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WEAK SHARP MINIMA ON RIEMANNIAN MANIFOLDS

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Abstract. This is the first paper dealing with the study of weak sharp minima for constrained optimization problems on Riemannian manifolds, which are important in many applications. We consider the notions of local weak sharp minima, boundedly weak sharp minima, and global weak sharp minima for such problems and obtain their complete characterizations in the case of convex problems on finite-dimensional Riemannian manifolds and their Hadamard counterparts. A number of the results obtained in this paper are also new for the case of conventional problems in linear spaces. Our methods involve appropriate tools of variational analysis and generalized differentiation on Riemannian and Hadamard manifolds developed and efficiently implemented in this paper.

Key words. Variational analysis and optimization, Weak sharp minima, Riemannian manifolds, Hadamard manifolds, Convexity, Generalized differentiability

AMS subject classifications. Primary 49J52; Secondary 90C31

1. Introduction.

A vast majority of problems considered in optimization theory are formulated in finite-dimensional or infinite-dimensional Banach spaces, where the *linear structure* plays a crucial role to employ conventional tools of variational analysis and (classical or generalized) differentiation to deriving optimality conditions and then develop numerical algorithms. At the same time many optimization problems arising in various applications cannot be posted in linear spaces and require a *Riemannian manifold* (in particular, a *Hadamard manifold*) structure for their formalization and study. Among various problems of this type we mention geometric models for human spine [2], eigenvalue optimization problems [15, 45, 58], nonconvex and nonsmooth problems of constrained optimization in $\mathbb{R}^n$ that can be reduced to convex and smooth unconstrained optimization problems on Riemannian manifolds as in [19, 27, 48, 54, 59], etc. We refer the reader to [2, 6, 24, 33, 45, 49, 58, 59] and the bibliographies therein for more examples and discussions.

It is worth recalling that a strong interest in optimization problems formulated on Riemannian manifolds goes back to the very beginning of modern variational analysis; it was one of the *crucial motivations* for developing the fundamental *Ekeland variational principle* [25] in the framework of complete metric spaces, with *no linear structure*. The seminal Ekeland's paper [25] contains applications of his variational principle to the existence of minimal geodesics on Riemannian manifolds; see also [26] for further developments. More recently, a number of important results have been obtained on various aspects of optimization theory and applications for problems formulated on Riemannian and Hadamard manifolds as well as on other spaces with nonlinear structures; see, e.g., [1, 2, 9, 6, 20, 24, 29, 33, 43, 44, 45, 58, 59] and the references therein. Let us particularly mention Newton's method, the conjugate gradient method, the trust-region method, and their modifications extended from optimization problems on linear spaces to their Riemannian counterparts.

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On the other hand, the maximal monotonicity notion in Banach spaces extended to Riemannian manifolds makes it possible to develop a proximal-type method to find singular points for multivalued vector fields on Riemannian manifolds with nonpositive sectional curvatures, i.e., on Hadamard manifolds; see, e.g., [44] with other references. Furthermore, various derivative-like and subdifferential constructions for nondifferentiable functions on spaces with no linear structure are developed in [3, 5, 28, 39, 46, 47, 51] and applied therein to the study of constrained optimization problems, nonclassical problems of the calculus of variations and optimal control, and generalized solutions to the first-order partial differential equations on Riemannian manifolds and other important classes of spaces with no linearity.

This paper is devoted to the study of *weak sharp minimizers* for constrained optimization problems on *Riemannian* and *Hadamard manifolds.* To the best of our knowledge, it is the first work concerning the notions of this type for optimization problems on spaces with no linear structure. Recall that the notion of *sharp minima* is introduced by Polyak [53] in the case of finite-dimensional Euclidean spaces for the analysis of perturbation behavior of optimization problems and the convergence analysis of some numerical algorithms; a related notion of "strongly unique local minimum" can be found in the paper by Cromme [18]. Then Ferris [30] introduces in the same framework the notion of *weak sharp minima* to describe an extension of sharp minimizers in order to include the possibility of multiple solutions. The later notion has been extensively studied by many authors in finite-dimensional and infinite-dimensional linear spaces. Primary motivations for these studies relate to sensitivity analysis [14, 15, 36, 41, 52, 61, 62, 63] and to convergence analysis of a broad range of optimization algorithms [10, 11, 16, 18, 31, 32, 34]. In particular, Burke and Ferris [10] derive necessary optimality conditions for weak sharp minimizers, obtaining also their full characterizations in the case of convex problems of unconstrained minimization, with applications to convex programming and convergence analysis in finite-dimensional Euclidean spaces. Then Burke and Deng [12] extend necessary optimality conditions and characterization results from [10] to problems of constrained optimization in Banach spaces, study asymptotic properties of weak sharp minima in terms of associated recession functions, and establish some new characterizations of local weak sharp minimizers and the so-called boundedly weak sharp minimizes. Furthermore, in [13] they explore relationships between the notions of weak sharp minima, linear regularity, and error bounds. Linear regularity has been extensively studied in [7, 8], where its importance for designing algorithms has been revealed. Note that linear regularity is closely related to metric regularity and error bounds for convex inequalities that have been comprehensively studied by many authors; see, e.g., [4, 21, 37, 38, 41, 42, 49, 50, 64, 65] and the references therein.

In the linear space setting the characterizations of weak sharp minimizers for convex optimization problems have been obtained in two interrelated terms: one via the directional derivative of convex functions and the other via the normal cone of convex analysis to the corresponding solution set $\overline{S}$; see [10, 12]. The key ingredients to derive these characterizations are the following well-known representations in convex analysis on Banach spaces: of the *subdifferential* of the *distance function* $d(\cdot;\overline{S})$ to $\overline{S}$ given by

$$\partial d_{\overline{S}}(x) = \mathbb{B} \cap N_{\overline{S}}(x) \quad \text{for all} \quad x \in \overline{S} \tag{1.1}$$

via the normal cone $N_{\overline{S}}(\cdot)$ to $\overline{S}$ and the unit ball $\mathbb{B}$ in the space in question, and of the *projection operator* $P(\cdot|\overline{S})$ associated with the above solution set by

$$y \in P(x|\overline{S}) \Longleftrightarrow \langle x-y, z-y\rangle \leq 0 \quad \text{for all} \quad z \in \overline{S}. \tag{1.2}$$

One of the *primary goals* of this paper is to develop the aforementioned characterizations for appropriately defined notions of weak sharp minima for convex problems on Riemannian manifolds. However, significant technical difficulties arise in this way from the very beginning: the underlying representations are not known to hold on Riemannian manifolds. In particular, the distance function $d_{\overline{S}}(\cdot)$ may *not be convex* when the solution set $\overline{S}$ is convex in the case Riemannian manifolds. Our approaches in this paper to derive the corresponding counterparts of representations (1.1) and (1.2) and apply them to characterizing weak sharp minimizers are largely different from those used under linear structures. We establish a Riemannian counterpart of (1.2) employing *variational fields*, which do not depend on local charts on Riemannian manifolds. Furthermore, an analog of equality (1.1) for convex sets is derived below for *totally convex* sets in *Hadamard manifolds* exploiting their *nonpositive sectional curvatures*. Based on these and other developments, we establish *full characterizations* of global, local, and boundedly weak sharp minima for convex constrained optimization problems on Riemannian and Hadamard manifolds. Some of the characterizations obtained in this paper are appropriate extensions of known ones for spaces with linear structures, while a number of our results are new even for the case of finite-dimensional Euclidean spaces.

The rest of the paper is organized as follows. In Section 2 we present some basic constructions and preliminaries in linear spaces, mostly for convex functions and sets, widely used in the sequel. Section 3 is devoted to the *Riemannian manifold theory* and contains, together with certain known constructions and facts important in what follows, some new results on Riemannian manifolds that play a crucial role for the subsequent characterizations of weak sharp minima. In particular, we establish new descriptions of *projections* for closed subsets in Riemannian manifolds, via verifiable conditions on minimizing geodesics, and obtain their complete characterizations and other useful consequences in the presence of convexity.

Sections 4 and 5 present the *main results* of the paper. In Section 4 we define the notions of local weak sharp minima, boundedly weak sharp minima, and global weak sharp minima on *general Riemannian manifolds* presenting also their equivalent descriptions in the case of *convex* problems of constrained optimization. Then we derive a number of their characterizations in terms of the appropriate directional derivative, subdifferential, and normal cone constructions of convex analysis on Riemannian manifolds. Section 5 is devoted to weak sharp minimizers and their modifications for convex constrained problems on *Hadamard manifolds*. We establish a Hadamard space counterpart of representation (1.1) and on its base derive new characterizations for the aforementioned notions of weak sharp minima. The final Section 6 contains *concluding discussions* of the main results obtained in the paper, their comparison with known results in the case of linear spaces, and also addresses some forthcoming developments for weak sharp minima in *nonconvex* problems on Riemannian and Hadamard manifolds extending recent results in this direction for spaces with linear structures.

2. Some preliminaries in linear spaces. For the reader's convenience we review in this section some conventional notions, notation, and facts from *convex and variational analysis* in linear spaces used in what follows; see, e.g., [9, 49, 55] for more details. Let X be a normed space with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between X and its topological dual X^*. The symbol $\mathbb{B}$ always stands for the closed unit ball in the space in question. Given a set $C\subset X$, we denote its *interior* and *closure* by $\operatorname{int} C$ and $\operatorname{cl} C$, respectively. The *conic hull* generated by C and the *polar* to X are defined, respectively, by

$$\operatorname{cone} C := \bigcup_{\lambda\geq 0}\{\lambda C\} \;\text{ and }\; C^\circ := \big\{x^*\in X^*\big|\ \langle x^*,x\rangle\leq 1 \;\text{ for all }\; x\in C\big\}.$$

The *indicator function* $\delta_C(\cdot)$ of the set $C\subset X$ is given by

$$\delta_C(x) := \begin{cases} 0 & x\in C,\\ \infty & \text{otherwise,}\end{cases}$$

and the *support function* $\sigma_C(\cdot)$ of C is defined by

$$\sigma_C(x^*) := \sup_{x \in C} \langle x^*, x \rangle \quad \text{for all} \quad x^* \in X^*.$$

By $d_C(x) := \inf\{\|x - c\| \mid c \in C\}$ we denote the *distance function* of the set C. Note that the support function is always convex, while the indicator and distance function associated with the set are convex if and only if the set is convex. The results presented in the next proposition are proved in [12].

PROPOSITION 2.1. **(Properties of support and distance functions of convex sets).** *Let E, F be two subsets of X, and let K be a nonempty closed convex cone in X. The following assertions hold:*

(i) *$\sigma_E(x) \le \sigma_F(x)$ for each $x \in K$ if and only if $E \subset \text{cl}(F + K^\circ)$.*

(ii) *For all $x \in X$ we have the relationship $d_K(x) = \sigma_{\mathbb{B} \cap K^\circ}(x)$.*

Consider now an extended-real-valued function $g\colon X \to \overline{\mathbb{R}} := (-\infty, \infty]$ with the *effective domain* $\text{dom}\, g := \{x \in X \mid g(x) < \infty\}$ and the *epigraph* of g defined by

$$\text{epi}\, g := \big\{(x, r) \in X \times \mathbb{R} \big| \, g(x) \le r\big\}.$$

Unless otherwise stated, we assume in what follows that g is *convex* and *proper*, i.e., $\text{dom}\, g \ne \emptyset$. The *conjugate function* to g is defined by

$$g^*(x^*) := \sup\big\{\langle x^*, x\rangle - g(x) \big| \, x \in X\big\} \quad \text{for all} \quad x^* \in X^*.$$

It easily follows from the definitions that we have

$$\delta_C^*(x^*) = \sigma_C(x^*) \quad \text{for all} \quad x^* \in X^*. \tag{2.1}$$

Recall further that a function $g\colon X \to \overline{\mathbb{R}}$ is *lower semicontinuous* (l.s.c.) on X if its epigraph $\text{epi}\, g$ is closed in $X \times \mathbb{R}$. The *lower semicontinuous hull*, or the *closure* of g, is the function $\text{cl}\, g\colon X \to \overline{\mathbb{R}}$ with

$$\text{epi}(\text{cl}\ g) = \text{cl}(\text{epi}\ g),$$

which is the greatest l.s.c. function not exceeded g. The following fundamental *duality relationship* in convex analysis involving the *biconjugate function* $g^{**}\colon X^{**} \to \overline{\mathbb{R}}$ follows from [56, Theorems 4 and 5] provided that the initial space X is reflexive:

$$g^{**}(x) = \text{cl}\, g(x) \quad \text{for all} \quad x \in X. \tag{2.2}$$

3. Auxiliary results on Riemannian manifolds. This section contains necessary material on Riemannian manifolds needed for obtaining the main results on weak sharp minima in the subsequent sections. We start with basic definitions and reviewing the required known facts referring the reader to [23, 35] for more details and then derive new results of their own interest that play a crucial role in what follows. For simplicity our considerations are confined to *finite-dimensional* Riemannian manifolds, while it is worth mentioning that the major results obtained below admit natural extensions to infinite-dimensional settings by using advanced variational principles and techniques of modern *variational analysis* in infinite-dimensional spaces; see, e.g., [25, 26, 49] and the references therein.

Let M be a complete connected m-dimensional Riemannian manifold. By ∇ we denote the *Levi-Civita connection* on M. The collection of all *tangent vectors* of M at p forms an m-dimensional vector space and is denoted by T_pM. The union $U_{p\in M}(p, T_pM)$ forms a new manifold, which is called the *tangent bundle* of M and is denoted by TM. Recall that a *Riemannian metric* on a smooth manifold M is a 2-tensor field that is symmetric and positively definite. Every Riemannian metric thus determines an inner product and a norm on each tangent space T_pM, which are typically written as $\langle\cdot,\cdot\rangle_p$ and $\|\cdot\|_p$, where the subscript p may be omitted if no confusion arises. In this way we can treat the tangent space T_pM for each $p \in M$ as a usual finite-dimensional space denoting by $\mathbb{B}_p$ the closed unit ball of T_pM, i.e., $\mathbb{B}_p := \{v \in T_pM \mid \|v\| \le 1\}$.

Given two points $p,\ q \in M$, let $\gamma\colon [0,1] \to M$ be a piecewise smooth curve connecting p and q. Then we define the *arc-length* $l(\gamma)$ of $\gamma(\cdot)$ and the *Riemannian distance* from p to q by, respectively,

$$l(\gamma) := \int_0^1 \|\gamma'(t)\|\,dt \quad\text{and}\quad d(p,q) := \inf_{\gamma} l(\gamma),$$

where the infimum is taken over all piecewise smooth curves $\gamma\colon [0,1] \to M$ connecting p and q. Thus (M,d) is a *complete metric space* by the Hopf-Rinow theorem; see, e.g., [23]. Taking into account that M is complete, the *exponential map* at p denoted by $\exp_p\colon T_pM \to M$ is well-defined on T_pM. Recall further that a geodesic $\gamma(\cdot)$ on M connecting p and q is called a *minimizing geodesic* if its arc-length equals to the Riemannian distance between p and q. It is easy to see that a curve $\gamma\colon [0,1] \to M$ is a minimizing geodesic connecting p and q *if and only if* there is a vector $v \in T_pM$ such that

$$\|v\| = d(p,q) \quad\text{and}\quad \gamma(t) = \exp_p(tv) \quad\text{for each}\quad t \in [0,1].$$

The symbols $\mathbf{B}(p,r)$ and $\overline{\mathbf{B}(p,r)}$ denote, respectively, the *open metric ball* and the *closed metric ball* centered at the point $p \in M$ with radius $r > 0$, i.e.,

$$\mathbf{B}(p,r) := \big\{q \in M \big|\ d(p,q) < r\big\} \quad\text{and}\quad \overline{\mathbf{B}(p,r)} := \big\{q \in M \big|\ d(p,q) \le r\big\}.$$

Given a nonempty subset D of M, define the *distance function* $d_D(\cdot)\colon M \to [0,\infty)$ associated with D by

$$d_D(x) := \inf\big\{d(x,y)\big|\ y \in D\big\}, \quad x \in M,$$

and consider the *projection* $P(x|D)$ of $x \in M$ on the set D formed by all points of D closest to x as measured by the corresponding distance, i.e.,

$$P(x|D) := \big\{y \in D\big|\ d(x,y) = d_D(x)\big\}.$$

Observe that $P(x|D) \ne \emptyset$ whenever D is closed due to the assumed finite dimensionality of M. It is not hard to show, similarly to the proof for the case of the standard distance function on linear spaces, that the Riemannian counterpart $d_D(x)$ is *globally Lipschitzian* on M; we omit the proof here.

PROPOSITION 3.1. **(Lipschitz continuity of the distance function on Riemannian manifolds.)** *Whenever $\emptyset \ne D \subset M$ in the Riemannian manifold M, the associated distance function $d_D(\cdot)$ satisfies the global Lipschitz condition on M with Lipschitz constant $\ell = 1$, i.e.,*

$$|d_D(x) - d_D(y)| \le d(x,y) \quad \textit{for all}\quad x,y \in M.$$

Next we define, following [59, 60], notions of *totally convex* and *strongly convex* subsets of Riemannian manifolds that play a significant role in the paper. Note to this end that the *uniqueness* of geodesics is always understood up to an equivalent parameter transformation.

DEFINITION 3.2. **(Totally convex and strongly convex subsets of Riemannian manifolds.)** *Let D be a nonempty subset of the Riemannian manifold M. We say that:*

(i) *D is* TOTALLY CONVEX *if it contains every geodesic $\gamma(\cdot)$ on M with endpoints $x, y \in D$.*

(ii) *D is* STRONGLY CONVEX *if for every two points $x, y \in D$ there is the only minimizing geodesic on M connecting x, y and entirely belonging to D.*

Recall further that the *convexity radius* at $x \in M$ is defined by

$$r(x) := \sup\left\{ r > 0 \;\middle|\; \begin{array}{l} \text{each ball in } \mathbf{B}(x,r) \text{ is strongly convex} \\ \text{and each geodesic in } \mathbf{B}(x,r) \text{ is minimizing} \end{array} \right\}. \tag{3.1}$$

The following lemma presents some important properties of the convexity radius. The first statement of it is taken from [57, Theorem 5.3, p. 169] while the second one is proved in [17, 60].

LEMMA 3.3. **(Properties of the convexity radius on Riemannian manifolds.)** *Let $r(x)$ be the convexity radius at $x \in M$ defined in* (3.1). *The following properties are satisfied:*

(i) *The function $r\colon M \to (0, \infty]$ is continuous on M. Furthermore, if $r(x) = \infty$ for some $x \in M$, then $r(y) = \infty$ for every point $y \in M$.*

(ii) *For any compact subset D of M there is a real number $0 < \rho(D) \le \inf_{x \in D} r(x)$ such that whenever $0 < r \le \rho(D)$ we have the implication: if $\gamma\colon [0,1] \to \mathbf{B}(x,r)$ is an arbitrary nonconstant geodesic and if the curve $\gamma_0\colon [0,1] \to \mathbf{B}(x,r)$ is a minimizing geodesic connecting x and $\gamma(0)$ with $\langle \gamma'(0), \gamma_0'(1)\rangle \ge 0$, then the function $s \mapsto d(\gamma(s), x)$ is strictly increasing on $[0,1]$.*

Throughout the paper we use the symbol Γ_{xy} to denote the set of *all geodesics* $\gamma\colon [0,1] \to M$ on M such that $\gamma(0) = x$ and $\gamma(1) = y$; the symbol Γ_{xy}^D stands for the set of geodesics $\gamma(\cdot)$ satisfying the conditions $\gamma \in \Gamma_{xy}$ and $\gamma \subset D$. The next theorem is new and plays a key technical role in this paper.

THEOREM 3.4. **(Characterizations of projections on convex subsets of Riemannian manifolds.)** *Let D be a closed subset of M, and let $y \in D$. The following assertions hold:*

(i) *Pick a point $x \in M$ and a minimizing geodesic $\gamma_{xy} \in \Gamma_{xy}$. Then the inclusion $y \in P(x|D)$ yields*

$$\langle \gamma_{xy}'(1), \gamma_{yz}'(0)\rangle \ge 0 \quad \text{for all } z \in D \text{ and } \gamma_{yz} \in \Gamma_{yz}^D. \tag{3.2}$$

(ii) *If furthermore D is totally convex, then there is $\varepsilon > 0$ such that for each $x \in \mathbf{B}(y, \varepsilon)$ we have the inclusion $y \in P(x|D)$ whenever* (3.2) *holds for some minimizing geodesic $\gamma_{xy} \in \Gamma_{xy}$.*

Proof. To justify asserting (i), observe first that it obviously holds when $x \in D$. Thus we need to verify it in the case of $x \notin D$. To proceed, take $y \in P(x|D)$ and consider a minimizing geodesic $\gamma_{xy} \in \Gamma_{xy}$ satisfying

$$l(\gamma_{xy}) = d(x, y) = d_D(x). \tag{3.3}$$

Arguing by contradiction, suppose that there exist a point $\overline{z} \in D$ and a geodesic $\gamma_{y\overline{z}} \in \Gamma_{y\overline{z}}^D$ such that

$$\langle \gamma_{xy}'(1), \gamma_{y\overline{z}}'(0)\rangle < 0. \tag{3.4}$$

We show now that (3.4) implies the existence of $\overline{y} \in D$ satisfying $d(x, \overline{y}) < d_D(x)$, which is a clear contradiction. Indeed, let $V\colon [0,1] \to TM$ be a differentiable vector field on the tangent bundle of the smooth manifold M considering along the geodesic γ_{xy} and satisfying the endpoint conditions

$$V(0) = 0 \text{ and } V(1) = \gamma'_{y\overline{z}}(0). \tag{3.5}$$

Take $\varepsilon \in (0,1)$ and define a variation $g\colon (-\varepsilon,\varepsilon) \times [0,1] \to M$ of γ_{xy} by

$$g(s,t) := \exp_{\gamma_{xy}(t)} sV(t) \text{ for each } (s,t) \in (-\varepsilon,\varepsilon) \times [0,1].$$

Then $V(t)$ is a *variational field* of g, i.e.,

$$V(t) = \frac{\partial g}{\partial s}(0,t) \text{ for all } t \in [0,1].$$

Define further a function $L\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ by

$$L(s) := \int_0^1 \left\| \frac{\partial g}{\partial t}(s,t) \right\| \mathrm{d}t \quad \text{for each } s \in (-\varepsilon,\varepsilon),$$

which means that $L(s)$ is the *arc-length* of the curve $g(s,\cdot)$. In particular, it gives

$$L(0) = l(\gamma_{xy}) = d_D(x) \tag{3.6}$$

by (3.3). Applying then the *first variational formula* from [40, Proposition 6.5, p. 99], we have

$$L'(0) = \frac{\mathrm{d}L(s)}{\mathrm{d}s}\Big|_{s=0} = \frac{1}{l(\gamma_{xy})}\left(\langle \gamma'_{xy}(1), V(1)\rangle - \langle \gamma'_{xy}(0), V(0)\rangle - \int_0^1 \langle V(t), \nabla_{\gamma'_{xy}(t)}\gamma'_{xy}(t)\rangle \,\mathrm{d}t \right). \tag{3.7}$$

Recall that γ_{xy} is a geodesic, and thus $\nabla_{\gamma'_{xy}(t)}\gamma'_{xy}(t) = 0$. This implies together with (3.5) and (3.7) that

$$L'(0) = \frac{1}{l(\gamma_{xy})}\langle \gamma'_{xy}(1), V(1)\rangle = \frac{1}{l(\gamma_{xy})}\langle \gamma'_{xy}(1), \gamma'_{y\overline{z}}(0)\rangle < 0.$$

The latter means that the function $L(\cdot)$ is *strictly decreasing* in a neighborhood of $s = 0$. Hence there is $s_0 \in (0,\varepsilon)$ satisfying $L(s_0) < L(0)$. Letting $\overline{y} := g(s_0, 1)$ we observe that

$$\overline{y} = g(s_0,1) = \exp_y s_0 V(1) = \gamma_{y\overline{z}}(s_0) \in \gamma_{y\overline{z}} \subset D,$$

which ensures that $\overline{y} \in D$ and therefore

$$d(x,\overline{y}) \le L(s_0) < L(0) = d_D(x)$$

by (3.6). This gives $d(x,\overline{y}) < d_D(x)$, which is a contradiction showing that assumption (3.4) was wrong, and thus we complete the proof of assertion (i) of the theorem.

It remains to justify the inverse implication (ii) under the additional assumption on the *total convexity* of the underlying set D. Taking $r > 0$ to be sufficiently small and applying Lemma 3.3(ii) to the compact set $\overline{\mathbf{B}(y,r)}$, we get the relationship

$$\varepsilon := \min\left\{r, \rho(\overline{\mathbf{B}(y,r)})\right\} > 0.$$

Pick now $x \in \mathbf{B}(y,\varepsilon) \subset \mathbf{B}(y,r)$ such that (3.2) holds. To prove (ii), we need to show that $y \in P(x|D)$. Assume on the contrary that there is $\overline{y} \in D$ satisfying

$$d(x,\overline{y}) < d(x,y). \tag{3.8}$$

Let $\gamma_{y\overline{y}} \in \Gamma_{y\overline{y}}$ be a minimizing geodesic. Since $y, \overline{y} \in D$ and D is *totally convex*, we have that $\gamma_{y\overline{y}} \subset D$. This implies, together with the fact that (3.2) holds for $x \in \mathbf{B}(y,\varepsilon)$, the inequality

$$\langle \gamma'_{xy}(1), \gamma'_{y\overline{y}}(0) \rangle \geq 0. \tag{3.9}$$

Observing that $y, \overline{y} \in \mathbf{B}(x,\varepsilon)$ and that $\mathbf{B}(x,\varepsilon)$ is *strongly convex* as $x \in \overline{\mathbf{B}(y,r)}$ and $\varepsilon \leq \rho(\overline{\mathbf{B}(y,r)})$, we conclude that $\gamma_{xy}, \gamma_{y\overline{y}} \subset \mathbf{B}(x,\varepsilon)$. This ensures by (3.9) and Lemma 3.3(ii) that the function $s \mapsto d(\gamma_{y\overline{y}}(s), x)$ is *strictly increasing* on $[0,1]$. Hence $d(y,x) < d(\overline{y},x)$, which contradicts (3.8). This completes the proof of assertion (ii) and of the whole theorem. □

Let D be a totally convex subset in M, and let $x \in D$. Recall that a vector $v \in T_xM$ is *tangent* to D if there is a curve $\gamma\colon [0,\varepsilon] \to D$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Then the collection T_xD of all tangent vectors to D at x is a *convex cone* in the space T_xM; see [59, p. 71]). This implies that $(N_D(x))^\circ = T_xD$ for the *dual/polar cone* $N_D(x) := (T_xD)^\circ$ to T_xD given by

$$N_D(x) = \big\{ w \in T_xM \big| \ \langle w, v \rangle \leq 0 \ \text{ for all } \ v \in T_xD \big\}. \tag{3.10}$$

Taking into account the normal cone construction (3.10), we can reformulate condition (3.2) of Theorem 3.4 in the following equivalent *dual form*: if $\gamma_{yx} \in \Gamma_{yx}$, then

$$(3.2) \text{ holds} \Longleftrightarrow \gamma'_{yx}(0) \in N_D(y) \Longleftrightarrow \langle \gamma'_{yx}(0), v \rangle \leq 0 \text{ for all } \ v \in T_yD. \tag{3.11}$$

Recall next that a function $f\colon M \to \overline{\mathbb{R}}$ on a Riemannian manifold is *convex* if for any $x, y \in M$ and $\gamma \in \Gamma_{xy}$ the composition $(f \circ \gamma)\colon [0,1] \to \overline{\mathbb{R}}$ is a convex function on $[0,1]$, i.e.,

$$f(\gamma(t)) \leq (1-t)f(x) + tf(y) \ \text{ for all } \ t \in [0,1].$$

The *effective domain* $\operatorname{dom} f$ and the *properness* of f are defined similarly to the standard case. It is easy to see that $\operatorname{dom} f$ is *totally convex* for any convex function $f\colon M \to \overline{\mathbb{R}}$. Furthermore, the *directional derivative* of f at the point $x \in \operatorname{dom} f$ in the direction $v \in T_xM$ is defined by

$$f'(x;v) := \lim_{t \to 0^+} \frac{f(\exp_x tv) - f(x)}{t},$$

while the *subdifferential* of f at $x \in \operatorname{dom} f$ is constructed by

$$\partial f(x) := \big\{ w \in T_xM \big| \ f(y) \geq f(x) + \langle w, \gamma'(0) \rangle \ \text{ for all } \ y \in M \ \text{ and } \ \gamma \in \Gamma_{xy} \big\}.$$

It is worth mentioning that the subdifferential set $\partial f(x)$ is nonempty, convex, and compact in the space T_xM for any point $x \in \operatorname{int}(\operatorname{dom} f)$.

The next proposition presents some useful properties of the directional derivative and subdifferential of convex functions on Riemannian manifolds that are similar to the known ones on linear spaces. Assertion (i) of this theorem is taken from [59, p. 71].

PROPOSITION 3.5. **(Properties of the directional derivative and subdifferential of convex functions on Riemannian manifolds.)** *Let $f\colon M \to \overline{\mathbb{R}}$ be a proper convex function on a Riemannian manifold, and let $x \in D := \operatorname{dom} f$. The following assertions hold:*

(i) *The directional derivative $f'(x;\cdot)\colon T_xD \to \overline{\mathbb{R}}$ is convex and positively homogeneous with respect to directions, i.e., it satisfies the conditions*

$$f'(x; v_1 + v_2) \leq f'(x; v_1) + f'(x; v_2) \ \text{ for all } \ v_1, v_2 \in T_xD,$$

$$f'(x;sv) = sf'(x;v) \ \ for\ all\ \ v \in T_xD \ \ and\ \ s > 0. \tag{3.12}$$

Moreover, it possesses the properties

$$f'(x;0) = 0 \ \ and\ \ -f'(x;-v) \le f'(x;v) \ \ whenever\ \ v \in T_xD.$$

(ii) *We have the subdifferential representation*

$$\partial f(x) = \big\{w \in T_xM \big|\ \langle w, v\rangle \le f'(x;v) \ \ for\ all\ \ v \in T_xM\big\}. \tag{3.13}$$

(iii) *The support function of the subdifferential $\sigma_{\partial f(x)}(\cdot)$ is the lower semicontinuous hull*

$$\sigma_{\partial f(x)}(\cdot) = \operatorname{cl} f'(x;\cdot) \tag{3.14}$$

of the directional derivative $f'(x;\cdot)$ of f at x.

Proof. We need to prove assertions (ii) and (iii). Starting with (ii), take any w belonging to the set on the right-hand side of (3.13). Then we have

$$\langle w, v\rangle \le f'(x;v) \ \ \text{for all} \ \ v \in T_xM.$$

Pick now an arbitrary element $y \in M$ and consider a geodesic $\gamma \in \Gamma_{xy}$. Then $\gamma'(0) \in T_xM$ and thus

$$\langle w, \gamma'(0)\rangle \le f'(x;\gamma'(0)). \tag{3.15}$$

Since f is convex, we have the relationships

$$f'(x;\gamma'(0)) = \inf_{t>0} \frac{f\big(\exp_x t\gamma'(0)\big) - f(x)}{t} \le f\big(\exp_x \gamma'(0)\big) - f(x) = f(y) - f(x).$$

This gives together with (3.15) that

$$\langle w, \gamma'(0)\rangle \le f'\big(x;\gamma'(0)\big) \le f(y) - f(x).$$

The latter implies by the subdifferential definition that $w \in \partial f(x)$, and thus the subdifferential $\partial f(x)$ contains the set on the right-hand side of (3.13).

To justify the opposite inclusion "$\subset$" in (3.13), take an arbitrary subgradient $w \in \partial f(x)$ and then pick some $v \in T_xM$ and $t \in (0,1)$. Define

$$c_t(s) := \exp_x(stv) \ \ \text{for all} \ \ s \in [0,1]$$

and get $c_t(1) = \exp_x(tv)$ and $c_t \in \Gamma_{x \exp_x(tv)}$. Since $c_t'(0) = tv$, we have by the subdifferential definition that

$$\langle w, tv\rangle = \langle w, c_t'(0)\rangle \le f\big(c_t(1)\big) - f(x) = f(\exp_x tv) - f(x).$$

Therefore, by using the directional derivative construction, we arrive at the relationships

$$\langle w, v\rangle \le \lim_{t\to 0^+} \frac{f(\exp_x tv) - f(x)}{t} = f'(x;v),$$

which show that the subgradient w belongs to the set on the right-hand side of (3.13) due to the arbitrary choice of $v \in T_xM$. This completes the proof of assertion (ii) of the proposition.

It remains to justify assertion (iii). To proceed, define an extended-real-valued function $h\colon T_xM \longrightarrow \overline{\mathbb{R}}$ by $h(v) := f'(x;v)$ as $v \in T_xM$. By (3.13) we have

$$\langle w, v\rangle \le f'(x;v) \quad \text{whenever} \quad w \in \partial f(x) \quad \text{and} \quad v \in T_xM$$

and using then the construction of the conjugate function arrive at

$$h^*(w) = 0 \quad \text{for all} \quad w \in \partial f(x). \tag{3.16}$$

Consider further the case of $w \in T_xM \setminus \partial f(x)$ and by (3.13) find $v_0 \in T_xM$ satisfying

$$\langle w, v_0\rangle - f'(x;v_0) > 0.$$

By (3.12) we have $f'(x;sv_0) = sf'(x;v_0)$ for each $s > 0$ and so

$$h^*(w) \ge \sup_{s>0}(\langle w, sv_0\rangle - f'(x;sv_0)) = \infty.$$

Combining the latter with (3.16) gives us $h^* = \delta_{\partial f(x)}$. Hence $h^{**} = (\delta_{\partial f(x)})^* = \sigma_{\partial f(x)}$ by (2.1). Consequently we have (3.14) due to (2.2) and thus complete the proof of (iii) and of the whole theorem. □

4. Weak sharp minima on Riemannian manifolds. Given a function $f\colon M \to \overline{\mathbb{R}}$ and a subset $S \subset M$ of a Riemannian manifold M, consider the *constrained optimization problem*

$$\mathcal{P}: \qquad \begin{array}{l} \text{minimize } \ f(x) \\ \text{subject to } \ x \in S. \end{array}$$

with the *cost function* f and the *constraint set* S. Our *standing assumptions* in Sections 4 and 5 are that the function f is a *proper and convex* on M and that the set S is *closed and totally convex* in M. Let $\overline{S}$ be the set of *optimal solutions* to $\mathcal{P}$, i.e.,

$$\overline{S} := \text{argmin}_S f = \{x \in S \,|\; f(x) = \min_{y\in S} f(y)\}. \tag{4.1}$$

It is easy to check that $\overline{S}$ is *totally convex* under the assumptions made. Throughout the paper we suppose that the solution set $\overline{S}$ in (4.1) is *closed* in M. The following definitions extend and modify the corresponding notions of weak sharp minima from linear spaces (cf. [12] with somewhat different terminology and also the discussion in Section 1) to the Riemannian manifold setting under consideration.

DEFINITION 4.1. **(Versions of weak sharp minima on Riemannian manifolds.)** *Let $\overline{S}$ be the solution set* (4.1) *for the constrained minimization problem $\mathcal{P}$. Then we say that:*

(i) *$\overline{x} \in \overline{S}$ is a* LOCAL WEAK SHARP MINIMIZER *for $\mathcal{P}$ with modulus $\alpha > 0$ if there is $\varepsilon > 0$ such that for all $x \in S \cap \mathbf{B}(\overline{x}, \varepsilon)$ we have the estimate*

$$f(x) \ge f(\overline{x}) + \alpha d_{\overline{S}}(x). \tag{4.2}$$

(ii) *$\overline{S}$ is the set of* LOCAL WEAK SHARP MINIMA *for problem $\mathcal{P}$ if each $\overline{x} \in \overline{S}$ is a local weak sharp minimizer for $\mathcal{P}$ with some modulus $\alpha > 0$.*

(iii) *$\overline{S}$ is the set of* BOUNDEDLY WEAK SHARP MINIMA *for $\mathcal{P}$ if for every bounded set $W \subset M$ with $W \cap \overline{S} \ne \emptyset$ there is $\alpha = \alpha_W > 0$ such that* (4.2) *holds with this modulus α for all $\overline{x} \in \overline{S}$ and $x \in S \cap W$.*

(iv) *$\overline{S}$ is the set of* GLOBAL WEAK SHARP MINIMA *for $\mathcal{P}$ with the uniform modulus $\alpha > 0$ if estimate* (4.2) *holds for all $\overline{x} \in S$ and $x \in S$.*

To conduct the study of all the versions of weak sharp minima from Definition 4.1, consider an extended-real-valued function $f_0 \colon M \to \overline{\mathbb{R}}$ given by

$$f_0(x) := f(x) + \delta_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ \infty & \text{otherwise} \end{cases} \tag{4.3}$$

and observe that the initial constrained optimization problem $\mathcal{P}$ can be rewritten in *unconstrained* form

$$\text{minimize } f_0(x) \text{ subject to } x \in M.$$

The following *qualification condition* plays an important role for deriving the subsequent results in this paper.

DEFINITION 4.2. **(Mild qualification condition.)** *Given a proper convex function $f \colon M \to \overline{\mathbb{R}}$ and a convex subset $S \subset M$ of a Riemannian manifold M, we say that the pair $\{f, S\}$ satisfies the* MILD QUALIFICATION CONDITION *(MQC) at $x \in (\operatorname{dom} f) \cap S$ if*

$$\partial(f + \delta_S)(x) = \operatorname{cl}\big(\partial f(x) + N_S(x)\big). \tag{4.4}$$

Condition (4.4) is a Riemannian manifold counterpart of the one used in [12] in the linear space setting. It is indeed a "mild" qualification condition ensuring a version of the *subdifferential sum rule* for the summation function f_0 in (4.3). Besides [12], we refer the reader to the recent paper [22] and the bibliographies therein for "closedness qualification conditions" of this type, their relationships with more conventional qualification conditions in convex analysis, and applications to various classes of optimization problems on linear spaces.

To proceed further, observe first the following obvious while useful relationships held for any $\alpha > 0$:

$$\alpha\|v\| = \sigma_{\alpha\mathbb{B}_z}(v) = \sigma_{\alpha\mathbb{B}_z \cap N_{\overline{S}}(z)}(v) \quad \text{for all } v \in N_{\overline{S}}(z). \tag{4.5}$$

The next two lemmas are important to establish the main results of this section.

LEMMA 4.3. **(Some relationships under the mild qualification condition.)** *Assume that the mild qualification condition of Definition 4.2 holds for the pair $\{f, S\}$ at a given point $z \in \overline{S}$ with the solution set $\overline{S}$ to problem $\mathcal{P}$ defined in* (4.1)*. Fix $\alpha > 0$ and consider the following conditions:*

$$f_0'(z; v) \geq \alpha\|v\| \quad \text{for all } v \in N_{\overline{S}}(z); \tag{4.6}$$

$$\alpha\mathbb{B}_z \subset \operatorname{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big); \tag{4.7}$$

$$\alpha\mathbb{B}_z \cap N_S(z) \subset \operatorname{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big); \tag{4.8}$$

$$f'(z; v) \geq \alpha\|v\| \quad \text{for all } v \in T_z S \cap N_{\overline{S}}(z); \tag{4.9}$$

$$\widehat{\alpha}\mathbb{B}_z \subset \partial f(z) + \big(T_z S \cap N_{\overline{S}}(z)\big)^\circ \quad \text{for all } \widehat{\alpha} \in (0, \alpha). \tag{4.10}$$

Then we have the relationships between these conditions:

$$(4.6) \Longleftrightarrow (4.7) \Longleftrightarrow (4.8) \Longrightarrow (4.9) \Longleftrightarrow (4.10).$$

Proof. First we justify the equivalencies:

$$(4.6) \Longleftrightarrow (4.7) \Longleftrightarrow (4.8).$$

Indeed, it follows from Proposition 3.5(iii) and the relationships in (4.5) that condition (4.6) is equivalent to

$$\sigma_{\partial f_0(z)}(v) \geq \alpha\|v\| = \sigma_{\alpha\mathbb{B}_z \cap N_{\overline{S}}(z)}(v) \quad \text{for all} \quad v \in N_{\overline{S}}(z). \tag{4.11}$$

Invoking Proposition 2.1 and the MQC assumption (4.4), we get the equivalence of condition (4.11) to

$$\alpha\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \mathrm{cl}\big(\partial f_0(z) + (N_{\overline{S}}(z))^\circ\big) = \mathrm{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big).$$

Hence (4.6) is equivalent to (4.8). Similarly we derive the equivalence of (4.6) to (4.7).

To check next the implication (4.6)$\Longrightarrow$(4.9), observe from (4.3) and the above constructions of nonsmooth analysis on Riemannian manifolds that $f_0'(z;v) = f'(z;v)$ for all $v \in T_zS \cap N_{\overline{S}}(z)$, which verifies the result.

It remains to prove the equivalence (4.9) $\Longleftrightarrow$ (4.10), which holds in fact without the MQC assumption. Indeed, by Proposition 3.5(iii) and the relationships in (4.5), condition (4.9) is equivalent to

$$\sigma_{\alpha\mathbb{B}_z}(v) = \alpha\|v\| \leq \sigma_{\partial f(z)}(v) \quad \text{for all} \quad v \in T_zS \cap N_{\overline{S}}(z).$$

Furthermore, by Proposition 2.1 the latter inequality is equivalent to the inclusion

$$\alpha\mathbb{B}_z \subset \big[\partial f(z) + \big(T_zS \cap N_{\overline{S}}(z)\big)^\circ\big],$$

which in turn is equivalent to the one

$$\mathrm{int}(\alpha\mathbb{B}_z) \subset \mathrm{int}\big[\partial f(z) + \big(T_zS \cap N_{\overline{S}}(z)\big)^\circ\big]$$

by the convexity of the sets involved. Therefore we conclude that (4.9) is equivalent to (4.10) and thus complete the proof of the lemma. □

The next crucial lemma establishes relationships between the generalized differential conditions of Lemma 4.3 and the underlying weak sharp inequality (4.2) in the setting under consideration. Its proof is largely based on the major Theorem 3.4 in Riemannian manifolds derived in the previous section.

LEMMA 4.4. **(Weak sharp inequality via generalized differentiation.)** *Fix arbitrary $\alpha > 0$, $0 < r \leq \infty$, and $\overline{x} \in \overline{S}$ and consider the following assertions:*

(i) *for each $x \in S \cap \mathbf{B}(\overline{x}, r)$ condition* (4.2) *holds;*

(ii) *for each $z \in \overline{S} \cap \mathbf{B}(\overline{x}, r)$ condition* (4.6) *holds;*

(iii) *for each $z \in \overline{S} \cap \mathbf{B}(\overline{x}, r)$ condition* (4.9) *holds;*

(iv) *for each $x \in S \cap \mathbf{B}\left(\overline{x}, \frac{r}{2}\right)$ condition* (4.2) *holds.*
Then we have the implications **(i)** $\Longrightarrow$ **(ii)** $\Longrightarrow$ **(iii)** $\Longrightarrow$ **(iv)**.

Proof. To justify implication (i)$\Longrightarrow$(ii), take arbitrary $z \in \overline{S} \cap \mathbf{B}(\overline{x}, r)$ and $v \in N_{\overline{S}}(z)$ and observe that

$$f_0(z) = f_0(\overline{x}) \tag{4.12}$$

due to construction (4.3). Define a geodesic $\gamma\colon [0,1] \to M$ by

$$\gamma(t) := \exp_z tv \;\text{ for all }\; t \in [0,1].$$

Since $z \in \overline{S}$ and $\overline{S}$ is a closed totally convex subset of M, we get from Theorem 3.4(ii) that there exists $\varepsilon_0 > 0$ such that for each $p \in \mathbf{B}(z,\varepsilon_0)$ the following implication holds:

$$\left[\langle \gamma'_{pz}(1), \gamma'_{zy}(0)\rangle \geq 0 \;\text{ for all }\; y \in \overline{S},\; \gamma_{zy} \in \Gamma^{\overline{S}}_{zy}\right] \Longrightarrow z \in P(p|\overline{S}), \tag{4.13}$$

where $\gamma_{pz} \in \Gamma_{pz}$ is a minimizing geodesic, and where the constructions involved in (4.13) are defined in Section 3. Choose $\eta \in (0,1)$ such that

$$\gamma([0,\eta]) \subset \mathbf{B}(z,\varepsilon_0) \;\text{ and }\; \gamma|_{[0,\eta]} \;\text{ is minimizing.} \tag{4.14}$$

Let us verify next the equality

$$d_{\overline{S}}(\exp_z tv) = t\|v\| \;\text{ for all }\; t \in [0,\eta]. \tag{4.15}$$

To this end we take $t \in [0,\eta]$, denote $p := \exp_z tv$, and define a geodesic $\gamma_{pz}\colon [0,1] \to M$ by

$$\gamma_{pz}(s) := \exp_z(1-s)tv \;\text{ for all }\; s \in [0,1].$$

Employing then the conditions in (4.14) held due to the choice of η, we get

$$p \in \mathbf{B}(z,\varepsilon_0) \;\text{ and }\; \gamma_{pz} \in \Gamma_{pz} \;\text{ is minimizing.} \tag{4.16}$$

Since $v \in N_{\overline{S}}(z)$, it follows from the above that

$$\langle \gamma'_{pz}(1), \gamma'_{zy}(0)\rangle = \langle -v, \gamma'_{zy}(0)\rangle \geq 0 \;\text{ for all }\; y \in \overline{S} \;\text{ and }\; \gamma_{zy} \in \Gamma^{\overline{S}}_{zy},$$

and thus $z \in P(p|\overline{S})$ by condition (4.13). This justifies that

$$d_{\overline{S}}(p) = d(z,p) = d(z, \exp_z tv),$$

which yields together with (4.16) that equality (4.15) holds by the above arbitrary choice of $t \in [0,\eta]$.

Note further that, since z belongs to the open ball $\mathbf{B}(\overline{x},r)$, there exists $\xi > 0$ such that $\gamma([0,\xi]) \subset \mathbf{B}(\overline{x},r)$. By assertion (i) assumed to hold we have condition (4.2). It follows from (4.12) that

$$f_0(\exp_z tv) - f_0(z) = f_0(\exp_z tv) - f_0(\overline{x}) \geq \alpha d_{\overline{S}}(\exp_z tv) \;\text{ for all }\; t \in [0,\xi]. \tag{4.17}$$

Letting now $\zeta := \min\{\eta,\xi\}$, we conclude from (4.15) and (4.17) that

$$f_0(\exp_z tv) - f_0(z) \geq \alpha d_{\overline{S}}(\exp_z tv) = \alpha t\|v\| \;\text{ for all }\; t \in [0,\zeta].$$

By the definition of the directional derivative in Section 3 the latter implies that

$$f'_0(z;v) = \lim_{t\to 0^+} \frac{f_0(\exp_z tv) - f_0(z)}{t} \geq \alpha\|v\|,$$

which gives (ii) and thus completes the proof of implication (i)$\Longrightarrow$(ii).

The next implication (ii)$\Longrightarrow$(iii) follows immediately from that of (4.6) $\Longrightarrow$ (4.9) in Lemma 4.3.

It remains to justify implication (iii)$\Longrightarrow$(iv). Pick $x \in S \cap \mathbb{B}\left(\overline{x}, \frac{r}{2}\right)$ and $\overline{y} \in P(x|\overline{S})$ and consider a minimizing geodesic $\gamma_{\overline{y}x} \in \Gamma_{\overline{y}x}$. Then we have

$$\|\gamma'_{\overline{y}x}(0)\| = l(\gamma_{\overline{y}x}) = d(x, \overline{y}) = d_{\overline{S}}(x). \tag{4.18}$$

It follows from Theorem 3.4(i) and the "dual" assertion (3.11) applied to $y = \overline{y}$ that

$$\langle \gamma'_{\overline{y}x}(0), v \rangle \leq 0 \ \text{ for all } \ v \in T_{\overline{y}}\overline{S}.$$

Noting that $\gamma'_{\overline{y}x}(0) \in T_{\overline{y}}S$ by $x, \overline{y} \in S$, we get

$$\gamma'_{\overline{y}x}(0) \in T_{\overline{y}}S \cap N_{\overline{S}}(\overline{y}). \tag{4.19}$$

It follows then from assertion (iii) and the obvious inequalities

$$d(\overline{y}, \overline{x}) \leq d(\overline{y}, x) + d(x, \overline{x}) \leq 2d(x, \overline{x}) < 2 \cdot \frac{r}{2} = r$$

that (4.9) holds with $z = \overline{y}$, i.e., we have

$$f'(\overline{y}; v) \geq \alpha \|v\| \ \text{ for all } \ v \in T_{\overline{y}}S \cap N_{\overline{S}}(\overline{y}).$$

This implies together with (4.18) and (4.19) that

$$f'\big(\overline{y}; \gamma'_{\overline{y}x}(0)\big) \geq \alpha \|\gamma'_{\overline{y}x}(0)\| = \alpha d_{\overline{S}}(x). \tag{4.20}$$

Remembering finally that f is a convex function on a Riemannian manifold, we get from (4.20) that

$$f(x) - f(\overline{x}) = f(x) - f(\overline{y}) = f\big(\gamma_{\overline{y}x}(1)\big) - f\big(\gamma_{\overline{y}x}(0)\big) \geq f'\big(\overline{y}, \gamma'_{\overline{y}x}(0)\big) \geq \alpha d_{\overline{S}}(x),$$

which justifies (4.2) and completes the proof of the lemma. □

In the rest of this section we employ the lemmas established above as well as auxiliary results from Section 3 and some related constructions to derive comprehensive characterizations of all the types of weak sharp minima from Definition 4.1 via the defined notions of generalized differentiation on Riemannian manifolds. Let us start with characterizing the set of *global weak sharp minima*.

THEOREM 4.5. **(Characterizations of the set of global weak sharp minima on Riemannian manifolds.)** *Let $\overline{S}$ be the solution set* (4.1) *for the constrained optimization problem $\mathcal{P}$ under the standing assumptions made. Suppose in addition that the pair $\{f, S\}$ satisfies the mild qualification condition* (4.4) *at every point $x \in \overline{S}$. Then, given a number $\alpha > 0$, the following assertions are equivalent:*

(i) *$\overline{S}$ is the set of global weak sharp minima for $\mathcal{P}$ with the unform modulus $\alpha > 0$.*

(ii) *For each $\overline{x} \in \overline{S}$ we have the inclusion*

$$\alpha \mathbb{B}_{\overline{x}} \subset \text{cl}\big(\partial f(\overline{x}) + N_S(\overline{x}) + T_{\overline{x}}\overline{S}\big).$$

(iii) *For each $\overline{x} \in \overline{S}$ we have the inclusion*

$$\alpha \mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x}) \subset \text{cl}\big(\partial f(\overline{x}) + N_S(\overline{x}) + T_{\overline{x}}\overline{S}\big).$$

(iv) *For each $\overline{x} \in \overline{S}$ we have the estimate*

$$f'(\overline{x}; v) \geq \alpha\|v\| \ \textit{whenever} \ v \in T_{\overline{x}}S \cap N_{\overline{S}}(\overline{x}).$$

(v) *For each $\overline{x} \in \overline{S}$ we have the inclusion*

$$\widehat{\alpha}\mathbb{B}_{\overline{x}} \subset \partial f(\overline{x}) + \big(T_{\overline{x}}S \cap N_{\overline{S}}(\overline{x})\big)^{\circ} \ \textit{whenever} \ \widehat{\alpha} \in (0, \alpha).$$

(vi) *For each $x \in S$ and each $\overline{x} \in P(x|\overline{S})$ we have the estimate*

$$f'\big(\overline{x}; \gamma'(0)\big) \geq \alpha d_{\overline{S}}(x),$$

where $\gamma \in \Gamma_{\overline{x}x}$ is any minimizing geodesic connecting these two points.

Proof. First we verify implication (i)$\Longrightarrow$(ii) of the theorem. By Definition 4.1(iv) the weak sharp inequality (4.2) is satisfied with the given $\alpha > 0$ for each $x \in S$ and $\overline{x} \in \overline{S}$. Thus it follows from implication (i)$\Longrightarrow$(ii) of Lemma 4.4 with $r = \infty$ that the directional derivative estimate (4.6) holds on $\overline{S}$. We now derive the validity of assertion (ii) of the theorem from the equivalency relationship (4.6) $\Longleftrightarrow$ (4.7) in Lemma 4.3.

Relationships (ii)$\Longleftrightarrow$(iii)$\Longrightarrow$(iv)$\Longleftrightarrow$(v) in the theorem follow from the corresponding relationships (4.7)$\Longleftrightarrow$(4.8)$\Longrightarrow$(4.9)$\Longleftrightarrow$(4.10) in Lemma 4.3.

To justify implication (iv)$\Longrightarrow$(vi), pick any points $x \in S$ and $\overline{x} \in P(x|\overline{S})$ and consider a minimizing geodesic $\gamma \in \Gamma_{\overline{x}x}$ connecting these points. Then we have the equalities

$$\|\gamma'(0)\| = d(x, \overline{x}) = d_{\overline{S}}(x). \tag{4.21}$$

Employing now Theorem 3.4(i) allows us to conclude that

$$\langle \gamma'(0), \gamma'_{\overline{x}z}(0)\rangle \leq 0 \ \text{ for all } z \in \overline{S} \ \text{ and } \ \gamma_{\overline{x}z} \in \Gamma^{\overline{S}}_{\overline{x}z},$$

which implies in turn the inclusion $\gamma'(0) \in N_{\overline{S}}(\overline{x})$. Combining the latter with the inclusion $\gamma \subset S$ gives us that $\gamma'(0) \in T_{\overline{x}}S \cap N_{\overline{S}}(\overline{x})$. Hence it follows from the assumed assertion (iv) that

$$f'(\overline{x}; \gamma'(0)) \geq \alpha\|\gamma'(0)\| = \alpha d_{\overline{S}}(x),$$

where the last equality holds due to (4.21). Thus we arrive at assertion (vi).

It remains to verify implication (vi)$\Longrightarrow$(i) of the theorem. Take $x \in S$, $\overline{x} \in P(x|\overline{S})$, and a minimizing geodesic $\gamma \in \Gamma_{\overline{x}x}$. Then by (vi) we get $f'\big(\overline{x}; \gamma'(0)\big) \geq \alpha d_{\overline{S}}(x)$. Since the function f is convex, the latter gives

$$f(x) - f(\overline{x}) = f\big(\gamma(1)\big) - f(\overline{x}) \geq f'\big(\overline{x}; \gamma'(0)\big) \geq \alpha d_{\overline{S}}(x),$$

which ensures (i) and completes the proof of the theorem. □

The next theorem provides similar characterizations of *boundedly weak sharp minima* for problem $\mathcal{P}$.

THEOREM 4.6. **(Characterizations of the set of boundedly weak sharp minima on Riemannian manifolds.)** *Let $\overline{S}$ be the solution set to problem $\mathcal{P}$, and let all the assumptions of Theorem 4.5 be satisfied. Then the following assertions are equivalent:*

(i) *$\overline{S}$ is the set of boundedly sharp minima for problem $\mathcal{P}$.*

(ii) *For every $\overline{x} \in \overline{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$\alpha(r)\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \mathrm{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big) \quad whenever \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r). \tag{4.22}$$

(iii) *For every $\overline{x} \in \overline{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$\alpha(r)\mathbb{B}_z \subset \mathrm{cl}\big(\partial f(\overline{x}) + N_S(z) + T_z\overline{S}\big) \quad whenever \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r). \tag{4.23}$$

(iv) *For every $\overline{x} \in \overline{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$f'(z; v) \geq \alpha(r)\|v\| \quad whenever \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r) \quad and \quad v \in T_zS \cap N_{\overline{S}}(z). \tag{4.24}$$

(v) *For every $\overline{x} \in \overline{S}$ and $r > 0$ there is $\alpha(r) > 0$ such that*

$$\alpha(r)\mathbb{B}_z \subset \partial f(z) + \big(T_zS \cap N_{\overline{S}}(z)\big)^\circ \quad whenever \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r).$$

(vi) *For every $\overline{x} \in \overline{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$f'\big(z; \gamma'(0)\big) \geq \alpha(r) d_{\overline{S}}(x) \quad whenever \quad x \in S \cap \mathbf{B}(\overline{x}, r) \quad and \quad z \in P(x|\overline{S}) \tag{4.25}$$

independently of the choice of a minimizing geodesic $\gamma \in \Gamma_{zx}$ connecting the points z and x.

Proof. We first justify implication (i)$\Longrightarrow$(ii) in the theorem. Observe that the Definition 4.1(iv) of the set of boundedly weak sharp minima can be equivalently formulated as follows: for every $\overline{x} \in \overline{S}$ and every $r > 0$ there is a modulus $\alpha(r) > 0$ such that

$$f(x) \geq f(\overline{x}) + \alpha(r) d_{\overline{S}}(x) \quad \text{whenever} \quad x \in S \cap \mathbf{B}(\overline{x}; r). \tag{4.26}$$

Using now implication (i)$\Longrightarrow$(ii) of Lemma 4.4, we conclude that assertion (i) of this theorem yields the validity of condition (4.6) for all $\overline{S} \cap \mathbf{B}(\overline{x}, r)$ with the same number $\alpha = \alpha(r)$ as in (4.26). Thus it follows from implication (4.6) $\Longrightarrow$ (4.8) in Lemma 4.3 that assertion (ii) of this theorem holds.

Observe further that relationships (ii)$\Longleftrightarrow$(iii)$\Longrightarrow$(iv)$\Longleftrightarrow$(v) of the theorem follow directly from the corresponding results of Lemma 4.3.

To verify next implication (iv)$\Longrightarrow$(vi) of the theorem, pick any $\overline{x} \in \overline{S}$ and $r > 0$ and find by (iv) a number $\alpha = \alpha(r) > 0$ such that

$$\big[z \in \overline{S} \cap \mathbf{B}(\overline{x}, 2r) \ \text{ and } \ v \in T_zS \cap N_{\overline{S}}(z)\big] \Longrightarrow \big[f'(z; v) \geq \alpha(r)\|v\|\big]. \tag{4.27}$$

Taking any $x \in S \cap \mathbf{B}(\overline{x}, r)$ and $z \in P(x|\overline{S})$, we have the equalities

$$\|\gamma'(0)\| = d(x, z) = d_{\overline{S}}(x), \tag{4.28}$$

where $\gamma \in \Gamma_{zx}$ is a minimizing geodesic connecting z and x. Applying then Theorem 3.4 gives us condition (3.2). It follows then from the equivalence in (3.11) and the inclusion $\gamma \subset S$ that

$$\gamma'(0) \in T_zS \cap N_{\overline{S}}(z). \tag{4.29}$$

Furthermore, we get the inclusion $z \in \overline{S} \cap \mathbf{B}(\overline{x}, 2r)$ from the obvious inequalities

$$d(z, \overline{x}) \le d(z, x) + d(x, \overline{x}) \le 2d(x, \overline{x}) \le 2r.$$

Combining this with relationships (4.27) and (4.29) allows us to conclude that

$$f'\big(z; \gamma'(0)\big) \ge \alpha(r)\|\gamma'(0)\| = \alpha(r) d_{\overline{S}}(x),$$

where the last equality holds due to (4.28). Thus we arrive at assertion (vi) of the theorem.

Let us finally justify the remaining implication (vi)$\Longrightarrow$(i). Take $\overline{x} \in \overline{S}$ and $r > 0$ and by (vi) find a number $\alpha(r) > 0$ such that

$$f'\big(z; \gamma'(0)\big) \ge \alpha(r) d_{\overline{S}}(x)$$

for each $x \in S \cap \mathbf{B}(\overline{x}, r)$, $z \in P(x|\overline{S})$, and the minimizing geodesic $\gamma \in \Gamma_{zx}$ connecting these points. It follows further from the convexity of the cost function f in $\mathcal{P}$ that

$$f(x) - f(\overline{x}) = f\big(\gamma(1)\big) - f(z) \ge f'\big(z; \gamma'(0)\big) \ge \alpha(r) d_{\overline{S}}(x),$$

which gives the underlying estimate (4.26) in (i) and thus completes the proof of the theorem. □

Our next step is to derive efficient characterizations of *local weak sharp minimizers* to $\mathcal{P}$ in the sense of Definition 4.1(i). To proceed, we need the following proposition taken from [60], which ensures the local Lipschitz continuity of the projection mapping.

PROPOSITION 4.7. **(Local Lipschitz continuity of projections on totally convex subsets of Riemannian manifolds.)** *Let D be a totally convex subset of a Riemannian manifold M. Then there are numbers $r > 0$ and $\ell \ge 1$ such that*

$$d\big(P(x|D), P(y|D)\big) \le \ell\, d(x, y)$$

for each pair $(x, y) \in M \times M$ with $d_D(x) \le r$ and $d_D(y) \le r$.

Combining this proposition with the methods and results developed above allows us to establish the following characterizations of local weak sharp minimizers.

THEOREM 4.8. **(Characterizations of local weak sharp minimizers for convex problems on Riemannian manifolds.)** *Let $\overline{x} \in \overline{S}$ be an optimal solution to the constrained optimization problem $\mathcal{P}$ under the standing assumptions on its initial data. Suppose in addition that the mild qualification condition* (4.4) *holds on a neighborhood of $\overline{x}$. Then, given $\alpha > 0$, the following assertions are equivalent:*

(i) *$\overline{x}$ is a local weak sharp minimizer for $\mathcal{P}$ with modulus α.*

(ii) *There is $\varepsilon > 0$ such that we have the inclusion*

$$\alpha \mathbb{B}_z \subset \operatorname{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big) \quad \textit{for all} \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, \varepsilon).$$

(iii) *There is $\varepsilon > 0$ such that we have the inclusion*

$$\alpha \mathbb{B}_z \cap N_{\overline{S}}(z) \subset \operatorname{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big) \quad \textit{for all} \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, \varepsilon).$$

(iv) *There is $\varepsilon > 0$ such that we have the estimate*

$$f'(z;v) \geq \alpha\|v\| \ \text{for all} \ z \in \overline{S} \cap \mathbf{B}(\overline{x},\varepsilon) \ \text{and} \ v \in T_zS \cap N_{\overline{S}}(z).$$

(v) *There is $\varepsilon > 0$ such that we have the inclusion*

$$\widehat{\alpha}\mathbb{B}_z \subset \partial f(z) + \big(T_zS \cap N_{\overline{S}}(z)\big)^{\circ} \ \text{for all} \ z \in \overline{S} \cap \mathbf{B}(\overline{x},\varepsilon) \ \text{and} \ 0 \leq \widehat{\alpha} < \alpha.$$

(vi) *There is $\varepsilon > 0$ such that we have the estimate*

$$f'\big(z;\gamma'(0)\big) \geq \alpha d_{\overline{S}}(x) \ \text{for all} \ x \in S \cap \mathbf{B}(\overline{x},\varepsilon) \ \text{and} \ z \in P(x|\overline{S}),$$

where $\gamma \in \Gamma_{zx}$ is a minimizing geodesic connecting the corresponding points.

Proof. Observe that the imposed mild qualification condition (4.4) on $\{f, S\}$ around $\overline{x}$ means there is a number $\varepsilon_1 > 0$ such that

$$\partial(f+\delta_S)(z) = \mathrm{cl}\big(\partial f(z) + N_S(z)\big) \ \text{for all} \ z \in \mathbf{B}(\overline{x},\varepsilon_1). \tag{4.30}$$

To verify implication (i)$\Longrightarrow$(ii) of the theorem, find by assertion(i) such a number $\varepsilon_2 > 0$ that the weak sharp inequality (4.2) holds on $S \cap \mathbf{B}(\overline{x},\varepsilon_2)$. Letting $\varepsilon := \min\{\varepsilon_1,\varepsilon_2\}$, we have by implication (i)$\Longrightarrow$(ii) of Lemma 4.4 that condition (4.6) is satisfied for all $z \in \overline{S} \cap \mathbf{B}(\overline{x},\varepsilon)$. Using now (4.30) and equivalence (4.6) $\Longleftrightarrow$ (4.7) from Lemma 4.3, we arrive at assertion (ii).

As above, relationships (ii)$\Longleftrightarrow$(iii)$\Longrightarrow$(iv)$\Longleftrightarrow$(v) in the theorem follow directly from the corresponding relationships in Lemma 4.3.

The proof of implication (iv)$\Longrightarrow$(vi) in this theorem requires some change in comparison with the similar implications of Theorem 4.5 and Theorem 4.6. To proceed, assume that assertion (iv) of the theorem holds, i.e., there is $\varepsilon_1 > 0$ such that

$$f'(z;v) \geq \alpha\|v\| \ \text{for all} \ z \in \overline{S} \cap \mathbf{B}(\overline{x},\varepsilon_1) \ \text{and} \ v \in T_zS \cap N_{\overline{S}}(z). \tag{4.31}$$

Since $\overline{S}$ is *totally convex*, Proposition 4.7 allows us to conclude that there are $r > 0$ and $\ell \geq 1$ such that

$$d\big(P(x|\overline{S}), P(y|\overline{S})\big) \leq \ell\, d(x,y) \ \text{whenever} \ d_{\overline{S}}(x) \leq r \ \text{and} \ d_{\overline{S}}(y) \leq r. \tag{4.32}$$

Take $x \in S \cap \mathbf{B}(\overline{x},\varepsilon)$ and $z \in P(x|\overline{S})$, where the number $\varepsilon > 0$ is defined by

$$\varepsilon := \min\left\{r, \frac{\varepsilon_1}{\ell}\right\}.$$

As in the proof of Theorem 4.6, we have the equalities in (4.28), where $\gamma \in \Gamma_{zx}$ is a minimizing geodesic connecting z and x. Applying further Theorem 3.4 with $D = \overline{S}$ and the equivalent dual description (3.11) of condition (3.2) ensures that $\gamma'(0) \in N_{\overline{S}}(z)$. Noting that $\gamma \subset S$, we get therefore that

$$\gamma'(0) \in T_zS \cap N_{\overline{S}}(z). \tag{4.33}$$

On the other hand, it follows from (4.32) by the choice of $x \in \mathbf{B}(\overline{x},\varepsilon)$ with $\varepsilon > 0$ defined above that

$$d(z,\overline{x}) \leq \ell\, d(x,\overline{x}) < \ell\varepsilon \leq \varepsilon_1.$$

Combining the latter with (4.31) and (4.33) yields that

$$f'(z;\gamma'(0)) \geq \alpha\|\gamma'(0)\| = \alpha d_{\overline{S}}(x),$$

where the last equality holds due to (4.28). Thus we arrive at assertion (vi) of the theorem.

Observing finally that the verification of implication (vi)$\Longrightarrow$(i) in this theorem is similar to the one in Theorem 4.6, we complete the proof of the result. □

The following characterizations of the set of *local weak sharp minima* are direct consequences of the corresponding characterizations of local weak sharp minimizers from Theorem 4.8 and Definition 4.1(ii).

COROLLARY 4.9. **(Characterizations of the set of local weak sharp minima on Riemannian manifolds.)** *Let all the assumptions of Theorem* 4.5 *be satisfied. The following assertions are equivalent:*

(i) $\overline{S}$ *is the set of local weak sharp minima for problem* $\mathcal{P}$.

(ii) *For every* $\overline{x} \in \overline{S}$ *there are* $r > 0$ *and* $\alpha(r) > 0$ *such that condition* (4.22) *holds.*

(iii) *For every* $\overline{x} \in \overline{S}$ *there are* $r > 0$ *and* $\alpha(r) > 0$ *such that condition* (4.23) *holds.*

(iv) *For every* $\overline{x} \in \overline{S}$ *there are* $r > 0$ *and* $\alpha(r) > 0$ *such that condition* (4.24) *holds.*

(v) *For every* $\overline{x} \in \overline{S}$ *there are* $r > 0$ *and* $\alpha(r) > 0$ *such that*

$$\widehat{\alpha}\mathbb{B}_z \subset \partial f(z) + \big(T_z S \cap N_{\overline{S}}(z)\big)^{\circ} \quad \text{for all} \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r) \quad \text{and} \quad 0 \leq \widehat{\alpha} < \alpha(r).$$

(vi) *For every* $\overline{x} \in \overline{S}$ *there are* $r > 0$ *and* $\alpha(r) > 0$ *such that condition* (4.25) *holds, where* $\gamma \in \Gamma_{zx}$ *is a minimizing geodesic connecting the points* z *and* x.

Comparing the results of Theorem 4.6 and of Corollary 4.9 and taking into account that the Riemannian manifold M considered in this section is *finite-dimensional*, we conclude by compactness arguments that sets of boundedly weak sharp minima and local weak sharp minima agree under the assumptions made.

COROLLARY 4.10. **(Sets of boundedly weak sharp minima and local weak sharp minima agree on finite-dimensional Riemannian manifolds.)** *Under the assumptions made in Theorem* 4.5 *the solution set* (4.1) *to problem* $\mathcal{P}$ *is the set of boundedly weak sharp minima for* $\mathcal{P}$ *if and only if it is the set of local weak sharp minima for this problem.*

Proof. It immediately follows from the definitions that the set of boundedly weak sharp minima is contained in the set of local ones. To justify the opposite implication, fix $\overline{x} \in \overline{S}$ and $r > 0$. Then for any $y \in \overline{S} \cap \overline{\mathbf{B}(\overline{x}, r)}$ we find by assertions (ii) of Corollary 4.9 numbers $r_y > 0$ and $\alpha(r_y) > 0$ such that

$$\alpha(r_y)\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \mathrm{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big) \quad \text{for all} \quad z \in \overline{S} \cap \mathbf{B}(y, r_y).$$

Since $\overline{S} \cap \overline{\mathbf{B}(\overline{x}, r)}$ is a *compact* subset of the finite-dimensional Riemannian manifold M and since

$$\bigcup_{y \in \overline{S} \cap \overline{\mathbf{B}(\overline{x}, r)}} \mathbf{B}(y, r_y) \supset \big(\overline{S} \cap \overline{\mathbf{B}(\overline{x}, r)}\big),$$

there exists a *finite covering* of the set $\overline{S} \cap \overline{\mathbf{B}(\overline{x}, r)}$ by the above balls, i.e., a natural number $n \geq 1$ such that

$$\bigcup_{i=1}^{n} \mathbf{B}(y_i, r_{y_i}) \supset \big(\overline{S} \cap \overline{\mathbf{B}(\overline{x}, r)}\big).$$

Letting now $\alpha(r) := \min\{\alpha(r_{y_i}) \mid 1 \le i \le n\}$, we have

$$\alpha(r)\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \mathrm{cl}\big(\partial f(z) + N_S(z) + T_z\overline{S}\big) \quad \text{for all} \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r),$$

which ensures the validity of condition (ii) in Theorem 4.6. The latter justifies that $\overline{S}$ is the set of boundedly weak sharp minima $\mathcal{P}$ and thus completes the proof of the corollary. □

5. Weak sharp minima on Hadamard manifolds. In this section we obtain new characterizations of all the versions of weak sharp minima under consideration for convex problems on Hadamard manifolds essentially exploiting their special structure; the new conditions obtained do not have appropriate analogs in the general Riemannian case. Recall that a *Hadamard manifold* is a complete connected m-dimensional Riemannian manifold with nonpositive sectional curvature. Throughout the whole section we assume that M is a Hadamard manifold. In this case the mapping $\exp_x\colon T_xM \to M$ is a *diffeomorphism* for each $x \in M$; see [23, p. 149]. The latter implies that for any two points $x, y \in M$ there is one and only one geodesic connecting x, y, which is a minimizing geodesic.

The following well known result (see, e.g., [57]) concerns some properties of *geodesic triangle* $\Delta(p_1p_2p_3)$ consisting by definition of three points p_1, p_2, p_3 and three minimizing geodesic segments γ_i that join p_i and p_{i+1} with $i = 1, 2, 3$ (mod 3).

PROPOSITION 5.1. **(Comparison result for geodesic triangles.)** *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle, and let $\gamma_i\colon [0,1] \to M$ be the corresponding geodesic segments joining p_i and p_{i+1} with $i = 1, 2, 3$* (mod 3). *Denote $l_i := l(\gamma_i)$, $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(1))$. Then we have the relationships*

$$\alpha_1 + \alpha_2 + \alpha_3 \le \pi \quad \textit{and} \quad l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \le l_{i-1}^2.$$

The next result reveals a significant specific feature of Hadamard manifolds in comparison with the general class of Riemannian manifolds: it shows namely that the distance function for *totally convex* subsets of Hadamard manifolds is *convex.*

LEMMA 5.2. **(Convexity of the distance function on Hadamard manifolds).** *Let D be a totally convex subset of a Hadamard manifold M. Then the distance function $d_D(\cdot)$ is convex on M.*

Proof. Take $x, y \in M$ and for any $\varepsilon > 0$ pick elements $c_x, c_y \in D$ satisfying the conditions

$$d(x, c_x) \le d_D(x) + \varepsilon \quad \text{and} \quad d(y, c_y) \le d_D(y) + \varepsilon.$$

Let $\gamma_1\colon [0,1] \to M$ be a geodesic connecting x and y, and let $\gamma_2\colon [0,1] \to M$ be another geodesic connecting c_x and c_y. It follows from the *total convexity* of the set D that $\gamma_2([0,1]) \subset D$. Since a function $h\colon [0,1] \to [0,\infty)$ defined by $h(t) := d(\gamma_1(t), \gamma_2(t))$ is *convex* on $[0,1]$ by [59, p. 105], we have the inequalities

$$\begin{aligned} d_D(\gamma_1(t)) &\le d\big(\gamma_1(t), \gamma_2(t)\big) \le (1-t)d(x, c_x) + td(y, c_y) \\ &\le (1-t)d_D(x) + td_D(y) + \varepsilon \quad \text{for all} \quad t \in [0,1]. \end{aligned}$$

This completes the proof of the lemma due to the arbitrary choice of $\varepsilon > 0$. □

The following theorem is certainly of independent interest for convex analysis on Hadamard manifolds, while it plays a crucial role in deriving the main characterizations of weak sharp minima in the rest of this section. Observe that results of this type relating subgradients of the distance function and normals to the corresponding set are known for various subdifferentials in general nonconvex settings of Banach spaces, being of great importance for many aspects of variational analysis and its applications; see [49, 50], particularly [49, Subsection 1.3.3], with the references and commentaries therein.

THEOREM 5.3. **(Subdifferential representation for the distance function of totally convex sets in Hadamard manifolds.)** *Let $D \subset M$ be a totally convex subset of a Hadamard manifold. Then the subdifferential of the convex distance function $d_D(\cdot)$ is computed by*

$$\partial d_D(x) = \mathbb{B}_x \cap N_D(x) \quad \textit{for all} \quad x \in D \tag{5.1}$$

via the normal cone $N_D(\cdot)$ to the set D at the corresponding point.

Proof. The convexity of the distance function $d_D(\cdot)$ on Hadamard manifolds is established in Lemma 5.2. Using the definitions of $d_D(x)$ and $N_D(x)$ on Riemannian manifolds given in Section 3, we easily conclude that inclusion "$\subset$" in (5.1) holds. Let us now prove the opposite inclusion in (5.1), i.e.,

$$\mathbb{B}_x \cap N_D(x) \subset \partial d_D(x)$$

for any fixed $x \in D$. To proceed, take $w \in \mathbb{B}_x \cap N_D(x)$ and check that $w \in \partial d_D(x)$. By Proposition 3.5(i) it is sufficient to verify the inequality

$$d'_D(x; v) \ge \langle w, v\rangle \quad \text{for all} \quad v \in T_x M. \tag{5.2}$$

Since $d'_D(x; v) \ge 0$ and $\langle \omega, v\rangle \le 0$ for each $v \in T_x D$, we have that (5.2) is satisfied on $T_x D$. It remains to show that (5.2) also holds on $T_x M \setminus T_x D$. Pick any $v \in T_x M \setminus T_x D$ and verify first that

$$d_D(\exp_x tv) \ge \inf_{u \in \exp_x^{-1} D} \|tv - u\| \quad \text{whenever} \quad t > 0. \tag{5.3}$$

Indeed, take arbitrary $t > 0$ and $u \in \exp_x^{-1} D$ and denote

$$p_1 := \exp_x tv, \quad p_2 := x, \quad \text{and} \quad p_3 := \exp_x u.$$

Let further $\gamma_i \colon [0, 1] \to M$ be a geodesic connecting p_i and p_{i+1} for $i = 1, 2, 3 \pmod 3$. Then

$$l(\gamma_1) = \|tv\|, \quad l(\gamma_2) = \|u\|, \quad \text{and} \quad \theta := \angle(-\gamma_1'(1), \gamma_2'(0)) = \angle(tv, u).$$

Applying Proposition 5.1 to the geodesic triangle $\triangle(p_1 p_2 p_3)$, we get

$$\begin{array}{rcl} d^2(\exp_x tv, \exp_x u) & = & d^2(p_1, p_3) \\ & \ge & l(\gamma_1)^2 + l(\gamma_2)^2 - 2l(\gamma_1)l(\gamma_2)\cos\theta \\ & = & \|tv\|^2 + \|u\|^2 - 2\|tv\| \cdot \|u\| \cos\theta \\ & = & \|tv - u\|^2. \end{array}$$

This allows us to conclude that

$$d_D(\exp_x tv) = \inf_{u \in \exp_x^{-1} D} d(\exp_x tv, \exp_x u) \ge \inf_{u \in \exp_x^{-1} D} \|tv - u\|,$$

and thus condition (5.3) is verified. Furthermore, it follows from $w \in \mathbb{B}_x \cap N_D(x)$ that

$$\begin{array}{rcl} \inf_{u \in \exp_x^{-1} D} \|tv - u\| & \ge & \inf_{u \in \exp_x^{-1} D} \langle \omega, tv - u\rangle \\ & = & \inf_{u \in \exp_x^{-1} D} (\langle w, tv\rangle - \langle w, u\rangle) \\ & = & \langle w, tv\rangle, \end{array} \tag{5.4}$$

where the last equality holds due to $u \in \exp_x^{-1} D$ and $\langle w, u \rangle \leq 0$. Combining consequently (5.3) and (5.4), we arrive at the relationship

$$d'_D(x;v) = \lim_{t\to 0^+} \frac{d_D(\exp_x tv) - d_D(x)}{t} \geq \langle w, v\rangle,$$

which justifies (5.2) and thus completes the proof of the theorem. □

To establish the main characterizations of weak sharp minima on Hadamard manifolds, we need one more auxiliary result involving the distance function of the solution set (4.1) to the optimization problem $\mathcal{P}$.

LEMMA 5.4. **(Some relationships involving the distance function of the solution set on Hadamard manifolds.)** *Let $\overline{S}$ be the solution set* (4.1) *to problem $\mathcal{P}$, and let $\overline{x} \in \overline{S}$. In addition to the standing assumptions made, suppose that the mild qualification condition* (4.4) *holds at $\overline{x}$. Consider the following relationships, where the function f_0 is defined in* (4.3)*:*

$$\alpha\mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x}) \subset \mathrm{cl}\big(\partial f_0(\overline{x})\big) = \mathrm{cl}\big(\partial f(\overline{x}) + N_S(\overline{x})\big); \tag{5.5}$$

$$f_0'(\overline{x};v) \geq \alpha d'_{\overline{S}}(\overline{x};v) \quad \textit{for all} \quad v \in T_{\overline{x}}M; \tag{5.6}$$

$$f'(\overline{x};v) \geq \alpha d_{T_{\overline{x}}\overline{S}}(v) \quad \textit{for all} \quad v \in T_{\overline{x}}S; \tag{5.7}$$

$$f'(\overline{x};v) \geq \alpha\|v\| \quad \textit{for all} \quad v \in T_{\overline{x}}S \cap N_{\overline{S}}(\overline{x}). \tag{5.8}$$

Then we have (5.5) $\Longleftrightarrow$ (5.6) $\Longrightarrow$ (5.7) $\Longrightarrow$ (5.8).

Proof. First we observe the validity of the following equalities:

$$d'_{\overline{S}}(\overline{x};v) = \sigma_{\partial d_{\overline{S}}(\overline{x})}(v) = \sigma_{\mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) = d_{T_{\overline{x}}\overline{S}}(v) \quad \text{for all} \quad v \in T_{\overline{x}}M. \tag{5.9}$$

Indeed, it follows from the condition $\mathrm{dom}\, d_{\overline{S}} = M$ that the function $d'_{\overline{S}}(\overline{x};\cdot)$ is sublinear and thus continuous on $T_{\overline{x}}M$. Then applying to $d'_{\overline{S}}(\overline{x};\cdot)$ equality (3.14) from Proposition 3.5 together with Proposition 2.1(ii) and the subdifferential representation (5.1) from Theorem 5.3, we see that all the equalities in (5.9) hold.

Returning now to the proof of this lemma, let us verify equivalence (5.5)$\Longleftrightarrow$(5.6). Since $d'_{\overline{S}}(\overline{x};\cdot)$ is continuous on $T_{\overline{x}}M$ as noted, we have that condition (5.6) is equivalent to

$$\mathrm{cl}\, f_0'(\overline{x};v) \geq \alpha d'_{\overline{S}}(\overline{x};v) \quad \text{for all} \quad v \in T_{\overline{x}}M. \tag{5.10}$$

Thus applying (3.14) to f_0 and then using (5.9) allows us to conclude that (5.10) is equivalent to

$$\sigma_{\partial f_0(\overline{x})}(v) \geq \sigma_{\partial(\alpha d_{\overline{S}})(\overline{x})}(v) = \sigma_{\alpha\mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) \quad \text{for all} \quad v \in T_{\overline{x}}M,$$

which is in equivalent in turn to (5.5) by Proposition 2.1 and the qualification condition (4.4).

To verify next implication (5.6)$\Longrightarrow$(5.7), take $v \in T_{\overline{x}}S$ and get the equalities $f'(\overline{x};v) = f_0'(\overline{x};v)$ and $d'_{\overline{S}}(\overline{x};v) = d_{T_{\overline{x}}\overline{S}}(v)$ due to (5.9), which thus justifies the result.

It remains to prove implication (5.7)$\Longrightarrow$(5.8). Taking $v \in T_{\overline{x}}S \cap N_{\overline{S}}(\overline{x})$ and applying the equalities in (4.5) at the point $z = \overline{x}$, we have

$$\sigma_{\mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x})}(v) = \|v\|.$$

The latter implies together with (5.9) that $d_{T_{\overline{x}}\overline{S}}(v) = \|v\|$, which justifies the claimed implication and completes the proof of the lemma. □

Now we are ready to establish characterizations of all the types of weak sharp minima from Definition 4.1 in the case of Hadamard manifolds. We start with *global weak sharp minima* deriving new characterizations in addition to those in Theorem 4.5.

THEOREM 5.5. **(Characterizations of the set of global weak sharp minima on Hadamard manifolds.)** *Suppose that all the assumptions of Theorem* 4.5 *are satisfied and that in addition the manifold* M *is Hadamard. Then assertions* (i)–(vi) *of Theorem* 4.5 *are equivalent to the following ones:*

(vii) *For each* $\overline{x} \in \overline{S}$ *we have the inclusion*

$$\alpha\mathbb{B}_{\overline{x}} \cap N_{\overline{S}}(\overline{x}) \subset \mathrm{cl}\big(\partial f(\overline{x}) + N_S(\overline{x})\big).$$

(viii) *For each* $\overline{x} \in \overline{S}$ *and each* $v \in T_{\overline{x}}S$ *we have the estimate*

$$f'(\overline{x}; v) \geq \alpha d_{T_{\overline{x}}\overline{S}}(v).$$

Proof. Assume that $\overline{S}$ is the set of global weak sharp minima for problem $\mathcal{P}$ and let $\overline{x} \in \overline{S}$. Then for all $t > 0$ and all $v \in T_{\overline{x}}M$ we have the inequality

$$f_0(\exp_{\overline{x}} tv) - f_0(\overline{x}) \geq \alpha d_{\overline{S}}(\exp_{\overline{x}} tv),$$

which implies in turn that

$$\frac{f_0(\exp_{\overline{x}} tv) - f_0(\overline{x})}{t} \geq \alpha \frac{d_{\overline{S}}(\exp_{\overline{x}} tv) - d_{\overline{S}}(\overline{x})}{t}. \tag{5.11}$$

Taking the limit as $t \downarrow 0$ on both sides in (5.11), we get the estimate

$$f_0'(\overline{x}; v) \geq \alpha d_{\overline{S}}'(\overline{x}; v).$$

Since $v \in T_{\overline{x}}M$ was chosen arbitrarily, condition (5.6) of Lemma 5.4 holds. The latter yields together with equivalence (5.5)$\Longleftrightarrow$(5.6) in Lemma 5.4 that condition (vii) is satisfied. Therefore assertion (i) of Theorem 4.5 implies that of (vii) in this theorem.

It easily follows from implications (5.5)$\Longrightarrow$(5.7)$\Longrightarrow$(5.8) in Lemma 5.4 that condition (vii) of this theorem implies that of (viii) and the latter implies in turn condition (iv) of Theorem 4.5. Thus all the aforementioned conditions (i)–(viii) are equivalent, and the proof of this theorem is complete. □

Next we establish new characterizations of *boundedly weak sharp minima* for constrained problem of convex optimization on Hadamard manifolds.

THEOREM 5.6. **(Characterizations of the set of boundedly weak sharp minima on Hadamard manifolds.)** *Let all the assumptions of Theorem* 5.5 *be satisfied. Then assertions* (i)–(vi) *in Theorem* 4.6 *and the following new assertions are equivalent:*

(vii) *For every* $\overline{x} \in \overline{S}$ *and every* $r > 0$ *there is* $\alpha(r) > 0$ *such that*

$$\alpha(r)\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \mathrm{cl}\big(\partial f(z) + N_S(z)\big) \quad \textit{whenever} \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, r). \tag{5.12}$$

(viii) *For every $\overline{x} \in \overline{S}$ and every $r > 0$ there is $\alpha(r) > 0$ such that*

$$f'(z; v) \geq \alpha(r) d_{T_z\overline{S}}(v) \tag{5.13}$$

whenever $z \in \overline{S} \cap \mathbf{B}(\overline{x}, r)$ and $v \in T_{\overline{x}}S$.

Proof. Let us assume according to assertion (i) of Theorem 4.6 that $\overline{S}$ is the set of boundedly weak sharp minima. Pick any $\overline{x} \in \overline{S}$ and $r > 0$, we find by Definition 4.1(iii) such $\alpha(r) > 0$ that

$$f(x) \geq f(\overline{x}) + \alpha(r) d_{\overline{S}}(x) \quad \text{for all} \quad x \in S \cap \mathbf{B}(\overline{x}, r). \tag{5.14}$$

Take further $z \in \overline{S} \cap \mathbf{B}(\overline{x}, r)$ and pick $v \in T_zM$ with $\|v\| \neq 0$. Then for any t satisfying $0 < t < \frac{r - d(z,\overline{x})}{\|v\|}$, we get from (5.14) and construction (4.3) of f_0 that

$$f_0(\exp_z tv) - f_0(z) = f_0(\exp_z tv) - f_0(\overline{x}) = f_0(\exp_z tv) - f(\overline{x}) \geq \alpha(r) d_S(\exp_z tv).$$

The latter implies by definition of the directional derivative that

$$f_0'(z; v) = \lim_{t>0, t\to 0} \frac{f_0(\exp_z tv) - f_0(z)}{t} \geq \alpha(r) d_{\overline{S}}'(z; v),$$

which gives assertion (vii) of the theorem by equivalence (5.5)$\Longleftrightarrow$(5.6) in Lemma 5.4.

The remaining implications (vii)$\Longrightarrow$(viii)$\Longrightarrow$[(iv) in Theorem 4.6] follow from those

$$(5.5) \Longrightarrow (5.7) \Longrightarrow (5.8)$$

in Lemma 5.4. This completes the proof of the theorem. □

The next result provides new characterizations of *local weak sharp minimizers* in Hadamard manifolds in addition to those obtained in Theorem 4.8 for the general Riemannian case.

THEOREM 5.7. **(Characterizations of local weak sharp minimizers on Hadamard manifolds.)** *In addition to all the assumptions of Theorem 4.8 suppose that M is a Hadamard manifold. Then assertions* (i)–(vi) *of Theorem 4.8 are equivalent to the following:*

(vii) *There exists a positive number ε such that*

$$\alpha\mathbb{B}_z \cap N_{\overline{S}}(z) \subset \operatorname{cl}\big(\partial f(z) + N_S(z)\big) \quad \textit{whenever} \quad z \in \overline{S} \cap \mathbf{B}(\overline{x}, \varepsilon).$$

(viii) *There exists a positive number ε such that*

$$f'(z; v) \geq \alpha d_{T_z\overline{S}}(v)$$

whenever $z \in \overline{S} \cap \mathbf{B}(\overline{x}, \varepsilon)$ and $v \in T_zS$.

Proof. Take an arbitrary local weak sharp minimizer $\overline{x}$ for problem $\mathcal{P}$. By Definition 4.1(i) there is $\varepsilon > 0$ such that we have the underlying weak sharp inequality

$$f(x) \geq f(\overline{x}) + \alpha d_{\overline{S}}(x) \quad \text{for all} \quad x \in S \cap \mathbf{B}(\overline{x}, \varepsilon).$$

Similarly to the proof of Theorem 5.6, pick any $z \in \overline{S} \cap \mathbf{B}(\overline{x}, \varepsilon)$ and $v \in T_zM$ with $\|v\| \neq 0$ and observe that

$$f_0(\exp_z tv) - f_0(z) = f_0(\exp_z tv) - f_0(\overline{x}) \geq \alpha d_S(\exp_z tv) \quad \text{whenever} \quad 0 < t < \frac{\varepsilon - d(z, \overline{x})}{\|v\|},$$

where the function f_0 is defined in (4.3). The latter inequality gives

$$f'_0(z;v) = \lim_{t>0,t\to 0} \frac{f_0(\exp_z tv) - f_0(z)}{t} \geq \alpha d'_{\overline{S}}(z;v),$$

which in turn ensures the validity of assertion (vii) of the theorem by equivalence (5.5)$\Longleftrightarrow$(5.6) in Lemma 5.4.

To conclude the proof of the theorem, observe that implications (vii)$\Longrightarrow$(viii)$\Longrightarrow$[(iv) in Theorem 4.8] follow from the corresponding ones (5.5)$\Longrightarrow$(5.7)$\Longrightarrow$(5.8) in Lemma 5.4. □

Finally in this section, we provide new characterizations of the set of *local weak sharp minima* on Hadamard manifolds. They follow directly from Theorem 5.7 by Definition 4.1(ii).

COROLLARY 5.8. **(Characterizations of the set of local weak sharp minima on Hadamard manifolds.)** *Under the assumptions of Theorem* 5.5 *we have the equivalence of the assertions of Corollary* 4.9 *with the following conditions:*

(vii) *For every* $\overline{x} \in \overline{S}$ *there exist* $r > 0$ *and* $\alpha(r) > 0$ *such that inclusion* (5.12) *is satisfied.*

(viii) *For every* $\overline{x} \in \overline{S}$ *there exist* $r > 0$ *and* $\alpha(r) > 0$ *such that estimate* (5.13) *is satisfied whenever* $z \in \overline{S} \cap \mathbf{B}(\overline{x}, r)$ *and* $v \in T_{\overline{x}}S$.

6. Concluding remarks. To the best of our knowledge, this paper is *the first one* in the literature dealing with weak sharp minima for constrained optimization problems on *Riemannian* and *Hadamard manifolds*. The main characterizations of global, boundedly, and local sharp minima for convex problems on Riemannian manifolds in assertions (ii) and (iii) of Theorems 4.5, 4.6, 4.8 and Corollary 4.9, respectively, are *new* even for the case of *finite-dimensional Euclidean spaces.* The other characterizations, including those for Hadamard manifolds from Section 5, are extensions of the corresponding results by Burke and Ferris [10] and Burke and Deng [12] obtained in the case of spaces with liner structures. Observe that to proceed with no linearity, we need to develop new methods and results of variational analysis on Riemannian and Hadamard manifolds including, in particular, the *characterization of projections* on convex subsets of Riemannian manifolds given in Theorem 3.4 and the normal cone representation for the *subdifferential* of the *distance function* for totally convex subsets of Hadamard manifolds derived in Theorem 5.3. These results seem to be of independent interest for various aspects of analysis on Riemannian and Hadamard manifolds regardless of their particular applications to the study of weak sharp minima.

Note that this paper mainly concerns *convex* problems of constrained optimization on Riemannian and Hadamard manifolds while some of our methods and results can be used for the further analysis of weak sharp minima in *nonconvex* optimization problems on spaces with no linear structure. In this respect we mention, in particular, the possibility of direct extending to the case of Hadamard manifolds necessary optimality conditions for weak sharp minima obtained in [52] for nonconvex optimization problems on Banach spaces in terms of Fréchet subgradients and Mordukhovich basic/limiting and singular subgradients of l.s.c. functions and the corresponding normals to closed subsets of Banach spaces. The latter constructions have been partly explored for other purposes in the recent study [39] in the case of smooth nonlinear manifolds. Among the key ingredients of the aforementioned developments in [52] is the subdifferential formula of type (5.1) for the distance function, which admits appropriate extensions to nonconvex frameworks in spaces with no linearity.

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