d\mu_\alpha(t) = Z_\alpha e^{-|t|^\alpha}dt,\ t \in \mathbb{R}$ where Z_α is a normalization constant. Using Example 2.2 and Proposition 3.6 one obtains that μ_α satisfies a **GBI** with $T(x) = C\,x^{\frac{2\alpha-2}{\alpha}}$ for $x \in (0,1)$. Then one can find $C(\alpha), C'(\alpha) > 0$ such that for all bounded density of probability f,

$$\mathbf{Ent}_\mu\big(f^2\big) \le C(\alpha)\,\Big(1 + \log^{(2/\alpha)-1}(\parallel f \parallel_\infty)\Big) \int |\nabla f|^2 d\mu\,.$$

As a consequence, for all $t \geqslant 0$,

$$\mathbf{Ent}_\mu(\mathbf{P_t}h) \le e^{-t\,/\,C'(\alpha)\,\left(1+\log^{(2/\alpha)-1}(\|h\|_\infty)\right)}\,\mathbf{Ent}_\mu(h)\,,$$

for any bounded density of probability h.

It seems very unlikely that one can derive such a result from a direct use of Proposition 4.1. As noticed in [CG05], these restricted logarithmic Sobolev inequalities (restricted to the ($\mathbf{P_t}$ stable) $\mathbb{L}^\infty$

balls) can be used to obtain modified (or restricted) transportation inequalities. We recall below a result taken from section 4.2 in [CG05]. If $\nu = h\mu$ is a probability measure, it can be shown

$$W_2^2(\nu,\mu) \;\leq\; \eta(0)\,\mathbf{Ent}_\mu(h) \,+\, \int_0^{+\infty} \eta'(t)\,\mathbf{Ent}_\mu(\mathbf{P_t}h)\;dt\,, \tag{26}$$

where η is a non-decreasing positive function such that $\int(1/\eta(t))dt = 1$, and W_2 denotes the (quadratic) Wasserstein distance between ν and μ. We may take here

$$\eta(t) = 2\,A(C_P, \|\, h\,\|_\infty^{\frac{1}{2}})\, e^{\frac{1}{2}\,t/A(C_P,\|h\|_\infty^{\frac{1}{2}})}$$

which yields

$$W_2(\nu,\mu) \;\leq\; D\,(1 + A^{\frac{1}{2}}(C_P, \|\, h\,\|_\infty^{\frac{1}{2}}))\,\sqrt{\mathbf{Ent}_\mu(h)}\,. \tag{27}$$

In the Latala-Oleszkiewicz situation, we recover, up to the constants, Theorem 1.11 in [CG05].
Using Marton's trick, (27) allows us to obtain a concentration result (a little bit less explicit than the one obtained via **GBI** in Proposition 29 of [BCR05b]) namely there exist r_0 and σ such that if $\mu(A) \geqslant 1/2$ and $A_r^c = \{x, d(x,A) \geqslant r\}$ one has

$$r - r_0 \;\leq\; \sigma\, A^{\frac{1}{2}}(C_P, (1/\mu^{1/2}(A_r^c)))\,\sqrt{\log(1/\mu(A_r^c))}\,.$$

In the Latala-Oleszkiewicz situation, we recover up to the constants, the same concentration function as μ_α, showing that our restricted logarithmic Sobolev inequality is (up to the constants) optimal. Note that another way to get the concentration result is to use the modified logarithmic Sobolev (and transportation) inequalities discussed in [GGM05a, GGM05b].

Let us finally note that even if the results obtained by the WLSI are always efficient in the regime between Poincaré and Gross inequality, it relies on the crucial assumption that h is a smooth bounded density. The goal of the next section is to get rid of these two assumptions.

5 Convergence to equilibrium for diffusion processes

In this section we shall discuss the rate of convergence to equilibrium for the diffusion process, both in total variation and in entropy. The main difference between the previous section is that we do not assume that the beginning of the diffusion processes is a density of probability with respect to symmetric measure μ. The initial entropy is not finite.
To clarify our statement in the introduction, we shall first define the diffusion process and the hypotheses we need. For simplicity we only consider the case when $M = \mathbb{R}^n$ and $\mu = e^{-2V}dx$. Hence our diffusion process is given by the stochastic differential equation

$$dX_t \,=\, dB_t \,-\, (\nabla V)(X_t)dt \quad , \quad Law(X_0) = \nu \tag{28}$$

where $B_.$ is a standard Brownian motion. We assume that V is C^3 and that there exists some ψ such that $\psi(x) \to +\infty$ as $|x| \to +\infty$ and $\frac{1}{2}\,\Delta\,\psi \,-\, \nabla V.\nabla\psi$ is bounded from above. This assumption ensures the existence of an unique non explosive strong solution for (28). If $\nu = \delta_x$ we will denote by X_t^x the associated process.
A remarkable consequence of Girsanov theory is that with our assumptions, for all ν and all $t > 0$ the law of X_t denoted by $\mathbf{P_t}\nu$ is absolutely continuous with respect to μ, its density will be denoted by h_t. Of course if $\nu = h\mu$, $\mathbf{P_t}\nu = (\mathbf{P_t}h)\mu$ and μ is a reversible measure.
In particular $\mathbf{P_t}\nu = (\mathbf{P}_{t-u}h_u)\mu$, (where $h_u = \mathbf{P}_u h$)and the rate of convergence of $\mathbf{P_t}\nu$ towards μ can be studied by using the semigroup properties only. But of course what is needed is the behaviour of $\mathbf{P_t}h$ for densities of probability (in a sense it is $\mathbf{P_t}f^2$ rather than $\mathbf{P_t}f$ which is interesting).

Of particular interest is the case when

$$|\nabla V|^2(x) \ - \ \Delta V(x) \ \geqslant \ -C_{min} \ > -\infty \tag{29}$$

for a nonnegative C_{min} since in this case one can show (see [Roy99, Theorem 3.2.7]) that $\mathbf{Ent}_\mu(\mathbf{P_t}\delta_x)$ is finite for all $t > 0$. Actually the proof of Royer can be used in order to get the following more general and precise result

Proposition 5.1 *With the previous hypotheses*

$$\int \mathbf{P_t}\delta_x \, \log_+^p(\mathbf{P_t}\delta_x)\, d\mu \ \leq$$
$$4^{p-1}\left(V_+^p(x) + \left(\frac{C_{min}t}{2}\right)^p + \left(\frac{n}{2}\,\log(\frac{1}{2\pi t})\right)^p + e^{V(x)+p(\log p-1)+\frac{1}{2}C_{min}t}\right) \tag{30}$$

for all $t \in]0, 1/2\pi[$ *and* $p \geqslant 1$.
If in addition

$$V_+(y) \leq D(V_+(x) + |y-x|^2 + D') \tag{31}$$

for some $D > 0$, D' *and all pair* (x, y), *then for all* $t \in]0, 1/2D \wedge 1/2\pi[$

$$\int \mathbf{P_t}\delta_x \, \log_+^p(\mathbf{P_t}\delta_x)\, d\mu \ \leq \ 4^{p-1}\left((1+D^p)\,(V_+(x)+D')^p + \left(\frac{C_{min}t}{2}\right)^p + \left(\frac{n}{2}\,\log(\frac{1}{2\pi t})\right)^p\right) . \tag{32}$$

In particular, if $\int e^{V_+}d\nu := M < +\infty$,

$$\left(\int \mathbf{P_t}\nu \, \log_+^p(\mathbf{P_t}\nu)\, d\mu\right)^{\frac{1}{p}} \ \leq \ p\, C(\nu, t_0) \tag{33}$$

for all $t \geqslant t_0 > 0$, *where* $C(\nu, t_0)$ *only depends on* t_0, M, C_{min} *and the dimension. If in addition* (31) *holds, it is enough to assume that* $\int e^{\lambda V_+}d\nu := M < +\infty$ *for some* $\lambda > 0$.

Proof
$\lhd$ Let

$$F = \exp\left(V(x) - V(W_t) - \frac{1}{2}\int_0^t \left(|\nabla V|^2 - \Delta V\right)(W_s)ds\right) ,$$

where W is a Brownian motion starting from x. Recall that F is a density of probability (with our hypotheses). If $I(t) = \int \mathbf{P_t}\delta_x \, \log_+^p(\mathbf{P_t}\delta_x)\, d\mu$ we may use the argument in [Roy99, Theorem 3.2.7] and the convexity of $u \mapsto u^p$ in order to get

$$I(t) \leq \mathbb{E}\left(F\, 4^{p-1}\left(V_+^p(x) + (V(W_t) - \frac{1}{2t}|W_t - x|^2)_+^p + (C_{min}t/2)^p + \log(\frac{1}{2\pi t})\right)^p\right) .$$

The first statement follows easily bounding $(V(W_t) - \frac{1}{2t}|W_t - x|^2)_+$ by $D(V(W_t) + D')_+$ and $u^p e^{-u}$ by $p^p\, e^{-p}$. The second one is immediate since (31) allows us to bound the same term by $V_+(x)$ for t small enough.
The last statements are obtained by using two arguments. First $u^p \leq p!\, e^u$ (or $u^p \leq p!(1/\lambda)^p e^{\lambda p}$), so that for a given t the result follows from $(p!)^{\frac{1}{p}} \leq cp$. The second one is standard, namely $t \mapsto \int \mathbf{P_t}h \, \log_+^p \mathbf{P_t}h\, d\mu$ is non-increasing. $\rhd$

It is important to notice that, a contrario, there is no tractable general sufficient condition for $\mathbf{P_t}\delta_x$ to belong to $\mathbb{L}^2(\mu)$. We shall come back to the condition (31) later on. Note however that such a condition is trivially verified for $V(x) = |x|^\gamma$, $\gamma \leq 2$.

Accordingly the logarithmic Sobolev inequality is particularly well suited for studying the convergence of $\mathbf{P_t}\delta_x$ towards μ in entropy. We shall see that simple manipulations allow us to obtain similar results with **WLSI** only and we shall also compare the role of **WLSI**, **WPI** and Poincaré inequality for the (weaker) convergence in total variation.

5.1 Rate of convergence

Theorem 5.2 *Let* $d\mu = e^{-2V}dx$ *be a probability measure which satisfies a* **WLSI** *with function* β_{WL} *and let* ξ *be defined as in* (24) *of Proposition 4.1. Assume that* (29) *holds and let* ν *be a probability measure such that* (33) *holds.*
Then for all $1 \geqslant \varepsilon > 0$ *and all* $k \in \mathbb{N}$, *there exist a constant* $C(\varepsilon, k)$ *depending (in addition) on* M, C_{min} *and the dimension only, and* $t_\varepsilon > 0$ *such that*

$$\mathbf{Ent}_\mu(\mathbf{P}_{kt}\nu) \leq \frac{C(\varepsilon, k)}{\log^{k(1-\varepsilon)}(1/\xi(t))},$$

for all $t > t_\varepsilon$.

Before proving the theorem we need a preliminary result. Recall first that for all non-negative functions f, g we have $\mathbf{Ent}_\mu(f+g) \leq \mathbf{Ent}_\mu(f) + \mathbf{Ent}_\mu(g)$. Then for $h \geqslant 0$, applying this with $f = \mathbf{P_t}(h\mathbb{1}_{h\leq K})$ and $g = \mathbf{P_t}(h\mathbb{1}_{h>K})$, and using the fact that entropy is decaying along the semi-group, we obtain that

$$\mathbf{Ent}_\mu(\mathbf{P_t}h) \;\leq\; \mathbf{Ent}_\mu(\mathbf{P_t}(h\mathbb{1}_{h\leq K})) + \mathbf{Ent}_\mu(h\mathbb{1}_{h>K}) \;, \tag{34}$$

for all $K > 0$. The next Lemma explains how control the second term of the right hand side of (34) using the estimate of the Proposition 5.1.

Lemma 5.3 *Let* h *be a density of probability with respect to* μ. *Assume that there exists* $c > 0$ *such that for all* $p > 1$,

$$\left(\int h \log_+^p h \, d\mu\right)^{\frac{1}{p}} \leq cp.$$

For $K \geqslant e^2$, *if* $\mathbf{Ent}_\mu(h) \leq \frac{1}{2e}\log K$ *then we get*

$$\mathbf{Ent}_\mu(h\mathbb{1}_{h>K}) \;\leq\; (ec+2)\,\frac{\mathbf{Ent}_\mu(h)}{\log K}\,\log\left(\frac{\log K}{\mathbf{Ent}_\mu(h)}\right). \tag{35}$$

Proof
$\triangleleft$ It is easily seen (see e.g. [CG05, Lemma 3.4]) that if $K \geqslant e^2$,

$$\int \mathbb{1}_{h>K} h d\mu \leq \frac{2}{\log K}\,\mathbf{Ent}_\mu(h) \;. \tag{36}$$

Hence

$$\begin{aligned}\int h\log h\,\mathbb{1}_{h>K}d\mu \;&\leq\; \left(\int h\,\mathbb{1}_{h>K}d\mu\right)^{\frac{p-1}{p}}\left(\int h\log_+^p(h)\,d\mu\right)^{\frac{1}{p}} \\ &\leq\; c\,p\left(\frac{\mathbf{Ent}_\mu(h)}{\log K}\right)^{\frac{p-1}{p}} \leq ce\,\frac{\mathbf{Ent}_\mu(h)}{\log K}\,\log\left(\frac{\log K}{\mathbf{Ent}_\mu(h)}\right)\end{aligned} \tag{37}$$

provided $\mathbf{Ent}_\mu(h) \leq \frac{1}{e}\log K$. The last inequality is obtained by an optimization upon p (for which we need $\mathbf{Ent}_\mu(h) \leq \frac{1}{e}\log K$).
If $\mathbf{Ent}_\mu(h) \leq \frac{1}{2e}\log K$,

$$-\left(\int \mathbb{1}_{h>K}hd\mu\right)\log\left(\int \mathbb{1}_{h>K}hd\mu\right) \leq -\left(\frac{2}{\log K}\,\mathbf{Ent}_\mu(h)\right)\log\left(\frac{2}{\log K}\,\mathbf{Ent}_\mu(h)\right),$$

using (36), so that we have finished the proof. $\triangleright$

Proof of Theorem 5.2

◁ Let $h = \mathbf{P_s}\nu$. According to (34), Proposition 4.1 and Lemma 5.3, it holds for all $t > s > 0$,

$$\mathbf{Ent}_\mu(\mathbf{P_t}\nu) \;\leq\; K\xi(t-s) \,+\, c_s\,\frac{H}{\log K}\,\log\left(\frac{\log K}{H}\right),$$

where $H = \mathbf{Ent}_\mu(h)$, provided K is large enough. Since H can be bounded from above by a quantity H_0 depending on M, C_{min} and the dimension only, we may choose $K > K_1$ independent of H. Choosing $K = c\,\frac{H_0}{\xi(t-s)}\,\frac{1}{1+\log_+\left(\frac{H}{\xi(t-s)}\right)}$, we obtain

$$\mathbf{Ent}_\mu(\mathbf{P_t}\nu) \leq C\,\frac{1+\log_+\left(\log_+(1/\xi(t-s))\right)}{1+\log_+(1/\xi(t-s))}\,. \tag{38}$$

It follows that, for all $1 \geqslant \varepsilon > 0$ there exists some t_ε such that for $t \geqslant t_\varepsilon$

$$\mathbf{Ent}_\mu(\mathbf{P_t}\nu) \leq \frac{C}{\log^{1-\varepsilon}(1/\xi(t))}\,. \tag{39}$$

Using again (34) and (35) (we may choose $c = c_s$ for all $t \geqslant s$) we may write

$$\begin{aligned}\mathbf{Ent}_\mu(\mathbf{P}_{2t}\nu) \;&\leq\; K\xi(t) \,+\, c\,\frac{\mathbf{Ent}_\mu(\mathbf{P_t}\nu)}{\log K}\,\log\left(\frac{\log K}{\mathbf{Ent}_\mu(\mathbf{P_t}\nu)}\right)\\ &\leq\; K\xi(t) \,+\, \frac{cc'}{\log K\,\log^{1-2\varepsilon}(1/\xi(t))} \,+\, \frac{c\,\log\log_+ K}{\log K\,\log^{1-\varepsilon}(1/\xi(t))}\end{aligned}$$

where we have used $y\log(1/y) \leq c'y^{1-\varepsilon}$ for $y \leq 1/e$. Hence choosing $K = 1/\xi(t)\log^2(1/\xi(t))$ we obtain a bound like

$$\mathbf{Ent}_\mu(\mathbf{P}_{2t}\nu) \leq \frac{C}{\log^{2-2\varepsilon}(1/\xi(t))}\,,$$

for t large enough. Note that C depends on ε. We may iterate the method and get the result. ▷

Of course this result is not totally satisfactory, but it indicates that the decay of entropy is faster than any $1/\log^{k(1-\varepsilon)}(1/\xi(t/k))$.

Let us study the two classical examples we already mentioned. To be rigorous $|t| := \sqrt{1+t^2}$ in what follows (to ensure the required regularity), so that (29) is satisfied.

- For $\alpha > 0$, the measure $dm_\alpha(t) = Z_\alpha(1+|t|)^{-1-\alpha}dt$, $t \in \mathbb{R}$ satisfies the weak logarithmic Sobolev inequality with

  $$\forall s \in (0,1), \qquad \beta_{WL}(s) = C\frac{(\log 1/s)^{1+2/\alpha}}{s^{2/\alpha}},$$

  for some constant $C > 0$. Hence,

  $$\xi(t) = \frac{c_\alpha}{t^{\alpha/2}\,\log^{1+\alpha}(t)}$$

  for large t, and

  $$\mathbf{Ent}_{m_\alpha}(\mathbf{P}_{kt}\nu) \leq \frac{C_{\alpha,k,\varepsilon}}{\log^{k(1-\varepsilon)}(t)}\,.$$

  Notice that, if roughly the rate of decay does not depend on α (it is faster than any $\log^k(t)$), the dependence on α of all constants shows that this regime is attained for smaller t when α increases.

- For $\alpha \in (0,2)$, the measure $d\mu_\alpha(t) = Z_\alpha e^{-|t|^\alpha} dt$, $t \in \mathbb{R}$, (Z_α is a normalization constant) satisfies the weak logarithmic Sobolev inequality with $\beta_{WL}(s) = C(\log 1/s)^{(2-\alpha)/\alpha}$, $C > 0$. Hence $\xi(t) = c\, e^{-dt^{\alpha/2}}$ and for t large enough,

$$\mathbf{Ent}_{\mu_\alpha}(\mathbf{P}_{kt}\nu) \leq \frac{C_{\alpha,k}}{1 + t^{(\alpha/2)(k-\varepsilon)}} .$$

  Of course this result is not satisfactory for $\alpha \geqslant 2$ where we know that the decay is exponential. See below for an improvement.

If we replace Proposition 4.1 or Proposition 4.2 we can greatly improve the previous results. Let us describe the latter situation.

Theorem 5.4 *In the situation of Example 4 (i.e. the Latala-Oleszkiewicz situation) and Theorem 5.2, there exists $s > 0$ such that for all $1 \geqslant \varepsilon > 0$ one can find T_ε in such a way that for $t \geqslant T_\varepsilon$,*

$$\mathbf{Ent}_\mu(\mathbf{P}_{t+s}\nu) \leq e^{1-t^{\frac{(1-\varepsilon)\alpha}{2-\varepsilon\alpha}}} .$$

In particular for $\alpha = 2$ relative entropy is exponentially decaying.

Proof
$\lhd$ The beginning of the proof is similar to the one of Theorem 5.2 but replacing the estimate of Proposition 4.1 by the one of Example 4 (in particular we may take $K = +\infty$ if $\alpha = 2$). The first step yields

$$H_t := \mathbf{Ent}_\mu(\mathbf{P}_{t+s}\nu) \leq \frac{C(1 + \log_+^{\frac{\alpha}{2-\alpha}}(t))}{1 + t^{\frac{\alpha}{2-\alpha}}} H \log(1/H) .$$

Let us choose s in such a way that $H \leq 1/e$, i.e. $H \log(1/H) \leq 1$. Then

$$H_{2t} \leq \frac{C(1 + \log_+^{\frac{\alpha}{2-\alpha}}(t))}{1 + t^{\frac{\alpha}{2-\alpha}}} H_t \log(1/H_t) \leq \left(\frac{C(1 + \log_+^{\frac{\alpha}{2-\alpha}}(t))}{1 + t^{\frac{\alpha}{2-\alpha}}}\right)^2 \log(1 + t^{\frac{\alpha}{2-\alpha}}) ,$$

provided $C \geqslant 1$ that we can assume. Iterating the procedure we get

$$\begin{aligned} H_{kt} &\leq \left(\frac{C(1 + \log_+^{\frac{\alpha}{2-\alpha}}(t))}{1 + t^{\frac{\alpha}{2-\alpha}}}\right)^k \prod_{j=1}^{k-1} \log\left((1 + t^{\frac{\alpha}{2-\alpha}})^j\right) \\ &\leq \left(\frac{C\,(1 + \log_+^{\frac{\alpha}{2-\alpha}}(t))\, \log(1 + t^{\frac{\alpha}{2-\alpha}})}{1 + t^{\frac{\alpha}{2-\alpha}}}\right)^k \frac{(k-1)!}{\log(1 + t^{\frac{\alpha}{2-\alpha}})} \end{aligned}$$

Now, we may find t_ε such that for $t \geqslant t_\varepsilon$,

$$\frac{C\,(1 + \log_+^{\frac{\alpha}{2-\alpha}}(t))\, \log(1 + t^{\frac{\alpha}{2-\alpha}})}{1 + t^{\frac{\alpha}{2-\alpha}}} \leq \frac{1}{t^{\frac{\alpha}{2-\alpha}(1-\varepsilon)}} ,$$

and $\log(1 + t^{\frac{\alpha}{2-\alpha}}) \geqslant 1$, so that

$$H_{kt} \leq \left(\frac{k}{e\, t^{\frac{\alpha}{2-\alpha}(1-\varepsilon)}}\right)^k$$

as soon as k is large enough (for $(k-1)! \leq (k/e)^k$). Choosing $t = k^{(2-\alpha)/\alpha(1-\varepsilon)}$ (hence k large enough for t to be greater than t_ε) we obtain that $H_u \leq e^{-k}$ for $u = k^{\frac{2-\varepsilon\alpha}{(1-\varepsilon)\alpha}}$, i.e. $H_t \leq e\, e^{-t^{\frac{(1-\varepsilon)\alpha}{2-\varepsilon\alpha}}}$.

$\triangleright$

Of course the statement of the Theorem is not sharp (we have bounded some logarithm by some power) but it is tractable and shows that (up to some ε) the decay is similar to ξ. Of course we are able to derive a similar (but not very explicit) result with the general bound (A) in Proposition 4.2.

It is interesting to see what can be done by using the usual Poincaré inequality. Indeed recall that $\mathbf{Ent}_\mu(g) \leq \mathbf{Var}_\mu(g)/\int g d\mu$ for a nonnegative g. Using this with $g = \mathbf{P_t}(h\mathbb{1}_{h\leq K})$, using also (34) and Poincaré yield a decay

$$\mathbf{Ent}_\mu(\mathbf{P_t}\nu) \leq C\frac{1+\log_+(t)}{1+t}$$

that is a slightly better result than the one we may obtain at the first step of the previous method (up to a $\log_+(t)$ factor) in this situation (corresponding to $\alpha = 1$). But iterating the procedure also yields a polynomial decay. Nevertheless if $\mathbf{P}_s\nu \in \mathbb{L}^2(\mu)$ for some s, we obtain an exponential decay. It is thus particularly interesting to study stronger integrability condition. This will be done on an example in the next subsection.

To finish this section we shall now discuss the weaker convergence in total variation distance. Denoting again $h = \mathbf{P_s}\nu$, we thus have for $K > 0$

$$\begin{aligned}\int |\mathbf{P_t}h - 1|d\mu &\leq \int |\mathbf{P_t}(h\wedge K) - \mathbf{P_t}h|d\mu + \int |\mathbf{P_t}(h\wedge K) - \int (h\wedge K)d\mu|d\mu + |\int (h\wedge K)d\mu - 1| \\ &\leq \int |\mathbf{P_t}(h\wedge K) - \int (h\wedge K)d\mu|d\mu + 2\int (h-K)\mathbb{1}_{h\geqslant K}d\mu \qquad (40)\end{aligned}$$

where we have used the fact that $\mathbf{P_t}$ is a contraction in L^1. The second term in the right hand sum is going to 0 when K goes to $+\infty$, while the first term can be controlled either by $\sqrt{\mathbf{Var}_\mu(\mathbf{P_t}(h\wedge K))}$ or by $\sqrt{2(\int (h\wedge K)d\mu)\,\mathbf{Ent}_\mu(\mathbf{P_t}(h\wedge K))}$ according respectively to Cauchy-Schwarz and to Pinsker inequality. In both cases, **WPI** or **WLSI** inequalities imply that $\mathbf{P_t}\nu$ goes to μ in total variation distance, for all initial ν.

If we want a rate of convergence, we immediately see that **WPI** will furnish a better rate than **WLSI** for the μ that do not satisfy Poincaré inequality. If μ satisfies a Poincaré inequality with constant C_P then

$$\mathbf{Var}_\mu(\mathbf{P_t}(h\wedge K)) \leq Ke^{-t/C_P}\,,$$

so that the optimal K is given (up to a factor 2) by $2\int (h-K)\mathbb{1}_{h\geqslant K}d\mu = K^{\frac{1}{2}}e^{-t/2C_P}$. In particular if (29) holds,

$$2\int (h-K)\mathbb{1}_{h\geqslant K}d\mu \leq \frac{2C(p)}{\log^p(K)}$$

for $K > 1$ and $p \geqslant 1$, so that we obtain $\parallel \mathbf{P}_{t+s}\nu - \mu \parallel_{TV} \leq \kappa(p)/t^p$ for all $s > 0$, $p \geqslant 1$, where κ depends on s, C_{min}, p, M and the dimension. But if we directly use Theorem 5.4 and Pinsker we have the much better $\parallel \mathbf{P}_{t+s}\nu - \mu \parallel_{TV} \leq \kappa\, e^{-\frac{1}{2}t^{\frac{(1-\varepsilon)\alpha}{2-\varepsilon\alpha}}}$ at least for s large enough. In particular for $\alpha = 1$ we obtain a faster decay. Once again, if $\|\mathbf{P}_s\nu\|_\infty$ is finite for some positive s then one should use the entropic convergence of Proposition 4.2 to get an exponential decay.

5.2 Example(s)

In the previous subsection, we have seen that finite entropy conditions are quite natural for the law of the diffusion at any positive time. If there is no general result ensuring a better integrability condition, it is however interesting to get such conditions on examples. Before to study such examples, we shall give a generic example showing that some natural measures ν never satisfy $\mathbf{P_s}\nu \in \mathbb{L}^2(\mu)$, but satisfy the conditions in Proposition 5.1.

Consider V such that for all $\lambda > 0$, $\int e^{-\lambda V} dx < +\infty$. Let $d\mu = e^{-2V} dx$ and $d\nu = e^{-(2-\varepsilon)V}/Z_\varepsilon \, dx$ so that $d\nu/d\mu := h = Z_\varepsilon \, e^{\varepsilon V} \notin \mathbb{L}^2(\mu)$ for $2 > \varepsilon > 1$, but $\int e^{\frac{2-\varepsilon}{2} V} d\nu < +\infty$. Set $G = e^V = h^{\frac{1}{\varepsilon}}$.
If $\mathbf{P_s} h \in \mathbb{L}^2(\mu)$ for some $s > 0$, then $\mathbf{P_s} G \in \mathbb{L}^{2\varepsilon}(\mu)$. If (29) holds, it follows from [Cat05, Theorem 2.8] that μ satisfies a logarithmic Sobolev inequality. Thus if it is not the case, $\mathbf{P_s} h \notin \mathbb{L}^2(\mu)$ for all $s \geqslant 0$, while if (31) is satisfied (for instance for $V(y) = |y|^\alpha$, $1 \leq \alpha < 2$ see below) ν satisfies the conditions in Proposition 5.1.
This example shows that the set of initial measures satisfying the conditions in the previous subsection but not the necessary conditions to simply apply Poincaré is non empty.

We shall go further, and for simplicity we shall only consider the measures μ_α for $\alpha \geqslant 1$, and essentially discuss the case $\alpha = 1$.
First of all notice that if $1 \leq \alpha \leq 2$,

$$|y|^\alpha \leq 2^{\alpha-1}(|x|^\alpha + |y-x|^2 + 1)$$

so that (31) is satisfied. Hence as soon as $\int e^{\lambda |x|^\alpha} \nu(dx) < +\infty$ for some $\lambda > 0$, we may apply all the results of the previous subsection. We shall now give a precise description of $h = \mathbf{P_s} \delta_x$. This will allow us to give a similar sufficient condition for $\mathbf{P_s} \nu$ to belong to $\mathbb{L}^2(\mu)$.

We thus consider (in one dimension)

$$dX_t \; = \; dB_t \; - \; \text{sign}(X_t) dt \quad , \quad X_0 = x \, , \tag{41}$$

corresponding to $\alpha = 1$. Elementary stochastic calculus (inspired by the first sections of [GHR01]) furnishes

$$\begin{aligned} \mathbb{E}[f(X_t)] \; &= \; \mathbb{E}\left[f(x+B_t) \, e^{-\frac{t}{2}} \, \exp\left(- \int_0^t \text{sign}(x+B_s) dB_s \right) \right] \\ &= \; e^{|x|} \, e^{-\frac{t}{2}} \, \mathbb{E}\left[f(x-W_t) \, \exp\left(-|W_t - x| + L_t^x \right) \right] \end{aligned}$$

where $W_s = -B_s$ is a new Brownian motion with local time at x denoted by L_s^x. Now as usual we introduce the hitting time of x of (W_s) denoted by T_x, and the supremum $S_t = \sup_{0 \leq s \leq t} W_s$. We also assume here that $x > 0$. Then

$$\begin{aligned} \mathbb{E}[f(X_t)] \; &= \; \mathbb{E}[f(X_t) \, \mathbb{I}_{t \leq T_x}] + \mathbb{E}[f(X_t) \, \mathbb{I}_{t > T_x}] \\ &= \; e^{|x|} \, e^{-\frac{t}{2}} \, \mathbb{E}[f(x - W_t) \, \mathbb{I}_{S_t \leq x} \, e^{W_t - x}] \; + \; e^{-\frac{t}{2}} \, \mathbb{E}[\mathbb{I}_{S_t > x} \mathbb{E}[f(B'_{t-T_x}) \, \exp\left(-|B'_{t-T_x}| + L'_{t-T_x} \right)]] \end{aligned}$$

where B' is a Brownian motion independent of W and L' its local time at 0.
For the first term, we know that the joint law of (W_t, S_t) is given by the density

$$(w, s) \mapsto \mathbb{I}_{w \leq s} \sqrt{2/\pi t^3} \, (2s - w) \, \exp(-(2s-w)^2/2t)$$

so that (recall $x > 0$)

$$\mathbb{E}[f(X_t) \, \mathbb{I}_{t \leq T_x}] = \int f(u) \left(\mathbb{I}_{u \geqslant 0} \sqrt{2/\pi t} \, e^{-\frac{t}{2}} \, e^x \, e^{-u} \, \left(e^{-(x-u)^2/2t} - e^{-(x+u)^2/2t} \right) \right) du \, .$$

For the second term, we know that the law of T_x is given by the density

$$T \mapsto x \sqrt{1/2\pi T^3} \, e^{-x^2/2T}$$

and that $(|B'_s|, L'_s)$ has the same law as $(S'_s - B'_s, S'_s)$ so that (noting that only the even part of f has to be considered)

$$\mathbb{E}[f(X_t) \, \mathbb{I}_{t > T_x}] = e^{-\frac{t}{2}} \iiint \mathbb{I}_{0 < T < t} \mathbb{I}_{u > 0} \mathbb{I}_{v > u} \left(\frac{f(u) + f(-u)}{2} \right) g(T, u, v) \, du dv dT \, ,$$

with
$$g(T,u,v) = \sqrt{1/2\pi T^3}\,\sqrt{2/\pi(t-T)^3}\,v\,e^v\,e^{-2u}\,e^{-v^2/2(t-T)}\,e^{-x^2/2T}\,.$$
But
$$Q := \int_0^t \int_u^{+\infty} \sqrt{1/2\pi T^3}\,\sqrt{2/\pi(t-T)^3}\,v\,e^v\,e^{-v^2/2(t-T)}\,e^{-x^2/2T}\,dvdT$$
is such that
$$\begin{aligned} Q &\leq \int_0^t \sqrt{1/2\pi T^3}\,\left(\sqrt{2/\pi(t-T)}\,e^u\,e^{-u^2/2(t-T)} + 2e^{t-T}\right)\,e^{-x^2/2T}\,dT \\ &\leq \int_0^t \sqrt{1/2\pi T^3}\,\left(\sqrt{2/\pi(t-T)}\,e^{t/2} + 2e^{t-T}\right)\,e^{-x^2/2T}\,dT \\ &\leq C(t) \end{aligned}$$
independently of x. The first inequality is obtained by performing an integration by parts in v, the second one by bounding $e^u\,e^{-u^2/2(t-T)}$ and the final one by bounding separately $\int_0^{t/2}$ and $\int_{t/2}^t$. We thus see that
$$\mathbb{E}[f(X_t)\,\mathbf{1}_{t>T_x}] = C'(t)\int f(u)\,e^{-2|u|}\,g(u)\,du$$
where g is bounded.
Putting all this together we have obtained the following
$$(\mathbf{P_t}\delta_x)(u) = c(t)\left(\mathbf{1}_{u\geqslant 0}\,e^x\,e^u\,\left(e^{-(x-u)^2/2t} - e^{-(x+u)^2/2t}\right)\right) + C'(t)g(u) \tag{42}$$
for all $x > 0$. A similar result holds for $x < 0$, while $\mathbf{P_t}\delta_0$ is bounded. Of course the previous (42) shows that for a fixed x, $\mathbf{P_t}\delta_x$ is bounded. This result is not so surprising. Indeed for $\alpha = 2$ (more precisely for the normalized gaussian measure i.e. the Ornstein-Uhlenbeck process) $(\mathbf{P_t}\delta_x)(u) = c(t)\,e^{(1-e^{-t})x^2/2(1-e^{-t})}\,e^{-(e^{-t/2}u-x)^2/2(1-e^{-t})}$ is bounded too. One may adapt our proof and Proposition 4 in [GHR01] in order to show that a similar result actually holds for all $1 \leq \alpha \leq 2$. But (42) allows us to look at more general $\mathbf{P_t}\nu$. In particular we see that $\mathbf{P_t}\nu \in \mathbb{L}^2(\mu)$ if and only if
$$\int_{u>0}\left(\int_{x>0} e^x\,e^{-(u-x)^2/2t}\,\nu(dx)\right)^2 du\ <\ +\infty \tag{43}$$
and a similar property is available on the negative real numbers. We then easily recover and complete the discussion at the beginning of this subsection, i.e. if $d\nu = e^{-\lambda|x|}dx/Z$, $\mathbf{P_t}\nu \notin \mathbb{L}^2(\mu)$ if $\lambda \leq 1$, but belongs to $\mathbb{L}^2(\mu)$ if $\lambda > 1$.
Let us finally give some discussion concerning the obtainable rate of entropic convergence depending on the initial measure:

i. if $\nu = \delta_x$, then $\|\mathbf{P}_{t_0}\delta_x\|_\infty < \infty$ and using respectively Proposition 4.1, Proposition 4.2 or Poincaré inequality, one gets
$$\mathbf{Ent}_\mu(\mathbf{P}_{t+t_0}\nu) \leq C\min\left(e^{-a\sqrt{t}}\|\mathbf{P}_{t_0}\delta_x\|_\infty, e^{-bt/(1+\log\|\mathbf{P}_{t_0}\delta_x\|_\infty)}, e^{-ct}\|\mathbf{P}_{t_0}\delta_x\|_\infty\right),$$
(note that it easily extends to the case where ν has compact support.)

ii. if ν does not satisfy (43) but for some positive λ, $\int e^{\lambda|x|}d\nu$ is finite then we can only use Theorem 5.4 to get that for all $\varepsilon > 0$, there exists T_ε such that for all $t \geq T_\varepsilon$ we have
$$\mathbf{Ent}_\mu(\mathbf{P}_t\nu) \leq e^{1-t^{\frac{1-\varepsilon}{2-\varepsilon}}}.$$

6 Classical properties of WLSI

6.1 Tensorization

Let us begin by the following naive procedure of tensorization.

Proposition 6.1 *Assume that for every $f : M \to \mathbb{R}$ and every $s > 0$ one has*

$$\mathbf{Ent}_\mu(f) \le \beta(s) \int |\nabla f|^2 d\mu + s\,\mathbf{Osc}^2(f).$$

Let $n \ge 1$, then the measure μ^n satisfies a **WLSI** *with function $\beta\left(\frac{s}{n}\right)$, for $s > 0$.*

Proof
$\lhd$ By the sub-additivity property of the entropy we get

$$\mathbf{Ent}_{\mu^n}(f) \le \sum_{i=1}^n \int \mathbf{Ent}_\mu(f(x_1,\dots,x_{i-1},\cdot,x_{i+1},\dots,x_n)) \prod_{j\ne i} d\mu(x_j).$$

For each i we get for all $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \in M^{n-1}$

$$\mathbf{Ent}_\mu(f(x_1,\dots,x_{i-1},\cdot,x_{i+1},\dots,x_n)) \le \beta(s) \int |\nabla_i f|^2(x_1,\dots,y_i,\dots,x_n) d\mu(y_i) + s\,\mathbf{Osc}(f(x_1,\dots,\cdot,\dots,x_n))^2,$$

It yields $\forall s > 0$, $\qquad \mathbf{Ent}_{\mu^n}(f) \le \beta(s) \int |\nabla f|^2 d\mu^n + ns\,\mathbf{Osc}^2(f)$. $\rhd$

The tensorization result above is of course the same as the one in [BCR05a] for weak Poincaré inequality. As explained in Section 5 of this paper, one cannot expect a better result beyond the exponential case. However as we have already seen, **WLSI** may take place between the exponential and the gaussian regime (when **GBI** holds), so that we obtain this corollary:

Corollary 6.2 *If μ_i ($1 \le i \le n$) satisfy a* **WLSI** *with the same function β_{WL} satisfying the hypotheses in Proposition 3.6, then the tensor product $\otimes_{i=1}^n \mu_i$ satisfies a* **WLSI** *with function*

$$\beta_{WL}^n(u) = C\,\beta_{WL}(C'u)$$

where C, C' are constants which don't depend on n.

Proof
$\lhd$ It is enough to use both parts of Proposition 3.6 and the (exact) tensorization property of **GBI**. One can see [LO00] for the proof of the tensorization of **GBI**. $\rhd$

Among the most important consequences of functional inequalities, one find concentration of measure and isoperimetric profile. Unfortunately weak inequalities are not easily tractable to derive results in this direction (due to the Oscillation term). However results for **WPI** are contained in [RW01, BCR05a] with a particular interest in dimension dependence in the latter. Actually we do not succeed in deriving similar estimates starting from **WLSI**, as Herbst's argument or Aida-Masuda-Shigekawa iteration argument are more intricate and we can only recover weak Poincaré unoptimal concentration rate.
The situation is still worse (from the **WLSI** point of view) when a **SPI** holds. In this case various (more or less explicit) results have been obtained. Let us mention on one hand [Wan00] Section 6, [GW02] Section 5 (using super Poincaré) and [Wan05] Corollary 2.4 (using **GBI**), on the other hand [BCR05b] Section 6 (using **GBI**) and Section 8 (using F-Sobolev inequalities) and [BCR05c] Theorem 12 for an improvement of [Wan00] Section 6. The previous result may be used in conjunction with the above mentioned results to get dimension free concentration (or isoperimetric) results, completing thus the transportation approach presented before.

6.2 Perturbation

Among the methods used to obtain functional inequalities, an efficient one is to perturb measures satisfying themselves some functional inequalities. The most known result in this direction was first obtained by Holley and Stroock who showed that a logarithmic Sobolev inequality is stable under a log-bounded perturbation. The same is true for a **SPI** (using the related **GBI** [Wan05, Proposition 2.5]), and actually one can replace the bounded assumption by a Lipschitz assumption (this was shown by Miclo for logarithmic Sobolev, and by Wang [Wan05, Proposition 2.6] for a **SPI**).
For the **WPI**, a similar result is shown in [RW01, Theorem 6.1]. Actually this result shows that one can consider non bounded perturbation, but with very strong integrability assumptions, the final result being far to be explicit. For **WLSI** we may state

Proposition 6.3 *Suppose that μ satisfies a* **WLSI** *with function β_{WL}. Let $\nu_V = e^V \mu / Z_V$, where $Z_V = \int e^V d\mu$ and assume that V is bounded on M.*
Then ν_V satisfies a **WLSI** *with function*

$$\beta^V_{WL}(u) = e^{2\mathbf{Osc}(V)}\, \beta_{WL}(ue^{-\mathbf{Osc}(V)})\,.$$

We may replace **WLSI** *by* **WPI** *replacing β_{WL} by β_{WP}, or by* **SPI** *with*

$$\beta^V_{SP}(u) = e^{2\mathbf{Osc}(V)}\, \beta_{SP}(ue^{-2\mathbf{Osc}(V)}).$$

Proof
$\lhd$ Recall that $\mathbf{Ent}_{\nu_V}(f^2) \le e^{\mathbf{Osc}(V)}\, \mathbf{Ent}_\mu(f^2)$. Applying **WLSI** for μ yields

$$\begin{aligned}\mathbf{Ent}_{\nu_V}(f^2) &\le e^{\mathbf{Osc}(V)} \left(\beta_{WL}(s) \int |\nabla f|^2 d\mu + s\, \mathbf{Osc}^2(f)\right) \\ &\le e^{2\mathbf{Osc}(V)}\, \beta_{WL}\Big(ue^{-\mathbf{Osc}(V)}\Big) \int |\nabla f|^2 d\nu_V + u\, \mathbf{Osc}^2(f)\,,\end{aligned}$$

which is exactly the first statement. The second one is similar since $\mathbf{Var}_{\nu_V}(f) \le e^{\mathbf{Osc}(V)} \mathbf{Var}_\mu(f)$. For **SPI** the proof is immediate. $\rhd$

The second way to get perturbation results is to use a natural isometry between $\mathbb{L}^2$ spaces. For notational convenience we assume now that $\nu_V = e^{-2V}\mu$. Then $g \mapsto f := e^{-V} g$ is an isometry between $\mathbb{L}^2(\nu_V)$ and $\mathbb{L}^2(\mu)$. It is thus immediate that on one hand

$$\mathbf{Ent}_{\nu_V}(g^2) \;=\; \mathbf{Ent}_\mu(f^2) \;+\; 2\int g^2 V\, d\nu_V\,. \tag{44}$$

On the other hand, an integration by parts yields

$$\int |\nabla f|^2 d\mu = \int |\nabla g|^2 d\nu_V \;+\; \int g^2 \, \Big(2LV - |\nabla V|^2\Big)\, d\nu_V\,, \tag{45}$$

where L is the generator of P_t reversible for μ.
Combining these two facts, yields perturbation results for logarithmic Sobolev inequalities (the idea goes back to Rosen [Ros76], and was used in [Car91, Cat05]). In order to see how to use it in our framework, we shall first introduce some notation.

Definition 6.4 *Let G be a positive continuous function defined on $\mathbb{R}^+$. We shall say that a smooth V is Witten (G,μ)-good, if $V(x) \to +\infty$ as $|x| \to +\infty$ and if there exists $A \geqslant 0$ such that one has for any x such that $V(x) \geqslant A$,*

$$|\nabla V|^2(x) - 2LV(x) \geqslant G(V(x))\,.$$

Our first general result is a bounded (but not log-bounded) perturbation result.

Proposition 6.5 *Let μ be a positive measure (not a necessarily probability measure) satisfying a* **WLSI** *with continuous function β_{WL}. Let V be Witten (G,μ)-good, such that $\nu_V = e^{-2V}\mu$ is a probability measure.*
Then for all $u > 0$ and $b \geqslant A$ the following inequality holds for any $g \in \mathcal{C}_b^1(M)$,

$$\mathbf{Ent}_{\nu_V}\left(g^2\right) \le C(u,b) \int |\nabla g|^2 d\nu_V + D(u,b)\, \mathbf{Osc}^2(g)\,,$$

with

$$C(u,b) = h(b) + (2 + 2A + M(V)h(b))\, \beta_{WP}^V(u)\,, \tag{46}$$

$$D(u,b) = s_b\, e^{-2\inf V} + (2 + 2A + M(V)h(b)))\, u + \int_{\{V \geqslant b\}} 2V d\nu_V\,, \tag{47}$$

where $h(b) := \sup_{\{A \le z \le b\}} \frac{2z}{G(z)}$, $s_b := \inf\{s > 0\,,\, \beta_{WL}(s) \le h(b)\}$,

$$M(V) := \sup_{\{V \le A\}} (2LV - |\nabla V|^2),$$

(which is finite) and β_{WP}^V is the best function such that ν_V satisfies **WPI** *(if it does not take $\beta_{WP}^V(u) = +\infty$ for small u).*

Proof
$\lhd$ First according to Rothaus inequality, we may assume that $\int g d\nu_V = 0$ up to $2\mathbf{Var}_{\nu_V}(g)$. Applying **WLSI** in (44) and (45) we get for all $s > 0$,

$$\mathbf{Ent}_{\nu_V}\left(g^2\right) \le \beta_{WL}(s) \int |\nabla g|^2 d\nu_V + \int g^2 \left(\beta_{WL}(s)\left(2LV - |\nabla V|^2\right) + 2V\right) d\nu_V + s\mathbf{Osc}^2(ge^{-V})\,. \tag{48}$$

Note that if β_{WL} is bounded, we may replace it by any $\beta(s) \geqslant \beta_{WL}(0)$.

- On $\{V \le A\}$, the second integrand is bounded by $(\beta_{WL}(s)M(V) + 2A)\ \mathbf{Var}_{\nu_V}(g)$, and can be controlled (together with the term $2\mathbf{Var}_{\nu_V}(g)$ coming from Rothaus inequality) with the **WPI** for the measure ν_V.
- On $\{b \geqslant V \geqslant A\}$, we choose $s = s_b$ then the second integrand is non-positive.
- On $\{b \le V\}$, $2LV - |\nabla V|^2$ is still non-positive, so that the second integrand is bounded by

$$\int_{\{V \geqslant b\}} 2V g^2\, d\nu_V \le \left(\int_{\{V \geqslant b\}} 2V d\nu_V\right) \mathbf{Osc}^2(g)\,,$$

since $\int g d\nu_V = 0$. $\rhd$

For this proposition to be useful, we must choose u and b in such a way that $D(u,b) \to 0$ as $b \to +\infty$. If μ is a probability measure, $\int e^{2V} d\nu_V = 1$ so that if $b > 1/2$,

$$\int_{\{V \geqslant b\}} 2V d\nu_V \le \mathbf{Ent}_{\nu_V}(\mathbf{I}_{V \geqslant b}) = \nu_V(V \geqslant b)\, \log\left(\frac{1}{\nu_V(V \geqslant b)}\right) \le b\, e^{-2b}$$

where we used Markov inequality and the fact that $x\log(1/x)$ is non decreasing on $[0, 1/e]$ for the latter.

If μ is not bounded, we assume in addition that $\int e^{-pV} d\mu = K(p) < +\infty$ for some $p < 2$, so that a similar argument (changing the constants) yields again

$$\int_{\{V \geqslant b\}} 2V d\nu_V \leq \nu_V(V \geqslant b)\,(2/2-p)\,\log\left(\frac{K(p)}{\nu_V(V \geqslant b)}\right) \leq (2K(p)/(2-p))\,b\,e^{(p-2)b}$$

if $b \geqslant (1 + \log(K(p))/(2-p)$.

In both cases, defining ε as the upper bound, one can find constants a and a' (depending on p if necessary) such that

$$b = a \log\left(\frac{a' \log(1/\varepsilon)}{\varepsilon}\right),$$

and the appropriate choice for u is then $u = \varepsilon/h(b)$, provided $\beta_{WL}(\varepsilon) \leq h(b)$.

Conversely, if $\beta_{WL}(\varepsilon) \geqslant h(b)$, s_b is greater than ε (up to multiplicative constants) and the good choice is then $u = s_b/h(b)$.

If $h(b) \geqslant Cb$ we obtain that $\beta^V_{WL}(s)$ behaves like a function greater than or equal to (up to some constants) $\log(1/s)\beta^V_{WP}(s/\log(1/s))$ in the first case, $\beta_{WL}(s)\beta^V_{WP}(s/\beta_{WL}(s))$ in the second case, with $\beta_{WL}(s)$ larger than $\log(1/s)$ in the latter case. Hence the result is not better (even worse) than (13) in Proposition 3.1.

If $h(b)/b \to 0$ as $b \to +\infty$ we obtain the same results, but replacing $\log(1/s)$ by $h(\log(1/s))$, provided β^V_{WP} is not bounded (otherwise $\beta^V_{WL}(s) = Ch(\log(1/s))$ for some C). Hence if $\beta_{WL}(s) \ll \log(1/s)$ we obtain a better result that the one in Proposition 3.1, namely ν_V satisfies **WPI** with a function

$$\beta(s) \geqslant \frac{h(\log(1/s))}{\log(1/s)}\,\beta^V_{WP}(cs)$$

provided this function is non-increasing. But if there exists M such that $\beta^V_{WP}(cs) \leq M\beta^V_{WP}(s)$, we may thus choose $\beta \leq (1/2)\beta^V_{WP}$, which leads to a contradiction since β^V_{WP} is assumed to be the best one. We have thus obtained (recall that we leave some constants away in the previous argument)

Corollary 6.6 *Let μ be a positive measure (not necessarily bounded) satisfying a* **WLSI** *with continuous function β_{WL}. Let V be Witten (G, μ)-good, such that $\nu_V = e^{-2V}\mu$ is a probability measure. If μ is not bounded, we assume in addition that there exists $p < 2$ such that $\int e^{-pV} d\mu < +\infty$. Assume in addition that*

- $h(b) := \sup_{\{A \leq z \leq b\}} \frac{2z}{G(z)}$ *is such that $h(b)/b \to 0$ as $b \to +\infty$,*
- $\beta_{WL}(s)/\log(1/s) \to 0$ *as $s \to 0$ (that is, if μ is bounded, μ satisfies some* **SPI** *which is stronger than the usual Poincaré inequality).*

Then ν_V satisfies a Poincaré inequality, and a **WLSI** *with function $\beta^V_{WL}(s) = ah(a'\log(1/s))$ for some constants a and a'.*

In particular if $G(z) \geqslant cz$ for large z, ν_V satisfies the usual logarithmic Sobolev inequality.

The previous result extends part of the results in [Cat05] since we do not assume that μ satisfies a logarithmic Sobolev inequality.

It has to be noticed that the conditions in Corollary 6.6 are far to be optimal for ν_V to satisfy Poincaré inequality. Indeed if $\mu = dx$ on the euclidean space, it is known that $G(b) \geqslant k > 0$ for large b is sufficient (i.e. h asymptotically linear) (see [Cat05] for a reference). In the general manifold case with μ the riemannian measure, Wang ([Wan99] Theorem 1.1 and Remark 1) has obtained a beautiful sufficient condition, namely $-L\rho(x) \geqslant k > 0$ for $\rho(x)$ large, when ρ is the riemannian distance to some point o. In the flat case, this condition reads $|\nabla V|(x) > k > 0$ for $|x|$ large. In the one dimensional case, it is easy to see that this condition is weaker than our $G(b) \geqslant k > 0$ for large b. Wang's condition thus appears as the best general one, though it is not necessary as shown

in one dimension by a potential $V(x) = x + \sin(x)$ for large x. But Wang's approach, based on Cheeger inequality and the control of local Poincaré inequality outside large balls, seems difficult to extend to more general functional inequalities (though it can be used in particular cases, see [RW01] section 3 and [Wan00]).
For $1 < \alpha \leq 2$ and $G(u) = u^{2(1-\frac{1}{\alpha})}$ we recover (here $d\mu = dx$) the same β_{WL} as the one corresponding to the measure μ_α studied at the end of section 2. This furnishes a new proof of some results in [BCR05b] section 7.2. For more general G the result is linked to the perturbation results in [BCR05c].

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