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# WEAK SOLUTIONS OF NAVIER-STOKES EQUATIONS

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**Introduction.** Consider the initial-value problem for the Navier-Stokes equation in a domain $\Omega$ of $\boldsymbol{R}^n$:

$$\text{(N-S)} \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u\cdot\nabla u + \nabla p = f\,; \quad \nabla\cdot u = 0\,, \quad x\in\Omega\,, \quad 0 < t < T\,. \\ u|_{\Gamma} = 0\,; \quad u|_{t=0} = a \end{cases}$$

($\Gamma$: the boundary of $\Omega$) where $u = u(x, t)$ is the unknown velocity vector $(u^1, u^2, \cdots, u^n)$; $p = p(x, t)$ is the unknown pressure; $a = a(x)$ is the initial velocity vector field; $f = f(x, t)$ is a given external force. Here we use the notation:

$$u\cdot\nabla v = \sum_{i=1}^{n} u^i \frac{\partial v}{\partial x_i}\,; \quad \nabla\cdot u = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x_i}$$

for vector functions $u$, $v$.

In his famous paper [8], E. Hopf showed the existence of the so-called Hopf's weak solution to the problem (N-S). The first purpose of the present paper is to show the existence of a weak solution, belonging to some class of functions introduced by J. L. Lions [14], which seems to have a somewhat stronger property than the Hopf's weak solution.

In the general case the uniqueness of a weak solution has been not known. Lions-Prodi [15] gave the uniqueness theorem when $n = 2$. C. Foias [15] introduced function spaces $L^{r,r'}$ (for the definition see the chapter 1 of this paper), and showed that if $\Omega = \boldsymbol{R}^n$, and if there is a weak solution $u$ in $L^{r,r'}$ with $r > n$, and with $n/r + 2/r' < 1$, then this $u$ is the only weak solution of (N-S). J. Serrin [23] gave a similar theorem under the assumptions that $\Omega$ is a general domain of $\boldsymbol{R}^n$ ($n = 2, 3, 4$), and that a pair of exponents $r$, $r'$ satisfies $r > n$ and $n/r + 2/r' \leqq 1$. The second purpose is to generalize the Foias-Serrin uniqueness theorem in two directions. First we shall remove the artificial restriction on the dimension $n$ imposed in the theorem of Serrin. Secondly, we shall show that if there is a weak solution $u$ in $L^{n,\infty}$ which is right continuous for $t$ as an $L^n$-valued function, then $u$ is the only weak solution. Recently von Wahl [26] obtained similar results (the uniqueness in the class $C([0, T); L^n)$) under the assumptions that the initial velocity and the external force are regular to some extent, and that $\Omega$ is a bounded domain, His method is however different from ours.

In the celebrated paper [13], J. Leray considered the case $\Omega = \boldsymbol{R}^3$, and constructed a weak solution. At the very end of the paper cited above he posed the problem whether or not the energy of the flow $(1/2)\int_{R^3} |u(x, t)|^2 dx$ tends to zero as $t \to \infty$. Our third purpose is to give an affirmative answer to this; the more general situations will be considered. T. Kato has obtained similar results on the decay of strong solutions with small initial value by a different method from ours.

## 1. Results.

**1.1.** Before stating our results we introduce some function spaces, and give our definition of weak solutions of (N-S). $C_{0,\sigma}^\infty$ is the set of all $C^\infty$ (vector) functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with support in $\Omega$, such that $\nabla \cdot \phi = 0$. $L_\sigma^2$ is the closure of $C_{0,\sigma}^\infty$ with respect to the $L^2$-norm $\|\cdot\|$; $(\cdot, \cdot)$ denotes the $L^2$-inner product. $L^p$ stands for the usual (vector-valued) $L^p$-space over $\Omega$, $1 \leqq p \leqq \infty$. $H_{0,\sigma}^1$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the norm

$$\|\phi\|_{H^1} = \|\phi\| + \|\nabla\phi\|$$

where $\nabla\phi = \partial_x\phi = (\partial\phi^i/\partial x_j;\ i, j = 1, 2, \cdots, n)$. $Y$ is the set of all $\phi$ in $H_{0,\sigma}^1 \cap L^n$. Equipped with the norm

$$\|\phi\|_Y = \|\phi\|_{H^1} + \|\phi\|_{L^n},$$

$Y$ is a Banach space.

When $X$ is a Banach space, its norm is a denoted by $\|\cdot\|_X$; $C^k([t_1, t_2]; X)$, $L^p((t_1, t_2); X)$ are then usual Banach spaces, where $t_1$, and $t_2$ are real numbers such that $t_1 < t_2$. $H^1((t_1, t_2); X)$ is the closure of $C^1([t_1, t_2]; X)$ with respect to the norm

$$\int_{t_1}^{t_2} (\|w(t)\|_X + \|w_t(t)\|_X) dt$$

$(w_t = \partial w/\partial t)$. In this paper we shall denote by $M$ various constants.

We can now introduce the assumptions on the initial function $a$ and the external force $f$, and state the definition of weak solutions of (N-S).

ASSUMPTION 1. The initial function $a = a(x)$ is in $L_\sigma^2$.

ASSUMPTION 2. The function $f = f(\cdot, t)$ is in $L^2$ for almost all $t$ in $(0, T)$, and $Pf(t)$ is an $L_\sigma^2$-valued integrable function on $(0, T)$. ($P$: the projection on $L_\sigma^2$ (in $L^2$)).

Throughout the present paper, we make the above assumptions. Our definition of a weak solution of (N-S) is as follows.

DEFINITION. Let $a$ and $f$ be as above. A measurable function $u$ on $\Omega \times (0, T)$ is called a weak solution of the initial-valued problem (N-S) if

( i ) $u \in L^2((0, T'); H^1_{0,\sigma})$ for any $T'$ with $0 < T' < T$;

( ii ) $u \in L^\infty((0, T); L^2_\sigma)$;

(iii)

$$(1.1) \quad \int_0^T \{-(u, \Phi_t) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} dt = \int_0^T (f, \Phi) dt + (a, \Phi(0))$$

for all $\Phi$ in $H^1((0, T); Y)$ such that for some $T_0 < T$, $\Phi(\cdot, t) = 0$ on $(T_0, T)$, $(\Phi(0) = \Phi(\cdot, 0))$.

The above definition is essentially due to J. Lions [14]. There are many other definitions of weak solutions. Concerning the relation between the Hopf's weak solution and the weak solution in our sense, we have

PROPOSITION 1. *Any weak solution in the above sense is a Hopf's weak solution. The converse is true when $C^\infty_{0,\sigma}$ is dense in $Y$. $C^\infty_{0,\sigma}$ is dense in $Y$ if one of the following conditions is satisfied:*

( a ) $2 \leqq n \leqq 4$ ;

( b ) *$\Omega$ is a star-shaped bounded domain;*

( c ) $\Omega = \boldsymbol{R}^n$ .

(For the proof, see the appendix).

Concerning the (weak) continuity (in $t$) of weak solutions, we have the result of G. Prodi [20] (see also J. Serrin [23]).

PROPOSITION 2. (Prodi) *Suppose that $u$ is a weak solution of* (N-S). *After suitable modification of its value of $u(t)$ on a set of measure zero of the time interval* $[0, T]$, *we have that $u(\cdot, t)$ is continuous for $t$ in the weak topology of $L^2_\sigma$, and that for any $0 \leqq s \leqq t < T$,*

$$(1.2) \quad \int_s^t \{-(u, \Phi_t) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} dt = \int_s^t (f, \Phi) dt - (u(t), \Phi(t)) + (u(s), \Phi(s))$$

*for every $\Phi$ in $H^1((s, t); Y)$. Here and in what follows we simply write $u(t)$, $\Phi(t)$ for $u(\cdot, t)$, $\Phi(\cdot, t)$.*

In what follows we shall mean by a weak solution a weak solution redefined as above.

**1.2.** Our result on the existence of weak solutions now reads:

THEOREM 1. *Let the assumptions* 1 *and* 2 *hold. Then there is a weak solution u of the problem* (N-S). *Moreover,*

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u\|^2 dt \leqq 2\int_0^t (f, u)dt + \|a\|^2 ; \quad (0 \leqq t < T) \tag{1.3}$$

$$\lim_{t\to 0} \|u(t) - a\| = 0 . \tag{1.4}$$

REMARKS. 1. The existence of a Hopf's weak solution is well-known. For the existence of our weak solution, see J. Lions [14].

2. For the existence of strong solutions, see Kiselev-Ladyzhenskaya [10], Fujita-Kato [5], Giga-Miyakawa [6].

3. If $\Omega$ is a bounded domain, then the energy inequality (of strong form)

$$\|u(t)\|^2 + 2\int_s^t \|\nabla u\|^2 dt \leqq 2\int_s^t (f, u)dt + \|u(s)\|^2$$

holds for almost all $s \geqq 0$, including $s = 0$, and all $t > s$. However, it is not known whether or not the above energy inequality of the strong form does hold for a general domain $\Omega$. Thus in the general case it is not known whether or not there is a weak solution of (N-S) with $f = 0$, such that $\|u(t)\|$ monotonously decreases with $t$.

**1.3.** We next proceed to our uniqueness results. To this end we first define a function space $L^{r,r'}$. If $w = w(x, t)$ is defined and measurable in a cylindrical domain $\Omega \times (t_1, t_2)$ of space-time, we set

$$\|w(t)\|_{L^r} = \left(\int_\Omega |w(x, t)|^r dx\right)^{1/r}$$

and

$$|w|_{r,r'} = \begin{cases} \left(\int_{t_1}^{t_2} \|w(t)\|_{L^r}^{r'} dt\right)^{1/r'} & (\text{if } 1 \leqq r' < \infty) \\ \sup_{t_1 \leqq t \leqq t_2} \|w(t)\|_{L^r} & (\text{if } r' = \infty) . \end{cases}$$

Here $r$ and $r'$ are considered to be independent exponents with $1 \leqq r, r' \leqq \infty$.

DEFINITION. We say that $w = w(x, t)$ is contained in the class $L^{r,r'}(\Omega \times (t_1, t_2))$ if $w$ is defined and measurable in $\Omega \times (t_1, t_2)$, and $|w|_{r,r'} < \infty$.

REMARK. It is easy to see that

$$L^{r,r'}(\Omega \times (t_1, t_2)) = L^{r'}((t_1, t_2); L^r) . \tag{1.5}$$

(see H. Rikimaru [21]).

Our uniqueness theorems read:

THEOREM 2. *Let the assumptions* 1 *and* 2 *hold. Let* $u$, $v$ *be weak solutions of the problem* (N-S). *Suppose also that*

$$\|v\|^2 + 2\int_0^t \|\nabla v\|^2 dt \leqq 2\int_0^t (f, v)dt + \|a\|^2 , \quad 0 < t < T , \tag{1.6}$$

*and that* $u \in L^{r,r'}(\Omega \times (0, T))$ *for a pair of exponents* $r$, $r'$ *satisfying*

$$\frac{n}{r} + \frac{2}{r'} \leqq 1 \tag{1.7}$$

*and also* $r > n$. *Then* $u = v$ *on* $[0, T)$.

THEOREM 3. *Let the assumptions* 1 *and* 2 *hold. Let* $u$, $v$ *be weak solutions of the problem* (N-S). *Suppose that* $v$ *satisfies the inequality* (1.6) *and that* $u \in L^\infty((0, T); L^n)$. *If there is an* $s$ $(0 \leqq s < T)$ *with* $u = v$ *on* $[0, s]$, *and if* $u$ *is right continuous for* $t$ *at* $t = s$ *in the norm of* $L^n$, *then there is a* $\delta > 0$ *such that* $u = v$ *on* $[0, s + \delta)$.

COROLLARY. *Let the assumptions* 1 *and* 2 *hold. Let* $u$, $v$ *be weak solutions of* (N-S). *Suppose that* $v$ *satisfies the inequality* (1.6) *and* $u \in L^\infty((0, T); L^n)$. *If* $u$ *is right continuous for all* $t$ *in* $[0, T)$ *in the norm of* $L^n$, *then* $u = v$ *on* $[0, T)$.

REMARKS. 1. If $n = 2$, then it can be shown that any weak solution $u$ in $L^\infty((0, T); L^2)$ is continuous for all $t$ in $(0, T)$. The uniqueness theorem for $n = 2$ due to Prodi-Lions [15] can be obtained.

2. C. Foias [4] first introduce function spaces $L^{r,r'}$ and showed that the uniqueness theorem (similar to Theorem 2 above) holds if $\Omega = \boldsymbol{R}^n$; $r > n$ and $n/r + 2/r' < 1$. On the other hand, J. Serrin [23] gave the uniqueness theorem under the assumptions that $\Omega$ is a general domain; $2 \leqq n \leqq 4$; $r > n$; $n/r + 2/r' \leqq 1$. Thus Theorem 2 may be considered as a generalization of the Foias-Serrin uniqueness theorem.

3. Recently von Wahl [26](1) gave the uniqueness theorem similar to Theorem 3 above. Under the assumptions that $a = a(x)$, and $f = f(x, t)$ are regular to some extent, and that $\Omega$ is a bounded domain, he showed that the uniqueness theorem holds in the class $C([0, T); L^n)$, by using the $a$-priori estimates due to Solonnikov [24]; the method is different from ours.

(1) After the completion of the present paper, Professor von Wahl kindly informed the author of their recent work [27], in which they independently showed similar results (Theorems 2 and 3 above) by using the Yosida approximation; the author would like to express his sincere thanks to Professor von Wahl for it.

**1.4.** We are next concerned with the problem whether or not $\|u(t)\| \to 0$ as $t \to \infty$. We first define the operator $A_0$ in $L^2_\sigma$. Let $A_0$ be the operator in $L^2_\sigma$ defined by: $A_0\phi = -\Delta\phi$; $D(A_0) = C^\infty_{0,\sigma}$. ($D(S)$; domain of $S$). Then the $A_0$ thus defined is clearly symmetric and positive in $L^2_\sigma$. Moreover we have $(A_0\phi, \phi) = \|\nabla\phi\|^2$. Hence $A_0$ admits the self-adjoint extension $A$ (called the Friedrichs extension of $A_0$) in $L^2_\sigma$. It is then easy to see that $A$ is positive and satisfies:

$$\|A^{1/2}\phi\| = \|\nabla\phi\| . \tag{1.8}$$

From the above identity it follows that the zero is not an eigenvalue of $A$. Thus $A$ is a strictly positive self-adjoint operator in $L^2_\sigma$. Now we make the following assumption on $A$.

ASSUMPTION 3. For some non-negative $\alpha$,

$$(I + A)^{-\alpha}\phi \in L^n \quad \text{for all} \quad \phi \quad \text{in} \quad L^2_\sigma .$$

In many cases the above assumption is satisfied:

PROPOSITION 3. *The above assumption is satisfied with* $\alpha = (n-2)/4$ *if one of the following conditions is satisfied.*

(i) $2 \leqq n \leqq 4$;

(ii) $\Omega = \boldsymbol{R}^n$, $n \geqq 2$.

PROOF. Define the operator $B$ in $L^2(\boldsymbol{R}^n)$ by: $B\phi = -\Delta\phi$, $D(B) = H^2(\boldsymbol{R}^n)$ (Sobolev space). By the Sobolev inequality

$$\|\phi\|_{L^n(\boldsymbol{R}^n)} \leqq M\|(I + B)^\alpha\phi\| , \quad \phi \in D(B^\alpha) \tag{1.9}$$

($\alpha = (n-2)/4$). If $\Omega = \boldsymbol{R}^n$, then $(I + A)^{-\alpha} = (I + B)^{-\alpha}P$. ($P$: the projection on $L^2_\sigma$). By (1.9), $(I + B)^{-\alpha}$ is a bounded operator from $L^2(\boldsymbol{R}^n)$ to $L^n(\boldsymbol{R}^n)$. Hence $(I + A)^{-\alpha}$ is a bounded operator from $L^2_\sigma(\boldsymbol{R}^n)$ to $L^n(\boldsymbol{R}^n)$. We next suppose that $2 \leqq n \leqq 4$. Let $E$ be the extension operator from $L^2_\sigma(\Omega)$ to $L^2(\boldsymbol{R}^n)$: $E\phi(x) = \phi(x)$ (if $x \in \Omega$); $= 0$ (if $x \notin \Omega$). Since $(I + B)^{1/2}E(I + A)^{-1/2}$ is a bounded operator by (1.8), it follows from the interpolation theorem that $(I + B)^\beta E(I + A)^{-\beta}$ is a bounded operator for $0 \leqq \beta \leqq 1/2$. Hence by (1.9) we see that $(I + A)^{-\alpha}$ is a bounded operator from $L^2_\sigma$ to $L^n$.

Our result on the decay of solutions reads:

THEOREM 4. *Let* $T = \infty$. *Let the assumptions* 1, 2 *and* 3 *hold. Let* $u$ *be a weak solution of* (N-S) *with* $\int_0^\infty \|\nabla u\|^2 dt < \infty$. *Then* $\|(I + A)^{-\alpha}u(t)\|$ *tends to zero as* $t \to \infty$.

The following two corollaries are immediate consequences of Theorem 4.

COROLLARY 1. *Under the assumptions of Theorem* 4,

$$\lim_{t\to\infty}\int_t^{t+1}\|u(s)\|^2 ds = 0 . \tag{1.10}$$

COROLLARY 2. *Let the assumptions of Theorem* 4 *hold. If* $\|u(t)\|$ *tends to some constant, say* $c$, *as* $t\to\infty$, *then we have* $c=0$.

J. Leray [13] considered the case $\Omega = \boldsymbol{R}^3$ and $f=0$, and constructed a weak solution $u$ that becomes smooth (in $x$ and $t$) for large $t$, say $t>T$; moreover, $\|u(t)\|$ monotonously decreases with $t>T$. He posed a problem whether or not $\|u(t)\|\to 0$ as $t\to\infty$. Corollary 2, together with Proposition 3, gives an affirmative answer to it. More generally, if $\Omega$ is a domain of $\boldsymbol{R}^3$, $f=0$, and if $u$ is a generalized solution (in the sense of Ladyzhenskaya [12]), then $\|u(t)\|$ monotonously decreases with $t$, and hence tends to zero as $t\to\infty$, by Corollary 2.

REMARKS. 1. We can construct a weak solution that $\|u(t)\|$ tends to zero as $t\to\infty$. (see K. Masuda [18]). T. Kato constructed a strong solution with $\|u(t)\|\to 0$ as $t\to\infty$ by a method different from ours. (His result was motivation for the present work.)

2. For the decay of $\|u(t)\|_{L^\infty}$ and $\|\nabla u(t)\|$ see Masuda [17, 18], J. G. Heywood [7], P. Maremonti [16].

3. Theorem 4 and the outline of its proof have been reported in Masuda [19].

## 2. Preliminaries.

**2.1.** We first recall elementary properties of the mollifier $J_h[w]$ of $w$, $h>0$. Let $\rho$ be a $C^\infty$ function in $\boldsymbol{R}^1$ with support in $|t|\leqq 1$, such that $\rho(t)=\rho(-t)$, $\rho(t)\geqq 0$, and $\int_{-\infty}^{\infty}\rho(t)dt=1$. We set $\rho_h(t)=h^{-1}\rho(t/h)$. Let $s$, $t$ be fixed numbers such that $0\leqq s<t<+\infty$. Let $X$ be a Banach space. For $w$ in $L^p((s,t);X)$, $1\leqq p<\infty$, we define the mollifier $J_h[w]$ of $w$ by

$$J_h[w](\tau)=\int_s^t \rho_h(\tau-\sigma)w(\sigma)d\sigma , \tag{2.1}$$

Then the following lemma is well-known, and is easy to prove.

LEMMA 2.1. *We have*

(i) *For each fixed* $h$, $J_h$ *is a bounded operator from* $L^p((s,t);X)$ *into* $C^1([s,t];X)$.

(ii) *For each fixed* $w$ *in* $L^p((s,t);X)$, $J_h[w]\to w$ *as* $h\to 0$ *in* $L^p((s,t);X)$;

(iii) *If* $w \in C([s, t]; X)$, *then* $J_h[w](t) \to (1/2)w(t)$ *and* $J_h[w](s) \to (1/2)w(s)$ *as* $h \to 0$ *in the norm of* $X$.

LEMMA 2.2. *Let* $X_0$ *be a dense subset of a Banach space* $X$. *Then any function* $\Phi \in H^1((s, t); X)$ *can be approximated by a sequence* $\{\Phi_N\}$, *in the topology of* $H^1((s, t); X)$, *such that each* $\Phi_N$ *has the form*

$$\Phi_N(\tau) = \sum_{\text{finite}} \lambda_j(\tau)\phi_j \tag{2.2}$$

*where* $\lambda_j$ *is some* $C^\infty$ *function on* $\boldsymbol{R}^1$ *and* $\phi_j$ *is some element of* $X_0$. *Similarly, any function in* $L^2((s, t); X)$ *can be approximated by a sequence of functions of the form* (2.2) *in the topology of* $L^2((s, t); X)$.

PROOF. Since $C^1([s, t]; X)$ is dense in $H^1((s, t); X)$, we may assume that $\Phi$ is in $C^1([s, t]; X)$. Since $X_0$ is dense in $X$ by hypothesis, for any positive integer $N$, there is a $\phi_{N,j}$ in $X_0$ with $\|\phi_{N,j} - \Phi(t_j)\| < 1/N^2$, $j = 0, 1, \cdots, N$. ($t_j = s + j\Delta_N$; $\Delta_N = (t-s)/N$). Set

$$\tilde{\Phi}_N(\tau) = \phi_{N,j} + \Delta_N^{-1}(\tau - t_j)(\phi_{N,j+1} - \phi_{N,j}) \,, \tag{2.3}$$

if $t_j \leqq \tau \leqq t_{j+1}$. It is easy to see that $\tilde{\Phi}_N \in H^1((s, t); X)$. Moreover $\tilde{\Phi}_N$ tends to $\Phi$ as $N \to \infty$ in $H^1((s, t); X)$. Indeed, we have

$$\tilde{\Phi}'_N(\tau) - \Phi'(\tau) = \Delta_N^{-1}\left[\phi_{N,j+1} - \Phi(t_{j+1}) - (\phi_{N,j} - \Phi(t_j)) + \int_{t_j}^{t_j+1} (\Phi'(\sigma) - \Phi'(\tau))d\sigma\right]$$

if $t_j \leqq \tau \leqq t_{j+1}$. Therefore

$$\|\tilde{\Phi}'_N(\tau) - \Phi'(\tau)\| \leqq 2\Delta_N + \sup_{|\sigma-\sigma'|<1/N} \|\Phi'(\sigma) - \Phi'(\sigma')\|$$

from which it follows that the integral

$$\int_s^t \|\tilde{\Phi}'_N(\tau) - \Phi'(\tau)\|^2 d\tau$$

tends to zero as $N \to \infty$. Thus we can see that $\tilde{\Phi}_N \to \Phi$ in $H^1((s, t); X)$; note that clearly $\tilde{\Phi}_N \to \Phi$ in $C([s, t]; X)$. Extend $\tilde{\Phi}_N$ to function on $\boldsymbol{R}^1$: $\tilde{\Phi}_N(\tau) = \phi_{N,0}$ (if $\tau \leqq t_0$); $= \phi_{N,N}$ (if $\tau \geqq t_N$). Then we mollify $\tilde{\Phi}_N$:

$$\Phi_N(\tau) \equiv \int_{-\infty}^{\infty} \rho_{1/N}(\tau - \sigma)\tilde{\Phi}_N(\sigma)d\sigma \,.$$

The $\Phi_N$ thus defined is a desired function of the form (2.2). The latter statement can be proved similarly.

**2.2.** In this subsection we shall give some estimates for $(w_1 \cdot \nabla w_2, w_3)$.

LEMMA 2.3. *Let* $\phi_1$, $\phi_2$ *be in* $H^1_{0,\sigma}$, $\phi_3 \in L^r$, *and* $\phi_4 \in Y$, *where* $n \leqq r \leqq \infty$. *Then*

(i) $\|\phi_1\phi_3\| \leqq M\|\nabla\phi_1\|^{n/r}\|\phi_1\|^{1-n/r}\|\phi_3\|_{L^r}$;

(ii) $|(\phi_1 \cdot \nabla\phi_2, \phi_3)| \leqq M\|\nabla\phi_1\|^{n/r}\|\phi_1\|^{1-n/r}\|\nabla\phi_2\|\,\|\phi_3\|_{L^r}$;
(iii) $|(\phi_3 \cdot \nabla\phi_2, \phi_1)| \leqq M\|\nabla\phi_1\|^{n/r}\|\phi_1\|^{1-n/r}\|\nabla\phi_2\|\,\|\phi_3\|_{L^r}$;
(iv) $(\phi_4 \cdot \nabla\phi_1, \phi_2) = -(\phi_4 \cdot \nabla\phi_2, \phi_1)$.

PROOF. By the Hölder inequality,

$$\|\phi_1\phi_3\| \leqq M\|\phi_1\|_{L^{n'}}^{n/r}\|\phi_1\|^{1-n/r}\|\phi_3\|_{L^r}\ , \quad \left(\frac{1}{n'} = \frac{1}{2} - \frac{1}{n}\right).$$

Hence the statement (i) follows from the Sobolev inequality:

$$\|\phi\|_{L^{n'}} \leqq M\|\nabla\phi\|\ .$$

The statements (ii), (iii) follow from (i). Let $\{\phi_{i,j}\}_{j=1}^{\infty}$ be a sequence in $C_{0,\sigma}^{\infty}$ such that $\phi_{i,j} \to \phi_i$ as $j \to \infty$ in $H_{0,\sigma}^1$, $i = 1, 2, 4$. Then by (ii) and (iii),

$$\begin{aligned}(\phi_4 \cdot \nabla\phi_2, \phi_1) &= \lim_{j\to\infty}\lim_{k\to\infty}\lim_{l\to\infty}(\phi_{4,l} \cdot \nabla\phi_{2,j}, \phi_{1,k}) \\ &= -\lim_{j\to\infty}\lim_{k\to\infty}\lim_{l\to\infty}(\phi_{2,j}, \phi_{4,l} \cdot \nabla\phi_{1,k}) \\ &= -(\phi_2, \phi_4 \cdot \nabla\phi_1)\ ,\end{aligned}$$

showing (iv).

LEMMA 2.4. *Let* $w_1 \in L^2((s, t); H_{0,\sigma}^1) \cap L^\infty((s, t); L^2)$, $w_2 \in L^2((s, t); H_{0,\sigma}^1)$, $w_3 \in L^{r'}((s, t); L^r)$, *and* $w_4 \in L^2((s, t); Y)$ *where* $n/r + 2/r' = 1$, $r \geqq n$. *If* $1 \leqq r' < \infty$, *we set*

$$g(s, t) = \left(\int_s^t \|w_1\|^2\|w_3\|_{L^r}^{r'}\,dt\right)^{1/r'}$$

*and if* $r' = \infty$, *we set*

$$g(s, t) = \operatorname*{ess\,sup}_{\tau}\|w_3(\tau)\|_{L^n} \quad (s \leqq \tau \leqq t)\ .$$

*Then*:

$$\text{(i)}\qquad \int_s^t |(w_1 \cdot \nabla w_2, w_3)|dt + \int_s^t |(w_3 \cdot \nabla w_2, w_1)|dt \leqq Mg(s, t)\left(\int_s^t \|\nabla w_1\|^{qn/r}\|\nabla w_2\|^q dt\right)^{1/q}\ ,$$

$(q = 2r/(n + r))$ $M$ *being a constant independent of* $w_1$, $w_2$, $w_3$, *and* $s$, $t$.

$$\text{(ii)}\qquad \int_s^t (w_4 \cdot \nabla w_1, w_2)dt = -\int_s^t (w_4 \cdot \nabla w_2, w_1)dt\ .$$

PROOF. The proofs of (i), (ii) follow from Lemma 2.3.

Let $\zeta$ be a monotone increasing $C^\infty$ function in $\boldsymbol{R}^1$ such that $0 \leqq \zeta \leqq 1$, $|\partial_s\zeta(s)| \leqq 1$ (for all $s$ in $\boldsymbol{R}^1$), and $\zeta(s) = 1$ (if $|s| \leqq 1$); $=0$ (if $|s| \geqq 4$). Set

$\zeta_k(x) = \zeta(|x|/k)$, $(x \in \boldsymbol{R}^m)$ $k = 1, 2, \cdots$. Then a sequence $\{\zeta_k\}_{k=1}^{\infty}$ will be called a sequence of $m$-dimensional cut-off functions. Then:

LEMMA 2.5. *For any* $\varepsilon > 0$ *and* $w_3$ *in* $C([0, T']; L^n)$, *there is a constant* $M$, *an integer* $N$, *and functions* $\psi_j(x)$ $(i = 1, \cdots, N)$ *in* $L^2$ *such that the inequality*

$$\text{(2.4)} \qquad \int_s^t |(w_1 \cdot \nabla w_2, w_3)| dt \leqq \varepsilon \int_s^t (\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \|w_1\| \|\nabla w_2\|) dt + M \sum_{i=1}^N \int_s^t |(w_1, \psi_i)|^2 dt$$

*holds for all* $w_1$, $w_2$ *in* $L^2((s, t); H^1_{0,\sigma})$, *and* $0 \leqq s < t \leqq T'$.

PROOF. We fix $w_1$, $w_2$; and define the linear functional on $C([s, t]; L^n)$.

$$I[w] = \int_s^t (w_1 \cdot \nabla w_2, w) dt .$$

Then we decompose $I[w_3]$ in the form:

$$\text{(2.5)} \qquad I[w_3] = I[w_{3,1}] + I[w_{3,2}] + I[w_{3,3}]$$

where

$$w_{3,1}(x, t) = (1 - \zeta_p(x)) w_3(x, t) ;$$
$$w_{3,2}(x, t) = \zeta_p(x)(1 - \eta_q(|w_3(x, t)|)) w_3(x, t) ;$$
$$w_{3,3}(x, t) = \zeta_p(x) \eta_q(|w_3(x, t)|) w_3(x, t) .$$

Here $\{\zeta_p\}$, $\{\eta_q\}$ be sequences of $n$-dimensional, 1-dimensional cut-off functions, respectively. We shall estimate each term on the RHS of (2.5). By Lemma 2.4 (i), (ii),

$$\text{(2.6)} \qquad |I[w_{3,i}]| \leqq M \int_s^t \|\nabla w_1\| \|\nabla w_2\| dt \sup_{0 \leqq \tau \leqq T'} \|w_{3,i}(\tau)\|_{L^n} , \quad i = 1, 2 .$$

We shall show that $\sup_{0 \leqq \tau \leqq T'} \|w_{3,i}(\tau)\|_{L^n}$ is sufficiently small for large $p$, and $q$. From hypothesis it easily follows that $\|w_{3,1}(\tau)\|_{L^n}$ is continuous for $\tau$. Moreover the family of continuous functions $\|w_{3,1}(\tau)\|_{L^n}$ on $[0, T']$ is monotone decreasing in $p$, and converges to zero for each fixed $\tau$ by Lebesgue convergence theorem. Hence it follows from the Dini theorem that $\|w_{3,1}(\tau)\|_{L^n}$ converges to zero as $p \to \infty$, uniformly on $[0, T']$. Hence we can take $p$ so large that

$$\text{(2.7)} \qquad |I[w_{3,1}]| \leqq \frac{\varepsilon}{4} \int_s^t \|\nabla w_1\| \|\nabla w_2\| dt , \quad 0 \leqq s < t \leqq T'$$

(see (2.6)): we fix such a $p$. Using the elementary inequality

$$|\eta_q(|\xi|)\xi - \eta_q(|\xi'|)\xi'| \leqq \sup_s (\eta(s) + |s\eta'(s)|) |\xi - \xi'|$$

for two vectors $\xi$, $\xi'$, we can see that $\|w_{3,2}(\tau)\|_{L^n}$ is continuous for $\tau$. Also the family of continuous functions $\|w_{3,2}(\tau)\|_{L^n}$ is monotone decrersing in $q$ and tends to zero as $q \to \infty$ for each fixed $\tau$ (and a fixed $p$). Hence by the Dini theorem, it converges to zero as $q \to \infty$, uniformly on $[0, T']$. Hence we can take $q$ so large that

$$(2.8) \qquad |I[w_{3,2}]| \leqq \frac{\varepsilon}{4}\int_s^t \|\nabla w_1\|\,\|\nabla w_2\|dt\,, \quad 0 \leqq s < t \leqq T'$$

(see (2.6)): we fix such a $q$.

We finally proceed to the estimate of $I[w_{3,3}]$. We have

$$(2.9) \qquad |I[w_{3,3}]| \leqq \int_s^t \|\zeta_p w_1\|\,\|\nabla w_2\|dt \sup_{s\leqq\tau\leqq t} \|\eta_q(|w_3(\tau)|)w_3(\tau)\|_{L^\infty}\,.$$

A trivial calculation gives:

$$(2.10) \qquad \|\eta_q(|w_3(\tau)|)w_3\|_{L^\infty} \leqq 4q\,, \quad 0 \leqq \tau \leqq T'\,.$$

On the other hand since $H_0^1(\Omega_p)$ is compactly imbedded in $L^2(\Omega_p)$ ($\Omega_p = \{x \in \Omega; |x| \leqq 4p\}$), and since $\zeta_p w_1 \in H_0^1(\Omega_p)$, it follows from the Friedrichs inequality (Courant-Hilbert [2; p. 489]) that for any $\varepsilon' > 0$ there is an integer $N$ and functions $\omega_i$ in $L^2(\Omega_p)$ ($i = 1, \cdots, N$) with

$$\|\zeta_p w_1(\tau)\| \leqq \varepsilon'\|\nabla(\zeta_p w_1(\tau))\| + M\sum_{i=1}^N |(\zeta_p w_1(\tau), \omega_i)_{L^2(\Omega_p)}| \quad \text{a.e. in } (s, t)\,.$$

($(\cdot, \cdot)_{L^2(\Omega_p)}$ denotes the $L^2$-inner product over $\Omega_p$.) Hence since $|\partial_x \zeta_p(x)| \leqq 1$, we have

$$(2.11) \qquad \|\zeta_p w_1(\tau)\| \leqq \varepsilon'\|\nabla w_1(\tau)\| + \varepsilon'\|w_1(\tau)\| + M\sum_{i=1}^N |(w_1(\tau), \psi_i)|$$

where $\psi_i(x) = \zeta_p(x)\omega_i(x)$ ($x \in \Omega_p$); $= 0$ ($x \in \Omega \setminus \Omega_p$). Thus, by the Schwarz inequality, (2.9), (2.10), (2.11), we have

$$(2.12) \qquad |I[w_{3,3}]| \leqq 4q\varepsilon'\int_s^t (\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \|w_1\|\,\|\nabla w_2\|)dt + M\sum_{i=1}^N \int_s^t |(w_1, \psi_i)|^2 dt\,.$$

Taking $\varepsilon'$ so small that $4q\varepsilon' < 1$, and collecting all the estimates (2.7), (2.9), (2.12), we obtain the desired estimate (2.4).

We wish to relax the assumption, made in Lemma 2.5, that $w_3$ is continuous on $[0, T']$ in the norm of $L^n$. To this end we prepare a lemma:

LEMMA 2.6. *Let $f$ be a non-negative and integrable function on $[s, T']$, and $\{g_k\}_{k=1}^\infty$ be a sequence of non-negative functions in $L^\infty(s, T')$. Suppose that $\int_s^t f(\tau)d\tau > 0$ for any $t$ in $(s, T')$. Suppose also that for each* *fixed t* $g_k(t)$ *decreases monotonously to zero as* $k\to\infty$, *and for each fixed* $k$ $g_k(t)$ *is right continuous for t at* $t=s$. *Then for any* $\varepsilon>0$, *there is an N such that*

$$\int_s^t f(\tau)g_k(\tau)d\tau \leqq \varepsilon\int_s^t f(\tau)d\tau$$

*for all t in* $(s, T')$ *and* $k>N$.

PROOF. Put

$$z_k(t)=\int_s^t fg_k dt\Big/\int_s^t fdt\ ,\quad t>s\ ,\quad k=1,\cdots .$$

If we define $z_k(s)=g_k(s)$, then $z_k$ is continuous for $t$ in $[s, T']$. Indeed, it is clearly continuous for $t$ in $(s, T']$. It is also easy to see that

$$|z_k(t)-g_k(s)|\leqq \sup_{s<\tau<t}|g_k(\tau)-g_k(s)|\ ,$$

from which it follows that $z_k$ is continuous on $[s, T']$. On the other hand for each fixed $t$ $z_k(t)$ decreases monotonously to zero as $k\to\infty$. Hence by the Dini theorem $z_k(t)$ converges to zero as $k\to\infty$, uniformly on $[s, T']$. This proves Lemma 2.6.

LEMMA 2.7. *Let* $w\in L^2((s, T'); H^1_{0,\sigma})$, *and* $u\in L^\infty((s, T'); L^n)$. *Suppose that* $\int_s^t\|w\|^2dt>0$ *for any t in* $(s, T')$. *Suppose also that u is right continuous for t at* $t=s$ *in the norm of* $L^n$. *Then for any* $\varepsilon>0$

$$\int_s^t|(w\cdot\nabla w, u)|dt\leqq\varepsilon\int_s^t\|\nabla w\|^2dt+M\int_s^t\|w\|^2dt\ ,\quad s\leqq t\leqq T'\ , \tag{2.13}$$

*M being a constant independent of t.*

PROOF. If we set

$$u_1=(1-\zeta_p(x))u(x, t)\ ;\qquad u_2=\zeta_p(x)(1-\eta_q(|u(x, t)|))u(x, t)\ ;$$
$$u_3=\zeta_p(x)\eta_q(|u(x, t)|)u(x, t)\ ,$$

then in the same way as in the proof of Lemma 2.5 we can get, by $u=u_1+u_2+u_3$,

$$\int_s^t|(w\cdot\nabla w, u)|dt\leqq M\int_s^t\|\nabla w\|^2(\|u_1\|_{L^n}+\|u_2\|_{L^n})dt$$
$$+4qM\int_s^t\|\zeta_p w\|\,\|\nabla w\|dt\ .$$

From hypothesis it follows that for each fixed $p$ $\|u_1(t)\|_{L^n}$ is right continuous for $t$ at $t=s$, and for each fixed $t$ $\|u_1(t)\|_{L^n}$ decreases monotonously to zero as $p\to\infty$, and that $\int_s^t\|\nabla w\|^2dt>0$ for $t$ in $(s, T')$. Thus by Lemma 2.6, there is a $p_0$ such that

$$\int_s^t \|\nabla w\|^2 \|u_1\|_{L^n} dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt\ , \quad p \geqq p_0\ , \quad s \leqq t \leqq T'\ .$$

Similarly we can see that for each fixed $p$ there is a $q_o$ with

$$\int_s^t \|\nabla w\|^2 \|u_2\|_{L^n} dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt\ , \quad q \geqq q_o\ , \quad s \leqq t \leqq T'\ .$$

By the Hölder inequality, for each fixed $p$ and $q$,

$$\int_s^t \|\zeta_p w\| \, \|\nabla w\| dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt + M\varepsilon^{-1} \int_s^t \|w\|^2 dt\ , \quad s \leqq t \leqq T'\ .$$

Collecting all the estimates above, we can get the desired estimate (2.13).

**3. Existence of weak solutions; Proof of Theorem 1.** Following Hopf [8], we first construct approximate solutions of the problem (N-S) by the well-known Galerkin method, in the Banach space $Y = H^1_{0,\sigma} \cap L^n$. To this end we need the following.

LEMMA 3.1. *The Banach space $Y$ is separable.*

PROOF. Define the extension $E$: $Y \to H^1(R^n) \cap L^n(\boldsymbol{R}^n)$ by $(Eu)(x) = u(x)$ (if $x \in \Omega$); $= 0$ (if $x \in \boldsymbol{R}^n \backslash \Omega$). By the identification $u \leftrightarrow Eu$, $Y$ can be regarded as a closed subspace of $H^1(R^n) \cap L^n(\boldsymbol{R}^n)$. By virtue of Lions [12; p. 6], $H^1(\boldsymbol{R}^n) \cap L^n(\boldsymbol{R}^n)$ is separable. Hence, $Y$ is separable.

Now by Lemma 3.1 just proved, there exists a sequence $\{\phi_k\}_{k=1}^{\infty}$ of linearly independent vectors which is total in $Y$. Since $C^{\infty}_{0,\sigma} \subset Y \subset L^2_{\sigma}$, and since $C^{\infty}_{0,\sigma}$ is dense in $L^2_{\sigma}$, it follows that $\{\phi_k\}_{k=1}^{\infty}$ is also total in $L^2_{\sigma}$; we may assume, without loss of generality, that it is a complete orthonormal system in $L^2_{\sigma}$. Using $\{\phi_k\}$, we construct approximate solution $u_m = u_m(x, t)$ of the problem (N-S) which has the form

$$u_m(x, t) = \sum_{l=1}^{m} c_{ml}(t)\phi_l(x)\ . \tag{3.1}$$

Here the coefficient $c_{ml} = c_{ml}(t)$ $(l = 1, 2, \cdots, m)$ is a solution of a system of ordinary differential equation

$$dc_{ml}/dt + \sum_{i=1}^{m} a_{il} c_{mi} + \sum_{i,p=1}^{m} a_{ipl} c_{mi} c_{mp} = f_l \quad (l = 1, 2, \cdots, m) \tag{3.2}$$

with the initial condition

$$c_{ml}(0) = c_{0,l} \quad (l = 1, 2, \cdots, m) \tag{3.3}$$

where

$$a_{il} = (\nabla\phi_i, \nabla\phi_l)\ ; \quad a_{ipl} = (\phi_i \cdot \nabla\phi_p, \nabla\phi_l)\ ; \quad f_l = (f, \phi_l)\ ; \quad c_{0,l} = (a, \phi_l)\ .$$

We note that $a_{ipl}$ is finite by Lemma 2.3. If $\lambda_l \in H^1((0, T))$ $(1 \leqq l \leqq m)$, then noting the relation

$$(u_m(t), \phi_l) = c_{ml} , \tag{3.4}$$

we multiply the both side of (3.2) by $\lambda_l(t)$ and integrate it in $t$ over the interval $(s, t)$; and there results:

$$\begin{aligned}\int_s^t \{-(u_m, \Phi_t) + (\nabla u_m, \nabla \Phi) + (u_m \cdot \nabla u_m, \Phi)\}dt \\ = \int_s^t (f, \Phi)dt - (u_m(t), \Phi(t)) + (u_m(s), \Phi(s)) ,\end{aligned} \tag{3.5}$$

where $\Phi = \lambda_l(t)\phi_l(x)$. Putting $\lambda_l(t) = c_{ml}(t)$ in the above identity, and taking the summation with respect to $l$, we find

$$\| u_m(t) \|^2 + 2\int_0^t \| \nabla u_m \|^2 dt = 2\int_0^t (u_m, f)dt + \| a_m \|^2 \tag{3.6}$$

where $a_m = u_m(0)$, since we have $(u_m \cdot \nabla u_m, u_m) = 0$ by Lemma 2.3. Since $\| a_m \| \leqq \| a \|$, it follows from the assumption 2 that

$$\| u_m(t) \|^2 + \int_0^t \| \nabla u_m \|^2 dt \leqq M_1 , \quad 0 \leqq t < T , \tag{3.7}$$

$M_1$ being a constant independent of $m$, $t$. (see Ladyzhenskaya [12; Chapter 6, Section 3]). As is well-known the above a priori estimate (3.7) guarantees the global existence of solution of (3.2), (3.3). Moreover, we have:

LEMMA 3.2. *For each fixed $j$, the family $\{(u_m(t), \phi_j)\}_{j=1}^{\infty}$ forms a uniformly bounded and equicontinuous family of continuous functions on* $[0, T]$.

PROOF. The uniform boundedness is an immediate consequence of (3.7). A simple calculation yields

$$\begin{aligned}(u_m(t), \phi_j) - (u_m(s), \phi_j) &= \int_s^t ((\partial/\partial\tau)u_m(\tau), \phi_j)dt \\ &= -\int_s^t (\nabla u_m, \nabla\phi_j)d\tau - \int_s^t (u_m \cdot \nabla u_m, \phi_j)d\tau + \int_s^t (f, \phi_j)d\tau \\ &(\equiv I_1 + I_2 + I_3) .\end{aligned}$$

We shall estimate $I_j$, $j = 1, 2, 3$. By the Schwarz inequality and (3.7),

$$|I_1| \leqq M(t-s)^{1/2} \tag{3.8}$$

and

$$|I_3| \leqq M\int_s^t \| Pf \| dt \tag{3.9}$$

$M$ being a constant independent of $m, s, t$. Applying to $I_2$ Lemma 2.5 with $w_1 = w_2 = u_m$ and $w_3 = \phi_j$, we see that for any $\varepsilon' > 0$, there holds

$$|I_2| \leqq \varepsilon' \int_s^t \|\nabla u_m\|^2 dt + M \int_s^t \|u_m\|^2 dt \ ;$$

and hence, by (3.7)

$$|I_2| \leqq M_1\varepsilon' + M|t - s| \tag{3.10}$$

$M$ being a constant independent of $m, s, t$. Therefore it follows from (3.8), (3.9), (3.10) that for any $\varepsilon > 0$ there is a $\delta > 0$ with

$$|(u_m(t), \phi_j) - (u_m(s), \phi_j)| < \varepsilon \quad \text{if} \quad |t - s| < \delta \ , \quad m = 1, 2, \cdots \ . \tag{3.11}$$

Since $\varepsilon$ is arbitrary positive number, (3.11) implies that the family $\{(u_m(t), \phi_j)\}$ is equicontinuous.

Now by the Ascoli-Arzelà theorem, and the usual diagonal argument, it follows from (3.7) and Lemma 3.2 that there is a subsequence $\{m_i\}$ of $\{m\}$ along which $\{u_m(t)\}$ converges to some $u(t)$, uniformly in $t \in [0, T]$, in the weak topology of $L^2_\sigma(\Omega)$: The uniform limit $u(t)$ of a sequence of continuous functions $u_m(t)$ is continuous for $t$, weakly (see Hopf [8]; and also Ladyzhenskaya [12]). On the other hand, since $\{u_m\}$ is bounded in $L^2((0, T); H^1_{0,\sigma})$ by (3.7), there is a subsequence of $\{m_i\}$ along which $\{u_{m_i}\}$ converges to some $\tilde{u}$ weakly in $L^2((0, T); H^1_{0,\sigma})$. It is easy to see that $\tilde{u} = u$; we shall assume that the original sequence $\{u_m(t)\}$ itself converges to $u$, for the sake of simplification of the notations. Since $\|a_m\| \leqq \|a\|$, taking the lim sup (in $m$) in (3.6), we see that the $u$ satisfies the energy inequality (1.3). To show that the $u$ is a desired solution, it remains only to show that it satisfies (1.2).

We claim:

$$\int_s^t (u_m \cdot \nabla u_m, \Phi) dt \to \int_s^t (u \cdot \nabla u, \Phi) dt \ , \quad \text{as} \quad m \to \infty \tag{3.12}$$

for every $\Phi$ in $\mathscr{F}_{s,t}$: $\mathscr{F}_{s,t}$ is the set of all $\Phi$ of the form

$$\Phi = \sum \lambda_l(\tau)\phi_l(x) \quad \text{(finite sum)} \tag{3.13}$$

where $\lambda_l(\tau)$ is arbitrary function in $H^1((s, t); \boldsymbol{R}^1)$. Indeed, we have

$$\int_s^t (u_m \cdot \nabla u_m, \Phi) dt - \int_s^t (u \cdot \nabla u, \Phi) dt$$
$$= \int_s^t ((u_m - u) \cdot \nabla u_m, \Phi) dt + \int_s^t (u \cdot \nabla (u_m - u), \Phi) dt \quad (\equiv I_1 + I_2) \ .$$

By (1.3), (3.7) and Lemma 2.5 (with $w_1 = w_m - u$, $w_2 = u_m$, $w_3 = \Phi$), we see that for any $\varepsilon > 0$ there is a constant $M = M_\varepsilon$, a positive integer

$N = N_\varepsilon$, and function $\psi_i(x)$ $(i = 1, 2, \cdots, N)$ in $L^2$, such that

$$|I_1| \leqq \varepsilon M' + M \sum_{i=1}^{N} \int_s^t (u_m - u, \psi_i)^2 dt \, . \tag{3.14}$$

$M'$ being a constant independent of $\varepsilon$, $m$. Hence, letting $m \to \infty$, we get

$$\limsup_{m\to\infty} |I_1| \leqq \varepsilon M'$$

since $u_m(t) \to u(t)$, uniformly in $t$, in the weak topology of $L^2_\sigma$. Since $\varepsilon$ is arbitrary, it follows that $I_1 \to 0$. We next show $I_2 \to 0$. If $w_i(x, t) = u^{(i)}(x, t)\Phi(x, t)$ ($u^{(i)}$: then $i$-th component of $u$), then $w_i \in L^2(\Omega \times (s, t))$ by Lemma 2.3. Hence there is a sequence $\{w_{i,k}\}_{k=1}^\infty$ $(i = 1, \cdots, n)$ in $C_0^\infty(\Omega \times (s, t))$ with $w_{i,k} \to w_i$ as $k \to \infty$ in $L^2(\Omega \times (s, t))$. For the $w_{i,k}$, we have, by partial integration,

$$|I_2| \leqq \sum_{i=1}^{n} \int_s^t |(u_m - u, \partial_i w_{i,k})| dt$$
$$+ \sum_{i=1}^{n} \left(\int_s^t \|\nabla u_m - \nabla u\|^2 dt\right)^{1/2} \left(\int_s^t \|w_{i,k} - w_i\|^2 dt\right)^{1/2} \quad (\partial_i = \partial/\partial x_i) \, .$$

Letting $m \to \infty$ and then $k \to \infty$ in the above inequality, we have, by (3.7), $I_2 \to 0$. Hence we have (3.12).

Taking a finite sum with respect to $l$ and then letting $m \to \infty$ in (3.5), we obtain, by (3.12),

$$\int_s^t \{-(u, \Phi_t) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} dt = \int_s^t (f, \Phi) dt - (u(t), \Phi(t)) + (u(s), \Phi(s)) \tag{3.15}$$

for every $\Phi$ in $\mathscr{F}_{s,t}$. We next show that (3.15) holds for every $\Phi$ in $C^1([s, t]; Y)$. Let $\Phi \in C^1([s, t]; Y)$. Let $\mathscr{F}_0$ be the set of all (finite) linear combination of the functions in the set $\{\phi_i\}$; $\mathscr{F}_0$ is dense in $Y$ by definition. Hence by (2.3), there is a sequence $\{\Phi_N\}$ such that $\Phi_N \to \Phi$ in $H^1((s, t); Y)$, and which has the form

$$\Phi_N(\tau) = \psi_j + \Delta_N^{-1}(\tau - t_j)(\psi_{j+1} - \psi_j) \quad \text{if} \quad t_j \leqq \tau \leqq t_{j+1}$$

where $t_j = s + j\Delta_N$ $(j = 0, \cdots, N)$; and $\psi_j \in \mathscr{F}_0$. Applying (3.15) with $s = t_j$, $t = t_{j+1}$, one finds

$$\int_{t_j}^{t_{j+1}} \{-(u, \Phi_{N,t}) + (\nabla u, \nabla \Phi_N) + (u \cdot \nabla u, \Phi_N)\} dt$$
$$= \int_{t_j}^{t_{j+1}} (f, \Phi_N) dt - (u(t_{j+1}), \Phi_N(t_{j+1})) + (u(t_j), \Phi_N(t_j)) \, .$$

Taking the summation with respect to $j$, we see that (3.15) holds for

$\Phi = \Phi_N$. Letting $N \to \infty$ in (3.15) with $\Phi = \Phi_N$, we can conclude that (3.15) holds for every $\Phi$ in $C^1([s, t]; Y)$. Since $C^1([s, t]; Y)$ is dense in $H^1((s, t); Y)$, it follows from Lemma 2.2 that (3.15) holds for every $\Phi$ in $H^1((s, t); Y)$. By taking $s = 0$, we can conclude that $u$ satisfies (1.2). This completes the proof of Theorem 1.

**4. The uniqueness of weak solutions; Proofs of Theorems 2 and 3.** We follow Serrin [23]. Suppose $u$ is a weak solution satisfying the assumptions of either Theorem 2 or Theorem 3. We then define

$$u_h(\tau) = \int_0^t \rho_h(\tau - \sigma)u(\sigma)d\sigma$$

for arbitrarily fixed $t$ $(0 < t < T)$ and the weak solution $u$. Then $u_h \in H^1((0, T); Y)$. Hence we can take the $u_h$ as a test function in (1.2) with $u$ replaced by $v$, and there results: $(u_{h,t} = \partial_t u_h)$

$$(4.1) \qquad \int_0^t \{-(v, u_{h,t}) + (\nabla v, \nabla u_h) + (v \cdot \nabla v, u_h)\}dt \\ = \int_0^t (f, u_h)dt - (v(t), u_h(t)) + (a, u_h(0)) .$$

On the other hand, since $v \in L^2((0, t); H^1_{0,\sigma})$ by hypothesis, and since $C^\infty_{0,\sigma}$ is dense in $H^1_{0,\sigma}$, it follows from Lemma 2.2 that there is a sequence $\{v^k\}$ in $H^1((0, T); Y)$ with $v^k \to v$ in $L^2((0, T); H^1_{0,\sigma})$: note $C^\infty_{0,\sigma} \subset Y$. We then define $v_h$, $v^k_h$:

$$v_h(\tau) = \int_0^t \rho_h(\tau - \sigma)v(\sigma)d\sigma \,; \qquad v^k_h(\tau) = \int_0^t \rho_h(\tau - \sigma)v^k(\sigma)d\sigma .$$

Then it follows from Lemma 2.1 that $v_h \in H^1((0, t); H^1_{0,\sigma})$, $v^k_h \in H^1((0, t); Y)$; and that $v_h \to v$ as $h \to 0$, $v^k_h \to v_h$ as $k \to \infty$ in the norm of $H^1((0, t); H^1_{0,\sigma})$. Now we take $v^k_h$ as a test function in (1.2), and there results

$$(4.2) \qquad \int_0^t \{-(u, v^k_{h,t}) + (\nabla u, \nabla v^k_h) + (u \cdot \nabla u, v^k_h)\}dt \\ = \int_0^t (f, v^k_h)dt - (u(t), v^k_h(t)) + (u(0), v^k_h(0)) .$$

Letting $k \to \infty$ in the above identity, we get, by Lemma 2.1 and Lemma 2.4,

$$(4.3) \qquad \int_0^t \{-(u, v_{h,t}) + (\nabla u, \nabla v_h) + (u \cdot \nabla u, v_h)\}dt \\ = \int_0^t (f, v_h)dt - (u(t), v_h(t)) + (a, v_h(0)) .$$

Now by virtue of Fubini's theorem and the symmetry of the kernel $\rho_h$, it is easy to see that

$$\int_0^t (u, v_{h,t})dt = -\int_0^t (u_{h,t}, v)dt .$$

Consequently, addition of (4.1) and (4.3) yields

$$\int_0^t \{(\nabla v, \nabla u_h) + (\nabla u, \nabla v_h) + (v\cdot\nabla v, u_h) + (u\cdot\nabla u, v_h)\}dt$$
$$= \int_0^t \{(f, u_h) + (f, v_h)\}dt - (v(t), u_h(t)) - (u(t), v_h(t)) + (a, u_h(0)) + (a, v_h(0)) .$$

In the above identity we let $h\rightarrow 0$. Then it follows from Lemma 2.1 and Lemma 2.4 that

$$\int_0^t \{2(\nabla u, \nabla v) + (v\cdot\nabla v, u) + (u\cdot\nabla u, v)\}dt = \int_0^t \{(f, u) + (f, v)\}dt - (u(t), v(t)) + (a, a) . \tag{4.4}$$

By the theorem of Prodi [18] and Serrin [20], the $u$ satisfies the energy equality:

$$\|n(t)\|^2 + 2\int_0^t \|\nabla u\|^2 dt = 2\int_0^t (f, u)dt + \|a\|^2 \tag{4.5}$$

since $u$ is a weak solution in the class $L^{r,r'}(\Omega\times(0, T))$. On the other hand, by (1.6), it satisfies the energy inequality:

$$\|v(t)\|^2 + 2\int_0^t \|\nabla v\|^2 dt \leqq 2\int_0^t (f, v)dt + \|a\|^2 . \tag{4.6}$$

Addition of (4.4) (multiplied by $-2$), (4.5) and (4.6) yields

$$\|w(t)\|^2 + 2\int_0^t \|\nabla w\|^2 dt \leqq 2\int_0^t (w\cdot\nabla w, u)dt \tag{4.7}$$

where $w(t) = v(t) - u(t)$. Here we made use of the identity:

$$\int_s^t \{(u\cdot\nabla w, u) + (w, u\cdot\nabla u)\}dt = 0 ,$$

which can be seen from Lemma 2.4.

Proof of Theorem 2. From Lemma 2.4 and the Hölder inequality it follows that for any $\varepsilon > 0$

$$\text{the RHS of (4.7)} \leqq \varepsilon\int_0^t \|\nabla w\|^2 dt + M\int_0^t \|u\|_{L^r}^{r'}\|w\|^2 dt ,$$

$M$ being a constant independent of $w$. If we take $\varepsilon$ so small that $\varepsilon \leqq 2$, then by (4.7) and the above inequality,

$$(4.8) \qquad \|w(t)\|^2 \leqq M\int_0^t \|u\|_{L^r}^{r'}\|w\|^2 dt\,, \quad 0 \leqq t < T\,.$$

Since $\|w(t)\|^2$ is locally integrable on $[0, T)$, the above inequality (4.8) implies $w(t)=0$, a.e. in $(0, T)$, by the Gronwall inequality. (see Beckenbach-Bellmann [1; p. 134]). This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Assume that there were not such a $\delta > 0$. Then $\int_s^t \|\nabla w\|^2 dt > 0$ for any $t > s$. Hence it follows from Lemma 2.7 that

$$\text{the RHS of (4.7)} \leqq \varepsilon\int_s^t \|\nabla w\|^2 dt + M\int_s^t \|w\|^2 dt\,,$$

$M$ being a constant independent of $w$. Hence, similarly to the proof of Theorem 2, we can get

$$\|w(t)\|^2 \leqq M\int_s^t \|w\|^2 dt\,, \quad s \leqq t < T\,.$$

Hence we must have $w = 0$ on $(s, T)$; a contradiction. This proves Theorem 3.

PROOF OF COROLLARY. Since $u$ and $v$ are both continuous in $t$ in the weak topology of $L^2_\sigma$, Corollary easily follows from Theorem 3.

### 5. The decay of solutions; Proof of Theorem 4.

**5.1.** The proof of Theorem 4 is based on the following estimate to be proved in the next subsection.

$$(5.1) \quad \|(I + A)^{-\alpha}u(t)\|^2 \leqq \|e^{-(t-s)A}(I + A)^{-\alpha}u(s)\|^2 + M\int_s^t \|\nabla u\|^2 dt + \|a\|\int_s^t \|Pf\| dt \quad (0 \leqq s < t)\,,$$

$M$ being a constant independent of $s$, $t$.

For the moment we assume that (5.1) holds true. If $Av = 0$, then $\nabla v = 0$ by (1.8), from which it follows that $v = 0$. Hence the zero is not an eigenvalue of the positive self-adjoint operator $A$ in $L^2_\sigma$. Thus

$$(5.2) \qquad \|e^{-tA}\phi\| \to 0 \quad \text{as} \quad t \to \infty$$

for every $\phi$ in $L^2_\sigma$. Hence, letting $t$ tend to infinity in (5.1), we see

$$(5.3) \qquad \limsup_{t\to\infty} \|(I + A)^{-\alpha}u(t)\|^2 \leqq M\int_s^\infty \|\nabla u\|^2 dt + M\int_s^\infty \|Pf\| dt\,.$$

Letting $s$ tend to infinity in the above inequality, we have Theorem 4 by hypothesis.

**5.2.** We shall show the estimate (5.1). Let $s$, $t$ be fixed numbers such that $0 \leqq s < t < +\infty$. For positive numbers $\varepsilon, h$, we define

$$\Phi_{\varepsilon,h}(\tau) = U_\varepsilon(\tau)\int_s^t \rho_h(\tau-\sigma)U_\varepsilon(\sigma)u(\sigma)d\sigma\ , \quad s \leqq \tau \leqq t\ , \tag{5.4}$$

where $u$ is a weak solution of the problem (N-S); $\rho_h = \rho_h(\tau)$ is a function defined in the section 4; and

$$U_\varepsilon(\tau) = e^{-(t+\varepsilon-\tau)A}(I+A)^{-\alpha}\ , \quad \tau \leqq t\ .$$

Then it is easy to see that for each fixed $\varepsilon$ and $h$, $\Phi_{\varepsilon,h}$ has the following properties (i), (ii), (iii):

(i) $\Phi_{\varepsilon,h} \in C^1([s,t]; L^2_\sigma)$ and

$$\|\Phi_{\varepsilon,h}(\tau)\| \leqq M_2 \quad (M_2 \equiv \sup_{t>0}\|u(t)\|)\ ; \tag{5.5}$$

(ii) $\Phi_{\varepsilon,h}(\tau) \in D(A)$ and $A\Phi_{\varepsilon,h}(\tau)$ is continuous for $\tau$ $(s \leqq \tau \leqq t)$ in the norm of $L^2_\sigma$;

(iii) $\Phi_{\varepsilon,h}$ satisfies

$$\partial_\tau\Phi_{\varepsilon,h}(\tau) - A\Phi_{\varepsilon,h}(\tau) = U_\varepsilon(\tau)\int_s^t \partial_\tau\rho_h(\tau-\sigma)\cdot U_\varepsilon(\sigma)u(\sigma)d\sigma\ , \quad s \leqq \tau \leqq t\ . \tag{5.6}$$

$(\partial_\tau = \partial/\partial\tau)$. Moreover we have:

(iv) $\Phi_{\varepsilon,h} \in C([s,t]; L^n)$ and

$$\|\Phi_{\varepsilon,h}(\tau)\|_{L^n} \leqq M_0M_2 \tag{5.7}$$

$M_0$ being a constant independent of $\varepsilon, h, u$.

Indeed, since by the closed graph theorem $(I+A)^{-2\alpha}$ is a bounded operator from $L^2_\sigma$ into $L^n$ (with a bound, say, $M_0$), it follows that $\Phi_{\varepsilon,h}(\tau)$ is continuous for $\tau$ in the norm of $L^n$, and that

$$\|\Phi_{\varepsilon,h}(\tau)\|_{L^n} \leqq M_0\int_s^t \rho_h(\tau-\sigma)\|u(\sigma)\|d\sigma \leqq M_0M_2$$

by hypothesis. Thus we have (iv).

Now we can take the $\Phi_{\varepsilon,h}$ as a test function $\Phi$ in (1.2) and there results

$$\int_s^t (u\cdot\nabla u, \Phi_{\varepsilon,h})dt = \int_s^t (f, \Phi_{\varepsilon,h})dt - (u(t), \Phi_{\varepsilon,h}(t)) + (u(s), \Phi_{\varepsilon,h}(s)) \tag{5.8}$$

since

$$\begin{aligned}
&\int_s^t \{-(u, \partial_\tau\Phi_{\varepsilon,h}) + (\nabla u, \nabla\Phi_{\varepsilon,h})\}dt \\
&\qquad = \int_s^t \{-(u, \partial_\tau\Phi_{\varepsilon,h}) + (u, A\Phi_{\varepsilon,h})\}dt \\
&\qquad = \int_s^t\int_s^t \partial_\tau\rho_h(\tau-\sigma)(U_\varepsilon(\tau)u(\tau), U_\varepsilon(\sigma)u(\sigma))d\sigma d\tau \quad \text{(by (5.6))} \\
&\qquad = 0 \quad \text{(by the symmetry of } \rho_h(t))\ .
\end{aligned}$$

We shall let $\varepsilon \to 0$ and then $h \to 0$ in (5.8). Since $((I + A)^{-2\alpha} e^{-(t-\sigma)A} u(t), u(\sigma))$ ($\equiv g(\sigma)$) is continuous for $\sigma$, we have, by Lemma 2.1,

$$\lim_{h\to 0} \lim_{\varepsilon\to 0} (u(t), \Phi_{\varepsilon,h}(t)) = \lim_{h\to 0} \int_s^t \rho_h(t-\sigma) g(\sigma) d\sigma = \frac{1}{2}(u(t), (I + A)^{-2\alpha} u(t)) .$$

Similarly,

$$\lim_{h\to 0} \lim_{\varepsilon\to 0} (u(s), \Phi_{\varepsilon,h}(s)) = \frac{1}{2}(u(s), e^{-2(t-s)A}(I + A)^{-2\alpha} u(s)) ;$$

and

$$\lim_{h\to 0} \lim_{\varepsilon\to 0} \int_s^t (f, \Phi_{\varepsilon,h}) dt = \int_s^t (f, e^{-2(t-\sigma)A}(I + A)^{-2\alpha} u(\sigma)) d\sigma$$
$$\leqq M_2 \int_s^t \| Pf \| d\sigma \quad \text{(by (5.5))} .$$

On the other hand, by Lemma 2.4 and (5.7)

$$\text{the LHS of (5.8)} \geqq -M \int_s^t \| \nabla u \|^2 \| \Phi_{\varepsilon,h} \|_{L^n} dt \geqq -M M_0 M_2 \int_s^t \| \nabla u \|^2 dt .$$

Noting all the results obtained above, we let $\varepsilon \to 0$, and then $h \to 0$ in (5.8). Then we get the desired estimate (5.1). This completes the proof of Theorem 4.

**5.3.** PROOF OF COROLLARY 1. By the interpolation theorem,

$$\| \phi \| \leqq \| (I + A)^{-\alpha} \phi \|^{\beta} \| (I + A)^{1/2} \phi \|^{1-\beta}$$

where $\beta = 1/(1 + 2\alpha)$. Hence

$$(5.9) \qquad \int_t^{t+1} \| u(s) \|^2 ds$$
$$\leqq \left( \int_t^{t+1} \| (I + A)^{-\alpha} u(s) \|^2 ds \right)^{\beta} \left( \int_t^{t+1} \| (I + A)^{1/2} u(s) \|^2 ds \right)^{1-\beta} .$$

Since

$$\int_t^{t+1} \| (I + A)^{1/2} u(s) \|^2 ds = \int_t^{t+1} \| u(s) \|^2 ds + \int_t^{t+1} \| \nabla u(s) \|^2 ds$$
$$\leqq M_2 + M_3 \quad \left( M_3 \equiv \int_0^{\infty} \| \nabla u \| dt < \infty \right)$$

by hypothesis, it easily follows from Theorem 4 that the RHS of (5.9) tends to zero as $t \to \infty$. This proves Corollary 1.

**5.4.** PROOF OF COROLLARY 2. By the change of the variable and Corollary 1,

$$c=\lim_{t\to\infty}\int_0^1\|u(s+t)\|^2ds=\lim_{t\to\infty}\int_t^{t+1}\|u(s)\|^2ds=0\ .$$

This proves Corollary 2.

**Appendix.** PROOF OF PROPOSITION 1. We first recall the definition of a Hopf's weak solution ([8], [23]). Let $\mathscr{V}$ be the set of all $C^\infty$ vector functions $\Phi=(\Phi^1,\cdots,\Phi^n)$ on $\Omega\times[0,T)$, which has its support in $\Omega\times[0,T)$, and are divergent free, i.e., $\sum_{i=1}^n(\partial/\partial x_i)\Phi^i(x,t)=0$. A function $u$ on $\Omega\times(0,T)$ is called a Hopf's weak solution if

(H-1) for each $T'$ $(0<T'<T)$, $u$ is in the closure $V_{T'}$ of $\mathscr{V}$ under the norm of $L^2((0,T');H^1_{0,\sigma})$;

(H-2) the norm $\|u\|$ is uniformly bounded in $t$;

(H-3)

$$\int_0^T\{(u,\Phi_t)+(u,\Delta\Phi)+(u,u\cdot\nabla\Phi)\}dt=-\int_0^T(f,\Phi)dt-(a,\Phi(0))$$

for all $\Phi$ in $\mathscr{V}$.

Suppose that $u$ is a weak solution in our sense. Since $C^\infty_{0,\sigma}$ is dense in $H^1_{0,\sigma}$, it follows from Lemma 2.2 that for any $T'$ $(<T)$ $u$ can be approximated by a sequence of functions $u_N$ of the form: $u_N=\sum\lambda_j(t)\psi_j$ (finite sum) in the norm of $L^2((0,T');H^1_{0,\sigma})$, where $\lambda_j\in C^\infty([0,T'])$, $\psi_j\in C^\infty_{0,\sigma}$. Hence it is easy to see that $u_N\in V_{T''}$ and so $u\in V_{T''}$ for all $T''$ $(<T')$. Thus $u$ satisfies the condition (H-1). Since (H-2), (H-3) are easily verified, $u$ is a Hopf's weak solution. Under the assumption that $C^\infty_{0,\sigma}$ is dense in $Y$, we next show that a Hopf's weak solution $u$ is a weak solution in our sense. By Lemma 2.2, any function $\Phi$ in $H^1((0,T);Y)$ such that for some $T_0$ $(<T)$ $\Phi(\cdot,t)=0$ on $(T_0,T)$, can be approximated by a sequence of functions of the form $\sum\lambda_j(t)\psi_j$ (finite sum) in the norm of $L^2((0,T);Y)$ where $\lambda_j\in C_0^\infty([0,T))$, $\psi_j\in C^\infty_{0,\sigma}$. Hence it follows from Lemma 2.4 and (H-3) that (1.1) holds for such a $\Phi$. It is now easy to see that a Hopf's solution is a weak solution in our sense. We next proceed to the proof of the latter part of Proposition 1. If $2\leqq n\leqq 4$, then by the Sobolev inequality, $H^1_{0,\sigma}\subset L^n$, and so $Y=H^1_{0,\sigma}$. Hence $C^\infty_{0,\sigma}$ is dense in $Y$. If $\Omega$ is a star-shaped bounded domain with respect to some point, say the origin, then for any $u$ in $Y$, $u_\lambda\in Y$ and $u_\lambda\to u$ as $\lambda\to 1$ $(\lambda>1)$ in $Y$ where $u_\lambda(x)=(Eu)(\lambda x)$ ($E$ is defined in Lemma 3.1). We mollify $u$:

$$u_{\lambda,h}(x)=\int_{R^n}\rho_h(x-y)u_\lambda(y)dy=\rho_h*u_\lambda(x)$$

where $\rho_h*$ is the usual mollifier on $\boldsymbol{R}^n$. Then $u_{\lambda,h}\in C^\infty_{0,\sigma}$ and $u_{\lambda,h}\to u_\lambda$ as $h\to 0$ in $Y$. Thus $C^\infty_{0,\sigma}$ is dense in $Y$. Finally we consider the case $\Omega=\boldsymbol{R}^n$. If $f\in Y$, then we mollify $f$: $f_h=\rho_h*f$, $h>0$. Let $B$ be the operator defined in Proposition 3, and $\{\zeta_N\}$ be a sequence of $n$-dimensional cut-off functions. We then set

$$f_{h,\mu,N}(x) = \left(-\delta_{jk}\Delta + \frac{\partial^2}{\partial x_j \partial x_k}\right)\zeta_N(x)(\mu + B)^{-1}f_h(x)\,, \quad \mu > 0\,.$$

It is easy to see that $f_{h,\mu,N} \in C_{0,\sigma}^{\infty}$. After letting $N \to \infty$, we let $\mu \to 0$, and then $h \to 0$; we see that $f_{h,\mu,N} \to f$ in $Y$. Thus $C_{0,\sigma}^{\infty}$ is dense in $Y$.

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## REFERENCES

[1] E. F. BECKENBACH AND R. BELLMAN, Inequalities. Second Revised Printing. Berlin-Heidelberg-New York: Springer 1965.
[2] R. COURANT AND D. HILBERT, Methoden der mathematischen Physik II. Berlin-Heidelberg-New York 1968.
[3] J. DIEUDONNÉ, Foundations of Modern Analysis. New York-London: Academic Press 1960.
[4] C. FOIAS, Une remarque sur l'unicité des solutions des equations de Navier-Stokes en dimension $n$. Bull. Soc. Math. France, 89 (1961), 1-8.
[5] H. FUJITA AND T. KATO, On the Navier-Stokes initial-value problem, I. Arch. Rat. Mech. Anal., 16 (1964), 269-315.
[6] Y. GIGA AND T. MIYAKAWA, Solutions in $L_r$ to the Navier-Stokes initial value problem, to appear in Arch. Rat. Math. Anal.
[7] J. HEYWOOD, The Navier-Stokes equations: on the existence, regularity and decay of solutions. Indiana Univ. Math. J., 29 (1980), 639-681.
[8] E. HOPF, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nach., 4 (1950/51), 213-231.
[9] T. KATO, Strong $L^p$-solutions of the Navier-Stokes equation in $\boldsymbol{R}^m$, with applications to weak solution, Math. Z. 187 (1984), 471-480.
[10] A. A. KISELEV AND O. A. LADYZHENSKAYA, On the existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluid. Izv. Akad. Nauk SSSR, Ser. Mat., 21 (1957), 655-680.
[11] O. A. LADYZHENSKAYA, Uniqueness and smoothness of solutions of Navier-Stokes equations, Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), 5 (1967), 169-187.
[12] O. A. LADYZHENSKAYA, The Mathematical Theory of Viscous Incompressible Flow., Moscow 1970 (Russian).
[13] J. LERAY, Sur le mouvement d'un liquids visqueux emplissant l'espace, Acta Math., 63 (1934), 193-248.
[14] J. L. LIONS, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Paris: Gauthier-Villars 1969.
[15] J. L. LIONS AND G. PRODI, Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2, C.R. Acad. Sci. Paris, 248 (1959), 3519-3521.
[16] P. MAREMONTI, Asymptotic stability theorems for viscous fluid motions in exterior domains. (to appear)
[17] K. MASUDA, On the stability of incompressible viscous fluid motions past objects. J.

Math. Soc. Japan., 27 (1975), 294–327.
[18] K. MASUDA, Non-stationary Navier-Stokes equations in the exterior domain. In the proceedings of the Japan-France Seminar on Functional Analysis and Numerical Analysis. 1976. Japan Soc. for the Promotion of Sciences (H. Fujita, ed.) (1978), 267–275.
[19] K. MASUDA, $L^2$-decay of solutions of the Navier-Stokes equations in the exterior domain. To appear in the Proceedings of the 1983 Summer Institute on Nonlinear Functional Analysis and Applications. Proceedings of Symposia in Pure Mathematics. Amer. Math. Soc: Providence, Rhode Island.
[20] G. PRODI, Uu teorema di unicità per le equazioni di Navier-Stokes, Annali di Mat., 48 (1959), 173–182.
[21] H. RIKIMARU, On the $(x, t)$-measurability of the functions $P \triangle u(x, t)$ related to the Navier-Stokes initial value problem. Memoir of the Faculty Sci. Kyushu Univ., ser. A. Math., 21 (1967), 194–240.
[22] J. SERRIN, On the interior regularity of weak solutions of the Navier-Stokes equations, Arch. Rat. Mech. Anal., 9 (1962), 187–195.
[23] J. SERRIN, The initial value problem for the Navier-Stokes equations. In: "Nonlinear problems", Univ. Wisconsin Press (R. E. Langer Ed.) (1963), 69–98.
[24] V. A. SOLONNIKOV, Estimates for solutions of nonstationary Navier-Stokes equations, J. Soviet Math., 8 (1977), 467–529.
[25] R. TÉMAM, Navier-Stokes Equations, Amsterdam-New York-Oxford: North Holland 1977.
[26] W. VON WAHL, Regularity of weak solutions of the Navier-Stokes equations. To appear in the Proceedings of the 1983 Summer Institute on Nonlinear Functional Analysis and Applications. Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc.: Providence Rhode Island.
[27] H. SOHR, AND W. WAHL, VON, On the singular set and the uniqueness of weak solutions of the Navier-Stokes equations. (to appear in manuscripta mathematica).
[28] H. SOHR, Zur Regularitätstheorie der instantionaren Gleichungen von Navier-Stokes, Math. Zeit., 184 (1983), 359–379.
[29] H. SOHR, Optimale lokale existenzsatze für die Gleichungen von Navier-Stokes, Math. Ann., 267 (1984), 107–123.
[30] W. VON WAHL, Über das Behlten fur $t \to 0$ der Lösungen nichtlinearer parabolischer Gleichungen, insbesondere der Gleichungen von Navier-Stokes. Sonderforschungsbereich, Universität Bonn (1983).

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ADDED IN PROOF. Professors J. Heywood and Y. Giga orally communicated to the author that $C_{0,\sigma}^{\infty}$ is dense in $Y$ if $\Omega$ is a bounded or exterior domain. (See Proposition 1.)