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WEAK SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS OVER THE FIELD OF p-ADIC NUMBERS

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Abstract. Study of stochastic differential equations on the field of p-adic numbers was initiated by the second author and has been developed by the first author, who proved several results for the p-adic case, similar to the theory of ordinary stochastic integral with respect to Lévy processes on Euclidean spaces. In this article, we present an improved definition of a stochastic integral on the field and prove the joint (time and space) continuity of the local time for p-adic stable processes. Then we use the method of random time change to obtain sufficient conditions for the existence of a weak solution of a stochastic differential equation on the field, driven by the p-adic stable process, with a Borel measurable coefficient.

1. Introduction. Stochastic processes on the field $\boldsymbol{Q}_p$ of p-adic numbers have been studied in many papers, for example, by Albeverio-Karwowski [1, 2], Albeverio-Zhao [3, 4, 5], Evans [11], Figà-Talamanca et al. [6, 9], Ismagilov [15], Kaneko [18, 19], Karwowski-Vilela Mendes [17], Kochubei [21, 22, 23, 24], Varadarajan [29] and Yasuda [32, 33, 34] (here we do not mention papers on processes with p-adic time and other related subjects). In particular, these authors constructed and studied wide classes of Markov processes on $\boldsymbol{Q}_p$ (most of the results can be extended easily to more general local fields). Their infinitesimal generators are usually hypersingular integral operators; the first and simplest example is Vladimirov's fractional differentiation operator D^α [30, 31] corresponding to the p-adic α-stable process.

Alike for the case of the Euclidean space, for the field $\boldsymbol{Q}_p$ of p-adic numbers, stochastic processes and analysis are tightly related. Indeed, we can find descriptions and facts in [30, 31, 23] which are transplanted for describing probablisitic concepts on $\boldsymbol{Q}_p$. In [22], the second author initiated the theory of stochastic differential equations on $\boldsymbol{Q}_p$, and in [18] the first author made several assertions on stochastic integrals on $\boldsymbol{Q}_p$ which are similar to ordinary Itô calculus on the Euclidean space. On the other hand, Yasuda [32] developed potential-theoretic notions related to p-adic Lévy processes, in particular the stable process. An analytic potential theory over $\boldsymbol{Q}_p$ was developed by Haran [12].

In [18, 22] we considered only strong solutions of stochastic differential equations on $\boldsymbol{Q}_p$ driven by the p-adic α-stable process. Note that in the conventional theory of stochastic differential equations (over $\boldsymbol{R}$), the study of weak solutions of such equations began only recently [35, 36, 10, 25].

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In this paper, we initiate a theory of weak solutions for the p-adic case. Since we need to discuss stochastic differential equations without continuity of the coeffficient as assumed in [18], we first provide a more general construction of stochastic integrals than the one in the first author's paper. Our improved definition of stochastic integrals admits predictable integrand satisfying a finiteness condition on its moment. Then, we follow the approach by Zanzotto [35] based on the method of random time change. In order to use this technique in our situation, we prove the joint (with respect to the space and time variables) continuity of the local time for the p-adic α-stable process, as well as some facts on the occupation time on mesurable sets from $\boldsymbol{Q}_p$ with positive Haar measure. After that, we apply the theory of the Lévy system of a Markov process on a general state space [28]. In fact, this is one of the first applications of the general theory of Markov processes outside the usual Euclidean realm. In Section 4, we give a sufficient condition for the existence of a weak solution of the stochastic differential equation

$$X(t) = x + \int_0^t b(X_{s-})dZ(s)$$

on $\boldsymbol{Q}_p$, with respect to the α-stable process $\{Z(t)\}_{t\geq 0}$, with a locally bounded coefficient b. The conditions are much less restritive than those guaranteeing the existence of strong solutions of such equations [18, 22].

2. Stochastic integral. In this section, we will establish the notion of stochastic integrals of predictable stochastic process with respect to the Lévy process $\{Z(t)\}_{t\geq 0}$ with $Z(0) = 0$ determined by the sequence $A = \{a(m)\}$ satisfying

$$a(m) \geq a(m+1)\,, \tag{1}$$

$$\lim_{m\to\infty} a(m) = 0\,, \qquad \lim_{m\to-\infty} a(m) > 0 \ \text{ or } \ = \infty\,, \tag{2}$$

and

$$\sum_{m=-\infty}^{\infty} a(m)p^{\gamma m} < \infty \tag{3}$$

(see [2, 23, 32] for the basic notions regarding Lévy processes on $\boldsymbol{Q}_p$). Then, we can choose a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying $\mathcal{F}_t \subset \sigma[Z(s) \mid s \leq t]$ for any t so that $\mathcal{F}_t$ is independent of $\sigma[Z(s+t) - Z(t) \mid s > 0]$ for every $t \geq 0$.

In this section, the smallest σ-field on $[0, \infty) \times \Omega$ based on which all left-continuous $\{\mathcal{F}_t\}$-adapted processes are measurable will be denoted by $\mathcal{S}$. The $\boldsymbol{Q}_p$-valued stochastic process defined on $[0, \infty) \times \Omega$ is said to be *predictable* if it is measurable with respect to the σ-field $\mathcal{S}$. As standard notation in the theory of Markov process, the starting point of $\{Z_t\}_{t\geq 0}$ will be indicated in the notation for the probability measure and the expectation as P_0 and E_0, respectively.

LEMMA 1. *The linear space Φ of the $\boldsymbol{Q}_p$-valued bounded stochastic processes defined on* $[0, \infty) \times \Omega$ *satisfying the following conditions* (i) *and* (ii):

(i) Φ contains all bounded left-continuous $\{\mathcal{F}_t\}$-adapted processes,

(ii) for any sequence $\{\phi^{(n)}\}_{n=1}^{\infty} \subset \Phi$, $\lim_{n\to\infty} \phi^{(n)}(t,\omega) = \phi(t,\omega)$ and $\|\phi^{(1)}(t,\omega)\|_p \leq \|\phi^{(2)}(t,\omega)\|_p \leq \|\phi^{(3)}(t,\omega)\|_p \leq \cdots$ imply $\phi \in \Phi$,

contains all bounded predictable processes.

PROOF. Since any bounded predictable process is described as the limit of bounded $\mathcal{S}$-measurable simple functions, it suffices to prove $1_E \in \Phi$ for any $E \in \mathcal{S}$. For the family $\mathcal{S}' = \{E \subset [0,\infty) \times \Omega \mid 1_E \in \Phi\}$, we can verify that

(i) $[0,\infty) \times \Omega \in \mathcal{S}'$,

(ii) $E_1, E_2 \in \mathcal{S}'$ and $E_1 \subset E_2$ imply $E_2 \setminus E_1 \in \mathcal{S}'$,

(iii) $E_n \in \mathcal{S}'$ and $E_1 \subset E_2 \subset \cdots$ imply $\cup E_n \in \mathcal{S}'$.

For any finite set $\{Y_1(t), \dots, Y_k(t)\}$ of bounded left-continuous $\{\mathcal{F}_t\}$-adapted processes and finite set of balls $\{B_1, \dots, B_k\}$ in $\boldsymbol{Q}_p$, we easily see $\bigcap\{(t,\omega) \mid Y_i(t,\omega) \in B_i\} \in \mathcal{S}'$. This is because $1_{B_i}(Y_i(t))$ is bounded $\{\mathcal{F}_t\}$-adapted left continuous process for any $i = 1, 2, \dots, k$.

Since the family $\mathcal{C}$ of the sets described as $\bigcap\{(t,\omega) \mid Y_i(t,\omega) \in B_i\}$ with finite set $\{Y_1(t), \dots, Y_k(t)\}$ of bounded left-continuous $\{\mathcal{F}_t\}$-adapted processes and finite set of balls $\{B_1, \dots, B_k\}$ in $\boldsymbol{Q}_p$ is closed under finite intersection and $\sigma[\mathcal{C}] = \mathcal{S}$, we can derive from Dynkin's theorem (e.g., Lemma 5.1 in [14]) that $\mathcal{S} \subset \mathcal{S}'$. □

For arbitrarily fixed $\gamma \geq 1$ and $T > 0$, we denote by $\mathcal{L}^\gamma$ the family of $\boldsymbol{Q}_p$-valued predictable processes each $\{\phi(t)\}_{t\geq 0}$ of which is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies $E_0[\int_0^T \|\phi(t,\omega)\|_p^\gamma dt] < \infty$. Here, we introduce a subfamily $\mathcal{L}^0$ of $\mathcal{L}^\gamma$ each element of which admits the following expression:

$$\phi(t,\omega) = f_0(\omega) 1_{\{0\}}(t) + \sum_{i=1}^{n-1} f_i(\omega) 1_{(t_i, t_{i+1}]}(t)$$

with some sequence $\{f_i\}$ of random variables satisfying $\|f_i\|_{L^\infty(\Omega;P)} < \infty$ and some $0 = t_0 < t_1 < \cdots < t_n = T$.

PROPOSITION 1. *$\mathcal{L}^0$ is a dense subfamily of $\mathcal{L}^\gamma$ with respect to the norm*

$$\|\phi\|_{\mathcal{L}^\gamma} = \left(\int_0^T E_0[\|\phi(t,\omega)\|_p^\gamma] dt \right)^{1/\gamma}.$$

PROOF. For any $\phi \in \mathcal{L}^\gamma$, the sequence $\{\phi^{(M)}\}_{M=1}^{\infty}$ of stochastic processes defined by $\phi^{(M)}(t,\omega) = \phi(t,\omega) \times 1_{B(0,p^M)}(\phi(t,\omega))$ satisfies $\lim_{M\to\infty} \|\phi - \phi^{(M)}\|_{\mathcal{L}^\gamma} = 0$. Therefore, we may assume that $\phi(t,\omega)$ vanishes outside $B(0,p^M)$.

We define Φ to be the set of $\phi \in \mathcal{L}^\gamma$ such that $\|\phi(t,\omega)\|_p \leq p^M$ for any (t,ω) and there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{L}^0$ satisfying $\lim_{n\to\infty} \|\phi - \phi_n\|_{\mathcal{L}^\gamma} = 0$. Then, the space Φ is a linear space and for any sequence $\{Z^{(n)}\}_{n=1}^{\infty} \subset \Phi$, $\lim_{n\to\infty} Z^{(n)}(t,\omega) = Z(t,\omega)$ and $\|Z^{(1)}(t,\omega)\|_p \leq \|Z^{(2)}(t,\omega)\|_p \leq \|Z^{(3)}(t,\omega)\|_p \cdots$ imply $\{Z(t)\}_{t\geq 0} \in \Phi$.

If $\{\phi(t)\}_{t\geq 0}$ is a left-continuous $\{\mathcal{F}_t\}$-adapted process satisfying $\|\phi(t,\omega)\|_p \leq p^M$ for any (t,ω), then

$$\phi_n(t,\omega) = \phi(0,\omega)1_{\{0\}}(t) + \sum_{i=0}^{2^n} \phi(iT/2^n,\omega)1_{(iT/2^n,(i+1)T/2^n]}(t)$$

is in $\mathcal{L}^0$ and satisfies $\lim_{n\to\infty}\|\phi_n - \phi\|_{\mathcal{L}^\gamma} = 0$. Accordingly, Lemma 1 shows that Φ contains all $\{\mathcal{F}_t\}$-adapted predictable processes enjoying $\|\phi(t,\omega)\|_p \leq p^M$ for any (t,ω). □

For any element $\phi \in \mathcal{L}^0$ given by $\phi(t,\omega) = f_0(\omega)1_{\{0\}}(t) + \sum_{i=1}^{n-1} f_i(\omega)1_{(t_i,t_{i+1}]}(t)$ with some sequence $\{f_i\}$ of random variables satisfying $\sup\|f_i\|_{L^\infty(\Omega;P_0)} < \infty$ and some $0 = t_0 < t_1 < \cdots < t_n = T$, the stochastic integral $\int_0^t \phi(s)dZ(s)$ with respect to $\{Z(t)\}_{t\geq 0}$ is defined by

$$\int_0^t \phi(s)dZ(s) = \sum_{i=1}^{n-1} f_i(Z(t_{i+1}\wedge t) - Z(t_i \wedge t)) \quad \text{for } 0 \leq t \leq T\,.$$

The stochastic processes $\{\int_0^t \phi(s)dZ(s)\}_{t\in[0,T]}$ can be regarded as a right continuous $\{\mathcal{F}_t\}$-adapted process.

As in [18], we can show the following.

LEMMA 2. *There exists a positive constant $C_{A,\gamma}$ satisfying the following conditions*:

(i) $E_0[\|Z(t)\|_p^\gamma] \leq C_{A,\gamma}t$ *for all* $t \geq 0$.

(ii) $$E_0\Big[\sup_{0\leq t\leq u}\Big\|\int_0^t \phi(s)dZ(s)\Big\|_p^\gamma\Big] \leq C_{A,\gamma}\int_0^u E_0[\|\phi(s)\|_p^\gamma]ds \quad \textit{for} \;\; 0 \leq u \leq T.$$

For any element $\phi \in \mathcal{L}^\gamma$, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset \mathcal{L}^0$, such that $\lim_{n\to\infty}\int_0^T E_0[\|\phi_n(t) - \phi(t)\|_p^\gamma]dt = 0$. Then one can derive from Proposition 1 that

$$\lim_{n,m\to\infty} E_0\Big[\sup_{0\leq t\leq T}\Big\|\int_0^t \phi_n(s)dZ(s) - \int_0^t \phi_m(s)dZ(s)\Big\|_p^\gamma\Big] = 0\,.$$

Therefore, the stochastic integral $\int_0^t \phi(s)dZ(s)$ of $\{\phi(t)_{t\geq 0}\}$ with respect to $\{Z(t)\}_{t\geq 0}$ can be defined as a unique $\boldsymbol{Q}_p$-valued process $\{Y(t)\}_{t\geq 0}$ satisfying

$$\lim_{n\to\infty} E_0\Big[\sup_{0\leq t\leq T}\Big\|Y(t) - \int_0^t \phi_n(s)dZ(s)\Big\|_p^\gamma\Big] = 0\,.$$

Let us denote by $\mathcal{D}([0,T] \to \boldsymbol{Q}_p)$ the space of all right continuous sample paths on the time interval $[0,T]$ to $\boldsymbol{Q}_p$ with left limit at every point. Since we already know that $\int_0^\cdot \phi_n(s)dZ(s)$ is a $\mathcal{D}([0,T] \to \boldsymbol{Q}_p)$-valued stochastic process, we immediately see that $\int_0^\cdot \phi(s)dZ(s)$ is a $\mathcal{D}([0,T] \to \boldsymbol{Q}_p)$-valued variable as well.

Again as in [18], we will obtain wider perspectives of p-adic stochastic integral so that it covers the one with respect to the α-stable processes. Indeed, we can consider a random walk corresponding to a sequence $A = \{a(m)\}$ satisfying (1), (2) and

$$\sum_{m=-\infty}^{0} a(m)p^{\gamma m} < \infty \quad \text{for a given real number } \gamma \geq 1 . \tag{4}$$

In this case, defining sequences $A(M) = \{a(M; m)\}$ with properties (1), (2) and (3) by

$$a(M; m) = \begin{cases} a(m) & \text{if } m < M , \\ 0 & \text{if } m \geq M , \end{cases}$$

we can prove the following proposition.

PROPOSITION 2. *For any Lévy process $\{Z(t)\}_{t\geq 0}$ corresponding to A with properties* (1), (2) *and* (4), *we have the following*:

(i) *The $\boldsymbol{Q}_p$-valued process $\{Z(M;t)\}_{t\geq 0}$ defined by $Z(M; t) = \int_{\boldsymbol{Q}_p} \int_0^t f_M(z) \Lambda(ds,dz)$ is a random walk corresponding to $A(M)$, where Λ stands for the Poisson random measure of $\{Z(t)\}_{t\geq 0}$ and f_M denotes the $\boldsymbol{Q}_p$-valued function defined by*

$$f_M(z) = \begin{cases} p^m z & \text{if } \|z\|_p = p^{m+M} \text{ with some integer } m \geq 0 , \\ z & \text{otherwise} . \end{cases}$$

(ii) *There exists a sequence $\{\Omega(M;T)\}_{M=0}^{\infty}$ of events satisfying* $\lim_{M\to\infty} P_0(\Omega(M;T)) = 1$ *and*

$$\int_0^t \phi(s) dZ(M; s) = \int_0^t \phi(s) dZ(M + k; s)$$

for any $t \in [0, T]$ and $k = 1, 2, \ldots$, a.s. on $\Omega(M; T)$ for any ϕ of $\mathcal{L}^\gamma$.

(iii) *For any sequence $\{\phi(M; t)\}_{M=0}^{\infty} \subset \mathcal{L}^\gamma$ enjoying $\phi(M; t) = \phi(M + k; t)$ for any $t \in [0, T]$ and $k = 1, 2, \ldots$, a.s. on $\Omega(M; T)$,*

$$\int_0^t \phi(M; s) dZ(M; s) = \int_0^t \phi(M + k; s) dZ(M + k; s)$$

for any $t \in [0, T]$ and $k = 1, 2, \ldots$, a.s. on $\Omega(M; T)$.

PROOF. The proof can be done as in the proof of Proposition 2 in [18]. □

Accordingly, if a $\boldsymbol{Q}_p$-valued predictable process $\{\phi(t)\}_{t\geq 0}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ admits predictable processes $\{\phi(M; t)\}_{t\geq 0}$ from $\mathcal{L}^\gamma$ satisfying $\phi(t) = \phi(M; t)$ on $\Omega(M; T)$ for any $t \in [0, T]$ and $M = 1, 2, \ldots$, we can define the stochastic integral $\int_0^t \phi(t) dZ(t)$ with respect to a stochastic process $\{Z(t)\}_{t\geq 0}$ determined by A satisfying (1), (2) and (4). Indeed, the stochastic integral is defined as a unique right continuous $\{\mathcal{F}_t\}$-adapted process $\{Y(t)\}_{t\geq 0}$ satisfying

$$Y(t) = \int_0^t \phi(M; s) dZ(M; s) \quad \text{a.s. on } \Omega(M; T) \text{ for all } M = 1, 2, \ldots .$$

3. Local time and related properties of p-adic stable processes. Let $\{Z(t)\}_{t\geq 0}$ be the symmetric α-stable process on $\boldsymbol{Q}_p$ with $\alpha > 1$. This stochastic process is characterized as the Hunt process $\{Z(t)\}_{t\geq 0}$ satisfying $Z(0) = 0$ with transition probability densities of the form

$$P(t, x-y) = \int_{\boldsymbol{Q}_p} \chi(-(x-y)\xi)e^{-t\|\xi\|_p^\alpha}\mu(d\xi)\,, \tag{5}$$

where χ is the canonical additive character on $\boldsymbol{Q}_p$ and μ is a Haar measure normalized by the requirement that the measure of a unit ball equals 1. The process $\{Z(t)\}_{t\geq 0}$ is given as the Lévy process determined by the sequence $A = \{a(m)\}$ of the form

$$a(m) = \frac{1-p^{-1}}{1-p^{-\alpha-1}}p^{-\alpha m}\,.$$

Below P_x will denote the probability law induced by $\{Z(t)+x\}_{t\geq 0}$ on the trajectory space $\mathcal{D}([0,\infty) \to \boldsymbol{Q}_p)$ under P_0, and E_x will be the corresponding expectation. When the probability measure P_x is provided, the α-stable process starting from x will be denoted again by $\{Z(t)\}_{t\geq 0}$.

Here, we can recall that Yasuda's result ([32]) shows that $\alpha > 1$ implies that every point $x \in \boldsymbol{Q}_p$ is regular for $\{x\}$. Accordingly, we have $P_x\{\tau_x = 0\} = 1$, where $\tau_x = \inf\{t > 0 \mid Z(t) = x\}$ (see [32]). Moreover, the Hunt process $\{Z(t)\}_{t\geq 0}$ satisfies Hunt's conditions (A) and (F) in [13], and $P_y\{\tau_x < \infty\} = 1$ for any $x, y \in \boldsymbol{Q}_p$ (see [32, 23]).

Consider the random Borel measure

$$\nu(t, B) = \int_0^t I_B(Z(s))ds$$

on $\boldsymbol{Q}_p$, called the occupation time measure, where I_B stands for the indicator of the Borel set B. If this random Borel measure admits a density L_t^x with respect to the Haar measure μ, that is

$$\int_B L_t^x\,\mu(dx) = \int_0^t I_B(Z(s))\,ds \quad \text{a.s.}$$

for any Borel set B, then $\{L_t^x\}$ is called the local time of the process $\{Z(t)\}_{t\geq 0}$.

It is known [7] that $\{Z(t)\}_{t\geq 0}$ in our situation admits the local time $\{L_t^x\}_{t\geq 0}$, for which the function L_t^x is jointly measurable in (t, x) and continuous and monotone non-decreasing in t for every x with probability one. We note that $L_t^0 > 0$ is satisfied a.s. for any $t > 0$ with probability one. Let us prove almost sure joint continuity of L_t^x in (t, x).

Let N, δ be arbitrary positive numbers. It is known (see (V.3.28) in [8]) that for any $x, a, b \in \boldsymbol{Q}_p$

$$P_x\left\{\sup_{0\leq t\leq N}|L_t^a - L_t^b| > 2\delta\right\} \leq 2e^N e^{-\delta/\gamma_{a,b}}\,, \tag{6}$$

where $\gamma_{a,b} = [1-\psi_a(b)\psi_b(a)]^{1/2}$, $\psi_a(x) = E_x(e^{-\tau_a})$ and $\psi_b(x) = E_x(e^{-\tau_b})$.

For the function

$$g^\lambda(x) = \int_0^\infty e^{-\lambda t}P(t, x)dt\,,$$

with positive parameter λ, we can derive from (5) that

$$g^{\lambda}(x)=\int_{\boldsymbol{Q}_p}\frac{\chi(-x\xi)}{\lambda+\|\xi\|_p^{\alpha}}\mu(d\xi)\,.$$

Since $\alpha>1$, the function $g^{\lambda}(x)$ is bounded and continuous in x, and $E_0(e^{-\tau_a})=g^{\lambda}(a)/g^{\lambda}(0)$ (see Lemma 3.2 in [32]).

Due to the spatial homogeneity, we have $\psi_a(b)=E_0(e^{-\tau_{a-b}})=g^1(a-b)/g^1(0)$ and $\psi_b(a)=g^1(b-a)/g^1(0)$. In our case, $g^{\lambda}(x)=g^{\lambda}(-x)$ is satisfied ($g^{\lambda}(x)$ depends actually on $\|x\|_p$ as the Fourier transform of a radial function) and so we have

$$\gamma_{a,b}=\left[1-\left(\frac{g^1(a-b)}{g^1(0)}\right)^2\right]^{1/2}\le\sqrt{2}\left[1-\frac{g^1(a-b)}{g^1(0)}\right]^{1/2}. \tag{7}$$

Next, we have the following lemma for the function

$$h(\|x\|_p)=1-\frac{g^1(x)}{g^1(0)}\,,\qquad x\in\boldsymbol{Q}_p\,.$$

LEMMA 3. *For any $N\in\boldsymbol{N}$, there exists a positive constant C_N such that*

$$0\le h(\|x\|_p)\le C_N\|x\|_p^{\alpha-1}\qquad\textit{for any } x \textit{ in } B(0,p^N)\,. \tag{8}$$

PROOF. We have

$$\begin{aligned}0\le h(\|x\|_p)&\le C\int_{\boldsymbol{Q}_p}\frac{1-\chi(-x\xi)}{1+\|\xi\|_p^{\alpha}}\mu(d\xi)\\&=C\int_{\|\xi\|_p>\|x\|_p^{-1}}\frac{1-\chi(-x\xi)}{1+\|\xi\|_p^{\alpha}}\mu(d\xi)=C\varphi(\|x\|_p)\end{aligned}$$

with some positive constant C, where

$$\varphi(\|x\|_p)=\int_{\|\xi\|_p>\|x\|_p^{-1}}\frac{1-\chi(-x\xi)}{1+\|\xi\|_p^{\alpha}}\mu(d\xi)\,.$$

For any $x\in\boldsymbol{Q}_p$ with $\|x\|_p=p^n$, from an integration formula in [31], we can derive that

$$\begin{aligned}\varphi(\|x\|_p)&=\sum_{j=-n+1}^{\infty}\int_{\|\xi\|_p=p^j}\frac{1-\chi(-x\xi)}{1+\|\xi\|_p^{\alpha}}\mu(d\xi)\\&=\sum_{j=-n+1}^{\infty}\frac{1}{1+p^{\alpha j}}\left[(1-p^{-1})p^j-\int_{\|\xi\|_p=p^j}\chi(-x\xi)\mu(d\xi)\right]\\&=\frac{1}{1+p^{\alpha(1-n)}}\left[(1-p^{-1})p^{-n+1}+p^{-n}\right]+(1-p^{-1})\sum_{j=-n+2}^{\infty}\frac{p^j}{1+p^{\alpha j}}\\&=\frac{p^{-n+1}}{1+p^{\alpha(1-n)}}+(1-p^{-1})\sum_{j=-n+2}^{\infty}\frac{p^j}{1+p^{\alpha j}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{p}{p^n + p^\alpha \cdot p^{-n(\alpha-1)}} + (1-p^{-1}) \sum_{k=2}^{\infty} \frac{p^{k-n}}{1+p^\alpha(k-n)} \\
&= \frac{p}{p^n + p^\alpha \cdot p^{-n(\alpha-1)}} + (1-p^{-1}) \sum_{k=2}^{\infty} \frac{p^{k}}{1+p^{\alpha k} \cdot p^{-n(\alpha-1)}} \\
&= \frac{p}{\|x\|_p + p^\alpha \|x\|_p^{-(\alpha-1)}} + (1-p^{-1}) \sum_{k=2}^{\infty} \frac{p^{k}}{1+p^{\alpha k}\|x\|_p^{-(\alpha-1)}} \\
&= \|x\|_p^{\alpha-1} \left\{ \frac{p}{\|x\|_p^\alpha + p^\alpha} + (1-p^{-1}) \sum_{k=2}^{\infty} \frac{p^{k}}{\|x\|_p^{-(\alpha-1)} + p^{\alpha k}} \right\}.
\end{aligned}
$$

As $\|x\|_p \to \infty$, the expression in the braces tends to

$$
p^{1-\alpha} + (1-p^{-1}) \sum_{k=2}^{\infty} p^{k(1-\alpha)} = p^{1-\alpha} + (1-p^{-1}) \frac{p^{2(1-\alpha)}}{1-p^{1-\alpha}}.
$$

Therefore, we obtain the inequality (8) for any x in $B(0, p^N)$. □

THEOREM 1. *The function $(t, x) \mapsto L_t^x$ from $(0, \infty) \times \boldsymbol{Q}_p$ to $(0, \infty)$ is continuous a.s. Moreover, for any κ with $0 < \kappa < (\alpha-1)/2$, any $T > 0$ and $M \in \boldsymbol{Z}_+$, there exists a random variable $C(\kappa, T, M) > 0$ such that*

$$
\sup_{0 \le t \le T} |L_t^a - L_t^b| \le C(\kappa, T, M) \|a-b\|_p^\kappa \quad a.s. \tag{9}
$$

for all a, b in $B(0, p^M)$.

PROOF. Let us fix a positive integer n and consider the set S_n of elements in $\boldsymbol{Q}_p$ of the form

$$
p^{-M}(\xi_0 + \xi_1 p + \cdots + \xi_{M+n} p^{M+n}), \quad \xi_j \in \{0, 1, \ldots, p-1\}.
$$

Let us first prove that, for any $a, b \in S_n$ satisfying $\|a-b\|_p = p^{-n}$, Inequality (9) holds almost surely with some constant independent of n. If $a, b \in S_n$ and $\|a-b\|_p = p^{-n}$, then it follows from (6), (7) and (8) that

$$
\begin{aligned}
&P_0\left\{ \sup_{0 \le t \le T} |L_t^a - L_t^b| > 2\delta_n \right\} \\
&\qquad \le 2e^N \exp\left(- \frac{\delta_n}{C\|a-b\|_p^{(\alpha-1)/2}} \right) = 2e^N \exp(-\sigma p^{n((\alpha-1)/2-\kappa)})
\end{aligned}
$$

for any $\sigma > 0$, where $\delta_n = p^{-\kappa n}$.

The number of pairs (a, b) of elements of S_n satisfying $\|a-b\|_p = p^n$ equals $p^{M+n+1}(p-1)/2$. This is because the quantity of all elements of S_n must be multiplied by the number $p-1$ of elements whose distance from a given element equals p^{-n}, and then divided by two, so that each pair to be counted only once. Since we have

$$\begin{aligned}\rho_n := P_0\Big\{ \sup_{\substack{a,b\in S_n\\ \|a-b\|_p=p^{-n}}} \sup_{0\le t\le T} |L_t^a - L_t^b| > 2\delta_n \Big\} \\ \le (p-1)p^{M+n+1}e^N \exp\left(-\sigma p^{n((\alpha-1)/2-\kappa)}\right),\end{aligned}$$

we have $\sum_{n=1}^{\infty} \rho_n < \infty$. By the Borel-Cantelli lemma, we see

$$\sup_{\substack{a,b\in S_n\\ \|a-b\|_p=p^{-n}}} \sup_{0\le t\le T} |L_t^a - L_t^b| \le 2\delta_n ,$$

except a finite number of values of n with probability one. In other words, almost surely, there exists a positive integer n_0 such that

(10) $$\sup_{0\le t\le T} |L_t^a - L_t^b| \le 2\|a-b\|_p^{\kappa}$$

for any a and b in S_n satisfying $\|a-b\|_p = p^{-n}$ with some integer $n \ge n_0$.

Now, suppose that a, b are arbitrary elements in $B(0, p^M)$ satisfying $\|a-b\|_p = p^{-n}$ with some integer $n \ge n_0$. Let us take canonical representations

$$\begin{aligned} a &= p^{-M}(\xi_0 + \xi_1 p + \cdots + \xi'_{M+n}p^{M+n} + \xi'_{M+n+1}p^{M+n+1} + \cdots),\\ b &= p^{-M}(\xi_0 + \xi_1 p + \cdots + \xi''_{M+n}p^{M+n} + \xi''_{M+n+1}p^{M+n+1} + \cdots)\end{aligned}$$

for a and b with the coefficients from $\{0, 1, \ldots, p-1\}$ enjoying $\xi'_{M+n} \ne \xi''_{M+n}$. For the sequences $\{a_k\}$ and $\{b_k\}$ given by

$$a_k = \begin{cases} p^{-M}(\xi_0 + \xi_1 p + \cdots + \xi'_{M+n}p^{M+n}) & k = 0,\\ a_0 + \sum_{j=1}^{k} \xi'_{M+n+j}p^{n+j} & k \ge 1,\end{cases}$$

and

$$b_k = \begin{cases} p^{-M}(\xi_0 + \xi_1 p + \cdots + \xi''_{M+n}p^{M+n}) & k = 0,\\ a_0 + \sum_{i=1}^{k} \xi''_{M+n+i}p^{n+i} & k \ge 1,\end{cases}$$

respectively, we have

$$\begin{aligned} L_t^a - L_t^b &= (L_t^a - L_t^{a_0}) + (L_t^{a_0} - L_t^{b_0}) + (L_t^{b_0} - L_t^b)\\ &= (L_t^a - L_t^{a_1}) + (L_t^{a_1} - L_t^{a_0}) + (L_t^{a_0} - L_t^{b_0}) + (L_t^{b_0} - L_t^{b_1}) + (L_t^{b_1} - L_t^b) = \cdots\\ &= (L_t^a - L_t^{a_k}) + \sum_{j=1}^{k}(L_t^{a_j} - L_t^{a_{j-1}}) + (L_t^{a_0} - L_t^{b_0}) + \sum_{i=1}^{k}(L_t^{b_{i-1}} - L_t^{b_i}) + (L_t^{b_k} - L_t^b).\end{aligned}$$

Since $a_k \to a$ and $b_k \to b$ as $k \to \infty$, we obtain that $L_t^a - L_t^{a_k} \to 0$ and $L_t^{b_k} - L_t^b \to 0$ in probability (see Corollary V.3.29 in [8]). Passing to the limit, we find that

$$L_t^a - L_t^b = (L_t^{a_0} - L_t^{b_0}) + \sum_{j=1}^{\infty}(L_t^{a_j} - L_t^{a_{j-1}}) + \sum_{i=1}^{\infty}(L_t^{b_{i-1}} - L_t^{b_i}) \quad \text{a.s.}$$

Hence, by (10), we get the inequality

$$\begin{aligned} |L_t^a - L_t^b| &\le 2\|a-b\|_p^\kappa + 2\sum_{j=1}^{\infty} p^{-(n+j)\kappa} + 2\sum_{i=1}^{\infty} p^{-(n+i)\kappa} \\ &= 2\|a-b\|_p^\kappa + \frac{4p^{-(n+1)\kappa}}{1-p^{-\kappa}} = \left(2 + \frac{4p^{-\kappa}}{1-p^{-\kappa}}\right)\|a-b\|_p^\kappa \quad \text{a.s.} \end{aligned}$$

for $a, b \in B(0, p^M)$ satisfying $\|a-b\|_p \le p^{-n_0}$. This is equivalent to (9). It is clear that the inequality (9) implies the required joint continuity. □

Let us consider the behavior of the occupation time measure $\nu(t, B)$ as $t \to \infty$.

THEOREM 2. *For every Borel set $B \subset \boldsymbol{Q}_p$ of a positive Haar measure,*

$$P_0\left\{\lim_{t\to\infty} \nu(t, B) = \infty\right\} = 1\,. \tag{11}$$

PROOF. Consider the transition probability semigroup $\{p_t\}$ given by

$$(p_t f)(x) = \int_{\boldsymbol{Q}_p} P(t, x-y) f(y)\mu(dy)\,.$$

It can be verified by a direct calculation that

$$\int_{\boldsymbol{Q}_p} (p_t f)(x)\mu(dx) = \int_{\boldsymbol{Q}_p} f(x)\mu(dx)$$

for the indicator function f of any ball in $\boldsymbol{Q}_p$. Thus this identity holds for any locally constant function f with a compact support, and therefore for any $f \in L^2(\boldsymbol{Q}_p; \mu)$. This shows that the Haar measure μ is an invariant measure for the α-stable process $\{Z(t)\}_{t\ge 0}$.

It is proved in [32] (see also [23]) that the α-stable process $\{Z(t)\}_{t\ge 0}$ with $\alpha \ge 1$ is recurrent, so that it hits any open set in $\boldsymbol{Q}_p$ after any large time (for these notions in the context of processes on general locally compact Abelian groups see [26]). Together with the existence of an invariant measure, this implies the Harris recurrence of $\{Z(t)\}_{t\ge 0}$ (Proposition X.3.9 in [27]), so that every Borel set B of a positive Haar measure is recurrent. Due to Propositions X.3.11 and X.2.2 in [27], for the proof of (11), it suffices to show that the functional

$$M_B(f) = \lim_{\beta\to\infty} \beta \int_{\boldsymbol{Q}_p} \mu(dx) E_x \int_0^{\infty} e^{-\beta t} f(x+Z(t)) I_B(x+Z(t))dt\,,$$

defined on bounded positive Borel measurable functions f on $\boldsymbol{Q}_p$, does not vanish. This follows from

$$M_B(f) = \lim_{\beta\to\infty} \beta \int_{\boldsymbol{Q}_p} \mu(dx) \int_0^\infty e^{-\beta t} p_t(I_B f)(x) dt = \int_B f(x)\mu(dx)$$

and $\mu(B) > 0$. □

The above properties of the local time make it possible to prove an analogue of the Engelbert-Schmidt zero-one law for our situation.

THEOREM 3. *Let f be a non-negative Borel function on $\boldsymbol{Q}_p$. The following properties are equivalent*:

(i) $P_0\{\int_0^t f(Z(s))\, ds < \infty,$ *for all* $t \geq 0\} > 0$.
(ii) $P_0\{\int_0^t f(Z(s))\, ds < \infty,$ *for all* $t \geq 0\} = 1$.
(iii) f *is locally integrable on* $\boldsymbol{Q}_p$.

4. Time change and moment estimate. In this section, we will discuss a weak solution of the stochastic differential equation

$$\begin{cases} dX(t) = b(X(t-))dZ(t)\,, \\ X(0) = x\,, \end{cases} \tag{12}$$

by taking some α-stable process on $\boldsymbol{Q}_p$. We will focus on the case that coefficient in the right-hand side is given by a $\boldsymbol{Q}_p$-valued Borel measurable function b defined on $\boldsymbol{Q}_p$.

DEFINITION 1. An $\{\mathcal{F}_t\}$-adapted stochastic process $\{X(t)\}_{t\geq 0}$ defined on a probability measure space $(\Omega, \mathcal{F}, P)$ is called a *solution of the stochastic differential equation* (12) if there exists an $\{\mathcal{F}_t\}$-adapted α-stable process $\{Z(t)\}_{t\geq 0}$ with $Z(0) = 0$ satisfying

$$X(t) = x + \int_0^t b(X_{s-})dZ(s)\,.$$

DEFINITION 2. A solution $\{X(t)\}_{t\geq 0}$ of equation (12) is said to be *trivial* if

$$P_0(X(t) = X(0) \text{ for all } t \geq 0) = 1\,.$$

DEFINITION 3. For $x \in \boldsymbol{Q}_p$, we say the coefficient b in (12) satisfies Condition (H) with respect to x if

$$\int_0^t \left(\int_{B(0,p^L)} \frac{1}{\|b(x+y)\|_p^\alpha} P(s,y)\mu(dy) \right) ds < \infty$$

for every integer L.

Here, we introduce an increasing process $\{C(t)\}_{t\geq 0}$ associated with the α-stable process $\{Z(t)\}_{t\geq 0}$ defined by

$$C(t) = \int_0^t \frac{1}{\|b(x+Z(s))\|_p^\alpha} ds\,.$$

LEMMA 4. *If $\alpha \geq 1$, then the increasing process $\{C(t)\}_{t\geq 0}$ satisfies the following*:

(i) $P_0(C(t) < \infty) = 1$ *for every* $t \geq 0$.
(ii) $P_0(\lim_{t\to\infty} C(t) = \infty) = 1$.

PROOF. For any integer M, we define an $\{\mathcal{F}_t\}$-stopping time $\sigma_M = \inf\{t > 0 \mid \|Z(t)\|_p \geq p^M\}$. Then we have

$$C(t) = C(t \wedge \sigma_M)1_{\{t<\sigma_M\}} + C(t)1_{\{t\geq\sigma_M\}} .$$

From the fact that $P(\sigma_M > 0) = 1$, we can derive

$$\begin{aligned} E_0[C(t \wedge \sigma_M)1_{\{t<\sigma_M\}}] &= E_0\left[1_{\{t<\sigma_M\}} \int_0^{t\wedge\sigma_M} \frac{1}{\|b(x + Z(s))\|_p^\alpha} ds\right] \\ &= E_0\left[1_{\{\sup_{0\leq s\leq t} \|Z(s)\|_p<p^M\}} \int_0^{t\wedge\sigma_M} \frac{1}{\|b(x + Z(s))\|_p^\alpha} ds\right] \\ &\leq \int_0^t \left(\int_{B(0,p^M)} \frac{1}{\|b(x + y)\|_p^\alpha} P(s, y)\mu(dy)\right) ds . \end{aligned}$$

Condition (H) imposed on the coefficient implies the finiteness of the right-hand side. Since $C(t \wedge \sigma_M)$ is equal to $C(t)$ on $\{t < \sigma_M\}$ and $\sigma_M \to \infty$ as $M \to \infty$, the first assertion has been proved.

Since $1/\|b(x + y)\|_p^\alpha > \varepsilon$ for all y with some $\varepsilon > 0$, we can take $B_\varepsilon = \{y \in \boldsymbol{Q}_p \mid 1/\|b(x + y)\|_p^\alpha \geq \varepsilon\}$ so that $\mu(B_\varepsilon) > 0$ is satisfied. Therefore, the second assertion follows from Theorem 2 and the estimate $\int_0^t 1/\|b(x + Z(s))\|_p^\alpha ds = \int_{\boldsymbol{Q}_p} L_t^y/\|b(x + y)\|_p^\alpha \mu(dy) \geq \varepsilon\nu(t, B_\varepsilon)$. □

Here, we define $\tau_t = \inf\{s \geq 0 \mid C(s) > t\}$ for every $t \geq 0$. Since Lemma 4 shows that $\|b(x + Z(s))\|_p > 0$ except Lebesgue measure zero set in $[0, \infty)$ for every $\omega \in \Omega$ and $x \in \boldsymbol{Q}_p$, we have

$$\tau_t = \int_0^{C_{\tau_t}} \|b(x + Z(\tau_s))\|_p^\alpha ds = \int_0^t \|b(x + Z(\tau_s))\|_p^\alpha ds .$$

Let us now introduce an $\{\mathcal{H}_t\}$-adapted time changed process $\{Y(t)\}_{t\geq0}$ defined by $Y(t) = Z(\tau_t)$, where $\{\mathcal{H}_t\}_{t\geq0}$ stands for the filtration given by $\mathcal{H}_t = \mathcal{F}_{\tau_t}$ for every $t \geq 0$. The objective of this section is demonstrating that $X(t) = x + Y(t)$ is a solution of the stochastic differential equation (12) by assuming b is locally bounded.

Since the ball $B(0, p^L)$ centered at the origin and with radius p^L is given as a disjoint union $\bigcup_i B(a_i, p^l)$ of balls centered at $a_i \in B(0, p^L)$, $i = 1, 2, \ldots, p^{L-l}$, we can define a Markov process $\{Z_{L,l}(t)\}_{t\geq0}$ by

$$Z_{L,l}(t) = \begin{cases} a_i & Z(t) \in B(a_i, p^l) , \\ p^{-(L+1)} & Z(t) \notin B(0, p^L) . \end{cases}$$

By applying [28, VI-28,3], we can obtain the Lévy system of the Markov process $\{Z_{L,l}(t)\}_{t\geq0}$. From this observation, we can derive that the Lévy system of the α-stable process is given by $H(t) = t$ and $N(x, dy) = (K/\|y - x\|_p^{1+\alpha})\mu(dy)$. This shows that, for any non-negative $\mathcal{B}_\Delta$ measurable function f on $(\boldsymbol{Q}_p)_\Delta$ satisfying $f(0) = 0$ and any initial probability measure m,

$$\tilde{A}^f(t) = K \int_{[0,t]} ds \int_{(\boldsymbol{Q}_p)_\Delta} \frac{f(y - Z(s-))}{\|y - Z(s-)\|_p^{1+\alpha}} \mu(dy)$$

is the dual predictable projection of $A_f(t) = \int_{[0,t]} \int_{(\boldsymbol{Q}_p)_\Delta} f(y)\lambda(\omega, ds, dy) = \sum_{s \le t} f(X(s) - X(s-))$ under P_m, where $(\boldsymbol{Q}_p)_\Delta$ stands for the one point compactification of $\boldsymbol{Q}_p$ with the topological Borel σ-field $\mathcal{B}_\Delta$ and $\lambda(\omega, dt, dy)$ stands for the jump measure $\sum_{s>0} 1_{\{Z(s,\omega)-Z(s-,\omega)\neq 0\}} \delta_{(s, Z(s,\omega)-Z(s-,\omega))}(dt, dy)$ of the α-stable process $\{Z(t)\}_{t\ge 0}$ with filtration $\{\mathcal{F}_t\}_{t\ge 0}$.

Consider a time changed jump measure

$$\tilde{\lambda} = \lambda(\omega, \cdot) \circ (\tilde{C}(t))^{-1} ,$$

with respect to $\tilde{C}(t) = (C(t), x)$ with $C(t) = \int_0^t 1/\|b(x + Z(s))\|^\alpha ds$. Then, another representation of $\{Y(t)\}_{t\ge 0}$ is obtained. Indeed, it is not difficult to see that $Y(t) = \int_{[0,t]} \int_{\boldsymbol{Q}_p} x \tilde{\lambda}(ds, dx)$.

Similarly to Lemma 2.20 in [35], we can show the following

LEMMA 5. *For almost all ω in Ω, $N_\omega = \{s \in [0, \infty) \mid b(X(s, \omega)) = b(x + Y(s, \omega)) = 0\}$ has Lebesgue measure zero.*

On the other hand, by performing time change given by $\tilde{C}(t)$, it turns out that the stochastic process determined by this jump measure $\tilde{\lambda}$ is characterized by the Lévy system given by $H_t = \int_0^t \|b(X(s))\|^\alpha ds$ and $N(x, dy) = (K/\|y - x\|_p^{1+\alpha})\mu(dy)$. Therefore, for any non-negative $\mathcal{B}_\Delta$ measurable function f on $(\boldsymbol{Q}_p)_\Delta$ and any initial probabaility measure m,

$$\tilde{A}_H^f(t) = K \int_{[0,t]} \|b(X(s))\|^\alpha ds \int_{(\boldsymbol{Q}_p)_\Delta} \frac{f(y - X(s-))}{\|y - X(s-)\|_p^{1+\alpha}} \mu(dy)$$

is the dual predictable projection of $A_H^f(t) = \sum_{s \le H_t} f(X(s) - X(s-))$ under P_m (see [28, VI-28,1]). By introducing a random measure $\tilde{\pi}(\omega, ds, dy) = \|b(X(s, \omega))\|_p^\alpha (K/\|y\|_p^{1+\alpha}) ds \mu(dy)$, we have $\tilde{A}_H^f(t) = \int_{[0,t]} \int_{(\boldsymbol{Q}_p)_\Delta} f(y)\tilde{\pi}(\omega, ds, dy)$. Accordingly, $\tilde{\pi}$ is the compensator of $\tilde{\lambda}$ in the sense given in [14].

Let us denote the topological Borel σ-field of $[0, \infty)$ by $\mathcal{B}[0, \infty)$ and the topological Borel σ-field of $\boldsymbol{Q}_p$ by $\mathcal{B}$. Now, we define a map β_ω from $([0, \infty) \times \boldsymbol{Q}_p, \mathcal{B}[0, \infty) \times \mathcal{B})$ to $([0, \infty) \times (\boldsymbol{Q}_p)_\Delta, \mathcal{B}[0, \infty) \times \mathcal{B}_\Delta)$ given by

$$\beta_\omega(s, y) = \left(\omega, s, \frac{y}{b(X(s-, \omega))}\right)$$

for every $\omega \in \Omega$.

LEMMA 6. *The random measure defined by $\Pi(\omega, dt, dy) = \beta_\omega(\tilde{\lambda}(\omega, dt, dy))$ is characterized by the compensator (intensity) $K\rho_\Delta$, where ρ_Δ stands for the measure on $(\boldsymbol{Q}_p)_\Delta$ given as $(1/\|y\|_p)\mu(dy)$ on $\boldsymbol{Q}_p$ and vanishing at Δ.*

PROOF. Thanks to Theorem 2.1(a) in [16], it suffices to show that $\beta_\omega(\tilde{\pi}) = (K/\|x\|^{1+\alpha})ds\mu(dx)$. Since $b(X(s-)) = b(X(s))$ is satisfied for almost all s with probability one, we have

$$\int_{[0,t]}\int_{(\boldsymbol{Q}_p)_\Delta} f(y)\beta_\omega(\tilde{\pi})(ds, dy)$$
$$= \int_{[0,t]} 1_{\{b(X(s-))\neq 0\}} \int_{(\boldsymbol{Q}_p)_\Delta} f\left(\frac{y}{b(X(s-,\omega))}\right)\|b(X(s-),\omega)\|_p^\alpha \frac{K}{\|y\|_p^{1+\alpha}}\mu(dy)$$

with probability one. By performing changing of the variables $z = y/b(X(s-,\omega))$ in the integral, we obatin

$$\int_{[0,t]}\int_{(\boldsymbol{Q}_p)_\Delta} f(y)\beta_\omega(\tilde{\pi})(ds, dy) = \int_{[0,t]} 1_{\{b(X(s-))\neq 0\}}ds \int_{(\boldsymbol{Q}_p)_\Delta} f(z)\frac{K}{\|z\|_p^{1+\alpha}}\mu(dz)$$
$$= \int_{[0,t]}\int_{(\boldsymbol{Q}_p)_\Delta} f(z)\frac{K}{\|z\|_p^{1+\alpha}}ds\mu(dz)\,. \qquad \square$$

Now, we define an $\{\mathcal{H}_t\}$-adapted stochastic process $\{Z^*(t)\}_{t\geq 0}$ given by

$$Z^*(t) = \int_{[0,t]}\int_{\boldsymbol{Q}_p} y\Pi(ds, dy)\,.$$

Then it turns out that $\{Z^*(t)\}_{t\geq 0}$ is an α-stable process. This is because the random measure Π admits the compensator $(K/\|z\|_p^{1+\alpha})ds\mu(dz)$.

For $\Omega_1(M;T) = \{\tau_T < \sigma_M\}$ and $\Omega_2(M;T) = \{\sup_{0\leq s\leq T}\|Z^*(s)\| \leq p^M\}$, we have $\lim_{M\to\infty} P_0(\Omega_1(M;T)) = \lim_{M\to\infty} P_0(\Omega_2(M;T)) = 1$. By modifying $\{b(X(t-)) \mid 0 \leq t \leq T\}$ and $\{Z^*(t) \mid 0 \leq t \leq T\}$ outside $\Omega(M;T) = \Omega_1(M;T) \cap \Omega_2(M;T)$ as the procedure established in Section 2, we can define the stochastic integral of $b(X(t-))$ with respect to the α-stable process $\{Z^*(t)\}_{t\geq 0}$. Similarly to Lemma 2.26 in [35], we have the following assertion on the stochastic integral $M(t) = \int_{[0,t]} b(X(s-))dZ^*(s)$:

LEMMA 7. $P_0(Y(t) = M(t)$ *for all* $t) = 1$.

Consequently, the following assertion on solutions of (12) is now concluded.

THEOREM 4. *If the coefficient b is locally bounded and if* (H) *is satisfied with respect to x, then the stochastic differential equation* (12) *admits a non-trivial solution.*

Here, we note that the α-stable process $\{Z(t)\}_{t\geq 0}$ on $\boldsymbol{Q}_p$ has an explicit description of the density $P(t,x)$ of transition probability kernel. Indeed, we firstly recall that

$$P_0(\|Z(t)\| \leq p^m) = \frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i}\exp(-p^{-\alpha m}t)\,,$$

is obtained by the expression of transition probability found by Yasuda (1996, [3]). Then, by denoting the right-hand side by $P_m(t)$, we immediately see that $P(t, y)$ is given as $((p-1)/p^{m+1})(P_m(t) - P_{m-1}(t))$ on $B(0, p^m) \setminus B(0, p^{m-1})$.

We have the following assertion which gives a sufficient condition for Condition (H).

PROPOSITION 3. *For the α-stable process with $\alpha \geq 1$ on $\boldsymbol{Q}_p$, the integrability condition*

$$\int_{B(0,1)} \frac{1}{\|b(x+y)\|_p^{\lambda}} \mu(dy) < \infty$$

with some positive number λ satisfying $\lambda > \alpha(1+\alpha)$ implies

$$\int_0^T \left\{ \int_{B(0,1)} \frac{1}{\|b(x+y)\|_p^{\alpha}} P(t, y)\mu(dy) \right\} dt < \infty .$$

PROOF. For any positive real number q, one sees that

$$\begin{aligned}
\int_{B(0,1)} P(t, y)^q \mu(dy) &= \sum_{m=-\infty}^{0} p^m (P_m(t) - P_{m-1}(t))^q \frac{1}{p^{mq}} \leq \sum_{m=-\infty}^{0} p^m P_m(t)^q \frac{1}{p^{mq}} \\
&\leq C_1 \sum_{m=-\infty}^{0} p^m \left(p^{mq} + \left(\sum_{i=0}^{-m} p^{-i} \exp(-p^{-\alpha(m+i)} t) \right)^q \right) \frac{1}{p^{mq}} \\
&\leq C_1 \sum_{m=-\infty}^{0} p^m \left(p^{mq} + \left(\sum_{i=0}^{-m} p^{-i} \frac{1}{1 + p^{-\alpha(m+i)} t} \right)^q \right) \frac{1}{p^{mq}} \\
&\leq C_1 \sum_{m=-\infty}^{0} p^m \left(p^{mq} + \left(\sum_{i=0}^{-m} p^{-i} \frac{p^{\alpha(m+i)}}{p^{\alpha(m+i)} + t} \right)^q \right) \frac{1}{p^{mq}} \\
&\leq C_1 \sum_{m=-\infty}^{0} p^m \left(p^{mq} + \left(\sum_{i=0}^{-m} \frac{p^{\alpha m}}{p^{\alpha m} + t} \right)^q \right) \frac{1}{p^{mq}} \\
&\leq C_2 \sum_{m=-\infty}^{0} p^m \left(p^{mq} + \left(\frac{\|m\|^q p^{mq}}{(p^{\alpha m} + t)^q} \right) \right) \frac{1}{p^{mq}} \\
&\leq C_2 \sum_{m=-\infty}^{0} p^m \left(1 + \left(\frac{\|m\|^q}{(p^{\alpha m} + t)^q} \right) \right) \\
&\leq C_2 \sum_{m=-\infty}^{0} p^m \left(1 + \left(\frac{1}{(p^{\alpha m} + t)^q} \right) \right)
\end{aligned}$$

with some positive constants C_1 and C_2, where $\alpha \geq 1$. Accordingly, it is not difficult to see that

$$\begin{aligned}\int_0^T \left\{ \int_{B(0,1)} P(t,y)^q \mu(dy) \right\}^{1/q} dt &\leq C_3 \int_0^T \left\{ \sum_{m=-\infty}^{0} p^m \left(1 + \left(\frac{1}{(p^{\alpha m}+t)^q} \right) \right) \right\}^{1/q} dt \\ &\leq C_3 \left(\int_0^T \sum_{m=-\infty}^{0} p^m \left(1 + \left(\frac{1}{(p^{\alpha m}+t)^q} \right) \right) dt \right)^{1/q} \\ &\leq C_3 \left(T + \sum_{m=-\infty}^{0} p^m \int_0^T \frac{1}{(p^{\alpha m}+t)^q} dt \right)^{1/q} \\ &\leq C_3 \left(T + \sum_{m=-\infty}^{0} p^{m(1-\alpha(q-1))} \right)^{1/q}\end{aligned}$$

with some positive constant C_3, where the right-hand side converges, when $(\alpha+1)/\alpha > q$.

Since $\lambda > \alpha(1+\alpha)$, a real number q' is introduced by $q' = \lambda/\alpha$ so that $q' > \alpha + 1$ is satisfied. Then, it turns out that

$$\begin{aligned}&\int_0^T \left\{ \int_{B(0,1)} \frac{1}{\|b(x+y)\|_p^{\alpha}} P(t,y)\mu(dy) \right\} dt \\ &\leq \left(\int_{B(0,1)} \frac{1}{\|b(x+y)\|_p^{\alpha q'}} \mu(dy) \right)^{1/q'} \left(\int_0^T \left\{ \int_{B(0,1)} P(t,y)^q \mu(dy) \right\}^{1/q} dt \right) \\ &\leq \left(\int_{B(0,1)} \frac{1}{\|b(x+y)\|_p^{\lambda}} \mu(dy) \right)^{1/q'} \left(\int_0^T \left\{ \int_{B(0,1)} P(t,y)^q \mu(dy) \right\}^{1/q} dt \right) < \infty\end{aligned}$$

with the positive number q given by $q = q'/(q'-1)$. This is because $\lambda > \alpha(1+\alpha)$ implies $(\alpha+1)/\alpha > q$. □

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