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Weak sharp minima in multiobjective optimization

by

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*Dedicated to Professor Stefan Rolewicz on the occasion of his 75th birthday**

Abstract: We extend some necessary and sufficient conditions for strict local Pareto minima of order m obtained by Jiménez (2002) to the case of weak ψ-sharp local Pareto minima, i.e., to the case when the local solution is not necessarily unique.

Keywords: weak ψ-sharp Pareto minima, nonsmooth functions, multiobjective optimization.

1. Introduction

The aim of this paper is to obtain necessary and sufficient conditions for weak ψ-sharp local Pareto minima in multiobjective optimization. The notion of a weak ψ-sharp local Pareto minimum, introduced below (see Definition 2), is a generalization of the well-known notion of a weak sharp local minimum of order m. Weak sharp (local or global) minima of order m are some special types of possibly non-isolated minima where the objective function is constant on a given set of minimizers and satisfies a certain "growth condition" outside this set. For scalar optimization problems, they have been studied in a number of papers, see e.g. Bonnans and Ioffe (1995), Burke and Deng (2002, 2005), Burke and Ferris (1993), Klatte (1994), Ng and Zheng (2003), Studniarski (1999, 2000), Studniarski and Taha (2003), Studniarski and Ward (1999), Ward (1994, 1998). Also for the scalar case, more general definitions of ψ-conditioning (see Cornejo, Jourani and Zălinescu, 1997) and well-conditioning (see Pallaschke and Rolewicz, 1997) have been considered, which are very similar to our definition of a weak ψ-sharp minimum.

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When the minimum point is locally unique, we obtain the concept of a sharp (or strict) local minimum. For this particular case, there were recently two different attempts to extend this notion to multiobjective constrained optimization problems: the first one by Jiménez (2002) and the second one by Zheng, Yang and Teo (2006). These two approaches use two definitions of sharp minima which are essentially different for constrained problems, but equivalent for unconstrained ones. The results of the present paper are motivated by the first approach. We will extend some optimality conditions from Jiménez (2002) to the case of non-unique local minimizers.

The first attempt to define weak sharp minima of order m in the context of vector optimization was made by Bednarczuk (2004). She used weak sharp minima to prove upper Hölder continuity and Hölder calmness of the solution set-valued mapping for a parametric optimization problem. For a comparison of her definitions with ours, see Remark 1 below.

2. Problems in normed spaces

Let X and Y be normed spaces, let Ω be an open subset of X, $S \subset \Omega$, $\bar{x} \in S$, and let $D \subset Y$ be a cone (containing 0). The cone D defines an *order structure on* Y, that is, a relation $\leq$ in $Y \times Y$ is defined by

$$y \leq y' \Leftrightarrow y' - y \in D.$$

Obviously, $\leq$ is reflexive. Moreover, $\leq$ is transitive if an only if D is convex, and $\leq$ is antisymmetric if and only if D is pointed (see Göpfert, Riahi, Tammer and Zălinescu, 2003, Definition 2.1.1 and Theorem 2.1.13). Following Jiménez (2002), we do not assume here that D is convex, pointed, or closed. This general order structure is sufficient for formulating the two definitions below and proving Proposition 1. In all further results D is the positive orthant in a finite-dimensional space (and then, of course, $\leq$ is both transitive and antisymmetric).

In the sequel, $B(x, \delta)$ denotes the open ball with center $x \in X$ and radius $\delta > 0$, $\mathcal{N}(x)$ is the family of all neighborhoods of x, and $\operatorname{dist}(x, W)$ is the distance from the point x to the set $W \subset X$. The symbols $\operatorname{cl} S$ and $\operatorname{bd} S$ denote, respectively, the closure and the boundary of S.

Given a function $f : \Omega \to Y$, the following abstract multiobjective optimization problem is considered:

$$\min \{f(x) : x \in S\}. \tag{1}$$

Definition 1 *We say that $\bar{x}$ is a* local Pareto minimizer *for (1), denoted $\bar{x} \in \operatorname{LMin}(f, S)$, if there exists $U \in \mathcal{N}(\bar{x})$ for which there is no $x \in S \cap U$ such that*

$$f(x) - f(\bar{x}) \in (-D) \backslash D. \tag{2}$$

If we can choose $U = X$, we will say that $\bar{x}$ is a Pareto minimizer *for (1), denoted $\bar{x} \in \operatorname{Min}(f, S)$.*

Note that (2) may be replaced by the simpler condition $f(x) - f(\bar{x}) \in (-D)\backslash\{0\}$ if we assume that D is pointed.

Definition 2 *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function with the property $\psi(t) = 0 \Leftrightarrow t = 0$ (such functions are called* admissible*; see Bednarczuk, 2004, p. 458). We say that $\bar{x}$ is a* weak ψ-sharp local Pareto minimizer *for (1), denoted $\bar{x} \in \mathrm{WSL}(\psi, f, S)$, if there exist $\alpha > 0$ and $U \in \mathcal{N}(\bar{x})$ such that*

$$(f(x) + D) \cap B\left(f(\bar{x}), \alpha\psi(\mathrm{dist}(x, W))\right) = \emptyset, \quad \forall x \in S \cap U\backslash W, \tag{3}$$

where

$$W := \{x \in S : f(x) = f(\bar{x})\}. \tag{4}$$

If we can choose $U = X$, we will say that $\bar{x}$ is a weak ψ-sharp Pareto minimizer *for (1), denoted $\bar{x} \in \mathrm{WS}(\psi, f, S)$. In particular, let $\psi_m(t) := t^m$ for $m = 1, 2, \ldots$ Then we say that $\bar{x}$ is a* weak sharp local Pareto minimizer of order m *for (1) if $\bar{x} \in \mathrm{WSL}(\psi_m, f, S)$, and we say that $\bar{x}$ is a* weak sharp Pareto minimizer of order m *for (1) if $\bar{x} \in \mathrm{WS}(\psi_m, f, S)$.*

Condition (3) can be expressed in the following equivalent forms:

$$f(x) \notin f(\bar{x}) + B\left(0, \alpha\psi(\mathrm{dist}(x, W))\right) - D, \quad \forall x \in S \cap U\backslash W, \tag{5}$$

$$\mathrm{dist}\left(f(x) - f(\bar{x}), -D\right) \geq \alpha\psi(\mathrm{dist}(x, W)), \quad \forall x \in S \cap U\backslash W. \tag{6}$$

Note that in (6) one can take, equivalently, $S \cap U$ instead of $S \cap U\backslash W$. In particular, if $Y = \mathbb{R}$, $D = [0, +\infty)$ and $\psi = \psi_m$, then (6) reduces to

$$f(x) - f(\bar{x}) \geq \alpha\, \mathrm{dist}^m(x, W), \quad \forall x \in S \cap U,$$

which is the well-known definition of a *weak sharp local minimizer of order m* for (1); see Studniarski and Ward (1999). On the other hand, if $W = \{\bar{x}\}$ and $\psi = \psi_m$ in Definition 2, we obtain the definition of a *strict local Pareto minimizer of order m* for (1) (respectively, if $U = X$, of a *strict Pareto minimizer of order m* for (1)); see Jiménez (2002), Definition 3.1.

Remark 1 *Under the assumption that D is closed, convex and pointed, Bednarczuk (2004, 2006) defines a* local weak sharp solution of order m *for a vector optimization problem in two different ways. In Bednarczuk (2004), Definition 4.2, the set $\mathrm{Min}(f, S)$ (in our notation) is used in place of the set W defined by (4), which gives a weaker condition than our Definition 2 with $\psi = \psi_m$. On the other hand, Definition 8.2.3 in Bednarczuk (2006) is equivalent to Definition 2 with $\psi = \psi_m$.*

The following result is a generalization of Proposition 3.4 in Jiménez (2002).

PROPOSITION 1 *$\bar{x} \notin \mathrm{WSL}(\psi, f, S)$ if and only if there exist sequences $x_k \in S \setminus \operatorname{cl} W$, $d_k \in D$, such that $x_k \to \bar{x}$ and*

$$\lim_{k\to\infty} \frac{f(x_k) - f(\bar{x}) + d_k}{\psi(\operatorname{dist}(x_k, W))} = 0. \tag{7}$$

Proof. Taking into account the equivalence of (3) and (6), we have that $\bar{x} \notin \mathrm{WSL}(\psi, f, S)$ if and only if there exists a sequence $x_k \in S \setminus \operatorname{cl} W$ such that

$$\lim_{k\to\infty} \frac{\operatorname{dist}(f(x_k) - f(\bar{x}), -D)}{\psi(\operatorname{dist}(x_k, W))} = 0. \tag{8}$$

But, clearly, (8) holds if and only if there exists a sequence $d_k \in D$ such that

$$\lim_{k\to\infty} \frac{\|f(x_k) - f(\bar{x}) + d_k\|}{\psi(\operatorname{dist}(x_k, W))} = 0,$$

which is equivalent to (7). ■

PROPOSITION 2 *Let $Y = \mathbb{R}^p$ and $D = \mathbb{R}^p_+ = [0, +\infty)^p$. Then the following conditions are equivalent:*

(i) *$\bar{x} \notin \mathrm{WSL}(\psi, f, S)$;*

(ii) *there exist a vector $\bar{d} \in [-\infty, 0]^p$ and a sequence $x_k \in S \setminus \operatorname{cl} W$ such that $x_k \to \bar{x}$ and*

$$\lim_{k\to\infty} \frac{f(x_k) - f(\bar{x})}{\psi(\operatorname{dist}(x_k, W))} = \bar{d}. \tag{9}$$

Proof. This result can be deduced from Proposition 1. The detailed proof can be obtained by replacing the term $\|x_k - \bar{x}\|^m$ with $\psi(\operatorname{dist}(x_k, W))$ in the proof of Proposition 3.5 in Jiménez (2002) (except for the second part of (b) which is not needed here). ■

We end this section with a "weak sharp" counterpart of Theorem 3.7 in Jiménez (2002).

THEOREM 1 *Let $Y = \mathbb{R}^p$ and $D = \mathbb{R}^p_+$. Denote $I := \{1, ..., p\}$ and suppose that W is closed.*

(a) *$\bar{x} \in \mathrm{WSL}(\psi, f, S)$ if and only if there exist $\alpha > 0$, $U \in \mathcal{N}(\bar{x})$ and at most p sets S_i, $i \in I' \subset I$, such that $\{S_i : i \in I'\}$ is a covering of $S \cap U$, and*

$$f_i(x) > f_i(\bar{x}) + \alpha\psi(\operatorname{dist}(x, W)), \quad \forall x \in S_i \setminus W, \forall i \in I'. \tag{10}$$

(b) *$\bar{x} \in \mathrm{LMin}(f, S)$ if and only if there exist $U \in \mathcal{N}(\bar{x})$ and at most p sets S_i, $i \in I' \subset I$, such that $\{S_i : i \in I'\}$ is a covering of $S \cap U$, and*

$$f_i(x) > f_i(\bar{x}), \quad \forall x \in S_i \setminus W, \forall i \in I'. \tag{11}$$

Proof. We will prove the theorem for the case where Y is endowed with the maximum norm $\|y\|_\infty := \max_{1\le i\le p}\left|y^i\right|$. Since every two norms on $\mathbb{R}^p$ are equivalent and the function ψ is nondecreasing, the theorem also holds for an arbitrary norm.

(a) First observe that, by the definition of $\|\cdot\|_\infty$, we have

$$\operatorname{dist}(y, -\mathbb{R}^p_+) = \max\left\{(y^1)^+, ..., (y^p)^+\right\}, \quad \forall y = (y^1, ..., y^p) \in \mathbb{R}^p, \tag{12}$$

where $\beta^+ := \max\{\beta, 0\}$ for $\beta \in \mathbb{R}$.

Part "only if": Let $\bar{x} \in \mathrm{WSL}(\psi, f, S)$ and let $\alpha > 0$ and $U \in \mathcal{N}(\bar{x})$ be given by Definition 2. Define

$$S_i := \{x \in S \cap U : f_i(x) - f_i(\bar{x}) \ge \alpha\psi(\operatorname{dist}(x, W))\}, \quad i \in I. \tag{13}$$

By (12), we have

$$\operatorname{dist}(f(x) - f(\bar{x}), -\mathbb{R}^p_+) = \max_{i\in I}\left(f_i(x) - f_i(\bar{x})\right)^+. \tag{14}$$

We will show that

$$S \cap U \subset \bigcup_{i\in I} S_i. \tag{15}$$

Let $x \in S \cap U$. If $x \in W$, then $f(x) = f(\bar{x})$ by (4), hence $x \in S_i$ for all i. If $x \notin W$, then by (6) and (14),

$$\max_{i\in I}\left(f_i(x) - f_i(\bar{x})\right)^+ \ge \alpha\psi(\operatorname{dist}(x, W)).$$

This inequality and (13) imply that $x \in S_j$ for some $j \in I$. We have thus proved that $\{S_i : i \in I\}$ covers $S \cap U$.

Now, let $i \in I$ and $x \in S_i\backslash W$. By (13), we have

$$f_i(x) - f_i(\bar{x}) \ge \alpha\psi(\operatorname{dist}(x, W)) > \frac{\alpha}{2}\psi(\operatorname{dist}(x, W))$$

because $\operatorname{dist}(x, W) > 0$. The conclusion follows by taking $I' = I$ and $\alpha/2$ instead of α.

Part "if": By assumption, there exist $\alpha > 0$, $U \in \mathcal{N}(\bar{x})$ and a covering $\{S_i : i \in I'\}$ of $S \cap U$ such that (10) holds. Let $x \in S \cap U\backslash W$. Then there exists $i \in I'$ such that $x \in S_i\backslash W$. By (10), we get $f_i(x) - f_i(\bar{x}) > \alpha\psi(\operatorname{dist}(x, W))$, and so

$$\operatorname{dist}(f(x) - f(\bar{x}), -\mathbb{R}^p_+) \ge \left(f_i(x) - f_i(\bar{x})\right)^+ > \alpha\psi(\operatorname{dist}(x, W)).$$

Hence, (6) holds.

(b) It is easy to verify that $\bar{x} \in \mathrm{LMin}(f, S)$ if and only if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$\max_{i\in I}\left(f_i(x) - f_i(\bar{x})\right)^+ > 0, \quad \forall x \in S \cap U\backslash W. \tag{16}$$

Part "only if": Let $\bar{x} \in \mathrm{LMin}(f, S)$ and let $U \in \mathcal{N}(\bar{x})$ be such that (16) holds. Define

$$S_i := \{x \in S \cap U : f_i(x) > f_i(\bar{x}) \text{ or } f(x) = f(\bar{x})\}, \quad i \in I. \tag{17}$$

We will demonstrate (15). Let $x \in S \cap U$. If $x \in W$, then $f(x) = f(\bar{x})$, hence $x \in S_i$ for all i. If $x \notin W$, then by (16), we have $\max_{i \in I} (f_i(x) - f_i(\bar{x}))^+ > 0$, and so $f_j(x) > f_j(\bar{x})$ for some $j \in I$; hence $x \in S_j$.

Now, let $i \in I$ and $x \in S_i \backslash W$. By (17), we have $f_i(x) > f_i(\bar{x})$, which means that (11) holds with $I' = I$.

Part "if": Let us take $U \in \mathcal{N}(\bar{x})$ and the sets S_i, $i \in I'$, which exist by assumption. We will verify condition (16). Let $x \in S \cap U \backslash W$. Then there exists $i \in I'$ such that $x \in S_i \backslash W$. By (11), we have $f_i(x) > f_i(\bar{x})$, hence $\max_{i \in I} (f_i(x) - f_i(\bar{x}))^+ > 0$. ∎

Remark 2 *By taking $U = X$ in part (a) (respectively, (b)) of Theorem 1, we obtain a necessary and sufficient condition for $\bar{x}$ to be in* $\mathrm{WS}(\psi, f, S)$ *(respectively,* $\mathrm{Min}(f, S)$*). This follows easily from the proof of the theorem.*

Example 1 *Let (a_k) be a strictly decreasing sequence of positive real numbers converging to 0, as for example $a_k = 1/k$; and let $A_1 := \{a_{2k+1} : k \in \mathbb{N}\}$, $A_2 := \{a_{2k} : k \in \mathbb{N}\}$. Define $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, by*

$$\varphi_i(t) := \begin{cases} -t & \text{if } t \in A_i, \\ |t| & \text{if } t \notin A_i, \end{cases}$$

and $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, where $f_i(x^1, x^2) := \varphi_i(x^1)$ for all $(x^1, x^2) \in \mathbb{R}^2$, $i = 1, 2$. Finally, let $\bar{x} = (0, 0)$, $D = \mathbb{R}^p_+$ and $S = \mathbb{R}^2$.

We will check by using Theorem 1(a) (and Remark 2) that $\bar{x}$ is a weak sharp Pareto minimizer of order 1 for (1), that is, $\bar{x} \in \mathrm{WS}(\psi_1, f, S)$. Let us observe that $W = \{0\} \times \mathbb{R}$ and $\mathrm{dist}((x^1, x^2), W) = |x^1|$. We choose $I' = \{1, 2\}$, $U = \mathbb{R}^2$, $\alpha = 1/2$ and $S_i := (\mathbb{R} \backslash A_i) \times \mathbb{R}$, $i = 1, 2$. Then one has $S_1 \cup S_2 = S$ and condition (10) holds because

$$f_i(x^1, x^2) = |x^1| > \frac{1}{2} |x^1| = f_i(\bar{x}) + \frac{1}{2} \mathrm{dist}((x^1, x^2), W)$$

for all $(x^1, x^2) \in S_i \backslash W$, $i = 1, 2$.

It is also possible to check that $\bar{x} \in \mathrm{WS}(\psi_1, f, S)$ by using directly Definition 2, but this requires some calculations which we do not include here.

3. Problems in finite-dimensional spaces

In this section we consider problem (1) in the case where $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$ and $D = \mathbb{R}^p_+$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. We now introduce a variant of the Mordukhovich normal cone.

DEFINITION 3 (Studniarski, 1999) *Let E and S be subsets of $\mathbb{R}^n$, and let $\bar{x} \in \operatorname{cl} E$. The* normal cone to E at $\bar{x}$ relative to S *is defined by*

$$N_S(E,\bar{x}) := \{y \in \mathbb{R}^n : \exists y_k \to y,\ x_k \to \bar{x},\ t_k \in (0,+\infty),\ w_k \in \mathbb{R}^n \\ \text{with } x_k \in S,\ w_k \in P(E,x_k) \text{ and } y_k = (x_k - w_k)/t_k\ (\forall k)\}, \tag{18}$$

where

$$P(E,x) := \{w \in \operatorname{cl} E : \|x - w\| = \operatorname{dist}(x,E)\} \tag{19}$$

is the metric projection of x onto E.

REMARK 3 (i) *If $S = \mathbb{R}^n$, then $N_S(E,\bar{x})$ is equal to to the* Mordukhovich normal cone *to E at $\bar{x}$ (see Mordukhovich, 2006, Vol. I, p. 8):*

$$N(E,\bar{x}) := \{y \in \mathbb{R}^n : \exists y_k \to y,\ x_k \to \bar{x},\ t_k \in (0,+\infty),\ w_k \in \mathbb{R}^n \\ \text{with } w_k \in P(E,x_k) \text{ and } y_k = (x_k - w_k)/t_k\ (\forall k)\}.$$

This cone has important applications in welfare economics; see Khan (1999) and Mordukhovich (2006), Vol. 2, Chapter 8.

(ii) *If $E = \{\bar{x}\}$, then $N_S(E,\bar{x})$ is equal to the well-known* contingent cone *to S at $\bar{x}$:*

$$K(S,\bar{x}) := \{y \in \mathbb{R}^n : \exists y_k \to y,\ x_k \to \bar{x},\ t_k \in (0,+\infty) \\ \text{with } x_k \in S \text{ and } y_k = (x_k - \bar{x})/t_k\ (\forall k)\}.$$

PROPOSITION 3 *Suppose that the set W defined by (4) is closed. Then the following conditions are equivalent:*

(i) $\bar{x} \notin \operatorname{WSL}(\psi, f, S)$;

(ii) *there exist a vector $\bar{d} \in [-\infty, 0]^p$ and sequences $x_k \in S \backslash W$ and $w_k \in P(W, x_k)$ such that $x_k \to \bar{x}$,*

$$\lim_{k\to\infty} \frac{x_k - w_k}{\|x_k - w_k\|} = y \tag{20}$$

for some $y \in N_S(W,\bar{x})$ with $\|y\| = 1$, and

$$\lim_{k\to\infty} \frac{f(x_k) - f(w_k)}{\psi\left(\|x_k - w_k\|\right)} = \bar{d}. \tag{21}$$

Proof. (i) $\Longrightarrow$ (ii): Let $\bar{x} \notin \operatorname{WSL}(\psi, f, S)$. By Proposition 2, there exist a vector $\bar{d} \in [-\infty, 0]^p$ and a sequence $x_k \in S \backslash W$ such that $x_k \to \bar{x}$ and condition (9) holds. Since W is closed, for each k, there exists a point $w_k \in P(W, x_k)$. It follows from (19) that $\|x_k - w_k\| = \operatorname{dist}(x_k, W)$, hence (9) implies (21). By taking subsequences of (x_k) and (w_k), if necessary, we can ensure the convergence, stated in (20), to some vector $y \in \mathbb{R}^n$. Now, using (18) with $t_k := \|x_k - w_k\|$, we can easily verify that $y \in N_S(W,\bar{x})$ and $\|y\| = 1$.

(i) $\Longleftarrow$ (ii): This follows immediately from (19) and Proposition 2. ∎

THEOREM 2 *Suppose that* W *is closed. Then the following conditions are equivalent:*

(i) $\bar{x} \in \mathrm{WSL}(\psi, f, S)$*;*

(ii) *for every* $y \in N_S(W, \bar{x})$ *with* $\|y\| = 1$ *and for every sequence* $x_k \in S \backslash W$ *and* $w_k \in P(W, x_k)$ *such that* $x_k \to \bar{x}$ *and (20) holds, we have*

$$\limsup_{k\to\infty} \frac{f_i(x_k) - f_i(w_k)}{\psi(\|x_k - w_k\|)} > 0, \quad \text{for some } i \in \{1, ..., p\}. \tag{22}$$

Proof. (i) $\Longrightarrow$ (ii): Suppose that (ii) is false, that is, there exist $y \in N_S(W, \bar{x})$, $\|y\| = 1$, and sequences $x_k \in S \backslash W$ and $w_k \in P(W, x_k)$ satisfying the following conditions: $x_k \to \bar{x}$, (20) and

$$\limsup_{k\to\infty} \frac{f_i(x_k) - f_i(w_k)}{\psi(\|x_k - w_k\|)} \leq 0, \quad \forall\, i \in \{1, ..., p\}. \tag{23}$$

We will prove that $\bar{x} \notin \mathrm{WSL}(\psi, f, S)$ by using Proposition 3 (and an argument similar to that used in the proof of Proposition 3.5(b) in Jiménez, 2002). To this end, we will show that, for suitable subsequences of (x_k) and (w_k), there exists a vector $\bar{d}$ satisfying (21). Let us define

$$d_k^i := \frac{f_i(x_k) - f_i(w_k)}{\psi(\|x_k - w_k\|)}, \quad i \in \{1, ..., p\},\ k \in \mathbb{N}.$$

Consider the first components of the vector $d_k = (d_k^1, ..., d_k^p)$. Let $d^1 := \limsup_{k\to\infty} d_k^1$. Then there exists an infinite set $K_1 \subset \mathbb{N}$ such that $d^1 = \lim_{K_1 \ni k \to\infty} d_k^1$. By (23), we have $d^1 \leq 0$ (it can be $d^1 = -\infty$). Now, let us consider the second components of the sequence $(d_k)_{k\in K_1}$. Again by (23), we have

$$d^2 := \limsup_{K_1 \ni k\to\infty} d_k^2 \leq \limsup_{k\to\infty} d_k^2 \leq 0.$$

Hence, there exists an infinite set $K_2 \subset K_1$ such that $d^2 = \lim_{K_2 \ni k\to\infty} d_k^2 \leq 0$ (it can be $d^2 = -\infty$). So, we have $\lim_{K_2 \ni k\to\infty}(d_k^1, d_k^2) = (d^1, d^2)$. Continuing this process, we obtain a vector $\bar{d} = (d^1, ..., d^p) \in [-\infty, 0]^p$ and an infinite set $K_p \subset \mathbb{N}$ such that

$$\bar{d} = \lim_{K_p \ni k\to\infty} d_k = \lim_{K_p \ni k\to\infty} \frac{f(x_k) - f(w_k)}{\psi(\|x_k - w_k\|)},$$

which means that (21) holds for the subsequences $(x_k)_{k\in K_p}$ and $(w_k)_{k\in K_p}$.

(i) $\Longleftarrow$ (ii): This follows immediately from Proposition 3. ∎

REMARK 4 *In the particular case where* $p = 1$, $S = \mathbb{R}^n$ *and* $\psi = \psi_m$*, Theorem 2 reduces to a variant of Theorem 2.2 in Studniarski and Ward (1999) with "*$\limsup$*" instead of "*$\liminf$*". Unfortunately, for* $p > 1$ *one cannot replace "*$\limsup$*" by "*$\liminf$*" in (22), as the following example shows.*

Example 2 *Let* (a_k), f, $\bar{x}$, D *and* S *be as in Example 1. We now choose the sequences* $x_k = (a_k, a_k) \to \bar{x}$, $t_k = a_k$, $w_k = (0, a_k) \in P(W, x_k)$. *Then*

$$y_k := \frac{x_k - w_k}{t_k} \to y := (1, 0) \in N_S(W, \bar{x}).$$

Let us note that $x_k \in S \backslash W$, $\|y\| = 1$ *and* $t_k = \|x_k - w_k\|$, *so (20) holds. However, we have*

$$\begin{aligned} \liminf_{k\to\infty} \frac{f_1(x_k) - f_1(w_k)}{\|x_k - w_k\|} &= \lim_{k\to\infty} \frac{-a_{2k+1}}{a_{2k+1}} = -1, \\ \liminf_{k\to\infty} \frac{f_2(x_k) - f_2(w_k)}{\|x_k - w_k\|} &= \lim_{k\to\infty} \frac{-a_{2k}}{a_{2k}} = -1. \end{aligned}$$

Therefore, condition (22) with "$\limsup$*" instead of "*$\liminf$*" is not satisfied.*

Definition 4 *Let* E *be a nonempty closed subset of* $\mathbb{R}^n$, *and let* $\varphi : \mathbb{R}^n \to \mathbb{R}$. *For* $x \in \operatorname{bd} E$ *and* $y \in \mathbb{R}^n$, *define*

$$\bar{d}_E^m \varphi(x; y) := \limsup_{\substack{\operatorname{bd} E \ni w \to x \\ (t,v)\to(0^+,y)}} \frac{\varphi(w + tv) - \varphi(w)}{t^m}. \tag{24}$$

(In particular, (x, y) *is an allowable choice of* (w, v)*.) For* $m = 1$, *we will write* $\bar{d}_E \varphi(x; y)$ *instead of* $\bar{d}_E^1 \varphi(x; y)$.

Theorem 3 *Suppose that* W *is closed. If* $\bar{x} \in \mathrm{WSL}(\psi_m, f, S)$, *then, for each* $y \in N_S(W, \bar{x})$ *with* $\|y\| = 1$, *there exists* $i \in \{1, ..., p\}$ *such that*

$$\bar{d}_W^m f_i(\bar{x}; y) > 0. \tag{25}$$

Proof. Let $\bar{x} \in \mathrm{WSL}(\psi_m, f, S)$, $y \in N_S(W, \bar{x})$ and $\|y\| = 1$. Then, by (18), there exist sequences $y_k \to y$, $x_k \to \bar{x}$, $t_k \in (0, +\infty)$, and $w_k \in \mathbb{R}^n$ such that $x_k \in S$, $w_k \in P(W, x_k)$ and

$$y_k = \frac{1}{t_k}(x_k - w_k), \quad \text{for all } k. \tag{26}$$

Since $\|y\| = 1$ and $y_k \to y$, we have $y_k \neq 0$ for k sufficiently large, and consequently, $\|x_k - w_k\| = \operatorname{dist}(x_k, W) > 0$, which implies that

$$x_k \in S \backslash W, \quad \text{for all } k. \tag{27}$$

Therefore, by taking a subsequence, if necessary, we can use (26) to obtain

$$\lim_{k\to\infty} \frac{x_k - w_k}{\|x_k - w_k\|} = \lim_{k\to\infty} \frac{t_k y_k}{\|t_k y_k\|} = \lim_{k\to\infty} \frac{y_k}{\|y_k\|} = \frac{y}{\|y\|} = y. \tag{28}$$

It follows from Theorem 2 and conditions (27) and (28) that there exists $i \in \{1, ..., p\}$ such that

$$\limsup_{k\to\infty} \frac{f_i(x_k) - f_i(w_k)}{\|x_k - w_k\|^m} > 0. \tag{29}$$

Observe that condition (26) implies

$$t_k = \frac{\|x_k - w_k\|}{\|y_k\|} \le \frac{\|x_k - \bar{x}\|}{\|y_k\|} \underset{k\to\infty}{\longrightarrow} 0. \tag{30}$$

Moreover, we have

$$\|w_k - \bar{x}\| \le \|w_k - x_k\| + \|x_k - \bar{x}\| \le 2\,\|x_k - \bar{x}\| \underset{k\to\infty}{\longrightarrow} 0. \tag{31}$$

Using (29)–(31), we conclude that

$$\begin{aligned}
\bar{d}^m_W f_i(\bar{x}; y) &= \limsup_{\substack{\mathrm{bd}\, W \ni w \to \bar{x} \\ (t,v)\to(0^+,y)}} \frac{f_i(w + tv) - f_i(w)}{t^m} \\
&\ge \limsup_{k\to\infty} \frac{f_i(w_k + t_k y_k) - f_i(w_k)}{t_k^m} \\
&= \limsup_{k\to\infty} \frac{f_i(x_k) - f_i(w_k)}{t_k^m} \\
&= \limsup_{k\to\infty} \left(\frac{f_i(x_k) - f_i(w_k)}{\|x_k - w_k\|^m} \|y_k\|^m \right) \\
&= \limsup_{k\to\infty} \frac{f_i(x_k) - f_i(w_k)}{\|x_k - w_k\|^m} \lim_{k\to\infty} \|y_k\|^m > 0. \qquad \blacksquare
\end{aligned}$$

REMARK 5 *In particular, if $W = \{\bar{x}\}$, then Theorem 3 reduces to Theorem 4.1(ii) in Jiménez (2002). Indeed, in this case, we have*

$$\bar{d}^m_{\{\bar{x}\}} f_i(\bar{x}; y) = \bar{d}^m f_i(\bar{x}; y) := \limsup_{(t,v)\to(0^+,y)} \frac{f_i(\bar{x} + tv) - f_i(\bar{x})}{t^m}$$

by (24), and $N_S(W, \bar{x}) = K(S, \bar{x})$ by Remark 3(ii).

For weak sharp local Pareto minimizers of order one, a necessary condition can be formulated in terms of *Clarke's generalized directional derivative*; see Clarke (1983). Recall that, for a locally Lipschitzian function $\varphi : \mathbb{R}^n \to \mathbb{R}$, this derivative is defined by

$$\varphi^\circ(x; y) := \limsup_{(t,w)\to(0^+,x)} \frac{\varphi(w + ty) - \varphi(w)}{t}.$$

PROPOSITION 4 *Let E be a nonempty closed subset of $\mathbb{R}^n$, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitzian. Then, for any $x \in \operatorname{bd} E$ and $y \in \mathbb{R}^n$, we have*

$$\varphi^\circ(x; y) \geq \bar{d}_E \varphi(x; y). \tag{32}$$

Proof. The local Lipschitz condition for φ implies that

$$\varphi^\circ(x; y) = \limsup_{(t,w,v) \to (0^+, x, y)} \frac{\varphi(w + tv) - \varphi(w)}{t}. \tag{33}$$

Inequality (32) now follows by comparison of (24) and (33). ■

From Theorem 3 and Proposition 4, we deduce the following corollary.

COROLLARY 1 *Suppose that W is closed. If f is locally Lipschitzian and $\bar{x} \in \mathrm{WSL}(\psi_1, f, S)$, then, for each $y \in N_S(W, \bar{x})$ with $\|y\| = 1$, there exists $i \in \{1, ..., p\}$ such that $f_i^\circ(\bar{x}; y) > 0$.*

REMARK 6 *The necessary conditions given in Theorem 3 and Corollary 1 are new even for the case of scalar optimization ($p = 1$). In Studniarski and Ward (1999), sufficient conditions for weak sharp local minima (for $p = 1$) were obtained in terms of the lower counterpart of derivative (24) (i.e., with "lim inf" in place of "lim sup"). Unfortunately, these sufficient conditions are not satisfied in many important situations; see Studniarski and Ward (1999), Example 2.1, and Studniarski (2000), Example 8. Obtaining better sufficient conditions for weak sharp local minima, for both scalar and vector optimization, is an open problem which will be the subject of further research.*

EXAMPLE 3 *Let $n = p = 2$, $S = \Omega = \mathbb{R}^2$, and let $f = (f_1, f_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by*

$$f_1(x^1, x^2) := \max\{0, \min\{x^1, x^2\}\} = \begin{cases} x^1, & \text{if } x^2 \geq x^1 > 0, \\ x^2, & \text{if } x^1 > x^2 > 0, \\ 0, & \text{if } x^1 \leq 0 \text{ or } x^2 \leq 0, \end{cases}$$

$$f_2(x^1, x^2) := \max\{0, \min\{-x^1, x^2\}\} = \begin{cases} -x^1, & \text{if } x^2 \geq -x^1 > 0, \\ x^2, & \text{if } -x^1 > x^2 > 0, \\ 0, & \text{if } x^1 \geq 0 \text{ or } x^2 \leq 0. \end{cases}$$

Using Theorem 1(a), we will show that $\bar{x} = (0, 0) \in \mathrm{WS}(\psi_1, f, S)$. Observe that

$$W = \{x : f(x) = (0, 0)\} = \{x : x^2 \leq 0\} \cup \{x : x^1 = 0\}.$$

We choose $U = \mathbb{R}^2$ and define $S_1 := W \cup \{x : x^1 > 0,\ x^2 > 0\}$ and $S_2 := W \cup \{x : x^1 < 0,\ x^2 > 0\}$. It is easy to verify that $f_i(x) = \operatorname{dist}(x, W)$ for all $x \in S_i \backslash W$, $i = 1, 2$, so that condition (10) with $\psi = \psi_1$ holds for any $\alpha \in (0, 1)$.

Now, we can see that the necessary condition of Corollary 1 is satisfied. We compute

$$N_S(W, \bar{x}) = N(W, \bar{x}) = \{x : x^2 = 0\} \cup \{x : x^1 = 0,\ x^2 > 0\}.$$

There are three unit vectors in this set: $(-1,0)$, $(0,1)$, $(1,0)$, *and for each of them, there is at least one component* f_i *of* f *with strictly positive value of the corresponding Clarke's derivative; namely,*

$$f_2^\circ(\bar{x};(-1,0)) = f_2^\circ(\bar{x};(0,1)) = f_1^\circ(\bar{x};(0,1)) = f_1^\circ(\bar{x};(1,0)) = 1. \tag{34}$$

However, the condition of Corollary 1 is not sufficient for a weak sharp local Pareto minimum of order one. To see this, we modify the previous example by adding the term $\max\{0,(x^2)^3\}$ *to each of the functions* f_i. *Then the new set* W *is* $\{x : x^2 \le 0\}$, *and the new normal cone* $N(W,\bar{x})$ *is* $\{x : x^1 = 0,\ x^2 > 0\}$. *The only unit direction in this normal cone is* $(0,1)$, *with Clarke's derivatives of both* f_i *in this direction being the same as in (34). But* $\bar{x} \notin \mathrm{WSL}(\psi_1, f, S)$ *since the condition of Theorem 1(a) is not satisfied (for any choice of* U *and* S_1, S_2*). Indeed, for each* $\alpha > 0$ *and* $i = 1,2$, *we have* $f_i(0,t) = t^3 < \alpha t = \alpha\,\mathrm{dist}((0,t),W)$ *for* $t > 0$ *sufficiently small, which contradicts (10).*

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