WEAK LOWER SEMICONTINUITY FOR POLYCONVEX INTEGRALS IN THE LIMIT CASE

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Abstract. We prove a lower semicontinuity result for polyconvex functionals of the Calculus of Variations along sequences of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ in $W^{1,m}$, $2 \le m \le n$ bounded in $W^{1,m-1}$ and convergent in L^1 under mild technical conditions but without any extra coercivity assumption on the integrand.

1. Introduction

Let n, m be positive integers, let Ω be an open set of $\mathbb{R}^n$ and let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a *map* in the *Sobolev space* $W^{1,p}(\Omega, \mathbb{R}^m)$ for some $p \ge 1$. We denote by ∇u the *gradient* of the map u, i.e., the $m \times n$ matrix of the first derivatives of u. The *energy functional* associated to the map u is an integral of the type

$$F(u) = \int_\Omega f(\mathcal{M}^\ell(\nabla u(x)))\, dx\,, \tag{1.1}$$

where $\ell = m \wedge n$ and $\mathcal{M}^\ell(\mathbb{A})$ denotes the vector with components are all the *minors* of order up to ℓ of the gradient matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, i.e.,

$$\mathcal{M}^\ell(\mathbb{A}) = (\mathbb{A}, \mathrm{adj}_2\mathbb{A}, \dots, \mathrm{adj}_i\mathbb{A}, \dots, \mathrm{adj}_\ell\mathbb{A})\,.$$

For instance, $\mathcal{M}^1(\mathbb{A}) = \mathbb{A}$ if $\ell = 1$, while if $\ell = m = n$ the "*last*" *component* of $\mathcal{M}^\ell(\mathbb{A})$ is the *determinant* $\det \mathbb{A}$ of the matrix $\mathbb{A}$.

Energy functional as in (1.1) are considered in *nonlinear elasticity*, when $\ell = m = n = 3$; in particular $\det \nabla u$ takes into account the contribution to the energy given by *changes of volume* of the deformation u. The integrand f in (1.1) is assumed to be a convex function; this makes the integral F consistent with the theory of *polyconvex* and *quasiconvex* integrals (see Morrey [23], Ball [4], see also the book by Dacorogna [6]). We assume that f is bounded below, say $f(\mathcal{M}^\ell(\mathbb{A})) \ge 0$ for all $\mathbb{A} \in \mathbb{R}^{m\times n}$.

To fix the ideas let us assume $m = n \ge 2$. Then well-known results by Morrey [23] (see also Acerbi-Fusco [2] and Marcellini [21]) imply that the functional in (1.1) is lower semicontinuous with respect to the weak convergence in $W^{1,n}(\Omega, \mathbb{R}^n)$.

The modelling of *cavitation phenomena* forces then naturally to consider maps in the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^m)$ for $p < n$. For instance, deformation maps of the type

$$u(x) = v(|x|)\frac{x}{|x|}\,, \tag{1.2}$$

with $v : [0, 1] \to \mathbb{R}$ an increasing function and $v(0) > 0$, belong to $W^{1,p}(\Omega, \mathbb{R}^n)$ for every $p < n$, but not to $W^{1,n}(\Omega, \mathbb{R}^n)$. An extension of the energy functional outside the space $W^{1,n}(\Omega, \mathbb{R}^n)$ is needed in this case. Different choices have been investigated. Referring to the prototype case of

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the determinant of ∇u in (1.2) (see Ball [4], Fonseca-Fusco-Marcellini [11], [12], Giaquinta-Modica-Souček [19] and Müller [24]), we recall the so called *distributional determinant* $\mathrm{Det}\nabla u$ as opposed to the *total variation* of the determinant, i.e.,

$$TV^p(u,\Omega) := \inf\left\{\liminf_j \int_\Omega |\det \nabla u_j|\, dx :\ (u_j)_j \subset W^{1,n},\ u_j \rightharpoonup u \text{ in } W^{1,p}\right\}.$$

TV^p is an extension of the orginal integral, i.e.,

$$TV^p(u,\Omega) = \int_\Omega |\det \nabla u|\, dx \quad \text{for } u \in W^{1,n},$$

if and only if for every sequence $(u_j)_j \subset W^{1,n}$ converging weakly to u in $W^{1,p}$

$$\int_\Omega |\det \nabla u|\, dx \le \liminf_j \int_\Omega |\det \nabla u_j|\, dx. \tag{1.3}$$

The lower semicontinuity inequality as in (1.3) for general integrands is the object of investigation in Theorem 1.1 below under the weak convergence in $W^{1,p}$ for $p < n$. Marcellini observed in [22] that lower semicontinuity inequality still holds below the critical exponent n. Later on Dacorogna-Marcellini [7] proved the lower semicontinuity for $p > n-1$ (see also [15]), while Malý [20] exhibited a counterexample in the case $p < n-1$. Finally, the limit case $p = n-1$ was addressed by Acerbi-Dal Maso [1], Dal Maso-Sbordone [8], Celada-Dal Maso [5] and Fusco-Hutchinson [17]. In particular, in [5] the integrand f can be any nonnegative convex function with no coercivity assumptions.

The situation significantly changes when an explicit dependence either on x and/or on u is also allowed, since the presence of these variables cannot be treated as a simple perturbation. Results in this context are due to Gangbo [18], under a structure assumption, and to Fusco-Hutchinson [17] and Fonseca-Leoni [13], assuming the coercivity of the integrand. More recently, Amar-De Cicco-Marcellini-Mascolo [3] studied the non-coercive case with $u-$dependence on the integrand, assuming the strict inequality $p > n-1$.

In this paper we deal with the limit case $p = n-1$ and consider integrals of the general form

$$F(u) = \int_\Omega f\left(x, u(x), \mathcal{M}^n(\nabla u(x))\right)\, dx.$$

Other new issues here are that we do not assume coerciveness of the integrand and we allow the dependence on lower order variables and minors.

Theorem 1.1. *Let $m = n \ge 2$ and $f = f(x,u,\xi) : \Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma \to [0,\infty)$ be such that*

(i) *$f \in C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma)$ and $f(x,u,\cdot)$ is convex for all (x,u);*

(ii) *denoting $\xi = (z,t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$, if $f(x_0,u_0,z_0,\cdot)$ is constant in $t \in \mathbb{R}$ for some (x_0,u_0,z_0), then for all $z \in \mathbb{R}^{\sigma-1}$*

$$f(x_0,u_0,z,t) = \inf\left\{f(y,v,z,s) :\ (y,v,s) \in \Omega \times \mathbb{R}^n \times \mathbb{R}\right\}. \tag{1.4}$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega,\mathbb{R}^n)$ satisfying

$$u_j \to u \text{ in } L^1 \text{ and } \sup_j \|u_j\|_{W^{1,n-1}} < \infty \tag{1.5}$$

we have

$$F(u) \le \liminf_j F(u_j).$$

Note that (1.5) assures that $u \in W^{1,n-1}(\Omega,\mathbb{R}^n)$ for $n \ge 3$, and $u \in BV(\Omega,\mathbb{R}^2)$ if $n = 2$; in the latter case ∇u denotes the *approximate gradient*.

To highlight the main ideas of our strategy we choose to study first autonomous functionals as in (1.1) for which the proof simplifies. Semicontinuity actually holds in a more general setting, i.e.

under the assumption $m \leq n$ (see Theorem 3.1). On the other hand, we are not able to remove the coerciveness assumption when $m > n$, a difficulty which was also present in the papers [5], [3] (details are in Remark 2.8). Further results for $m \leq n$ are discussed for some special integrands (see Section 3).

The main tools used in the proof of Theorem 1.1 are: *(i)* the De Giorgi's approximation theorem for convex integrands; *(ii)* a suitable generalization of a truncation lemma by Fusco-Hutchinson [17] (see Proposition 2.7); *(iii)* a measure-theoretic lemma by Celada-Dal Maso [5] (see Lemma 2.4). All these tools are carefully combined in an argument which exploits assumption (1.4) in the statement above and the blow-up technique. Note though, that a less general version of the above theorem is proved in the final Section with a suitable new approximation technique, which we believe may be of interest herself.

Let us now comment on assumption (1.4), that it is trivially satisfied when f is coercive in t. We consider a simple functional with integrand a positive piecewise affine function

$$I(u) = \int_\Omega (a(x,u) + b(x,u) \det \nabla u)^+ \, dx.$$

In this particular case condition (1.4) becomes

$$b(x,u) = 0 \Longrightarrow a(x,u) \leq 0. \tag{1.6}$$

We do not know whether (1.6) is just a technical assumption or not. In fact, Müller's isoperimetric inequality [25] implies that (1.6) is not needed when dealing with sequences with nonnegative determinants (see Proposition 3.5). Thus, if one would show that (1.4) is a necessary condition for lower semicontinuity one should work with sequences whose determinants change sign rapidly. However, all the more or less standard constructions of such sequences fail.

The research presented in this paper took origin by the work of two different groups, one in Firenze and the other one in Napoli. Before Summer 2012 the two groups independently reached quite similar results. Then our colleague Bernard Dacorogna, talking separately with some of us, pointed out the similarities. What to do: *cooperation or competition?* We decided to continue to study together the problem, and this the reason why the paper has six authors.

2. Definitions and preliminary results

The aim of this section is to introduce some notations and to recall some basic definition and results which will be used in the sequel.

We begin with some algebraic notation.

Let $n, m \geq 2$ and $\mathbb{M}^{m\times n}$ be the linear space of all $m \times n$ real matrices. For $\mathbb{A} \in \mathbb{M}^{m\times n}$, we denote $\mathbb{A} = (\mathbb{A}^i_j)$, $1 \leq i \leq m$, $1 \leq j \leq n$, where upper and lower indices correspond to rows and columns respectively.

The euclidean norm of $\mathbb{A}$ will be denoted by $|\mathbb{A}|$. The number of all minors up to the order m of any matrix in $\mathbb{M}^{m\times n}$ is given by

$$\sigma := \sum_{i=1}^{m\wedge n} \binom{m}{i}\binom{n}{i}.$$

We shall also adopt the following notations. We set $I_{l,k} = \{\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l : 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l \leq k\}$, where $1 \leq l \leq k$. If $\alpha \in I_{l,m}$ and $\beta \in I_{l,n}$, then $M_{\alpha,\beta}(\mathbb{A}) = \det(\mathbb{A}^{\alpha_i}_{\beta_j})$.

By $\mathcal{M}_l(\mathbb{A})$ we denote the vector whose components are all the minors of order l, and by $\mathcal{M}^l(\mathbb{A})$ the vector of all minors of order up to l.

As usual, $Q_r(x)$, $B_r(x)$ denote the open euclidean cube, ball in $\mathbb{R}^n$, $n \geq 2$, with side r, radius r and center the point x, respectively. The center shall not be indicated explicitly if it coincides with the origin.

2.1. **Approximation of convex functions.** We survey now on an approximation theorem for convex functions, due to De Giorgi, that plays an important role in the framework of lower semicontinuity problems (see [9]). Given a convex function $f : \mathbb{R}^k \to \mathbb{R}$, $k \geq 1$, consider the affine functions $\xi \to a_j + \langle b_j, \xi \rangle$, with $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R}^k$, given by

$$a_j := \int_{\mathbb{R}^k} f(\eta)\big((\nu+1)\alpha_j(\eta) + \langle \nabla\alpha_j(\eta), \eta\rangle\big)d\eta \tag{2.1}$$

$$b_j := -\int_{\mathbb{R}^k} f(\eta)\nabla\alpha_j(\eta)d\eta, \tag{2.2}$$

where $\alpha_j \in C_0^1(\mathbb{R}^k)$, $j \in \mathbb{N}$, is a non negative function such that $\int_{\mathbb{R}^k} \alpha_j(\eta)d\eta = 1$.

Lemma 2.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function and a_j, b_j be defined as in* (2.1)*. Then,*

$$f(\xi) = \sup_{j\in\mathbb{N}} \left(a_j + \langle b_j, \xi\rangle\right), \quad \textit{for all } \xi \in \mathbb{R}^k.$$

The main feature of the approximation above is the explicit dependence of the coefficients a_j and b_j on f. In particular, if f depends on the lower order variables (x, u) regularity properties of the coefficients a_j and b_j with respect to (x, u) can be easily deduced from related hypotheses satisfied by f thanks to formulas (2.1) and Lemma 2.1.

In particular, the following approximation result holds.

Theorem 2.2. *Let $f = f(x, u, \xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0, \infty)$, be a continuous function, convex in the last variable ξ. Then, there exist two sequences of compactly supported continuous functions $a_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $b_i : \Omega \times \mathbb{R}^m \to \mathbb{R}^\sigma$ such that, setting for every $i \in \mathbb{N}$,*

$$f_i(x, u, \xi) := \left(a_i(x, u) + \langle b_i(x, u), \xi\rangle\right)^+,$$

then

$$f(x, u, \xi) = \sup_i f_i(x, u, \xi).$$

Moreover, for every $i \in \mathbb{N}$ there exists a positive constant C_i such that

(a) *f_i is continuous, convex in ξ and*

$$0 \leq f_i(x, u, \xi) \leq C_i(1 + |\xi|) \quad \textit{for all } (x, u, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma; \tag{2.3}$$

(b) *if γ_i denotes a modulus of continuity of $a_i + |b_i|$ we have*

$$|f_i(x, u, \xi) - f_i(y, v, \eta)| \leq C_i|\xi - \eta| + \gamma_i(|x - y| + |u - v|)(1 + |\xi| \wedge |\eta|) \tag{2.4}$$

for all (x, u, ξ) and $(y, v, \eta) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma$.

Let us finally recall the following localization lemma established in [10].

Lemma 2.3. *Let μ be a positive Radon measure defined on an open set $\Omega \subset \mathbb{R}^n$. Consider a sequence ϕ_j of Borel positive functions defined on Ω. Then*

$$\int_\Omega \sup_{j\in\mathbb{N}} \phi_j \, d\mu = \sup_{j\in\mathbb{N}} \left\{ \sum_{i=1}^{j} \int_{U_i} \phi_i \, d\mu : U_i \subset\subset \Omega \quad \textit{open and pairwise disjoint} \right\}.$$

2.2. **A truncation method for minors.** We first recall a lemma proved in [5, Lemma 3.2].

Lemma 2.4. *Let $(\mu_k)_k$ be a sequence of Radon measures on Ω. Assume that*

(a) *there exists $T \in \mathcal{D}'(\Omega)$ such that $\mu_k \to T$ in the sense of distributions on Ω;*
(b) *there exists a positive Radon measure ν such that $\mu_k^+ \rightharpoonup \nu$ weakly* in the sense of measures on Ω.*

Then, there exists a Radon measure μ such that $T = \mu$ on Ω and $\mu_k \rightharpoonup T$ weakly in the sense of measures on Ω.*

An immediate consequence is the following corollary.

Corollary 2.5. *Let $2 \le l \le m \wedge n$, u_k, $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ be maps such that*

(a) *$(u_k)_k$ is bounded in $W^{1,l-1}(\Omega, \mathbb{R}^m)$;*
(b) *$(u_k)_k$ converges to u in $L^\infty(\Omega, \mathbb{R}^m)$;*
(c) *there exists $c \in \mathbb{R}^\tau$, $\tau = \binom{m}{l}\binom{n}{l}$, such that*

$$\sup_k \int_\Omega \langle c, \mathcal{M}_l(\nabla u_k)\rangle^+ \, dx < \infty.$$

Then, $\mathcal{M}^{l-1}(\nabla u_k) \rightharpoonup \mathcal{M}^{l-1}(\nabla u)$ and $\langle c, \mathcal{M}_l(\nabla u_k)\rangle \rightharpoonup \langle c, \mathcal{M}_l(\nabla u)\rangle$ weakly in the sense of measures on Ω.*

Proof. Passing to a subsequence, not relabeled for convenience, we may suppose that for any $1 \le |\alpha| = |\beta| = k \le l-1$

$$M_{\alpha,\beta}(\nabla u_k) \rightharpoonup \mu_{\alpha,\beta} \quad \text{weakly}^* \text{ in the sense of measures.}$$

We claim that for any h and $|\alpha| = |\beta| = h \le l-1$ then

$$\mu_{\alpha,\beta} = M_{\alpha,\beta}(\nabla u)\mathcal{L}^n \llcorner \Omega.$$

The case $h = 1$ is obvious. Then, by induction, suppose that the result is true for any $\alpha \in I_{h-1,m}$, $\beta \in I_{h-1,n}$ with $2 \le h \le l-1$. Fix now $\alpha \in I_{h,m}$, $\beta \in I_{h,n}$ and a test function $\varphi \in C_c^\infty(\Omega)$, then

$$\begin{aligned}\int_\Omega \varphi d\mu_{\alpha,\beta} &= \lim_k \int_\Omega \varphi \det\left(\frac{\partial u_k^{\alpha_i}}{\partial x_{\beta_j}}\right) dx \\ &= -\lim_k \int_\Omega u_k^{\alpha_1} \frac{\partial(\varphi, u_k^{\alpha_2}, \ldots, u_k^{\alpha_h})}{\partial(x_{\beta_1}, \ldots, x_{\beta_h})} = -\lim_k \sum_{j=1}^h (-1)^{j-1} \int_\Omega u_k^{\alpha_1} \frac{\partial \varphi}{\partial x_{\beta_j}} M_{\hat{\alpha_1},\hat{\beta_j}}(\nabla u_k)\, dx,\end{aligned}$$

where $\hat{\alpha_1} = (\alpha_2, \ldots, \alpha_h)$, $\hat{\beta_j} = (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_h)$. Since $u_k \to u$ in L^∞ and by induction assumption $M_{\hat{\alpha_1},\hat{\beta_j}}(\nabla u_k) \rightharpoonup M_{\hat{\alpha_1},\hat{\beta_j}}(\nabla u)$ weakly* in the sense of measures, we have

$$\int_\Omega \varphi d\mu_{\alpha,\beta} = -\lim_k \sum_{j=1}^h (-1)^{j-1} \int_\Omega u_k^{\alpha_1} \frac{\partial \varphi}{\partial x_{\beta_j}} M_{\hat{\alpha_1},\hat{\beta_j}}(\nabla u_k)\, dx = \int_\Omega \varphi M_{\alpha,\beta}(\nabla u)\, dx,$$

that gives the claim.

To prove that $(\langle c, \mathcal{M}_l(\nabla u_k)\rangle)_k$ converges to $\langle c, \mathcal{M}_l(\nabla u)\rangle$ weakly* in the sense of measures, we first observe that arguing as before, assumptions in items (a) and (b) ensure that $(\mathcal{M}_l(\nabla u_k))_k$ converges in $\mathcal{D}'(\Omega, \mathbb{R}^\tau)$ to some T. In particular, $(\langle c, \mathcal{M}_l(\nabla u_k)\rangle)_k$ converges to $\langle c, T\rangle$ in $\mathcal{D}'(\Omega)$.

Then, using assumption (c), Lemma 2.4 yields that up to a subsequence $(\langle c, \mathcal{M}_l(\nabla u_k)\rangle)_k$ converges to $\langle c, T\rangle$ weakly* in the sense of measures to some Radon measure $\mu = \langle c, T\rangle$. The equality $\mu = \langle c, \mathcal{M}_l(\nabla u)\rangle$ follows by an integration by part argument as before.

Since the limit is independent of the extracted subsequence the convergence for the whole sequence follows immediately. ☐

Remark 2.6. If condition (c) above is strengthened to $\sup_k \|\mathcal{M}^l(\nabla u_k)\|_{L^1} < \infty$, [17, Lemma 2.2] establishes the weak* convergence in the sense of measures of $(\mathcal{M}^l(\nabla u_k))_k$ to $\mathcal{M}^l(\nabla u)$ without assuming (a).

The next result is inspired to [17, Proposition 2.5].

Proposition 2.7. *Let $n \geq m \geq 2$ and let u_j be $W^{1,m}(\Omega, \mathbb{R}^m)$ functions and u in $W^{1,\infty}(\Omega, \mathbb{R}^m)$. Suppose that $u_j \to u$ in $L^1(\Omega, \mathbb{R}^m)$ and that $\sup_j(\|\mathcal{M}^{m-1}(\nabla u_j)\|_{L^1} + \|\langle c, \mathcal{M}_m(\nabla u_j)\rangle\|_{L^1}) < \infty$, for some $c \in \mathbb{R}^\tau$.*

Then, there exists a sequence $(v_j)_j \in W^{1,m}(\Omega, \mathbb{R}^m)$ converging to u in $L^\infty(\Omega, \mathbb{R}^m)$, such that

$$\mathcal{M}^{m-1}(\nabla v_j) \rightharpoonup \mathcal{M}^{m-1}(\nabla u), \quad \langle c, \mathcal{M}_m(\nabla v_j)\rangle \rightharpoonup \langle c, \mathcal{M}_m(\nabla u)\rangle \tag{2.5}$$

weakly in the sense of measures. Moreover, an infinitesimal sequence of positive numbers s_j exists such that*

$$\{x \in \Omega : u_j(x) \neq v_j(x)\} \subset A_j := \{x \in \Omega : |u_j(x) - u(x)| > s_j\} \tag{2.6}$$

and

$$\lim_j \int_{A_j} \left(1 + |\mathcal{M}^{m-1}(\nabla v_j)| + |\langle c, \mathcal{M}_m(\nabla v_j)\rangle|\right) dx = 0. \tag{2.7}$$

Proof. Let us first fix some notation that shall be used throughout this proof. If $\mathbb{A}$, $\mathbb{B}$ are in $\mathbb{M}^{m\times n}$ and $1 \leq k \leq m$ the components of $\mathcal{M}_k(\mathbb{A} + \mathbb{B})$ can be written as certain linear combinations of products of components of the vectors $\mathcal{M}_i(\mathbb{A})$ and $\mathcal{M}_{k-i}(\mathbb{B})$. We denote these linear combinations writing

$$\mathcal{M}_k(\mathbb{A} + \mathbb{B}) = \sum_{i=0}^{k} \mathcal{M}_i(\mathbb{A}) \odot \mathcal{M}_{k-i}(\mathbb{B}), \tag{2.8}$$

with the convention that $\mathcal{M}_0(\mathbb{A}) = \mathcal{M}_0(\mathbb{B}) = 1$.

Let $0 < s < t$ and denote by

$$\varphi_{s,t}(r) := \begin{cases} 1 & r \leq s \\ \dfrac{t-r}{t-s} & s \leq r \leq t \\ 0 & r \geq t, \end{cases}$$

and by $\Phi_{s,t}$ the function

$$\Phi_{s,t}(y) := y\, \varphi_{s,t}(|y|) \quad \text{for all } y \in \mathbb{R}^m.$$

We set

$$v^j_{s,t} := u + \Phi_{s,t}(u_j - u).$$

Note that

$$\mathcal{L}^n(\{x \in \Omega : v^j_{s,t} \neq u_j\}) \leq \mathcal{L}^n(\{x \in \Omega : |u_j - u| \geq s\}) \leq s^{-1}\|u_j - u\|_{L^1}, \quad \|v^j_{s,t} - u\|_{L^\infty} \leq t, \tag{2.9}$$

and in addition

$$\nabla v^j_{s,t} = \nabla u + D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u)$$

where $D\Phi_{s,t}(y) = \varphi_{s,t}(|y|)Id + \varphi'_{s,t}(|y|)\frac{y\otimes y}{|y|}$. Note that if $1 \leq i \leq m$, and α, $\lambda \in I_{i,m}$, since $y \otimes y$ is a rank one matrix one easily infers that

$$|M_{\alpha,\lambda}(D\Phi_{s,t}(y))| \leq (\varphi_{s,t}(|y|))^i + C|y||\varphi'_{s,t}(|y|)|(\varphi_{s,t}(|y|))^{i-1}. \tag{2.10}$$

Finally, we define

$$C_j := \{x \in \Omega : \nabla(|u_j - u|(x)) = 0\},$$

and it is easy to check that

$$\nabla v^j_{s,t}(x) = \nabla u(x) + \varphi_{s,t}(|u_j(x) - u(x)|)(\nabla u_j(x) - \nabla u(x))$$

for $\mathcal{L}^n$ a.e. x in C_j.

We now estimate the linear combination of the m-th order minors on the set $E^j_{s,t} := \{x \in \Omega : s < |u_j(x) - u(x)| < t\}$ by taking into account (2.8)

$$\int_{E^j_{s,t}} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\, dx = \int_{E^j_{s,t}\cap C_j} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\, dx + \int_{E^j_{s,t}\setminus C_j} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\, dx \tag{2.11}$$

$$\leq \sum_{i=0}^{m} \int_{E^j_{s,t}\cap C_j} |\langle c, \mathcal{M}_{m-i}(\nabla u) \odot \mathcal{M}_i(\varphi_{s,t}(|u_j - u|)(\nabla u_j - \nabla u))\rangle|\, dx$$

$$+ \sum_{i=0}^{m} \int_{E^j_{s,t}\setminus C_j} |\langle c, \mathcal{M}_{m-i}(\nabla u) \odot \mathcal{M}_i(D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u))\rangle|\, dx$$

$$\leq C \int_{E^j_{s,t}} \left(1 + |\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\right)\, dx$$

$$+ C \int_{E^j_{s,t}\setminus C_j} \left(|\mathcal{M}^{m-1}(D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u))| + |\langle c, \mathcal{M}_m(D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u))\rangle|\right)\, dx,$$

where $C = C(m, n, \|\nabla u\|_{L^\infty})$.

An elementary but lengthy algebraic computation shows that if $\mathbb{A} \in \mathbb{M}^{m\times m}$ and $\mathbb{B} \in \mathbb{M}^{m\times n}$ then

$$|\mathcal{M}_i(\mathbb{A} \circ \mathbb{B})| \leq \sum_{\alpha,\lambda \in I_{i,m}} \sum_{\beta \in I_{i,n}} |M_{\alpha,\lambda}(\mathbb{A})||M_{\lambda,\beta}(\mathbb{B})| \quad \text{for } 1 \leq i \leq m-1$$

$$\mathcal{M}_m(\mathbb{A} \circ \mathbb{B}) = (\det \mathbb{A})\mathcal{M}_m(\mathbb{B}).$$

By taking into account this estimate and (2.10) we conclude that

$$\int_{E^j_{s,t}\setminus C_j} |\mathcal{M}^{m-1}(D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u))|\, dx$$

$$\leq \int_{E^j_{s,t}\setminus C_j} \left(1 + \frac{C}{t-s}|u_j - u|\right) |\mathcal{M}^{m-1}(\nabla u_j - \nabla u))|\, dx,$$

and

$$\int_{E^j_{s,t}\setminus C_j} |\langle c, \mathcal{M}_m(D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u))\rangle|\, dx$$

$$= \int_{E^j_{s,t}\setminus C_j} |\det(D\Phi_{s,t}(u_j - u))||\langle c, \mathcal{M}_m(\nabla u_j - \nabla u)\rangle|\, dx.$$

Therefore, recalling (2.11) we get

$$\int_{E^j_{s,t}} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\, dx \leq C \int_{E^j_{s,t}} \left(1 + |\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\right)\, dx$$

$$+ \frac{C}{t-s} \int_{E^j_{s,t}\setminus C_j} \left(|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\right) |u_j - u|\, dx.$$

Repeating for the lower order minors the argument used to infer the previous estimate we get

$$\int_{E^j_{s,t}} \Big(|\mathcal{M}^{m-1}(\nabla v^j_{s,t})| + |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\Big) \le C\int_{E^j_{s,t}} \big(1 + |\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\,\big)\,dx$$
$$+ \frac{C}{t-s}\int_{E^j_{s,t}\setminus C_j} \big(|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\,\big)\,|u_j - u|\,dx. \quad (2.12)$$

To deal with the last integral in (2.12) we recall that $\nabla(|u_j - u|) \neq 0$ on $E^j_{s,t} \setminus C_j$, then we use the coarea formula to get for $\mathcal{L}^1$ a.e. $t > 0$

$$\lim_{s\uparrow t} \frac{1}{t-s}\int_{\{x\in\Omega\setminus C_j:\, s<|u_j-u|<t\}} \big(|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\,\big)\,|u_j - u|\,dx$$
$$= \lim_{s\uparrow t} \frac{1}{t-s}\int_s^t dr \int_{\{x\in\Omega\setminus C_j:\, |u_j-u|=r\}} \big(|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\,\big)\,\frac{|u_j - u|}{|\nabla(|u_j - u|)|}\,d\mathcal{H}^{n-1}$$
$$= t\int_{\{x\in\Omega\setminus C_j:\, |u_j-u|=t\}} \frac{|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|}{|\nabla(|u_j - u|)|}\,d\mathcal{H}^{n-1} \quad (2.13)$$

Let us now denote by C_0 a constant such that for all $j \in \mathbb{N}$

$$\int_0^\infty dr \int_{\{x\in\Omega\setminus C_j:\, |u_j-u|=r\}} \frac{|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|}{|\nabla(|u_j - u|)|}\,d\mathcal{H}^{n-1}$$
$$= \int_{\Omega\setminus C_j} \big(|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\big)\,dx < C_0.$$

We now recall the following elementary fact: If g is a nonnegative measurable function in $[0,\infty)$ with $\int_0^\infty g(r)\,dr \le C_0$, then for every $0 < r_1 < r_2$ there exists a set J of positive measure in (r_1, r_2) such that for all $r \in J$

$$rg(r) \le \frac{C_0}{\ln(r_2/r_1)}.$$

By applying for j sufficiently large this inequality with

$$0 < r_1 = \|u_j - u\|^{1/2}_{L^1} < r_2 = \|u_j - u\|^{1/4}_{L^1} < 1$$

we find $t_j \in (\|u_j - u\|^{1/2}_{L^1}, \|u_j - u\|^{1/4}_{L^1})$ so that $\mathcal{L}^n(\{x \in \Omega :\, |u_j - u| = t_j\}) = 0$ and

$$t_j\int_{\{x\in\Omega\setminus C_j:\, |u_j-u|=t_j\}} \frac{|\mathcal{M}^{m-1}(\nabla u_j)| + |\langle c, \mathcal{M}_m(\nabla u_j)\rangle|}{|\nabla(|u_j - u|)|}\,d\mathcal{H}^{n-1} \le \frac{4C_0}{-\ln\|u_j - u\|_{L^1}}.$$

The latter estimate and (2.12) and (2.13) imply that for every $j \in \mathbb{N}$ there exist $s_j \in (\|u_j - u\|^{1/2}_{L^1}, t_j)$ such that

$$\int_{\{x\in\Omega:\, s_j<|u_j-u|<t_j\}} \Big(|\mathcal{M}^{m-1}(\nabla v^j_{s_j,t_j})| + |\langle c, \mathcal{M}_m(\nabla v^j_{s_j,t_j})\rangle|\Big)\,dx \le \frac{5C_0}{-\ln\|u_j - u\|_{L^1}}.$$

Therefore, we may conclude by choosing $v_j := v^j_{s_j,t_j}$. Indeed, from (2.9), setting

$$A_j := \{x \in \Omega :\, |u_j - u| > s_j\},$$

we have that

$$\mathcal{L}^n(\{x \in \Omega :\, v_j \neq u_j\}) \le \mathcal{L}^n(A_j) \le \frac{1}{s_j}\|u_j - u\|_{L^1} \le \|u_j - u\|^{1/2}_{L^1},$$

$$\|v_j - u\|_{L^\infty} \le t_j \le \|u_j - u\|^{1/4}_{L^1},$$

and

$$
\begin{aligned}
\int_{A_j} & \left(|\mathcal{M}^{m-1}(\nabla v_j)| + |\langle c, \mathcal{M}_m(\nabla v_j)\rangle|\right)\, dx \\
&= \int_{\{x\in\Omega:\, s_j<|u_j-u|<t_j\}} \left(|\mathcal{M}^{m-1}(\nabla v_j)| + |\langle c, \mathcal{M}_m(\nabla v_j)\rangle|\right)\, dx \\
&\quad + \int_{\{x\in\Omega:\, |u_j-u|\geq t_j\}} \left(|\mathcal{M}^{m-1}(\nabla u)| + |\langle c, \mathcal{M}_m(\nabla u)\rangle|\right)\, dx \\
&\leq \frac{5C_0}{-\ln \|u_j-u\|_{L^1}} + C\int_{A_j} |\mathcal{M}^m(\nabla u)|\, dx \to 0.
\end{aligned}
$$

Finally, the weak* convergence stated in (2.5) follows from Corollary 2.5. □

Remark 2.8. Let us point out that the assumption $m \leq n$ plays a crucial role in the proof of Proposition 2.7. The reason is essentially of algebraic nature. In fact if $m \leq n$ and $\mathbb{A} \in \mathbb{M}^{m\times m}$ and $\mathbb{B} \in \mathbb{M}^{m\times n}$ from the algebraic equality

$$\mathcal{M}_m(\mathbb{A}\circ\mathbb{B}) = (\det\mathbb{A})\mathcal{M}_m(\mathbb{B}) \tag{2.14}$$

we can trivially estimate $\langle c, \mathcal{M}_m(\mathbb{A}\circ\mathbb{B})\rangle$ with $\langle c, \mathcal{M}_m(\mathbb{B})\rangle$.

Instead, when $m > n$ equality (2.14) is replaced by a more complicate expression that involves suitable linear combinations of higher order minors and it is no longer true that $|\langle c, \mathcal{M}_m(\mathbb{A}\circ\mathbb{B})\rangle| \leq C|\langle c, \mathcal{M}_m(\mathbb{B})\rangle|$, for some positive constant C depending on $\mathbb{A}$.

We do not know if Proposition 2.7 fails to be true if $m > n$.

Remark 2.9. For every m and $n \in \mathbb{N}$, if the second condition in the statement of Proposition 2.7 is replaced with $\sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty$, $\ell = m\wedge n$, then [17, Proposition 2.5] establishes a stronger result. More precisely, if $\sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty$, then the sequence $(v_j)_j$ defined accordingly, converges to u in L^∞ and satisfies $\sup_j \|\mathcal{M}^\ell(\nabla v_j)\|_{L^1} < \infty$, (2.6), and

$$\lim_j \int_{A_j} (1 + |\mathcal{M}^\ell(\nabla v_j)|)\, dx = 0. \tag{2.15}$$

Furthermore, $(\mathcal{M}^\ell(\nabla v_j))_j$ weakly* converges in the sense of measures to $\mathcal{M}^\ell(\nabla u)$.

2.3. **A blow-up type lemma.** Let $f : \Omega\times\mathbb{R}^n\times\mathbb{R}^\sigma \to [0,\infty)$ be a continuous function, and consider

$$F(v,U) := \int_U f(x, v(x), \mathcal{M}^\ell(\nabla v(x)))dx,$$

where U is any open subset in Ω, $v \in W^{1,\ell-1}(\Omega,\mathbb{R}^m)$, if $\ell = m\wedge n \geq 3$, $v \in BV(\Omega,\mathbb{R}^m)$ for $\ell = 2$. In the latter case ∇v is the density of the absolutely continuous part of the distributional gradient of v.

We shall show that to infer the lower semicontinuity inequality

$$F(u) \leq \liminf_j F(u_j), \tag{2.16}$$

along sequences $(u_j)_j \subset W^{1,\ell}(\Omega,\mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1(\Omega,\mathbb{R}^m), \qquad \text{and } \sup_j \|u_j\|_{W^{1,\ell-1}} < \infty,$$

we can always reduce ourselves to affine target maps thanks to the next lemma.

Lemma 2.10. *Suppose that for $\mathcal{L}^n$ a.e. $x_0 \in \Omega$, and for all sequences $\varepsilon_k \downarrow 0$ and $(v_k)_k \subset W^{1,\ell}(\Omega, \mathbb{R}^m)$ such that*

$$v_k \to v_0 := \nabla u(x_0) \cdot y \text{ in } L^1(Q_1, \mathbb{R}^m), \qquad \text{and } \sup_k \|v_k\|_{W^{1,\ell-1}} < \infty,$$

we have

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^\ell(\nabla v_k))\, dy \geq f(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))), \tag{2.17}$$

then the lower semicontinuity inequality (2.16) *holds.*

Proof. We employ the blow-up technique introduced by Fonseca & Müller [16]. Without loss of generality we may assume that

$$\liminf_j F(u_j) = \lim_j F(u_j) < +\infty,$$

and define the (traces of the) non negative Radon measures $\mu_j(U) := F(u_j, U)$, $U \subseteq \Omega$ open. Then, as $\sup_j \mu_j(\Omega) < +\infty$, by passing to a subsequence if necessary, there exists a non negative Radon measure μ such that $\mu_j \rightharpoonup \mu$ weakly* in the sense of measures.

In what follows, by using (2.17) we shall prove that

$$\frac{d\mu}{d\mathcal{L}^n}(x) = \lim_{\varepsilon \downarrow 0} \frac{\mu\,(Q_\varepsilon(x))}{\mathcal{L}^n(Q_\varepsilon(x))} \geq f(x, u(x), \mathcal{M}^\ell(\nabla u(x))), \tag{2.18}$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$. Clearly, given (2.18) for granted, the conclusion easily follows as

$$\liminf_j F(u_j) = \liminf_j \mu_j(\Omega) \geq \mu(\Omega) \geq F(u).$$

To this aim we consider the Radon measures $\nu_j(U) := \|u_j\|^{\ell-1}_{W^{1,\ell-1}(U,\mathbb{R}^n)}$ and suppose that $(\nu_j)_j$ converges weakly* in the sense of measures to a Radon measure ν.

The ensuing properties are satisfied for all x in a set Ω_0 of full measure in Ω

$$\frac{d\mu}{d\mathcal{L}^n}(x),\ \frac{d\nu}{d\mathcal{L}^n}(x) \quad \text{exist finite,}$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n+1}} \int_{Q_\varepsilon(x)} |u(y) - u(x) - \nabla u(x) \cdot (y - x)|\, dy = 0. \tag{2.19}$$

We shall establish (2.18) for all points in Ω_0. Thus, with fixed $x_0 \in \Omega_0$, let $\varepsilon_k \downarrow 0$ be any sequence such that for every $k \in \mathbb{N}$ we have

$$\mu(\partial Q_{\varepsilon_k}(x_0)) = \nu(\partial Q_{\varepsilon_k}(x_0)) = 0. \tag{2.20}$$

By changing variables, (2.19) rewrites for $x = x_0$ as

$$\lim_k \int_{Q_1} \left| \frac{u(x_0 + \varepsilon_k\, y) - u(x_0)}{\varepsilon_k} - \nabla u(x_0) \cdot y \right| dy = 0.$$

The choice of $(\varepsilon_k)_k$ in (2.20) yields

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^n}(x_0) = \lim_k \frac{\mu(Q_{\varepsilon_k}(x_0))}{\varepsilon_k^n} &= \lim_k \lim_j \frac{1}{\varepsilon_k^n} \int_{Q_{\varepsilon_k}(x_0)} f(x, u_j, \mathcal{M}^\ell(\nabla u_j))dx \\ &= \lim_k \lim_j \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_{j,k}, \mathcal{M}^\ell(\nabla v_{j,k}))dy, \end{aligned} \tag{2.21}$$

where we have set

$$v_{j,k}(y) := \frac{u_j(x_0+\varepsilon_k\, y) - u(x_0)}{\varepsilon_k}.$$

Clearly, $v_{j,k} \in W^{1,\ell}$ and by denoting

$$v_0(y) := \nabla u(x_0)\cdot y \tag{2.22}$$

we have that

$$\lim_k \lim_j \|v_{j,k} - v_0\|_{L^1} = 0.$$

A standard diagonalization argument provides a subsequence $j_k \uparrow \infty$ for which (2.21) and (2.22) become

$$v_k := v_{j_k,k} \to v_0 \quad L^1, \text{ and } \sup_k \|v_k\|_{W^{1,\ell-1}} < +\infty,$$

$$+\infty > \frac{d\mu}{d\mathcal{L}^n}(x_0) = \lim_k \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^\ell(\nabla v_k))\, dy,$$

and thus inequality (2.18) is implied by (2.17). □

3. The model case

In this section we discuss, for $m \le n$, the model case of autonomous functionals of the form

$$F(v) := \int_\Omega f\left(\mathcal{M}^m(\nabla v(x))\right) dx.$$

where $v \in W^{1,m-1}(\Omega,\mathbb{R}^n)$, if $n \ge m \ge 3$, $v \in BV(\Omega,\mathbb{R}^2)$ for $n \ge m = 2$. In the latter case ∇v is the density of the absolutely continuous part of the distributional gradient of v.

Our result improves upon [5, Theorems 3.1 and 4.1] (see also [14, Theorem 10]).

Theorem 3.1. *Let $2 \le m \le n$ and $f : \mathbb{R}^\sigma \to [0,+\infty)$ be a convex function.*

Then, for every sequence $(u_j)_j \subset W^{1,m}(\Omega,\mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \text{and } \sup_j \|u_j\|_{W^{1,m-1}} < \infty \tag{3.1}$$

we have

$$F(u) \le \liminf_j F(u_j).$$

Proof. By Lemma 2.10 it is sufficient to show that

$$\liminf_k \int_{Q_1} f\left(\mathcal{M}^m(\nabla v_k)\right) dy \ge f\left(\mathcal{M}^m(\nabla u(x_0))\right), \tag{3.2}$$

for all points x_0 of approximate differentiability of u and for all sequences

$$v_k \to v_0 := \nabla u(x_0)\cdot y \quad \text{in } L^1, \text{ and } \sup_k \|v_k\|_{W^{1,m-1}} < +\infty.$$

Without loss of generality we may assume that the inferior limit in (3.2) is finite.

Furthermore, to infer (3.2) we are left with proving for all $i \in \mathbb{N}$

$$\liminf_k \int_{Q_1} f_i\left(\mathcal{M}^m(\nabla v_k)\right) dy \ge f_i\left(\mathcal{M}^m(\nabla u(x_0))\right), \tag{3.3}$$

where $f_i(\xi) = (a_i + \langle b_i, \xi\rangle)^+$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}^\sigma$, are the functions in Lemma 2.1.

Fix now $M \geq \|v_0\|_{L^\infty} + 1$ and set

$$v_{k,M}(x) := \begin{cases} v_k(x) & \text{if } |v_k(x)| \leq M \\ M\dfrac{v_k(x)}{|v_k(x)|} & \text{otherwise.} \end{cases}$$

The sequence $(v_{k,M})_k$ is bounded in $W^{1,m-1} \cap L^\infty$ and as $f_i \geq 0$, we have for all $k \in \mathbb{N}$

$$\int_{Q_1} f_i(\mathcal{M}^m(\nabla v_{k,M}))\,dy \leq \int_{Q_1} f_i(\mathcal{M}^m(\nabla v_k))\,dy + |a_i|\,\mathcal{L}^n(\{y \in Q_1 : |v_k| > M\}). \tag{3.4}$$

As $0 \leq f_i \leq f$ and being the liminf in (3.2) finite by assumption, inequality (3.4) implies that

$$\sup_k \int_{Q_1} (\langle b_i, \mathcal{M}^m(\nabla v_{k,M})\rangle)^+\,dy < \infty.$$

Then, in view of Lemma 2.4, Proposition 2.7 provides a new sequence $(w_k)_k$ satisfying all the conclusions there. In particular, we have

$$\int_{Q_1} f_i\left(\mathcal{M}^m(\nabla w_k)\right)dy \leq \int_{Q_1\setminus A_k} f_i\left(\mathcal{M}^m(\nabla v_{k,M})\right)dy + \int_{A_k} (|a_i| + |\langle b_i, \mathcal{M}^m(\nabla w_k)\rangle|)dy. \tag{3.5}$$

Recalling the choice of M, for k sufficiently large we have

$$\{y \in Q_1 : |v_k(y)| > M\} \subseteq A_k = \{y \in Q_1 : |v_k(y) - v_0(y)| > s_k\}.$$

Therefore, thanks to (2.7), (3.4) and (3.5), we get

$$\liminf_k \int_{Q_1} f_i\left(\mathcal{M}^m(\nabla v_k)\right)dy \geq \liminf_k \int_{Q_1} f_i\left(\mathcal{M}^m(\nabla w_k)\right)dy. \tag{3.6}$$

From this inequality, (3.3) follows at once recalling the weak* convergence of $(\mathcal{M}^{m-1}(\nabla w_k))_k$ to $\mathcal{M}^{m-1}(\nabla u(x_0))$ and of $(\langle b_i, \mathcal{M}_m(\nabla w_k)\rangle)_k$ to $\langle b_i, \mathcal{M}_m(\nabla u(x_0))\rangle$. $\square$

A simple variant of the proof of Theorem 3.1 allows us to treat also some special cases when a dependence on x and u appears (cp. with [18] for $m = n$ and $p > n-1$). To be precise, let us consider the functional

$$F(u) = \int_\Omega h(x,u) f(\mathcal{M}^m(\nabla u(x)))\,dx. \tag{3.7}$$

Proposition 3.2. *Let m, n, f be as in Theorem 3.1 and F defined by* (3.7)*. If h is a nonnegative continuous function in $\Omega \times \mathbb{R}^m$ such that either $h \geq c_0 > 0$ or $h \equiv h(x)$, then*

$$F(u) \leq \liminf_j F(u_j)$$

for every sequence $(u_j)_j \subset W^{1,m}(\Omega, \mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1, \qquad \text{and } \sup_j \|u_j\|_{W^{1,m-1}} < \infty.$$

Proof. Let us first assume that $h \equiv h(x)$ is nonnegative and continuous. In this case the result follows from Theorem 3.1 by a simple continuity and localization argument.

If instead $h \equiv h(x,u) \geq c_0 > 0$ we use Lemma 2.10 and reduce ourselves to show that

$$\liminf_k \int_{Q_1} h(x_0 + \varepsilon_k\,y, u(x_0) + \varepsilon_k\,v_k) f(\mathcal{M}^m(\nabla v_k))\,dy \geq h(x_0, u(x_0)) f(\mathcal{M}^m(\nabla u(x_0))),$$

where $\varepsilon_k \downarrow 0$ and $v_k \to v_0 = \nabla u(x_0)\cdot y$ in $L^1(Q_1)$, and the liminf on the left hand side can be taken finite.

Then the inequality above is proved if we show that for all $i \in \mathbb{N}$ we have

$$\liminf_k \int_{Q_1} h(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k) f_i(\mathcal{M}^m(\nabla v_k))\, dy \geq h(x_0, u(x_0)) f_i(\mathcal{M}^m(\nabla u(x_0))), \tag{3.8}$$

where $f_i(\xi) = (a_i + \langle b_i, \xi\rangle)^+$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}^\sigma$, are the functions in Lemma 2.1. Note that, since the above liminf is finite and $h \geq c_0$, we infer

$$\sup_k \int_{Q_1} (\langle b_i, \mathcal{M}^m(\nabla v_k)\rangle)^+\, dy < \infty. \tag{3.9}$$

For $M \geq \|v_0\|_{L^\infty} + 1$ consider the truncated functions

$$v_{k,M}(x) = \begin{cases} v_k(x) & \text{if } |v_k(x)| \leq M \\ M\dfrac{v_k(x)}{|v_k(x)|} & \text{otherwise,} \end{cases}$$

the sequence $(v_{k,M})_k$ turns out to be bounded in $W^{1,m-1} \cap L^\infty$ and it satisfies

$$\int_{Q_1} h(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_{k,M})\, f_i\, (\mathcal{M}^m(\nabla v_{k,M}))\, dy$$
$$\leq \int_{Q_1} h(x_0{+}\varepsilon_k\, y, u(x_0){+}\varepsilon_k\, v_k) f_i\, (\mathcal{M}^m(\nabla v_k))\, dy{+}|a_i| \int_{\{y\in Q_1:\, |v_k|>M\}} h(x_0{+}\varepsilon_k\, y, u(x_0){+}\varepsilon_k\, v_{k,M})\, dy.$$

Therefore, arguing as in Theorem 3.1 and taking into account (3.9) and Lemma 2.4, Proposition 2.7 gives a new sequence $(w_k)_k$ satisfying all the conclusions there. Thus, using (2.6) and (2.7) as in the proof of (3.6), we get

$$\liminf_k \int_{Q_1} h(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k) f_i\, (\mathcal{M}^m(\nabla v_k))\, dy$$
$$\geq \liminf_k \int_{Q_1} h(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, w_k) f_i\, (\mathcal{M}^m(\nabla w_k))\, dy.$$

From this inequality we easily get (3.8), since $(h(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, w_k))_k$ converges uniformly in Q_1 to $h(x_0, u(x_0))$, and since for any open subset $U \subseteq Q_1$

$$\mathcal{L}^n(U) f_i(\mathcal{M}^m(\nabla u(x_0))) \leq \liminf_k \int_U f_i\, (\mathcal{M}^m(\nabla w_k))\, dy$$

thanks to the weak* convergence of $(\mathcal{M}^{m-1}(\nabla w_k))_k$ to $\mathcal{M}^{m-1}(\nabla u(x_0))$ and of $(\langle b_i, \mathcal{M}_m(\nabla w_k)\rangle)_k$ to $\langle b_i, \mathcal{M}_m(\nabla u(x_0))\rangle$. □

When trying to extend Theorem 3.1 to general integrands depending on lower order variables some difficulties arise. We have not been able neither to prove such a statement nor to find counterexamples in such a generality. Instead, we have established lower semicontinuity either strengthening condition (3.1) (cp. with Proposition 3.3) or adding a further condition on the integrand (cp. with Theorems 4.1 and 5.1).

In Sections 4 and 5 we shall establish the lower semicontinuity property under condition (3.1) adding a mild technical assumption on the integrand f provided the domain and the target space have equal dimension, that is $m = n$.

Instead, for all values of m and n, we shall prove below lower semicontinuity without any further technical assumption on the integrand along sequences that have all the set of minors bounded in L^1. Actually, the latter condition holding true, we need only to suppose convergence in L^1, no bound on any Sobolev norm for the relevant sequence is needed (see [1, Theorem 3.5], [5, Theorem 2.2, Corollary 2.3], [8, Theorem 3.1], [17, Theorem 3.3], and [13, Theorem 1.4, Corollary 1.5] for related results).

Proposition 3.3. *Let m and $n \geq 2$, $\ell = m \wedge n$, and $f = f(x,u,\xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0,\infty)$, $\Omega \subset \mathbb{R}^n$ open, be in $C^0(\Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma)$, and such that $f(x,u,\cdot)$ is convex for all $(x,u) \in \Omega \times \mathbb{R}^m$.*

Then, for every sequence $(u_j)_j \subset W^{1,\ell}(\Omega,\mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \text{and} \quad \sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty, \tag{3.10}$$

we have

$$F(u) \leq \liminf_j F(u_j).$$

Proof. We can argue analogously to Lemma 2.10 and infer that to conclude we need to show

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^\ell(\nabla v_k))\, dy \geq f(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))), \tag{3.11}$$

along sequences $(v_k)_k \subset W^{1,\ell}$ satisfying

$$v_k \to v_0 = \nabla u(x_0) \cdot y \quad \text{in } L^1, \text{ and } \sup_k \|\mathcal{M}^\ell(\nabla v_k)\|_{L^1} < +\infty.$$

Thus, we can apply [17, Proposition 2.5] (cp. with Remarks 2.6 and 2.9) that provides a sequence $(w_k)_k \subset W^{1,\infty}$ converging to v_0 uniformly in Q_1, such that $\mathcal{M}^\ell(\nabla w_k) \rightharpoonup \mathcal{M}^\ell(\nabla u(x_0))$ weakly* in the sense of measures and satisfying (2.15).

In particular, by assuming the left hand side in (3.11) is finite, for all functions f_i in Theorem 2.2, estimate (2.3) yields

$$\begin{aligned}\int_{Q_1} & f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, w_k, \mathcal{M}^\ell(\nabla w_k))\, dy \\ &\leq \int_{Q_1} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^\ell(\nabla v_k))\, dy + C_i \int_{\{v_k \neq w_k\}} (1 + |\mathcal{M}^\ell(\nabla w_k)|)\, dy,\end{aligned}$$

and moreover (2.4) gives

$$\begin{aligned}&\int_{Q_1} f_i(x_0, u(x_0), \mathcal{M}^\ell(\nabla w_k))\, dy \\ &\leq \int_{Q_1} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, w_k, \mathcal{M}^\ell(\nabla w_k))\, dy + \gamma_i(\varepsilon_k(1 + \|w_k\|_{L^\infty})) \int_{Q_1} (1 + |\mathcal{M}^\ell(\nabla w_k)|)\, dy.\end{aligned}$$

Thus, the convexity of $f_i(x_0, u(x_0), \cdot)$, the weak* convergence of the minors and (2.15) imply that

$$\begin{aligned}\liminf_k & \int_{Q_1} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^\ell(\nabla v_k))\, dy \\ &\geq \liminf_k \int_{Q_1} f_i(x_0, u(x_0), \mathcal{M}^\ell(\nabla w_k))\, dy \geq f_i(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))).\end{aligned}$$

Hence, inequality (3.11) follows. □

Remark 3.4. Note that the counterexample constructed in [18, Theorem 3.1] shows that a continuous, or better a lower-semicontinuous, dependence of the integrand f on the lower order variables is needed.

We point out that if we restrict to maps from $\mathbb{R}^n$ to itself with positive determinants, assumption (3.1) implies that (3.10) holds locally thanks to Müller's isoperimetric inequality. Indeed, [25, Lemma 1.3] states that for every $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $x_0 \in \Omega$ and for a.e. $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ one has

$$\left| \int_{B_r(x_0)} \det \nabla u \, dx \right|^{\frac{n-1}{n}} \leq C(n) \int_{\partial B_r(x_0)} |\mathcal{M}_{n-1}(\nabla u)| \, d\mathcal{H}^{n-1}.$$

Thus, by integrating this inequality on $(0, R)$, with $R < \mathrm{dist}(x_0, \partial\Omega)$, we get

$$\left| \int_{B_{\frac{R}{2}}(x_0)} \det \nabla u \, dx \right|^{\frac{n-1}{n}} \leq \frac{C(n)}{R} \int_{B_R(x_0)} |\mathcal{M}_{n-1}(\nabla u)| \, dx.$$

Therefore, given a sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ with positive determinants and bounded in $W^{1,n-1}$ we infer that for every $\Omega' \subset\subset \Omega$

$$\sup_j \|\mathcal{M}^n(\nabla u_j)\|_{L^1(\Omega', \mathbb{R}^\sigma)} < \infty.$$

This observation leads to the following lower semicontinuity result for which we introduce some further notation: if $\xi \in \mathbb{R}^\sigma$ we write $\xi = (z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$.

Proposition 3.5. *Let $m = n \geq 2$, $\Omega \subset \mathbb{R}^n$ open, and $f = f(x, u, z, t) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1} \times (0, \infty) \to [0, \infty)$ be in $C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1} \times [0, \infty))$, and such that $f(x, u, \cdot)$ is convex for all $(x, u) \in \Omega \times \mathbb{R}^n$.*

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \det \nabla u_j > 0 \quad \mathcal{L}^n \text{ a.e. in } \Omega, \text{ and } \quad \sup_j \|\nabla u_j\|_{W^{1,n-1}} < \infty,$$

we have

$$F(u) \leq \liminf_j F(u_j).$$

4. The general case for $m = n$

In this section we address the case of integrands depending on the full set of variables in case $m = n$. More precisely, we consider the functional

$$F(v) := \int_\Omega f\left(x, v(x), \mathcal{M}^n(\nabla v(x))\right) dx,$$

where $v \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, if $n = m \geq 3$, $v \in BV(\Omega, \mathbb{R}^2)$ for $n = m = 2$. In the latter case ∇v is the density of the absolutely continuous part of the distributional gradient of v.

The ensuing result improves upon [3, Theorem 4.2]. We recall the notation $\xi = (z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$ if $\xi \in \mathbb{R}^\sigma$.

Theorem 4.1. *Let $f = f(x, u, \xi) : \Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma \to [0, \infty)$ be such that*

(i) *$f \in C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma)$, and $f(x, u, \cdot)$ is convex for all (x, u),*

(ii) *if $f(x_0, u_0, z_0, \cdot)$ is constant with respect to $t \in \mathbb{R}$ for some point (x_0, u_0, z_0), then*

$$f(x_0, u_0, z, t) = \inf\{f(y, v, z, s) : (y, v, s) \in \Omega \times \mathbb{R}^n \times \mathbb{R}\} := g(z) \quad \text{for all } z \in \mathbb{R}^{\sigma-1}. \tag{4.1}$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \text{and } \sup_j \|u_j\|_{W^{1,n-1}} < \infty \tag{4.2}$$

we have

$$F(u) \leq \liminf_j F(u_j). \tag{4.3}$$

Again, condition (4.2) assures that $u \in BV(\Omega, \mathbb{R}^2)$ if $n = 2$, and $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ for $n \geq 3$.

Note that, by the very definition g is a convex function.

In addition, we remark that if f is coercive with respect to the t variable, then condition (ii) is trivially verified as no such point (x_0, u_0, z_0) exists.

Let us now prove Theorem 4.1.

Proof of Theorem 4.1. We split the proof into several intermediate steps.

Step 1. *Reduction to affine target maps*

By Lemma 2.10 to infer (4.3) we are left with proving

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy \geq f(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))). \tag{4.4}$$

along sequences satisfying

$$v_k \to v_0 := \nabla u(x_0) \cdot y \quad L^1, \text{ and } \sup_k \|v_k\|_{W^{1,n-1}} < +\infty,$$

for all points x_0 of approximate differentiability of u. As usual we can assume that the left hand side in (4.4) is finite.

Let us now distinguish two cases:

(a) there exists $z_0 \in \mathbb{R}^{\sigma-1}$ such that $t \mapsto f(x_0, u(x_0), z_0, t)$ is constant;

(b) no such a point exists.

Step 2. Proof in case (a)

In this case we apply assumption (ii) in the statement and use at once Theorem 3.1 to get

$$\lim_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy$$
$$\geq \liminf_k \int_{Q_1} g(\mathcal{M}^{n-1}(\nabla v_k))\, dy \geq g(\mathcal{M}^{n-1}(\nabla u(x_0))) = f(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))).$$

Step 3. Proof in case (b)

Let us now recall that by the approximation Theorem 2.2 there exist three sequences of continuous function with compact support, a_i, $\gamma_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $\beta_i : \Omega \times \mathbb{R}^n \to \mathbb{R}^{\sigma-1}$ such that, setting for all $i \in \mathbb{N}$ and $(x, u, z, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1} \times \mathbb{R}$,

$$f_i(x, u, z, t) = (a_i(x, u) + \langle \beta_i(x, u), z \rangle + \gamma_i(x, u)\, t)^+ \,,$$

we have

$$f(x, u, z, t) = \sup_{i \in \mathbb{N}} f_i(x, u, z, t).$$

Therefore, to prove (4.4) it is enough to show that for all $i \in \mathbb{N}$

$$\lim_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy \geq f_i(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))). \tag{4.5}$$

To this aim note that there exists $j \in \mathbb{N}$ such that $\gamma_j(x_0, u(x_0)) \neq 0$ since otherwise we would fall in case (a).

Without loss of generality we may assume $\gamma_j(x_0, u(x_0)) > 0$. Otherwise, we replace the functions $v_k = (v_k^1, \ldots, v_k^n)$ with $(-v_k^1, v_k^2, \ldots, v_k^n)$, the coefficient $\gamma_j(x, u)$ with $-\gamma_j(x, -u^1, \ldots, u^n)$ and the remaining coefficients a_j and β_j accordingly.

Fix now $M > \|v_0\|_{L^\infty} + 1$ and set

$$v_{k,M}(x) := \begin{cases} v_k(x) & \text{if } |v_k(x)| \le M \\ M \dfrac{v_k(x)}{|v_k(x)|} & \text{otherwise.} \end{cases} \tag{4.6}$$

Then, as $0 \le f_j \le f$, for all k we have

$$\begin{aligned} &\int_{\{y\in Q_1:\,|v_k|\le M\}} f_j(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_{k,M}, \mathcal{M}^n(\nabla v_{k,M}))\, dy \\ &\le \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy. \end{aligned} \tag{4.7}$$

Therefore, since the sequence $(v_{k,M})_k$ is bounded in $W^{1,n-1}(Q_1, \mathbb{R}^n)$ we deduce that

$$\sup_k \int_{\{y\in Q_1:\,|v_k|\le M\}} (\gamma_j(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k) \det \nabla v_{k,M})^+ \; dy < \infty.$$

Recalling the choice $\gamma_j(x_0, u(x_0)) > 0$, the continuity of γ_j yields for k sufficiently large

$$\sup_k \int_{\{y\in Q_1:\,|v_k|\le M\}} (\det \nabla v_{k,M})^+ \; dy < \infty,$$

in turn implying

$$\sup_k \int_{Q_1} (\det \nabla v_{k,M})^+ \; dy < \infty.$$

An application of Lemma 2.4 gives that, up to a subsequence not relabeled for convenience, the sequence $(\det \nabla v_{k,M})_k$ converges weakly* in the sense of measure in Q_1. In particular, $(\det \nabla v_{k,M})_k$ is bounded in L^1. Hence, Proposition 2.7 provides sequences $s_k \downarrow 0$ and $(w_k)_k$ in $W^{1,n}(\Omega, \mathbb{R}^n)$ satisfying conclusions (2.5), (2.6) and (2.7) there. Note that, for k sufficiently large, recalling the choice of M, we have

$$\{y \in Q_1 : \, |v_k(y)| > M\} \subseteq A_k := \{y \in Q_1 : \, |v_k(y) - v_0(y)| > s_k\}.$$

Therefore, estimate (2.3) and equation (4.7) imply for all $i \in \mathbb{N}$

$$\begin{aligned} &\int_{Q_1} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, w_k, \mathcal{M}^n(\nabla w_k))\, dy \\ &\le \int_{Q_1\setminus A_k} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_{k,M}, \mathcal{M}^n(\nabla v_{k,M})\, dy + C_i \int_{A_k} (1 + |\mathcal{M}^n(\nabla w_k)|)\; dy \\ &\le \int_{\{y\in Q_1:\,|v_k|\le M\}} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_{k,M}, \mathcal{M}^n(\nabla v_{k,M}))\, dy + C_i \int_{A_k} (1 + |\mathcal{M}^n(\nabla w_k)|)\; dy \\ &\le \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy + C_i \int_{A_k} (1 + |\mathcal{M}^n(\nabla w_k)|)\; dy. \end{aligned}$$

The convergence of $(w_k)_k$ to v_0 in L^∞, the latter inequality, (2.7) and (2.4) imply

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy \ge \liminf_k \int_{Q_1} f_i(x_0, u(x_0), \mathcal{M}^n(\nabla w_k))\, dy.$$

In turn, from this and by taking into account that $(\mathcal{M}^n(\nabla w_k))_k$ converges to $\mathcal{M}^n(\nabla u(x_0))$ weakly* in the sense of measures, by the convexity of $f_i(x_0, u(x_0), \cdot)$ we get (4.5), thus concluding the proof. □

In case the integrand depends only on the determinant, the same conclusion holds under a local version of the condition in (4.1).

Proposition 4.2. *Let $f = f(x,u,t) : \Omega \times \mathbb{R}^n \times \mathbb{R} \to [0,\infty)$ be such that*

(i) *$f \in C^0(\Omega \times \mathbb{R}^n \times \mathbb{R})$, and $f(x,u,\cdot)$ is convex for all (x,u),*

(ii) *if $f(x_0,u_0,\cdot)$ is constant with respect to $t \in \mathbb{R}$ for some point (x_0,u_0), then*

$$f(x_0,u_0,t) = h(x_0,u_0) \tag{4.8}$$

where

$$h(x,u) := \sup_{\delta>0} h_\delta(x,u) \tag{4.9}$$

and for all $\delta > 0$

$$h_\delta(x,u) := \inf\left\{f(y,v,t) : (y,v,t) \in B_\delta(x,u) \times \mathbb{R}\right\}. \tag{4.10}$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega,\mathbb{R}^n)$ satisfying

$$u_j \to u \text{ in } L^1, \qquad \text{and } \sup_j \|u_j\|_{W^{1,n-1}} < \infty$$

we have

$$F(u) \le \liminf_j F(u_j).$$

Proof. We argue as in Theorem 4.1: first using the blow-up type Lemma 2.10 to reduce to inequality (4.4). At this point we distinguish as before the two cases (a) and (b). The latter is dealt with as before, while for case (a) we argue as follows.

Fix $M > 0$ and consider the function $v_{k,M}$ in (4.6). Then,

$$\int_{Q_1} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_{k,M}, \det\nabla v_{k,M})dy$$
$$\le \int_{Q_1} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_k, \det\nabla v_k)dy + C\,\mathcal{L}^n(\{y \in Q_1 : |v_k| \ge M\}).$$

In turn, for any $\delta > 0$, by definition of h_δ we get

$$\liminf_k \int_{Q_1} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_{k,M}, \det\nabla v_{k,M})dy \ge h_\delta(x_0,u(x_0)),$$

and the conclusion then follows as $f(x_0,u(x_0),t) = \sup_\delta h_\delta(x_0,u(x_0))$ for all $t \in \mathbb{R}$. □

5. An alternative approach

In this section we give an alternative approach without using the blow-up technique exploited to prove Theorem 4.1. This approach relies on an approximation argument (see also [18] for similar ideas) that works in the simplified case treated in Proposition 4.2.

More precisely, we consider the case $n = m$ and the functional

$$F(v) := \int_\Omega f\left(x, v(x), \det\nabla v(x)\right) dx,$$

where $v \in W^{1,n-1}(\Omega,\mathbb{R}^n)$, if $n \ge 3$, $v \in BV(\Omega,\mathbb{R}^2)$ for $n = 2$. In the latter case ∇v is the density of the absolutely continuous part of the distributional gradient of v.

We state and prove the result under a slightly stronger technical assumption than the one used in Proposition 4.2 (cp. with Lemma 5.3).

Theorem 5.1. *Let $f = f(x,u,t) : \Omega \times \mathbb{R}^n \times \mathbb{R} \to [0,\infty)$ be a continuous function such that $f(x,u,\cdot)$ is convex for all $(x,u) \in \Omega \times \mathbb{R}^n$. Assume also that the function*

$$(x,u) \in \Omega \times \mathbb{R}^n \mapsto \inf_{t\in\mathbb{R}} f(x,u,t) \text{ is continuous.} \tag{5.1}$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega,\mathbb{R}^n)$ satisfying

$$u_j \to u \text{ in } L^1, \qquad \text{and } \sup_j \|u_j\|_{W^{1,n-1}} < \infty \tag{5.2}$$

we have

$$F(u) \le \liminf_j F(u_j). \tag{5.3}$$

Note that condition (5.2) assures that $u \in BV(\Omega,\mathbb{R}^2)$ if $n=2$, and $u \in W^{1,n-1}(\Omega,\mathbb{R}^n)$ for $n \ge 3$.

The proof of Theorem 5.1 hinges upon the following approximation result that will be proved at the end of the section.

Lemma 5.2. *Let $g \in C_c(\mathbb{R}^k)$, $k \ge 2$, be nonnegative. Then for every $\varepsilon > 0$ there exists a function g_ε of the type*

$$g_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} b_{i,1}(x_1)\cdot\ldots\cdot b_{i,n}(x_n) \tag{5.4}$$

where the $b_{i,j}$'s are nonnegative functions in $C_c(\mathbb{R})$ and

$$g(x) - \varepsilon \le g_\varepsilon(x) \le g(x)$$

for all $x \in \mathbb{R}^k$.

Proof of Theorem 5.1. We divide the proof in two steps.

Step 1. *Algebraic simplifications of the integrand.*

Possibly replacing the integrand f with $f - \inf_t f(\cdot,\cdot,t)$, without loss of generality, we may take

$$\inf_{t\in\mathbb{R}} f(x,u,t) = 0 \qquad \text{for all } (x,u) \in \Omega\times\mathbb{R}^n. \tag{5.5}$$

Moreover, Theorem 2.2, a monotone approximation argument and the localization Lemma 2.3 show that we may reduce ourselves to prove (5.3) for integrands $f(x,u,t) = (a(x,u) + b(x,u)t)^+$, where a and b are continuous functions with compact support in $\Omega \times \mathbb{R}^n$. Assumption (5.5) then yields that

$$b(x,u) = 0 \quad \Longrightarrow \quad a^+(x,u) = 0. \tag{5.6}$$

Notice also that we may also assume that $b \ge 0$. Otherwise, we observe that for all (x,u,t)

$$(a(x,u) + b(x,u)t)^+ = \big(a(x,u) + b^+(x,u)t\big)^+ + \big(a(x,u) - b^-(x,u)t\big)^+ - a^+(x,u),$$

that the functional

$$u \in L^1(\Omega,\mathbb{R}^n) \mapsto \int_\Omega a^+(x,u(x))\,dx$$

is continuous with respect to the L^1 convergence and that for all $u \in BV(\Omega,\mathbb{R}^n)$

$$\int_\Omega \big(a(x,u) - b^-(x,u)\det\nabla u\big)^+\,dx = \int_\Omega \big(\tilde a(x,\tilde u) + \tilde b^-(x,\tilde u)\det\nabla\tilde u\big)^+\,dx,$$

where $\tilde u = (-u^1, u^2, \ldots, u^n)$, $\tilde a(x,\tilde u) = a(x,u)$ and $\tilde b(x,\tilde u) = b(x,u)$.

Finally, we claim that we may further reduce to the case where

$$b(x,u) = \sum_{i=1}^k \beta_i(x) \prod_{j=1}^n \gamma_{ij}(u^j), \tag{5.7}$$

for some continuous functions $\beta_i : \Omega \to [0,\infty)$, $\gamma_{ij} : \mathbb{R} \to [0,\infty)$ with compact supports.

In fact, recall that from Lemma 5.2, there exists a sequence of functions s_j of the type of the functions appearing on the right hand side of (5.7), converging uniformly from below to b in $\Omega \times \mathbb{R}^n$. Then, if ε_j is a sequence of strictly positive numbers converging to zero, observe that for all (x,u,t)

$$(a(x,u)+b(x,u)t)^+ \geq \Big(a^+(x,u)\frac{s_j(x,u)}{b(x,u)+\varepsilon_j} - a^-(x,u) + s_j(x,u)t\Big)^+. \tag{5.8}$$

In fact, if $b(x,u)=0$, then also $s_j(x,u)=0$ and the inequality above is trivial, while if $b(x,u)>0$ we only need to show the inequality when

$$a^+(x,u)\frac{s_j(x,u)}{b(x,u)+\varepsilon_j} - a^-(x,u) + s_j(x,u)t > 0,$$

i.e. when $s_j(x,u)>0$ and

$$t > \frac{a^-(x,u)}{s_j(x,u)} - \frac{a^+(x,u)}{b(x,u)+\varepsilon_j}. \tag{5.9}$$

To this aim, note that

$$a+bt = \Big(\frac{a^+ s_j}{b+\varepsilon_j} - a^- + s_j t\Big) + \Big[(b-s_j)t + \frac{a^+(b+\varepsilon_j - s_j)}{b+\varepsilon_j}\Big].$$

So the proof of (5.8) will follow if we show that the quantity in square brackets is nonnegative. This is in turn a consequence of (5.9), since

$$(b-s_j)t + \frac{a^+(b+\varepsilon_j-s_j)}{b+\varepsilon_j} \geq -\frac{a^+(b-s_j)}{b+\varepsilon_j} + \frac{a^+(b+\varepsilon_j-s_j)}{b+\varepsilon_j} \geq 0.$$

Then, the convergence of the right hand side of (5.8) to the left hand follows from the convergence of $s_j(x,u)$ to $b(x,u)$ and from (5.6).

Finally, replacing β_i with $\beta_i \vee \delta - \delta$ if necessary, $\delta>0$, in such a case note that $0 \leq \beta_i \vee \delta - \delta \leq \beta_i$, and arguing as in the proof of the previous claim, we may also assume that for all $i=1,\dots,k$

$$\mathcal{L}^n(\partial\Omega_i^+) = 0, \quad \text{where} \quad \Omega_i^+ = \{x \in \Omega : \beta_i(x) > 0\}.$$

Step 2. *The lower semicontinuity property.*

Let us now assume $f(x,u,t) = (a(x,u)+b(x,u)t)^+$ with b satisfying (5.6) and (5.7), and let us consider a sequence of functions u_j satisfying (5.2). We may also suppose that $\sup_j \|u_j\|_\infty < \infty$. Otherwise, set for $M>0$

$$v_{j,M}(x) = \begin{cases} u_j(x) & \text{if } |u_j(x)| \leq M \\ M\dfrac{u_j(x)}{|u_j(x)|} & \text{otherwise,} \end{cases}$$

and define u_M similarly. Then, being the functions a and b with compact support in $\Omega \times \mathbb{R}^n$, we have $F(v_{j,M}) = F(u_j)$, $F(u_M) = F(u)$ if M is large enough, and then it is sufficient to establish the lower semicontinuity result for the sequence $(v_{j,M})_j$.

As usual, in order to prove (5.3) we may assume that the inferior limit on the right hand side is a limit and that this limit is finite. Therefore, we have that

$$\sum_{i=1}^k \int_\Omega \beta_i(x)\Big(\prod_{h=1}^n \gamma_{ih}(u_j^h)\Big)\big(\det \nabla u_j\big)^+ \, dx \leq C,$$

for some positive constant C. Let us now set for any $j \in \mathbb{N}$ and $i=1,\dots,k$

$$v_{j,i} = (B_{i1}(u_j^1), \dots, B_{in}(u_j^n)), \quad \text{where} \quad B_{ih}(r) = \int_0^r \gamma_{ih}(\varrho)\, d\varrho.$$

The functions v_i are defined similarly, by substituting u_j with u in the equality above. We then have that $(v_{j,i})_j$ converges to v_i in $L^1(\Omega,\mathbb{R}^n)$ and

$$\sup_j \sum_{i=1}^k \int_{\Omega_i^+} \beta_i(x)\big(\det \nabla v_{j,i}\big)^+ \, dx < \infty. \tag{5.10}$$

In addition, $(v_{j,i})_j$ is bounded in L^∞ and in $W^{1,n-1}$, so that, up to a subsequence not relabeled for convenience, $(\det \nabla v_{j,i})_j$ converge, in the sense of distributions on Q_1. Therefore, in view of (5.10), Lemma 2.4 and a diagonal argument imply that $(\det \nabla v_{j,i})_j$ locally weakly* converges in the sense of measures on Ω_i^+. Then, [5, Lemma 1.2] (see also [14, Theorem 1.2]) yields that in fact $(\det \nabla v_{j,i})_j$ locally weakly* converge in Ω_i^+ to a Radon measure μ_i whose absolutely continuous part with respect to the Lebesgue measure is $\det \nabla v_i$.

Let us now fix a nonnegative continuous function φ, $0 \le \varphi \le 1$, with compact support in Ω and a nonnegative continuous function ψ, $0 \le \psi \le 1$, with compact support in $\Omega \setminus \cup_{i=1}^k \partial\Omega_i^+$. Then, we get

$$\begin{aligned}
\liminf_j \int_\Omega f(x,u_j,\det\nabla u_j)\,dx &\ge \liminf_j \int_\Omega (a(x,u_j)+b(x,u_j)\det\nabla u_j)\,\varphi\,\psi\,dx \\
&= \lim_j \int_\Omega a(x,u_j)\,\varphi\,\psi\,dx + \sum_{i=1}^k \liminf_j \int_{\Omega_i^+} \beta_i(x)\,\varphi\,\psi\, \det\nabla v_{j,i}\,dx \\
&= \int_\Omega a(x,u)\,\varphi\,\psi\,dx + \sum_{i=1}^k \int_{\Omega_i^+} \beta_i(x)\,\varphi\,\psi\, \det\nabla v_i\,dx + \sum_{i=1}^k \int_{\Omega_i^+} \varphi\,\psi\,d\mu_i^s \\
&= \int_\Omega (a(x,u)+b(x,u)\det\nabla u)\varphi\,\psi\,dx + \sum_{i=1}^k \int_{\Omega_i^+} \varphi\,\psi\,d\mu_i^s,
\end{aligned}$$

recalling the definition of v_i.

Thus, taking the supremum over all the possible positive continuous functions φ, $0 \le \varphi \le 1$, we get

$$\liminf_j \int_\Omega f(x,u_j,\det\nabla u_j)\,dx \ge \nu_\psi^+(\Omega)$$

where for all Borel subset E in Ω

$$\nu_\psi(E) := \int_E (a(x,u)+b(x,u)\det\nabla u)\ \psi\,dx + \sum_{i=1}^k \int_{E\cap\Omega_i^+} \psi\,d\mu_i^s.$$

Since the measure $\sum_{i=1}^k \psi\mu_i^s$ is singular with respect to the Lebesgue measure, we have

$$\nu_\psi^+(\Omega) \ge \int_\Omega (a(x,u)+b(x,u)\det\nabla u)^+\ \psi\,dx.$$

Letting $\psi \uparrow 1$ in $\Omega \setminus \cup_{i=1}^k \partial\Omega_i^+$

$$\liminf_j \int_\Omega f(x,u_j,\det\nabla u_j)\,dx \ge \int_\Omega (a(x,u)+b(x,u)\det\nabla u)^+\ dx.$$

□

We next discuss the relationship between assumption (4.8) in Proposition 4.2 and assumption (5.1) in Theorem 5.1.

Lemma 5.3. *Let $f \in C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma)$, and $f(x,u,\cdot)$ is convex for all (x,u), and let $g : \Omega \times \mathbb{R}^m \to [0,\infty)$ be defined by*

$$g(x,u) := \inf\{f(x,u,t) : t \in \mathbb{R}\} \qquad \text{for all } (x,u),$$

and assume that g is continuous. Then, f satisfies equality (4.8).

Proof. Let h be the function defined in (4.9) and (4.10), then note that by the very definition of g and h, we have that

$$h(x,u) \le g(x,u) \le f(x,u,t) \quad \text{for all } (x,u,t). \tag{5.11}$$

Let (x_0,u_0) be as in condition (ii) of Proposition 4.2, then by definition

$$g(x_0,u_0) = f(x_0,u_0,t) \quad \text{for all } t. \tag{5.12}$$

Note that by the continuity assumption, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|g(y,v) - g(x_0,u_0)| \le \varepsilon \quad \text{for all } (y,v) \in B_\delta(x_0,u_0).$$

For such points (y,v) we have

$$f(y,v,t) \ge g(y,v) \ge g(x_0,u_0) - \varepsilon, \quad \text{for all } t,$$

that gives

$$h_\delta(x_0,u_0) \ge g(x_0,u_0) - \varepsilon,$$

and then $h(x_0,u_0) = g(x_0,u_0)$ by (5.11). The conclusion then follows from equality (5.12). □

To prove Lemma 5.2 we need first some technical results. We start with an elementary algebraic lemma.

Lemma 5.4. *For any integer $k \ge 2$ there exist $N \in \mathbb{N}$, $q_r > 0$, $p_{i,r} > 0$, $\sigma_{i,r} \in \{0,-1,1\}$, $1 \le r \le N$ and $1 \le i \le k$, such that*

$$1 - \prod_{i=1}^k x_i = \sum_{r=1}^N q_r \prod_{i=1}^k (p_{i,r} - \sigma_{i,r} x_i). \tag{5.13}$$

Note that from (5.13) a similar formula holds for $1 + \prod_{i=1}^k x_i$ with the same q_r's, $p_{i,r}$'s and changing the sign of only one of the $\sigma_{i,r}$'s.

Proof. For $k = 2$ we have

$$1 - x_1 x_2 = \frac{1}{3} - x_1 + \frac{1}{3}(1 + x_2) + \left(\frac{1}{3} + x_1\right)(1 - x_2), \tag{5.14}$$

thus proving formula (5.13). Suppose inductively that (5.13) holds up to some k, then by (5.14) we have

$$1 - (\prod_{i=1}^k x_i) x_{k+1} = \frac{1}{3} - \prod_{i=1}^k x_i + \frac{1}{3}(1 + x_{k+1}) + \left(\frac{1}{3} + \prod_{i=1}^k x_i\right)(1 - x_{k+1}),$$

the result follows by applying (5.13) to the products of the first k variables. □

Lemma 5.5. *For every $\varepsilon > 0$ there exist N_ε nonnegative functions $g_i \in C_c(\mathbb{R}^k)$ such that*

$$\operatorname{diam}(\operatorname{supp} g_i) < \varepsilon, \quad 1 - \varepsilon \le \sum_{i=1}^{N_\varepsilon} g_i(x) \le 1,$$

for all $x \in Q_1$, and each function g_i is the finite sum of products of continuous, nonnegative functions of one variable with compact supports.

Proof. Fix an integer $p \geq 3$ and divide Q_1 into p^k subcubes $Q_{1/p}(x_h)$ where $x_h = (x_{h,1}, \ldots, x_{h,k})$.

Fix $\varphi \in C^1_c(\mathbb{R})$ such that $\varphi = 1$ on $|t| \leq 1/2$, $0 \leq \varphi \leq 1$, and $\operatorname{supp}\varphi \subset (-1/2 - 1/p, 1/2 + 1/p)$. Define, for $1 \leq h \leq p^k$,

$$\varphi_h(x) := \prod_{i=1}^{k} \varphi((x_i - x_{h,i})p).$$

Note that $\operatorname{supp}\varphi_h$ is contained in the cube with center x_h and side $1/p + 2/p^2$ and intersects at most $3^k - 1$ supports of the remaining functions. Therefore, setting

$$\psi(x) := \sum_{h=1}^{p^k} \varphi_h(x)$$

we have that

$$1 \leq \psi(x) \leq 3^k \quad \text{for every } x \in \mathbb{R}^k.$$

Take now a large integer $M > 3^k$ to be chosen later and set for every $x \in \mathbb{R}^k$ and $1 \leq h \leq p^k$

$$\psi_h(x) := \frac{\varphi_h(x)}{\psi(x)} = \frac{\frac{\varphi_h(x)}{M}}{1 - (1 - \frac{\psi(x)}{M})}$$

The family $(\psi_h)_h$ provides a partition of unity in Q_1, though each of its members is not a finite sum of products of functions depending on one variable.

To this aim, observe that

$$\psi_h(x) = \frac{\varphi_h(x)}{M} \sum_{i=0}^{\infty} \left(1 - \frac{\psi(x)}{M}\right)^i,$$

therefore if we choose $M \geq 3^k \vee 2\|\psi\|_{L^\infty}$, we can find an integer q such that setting

$$g_h(x) := \frac{\varphi_h(x)}{M} \sum_{i=0}^{q} \left(1 - \frac{\psi(x)}{M}\right)^i,$$

we have that

$$1 - \varepsilon \leq \sum_{h=1}^{p^k} g_h(x) \leq 1$$

for all $x \in Q_1$. To conclude we are left with proving that the functions $x \to 1 - \frac{\psi(x)}{M}$ can be written as finite sum of products of functions of one variable. In fact, observe that

$$\begin{aligned} 1 - \frac{\psi(x)}{M} &= 1 - \frac{1}{M} \sum_{h=1}^{p^k} \varphi_h(x) = 1 - \frac{1}{M} \sum_{h=1}^{p^k} \prod_{i=1}^{k} \varphi((x_i - x_{h,i})p) \\ &= \frac{1}{p^k} \sum_{h=1}^{p^k} \left[1 - \prod_{i=1}^{k} \frac{p}{\sqrt[k]{M}} \varphi((x_i - x_{h,i})p)\right]. \end{aligned}$$

By applying Lemma 5.4 we get for every h

$$1 - \prod_{i=1}^{k} \frac{p}{\sqrt[k]{M}} \varphi((x_i - x_{h,i})p) = \sum_{r=1}^{N} q_r \prod_{i=1}^{k} \left[p_{i,r} - \frac{p}{\sqrt[k]{M}} \sigma_{i,r} \varphi((x_i - x_{h,i})p)\right]$$

and the result is proved if we choose M satisfying in addition

$$\frac{p}{\sqrt[k]{M}} \leq \min\{p_{i,r} : 1 \leq i \leq k,\, 1 \leq r \leq N\}.$$

□

Proof of Lemma 5.2. Without loss of generality we may assume that g has support contained in Q_1. With fixed $\varepsilon > 0$ choose $\delta > 0$ so that if $|x-y| < \delta$ then $|g(x) - g(y)| < \varepsilon$. Consider a finite family Lemma 5.4 provides a finite family of functions g_i, $1 \le i \le N_\delta$, with $\mathrm{diam}(\mathrm{supp}\, g_i) < \delta$, $1 - \delta \le \sum_{i=1}^{N_\delta} g_i(x) \le 1$.

Then the required approximation is obtained setting

$$g_\varepsilon(x) := \sum_{i=1}^{N_\delta} m_i\, g_i(x) \quad \text{with} \quad m_i = \min_{x \in \mathrm{supp} g_i} g.$$

□

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